Publ. RIMS Kyoto Univ. **59** (2023), 57–88
DOI 10.4171/PRIMS/59-1-2

On Auslander-Type Conditions of Modules

by

Zhaoyong Huang

Abstract

For a left and right Noetherian ring R, we give some equivalent characterizations for ${}_RR$ satisfying the Auslander condition in terms of the flat (resp. injective) dimensions of the terms in a minimal injective coresolution (resp. flat resolution) of left R-modules. Furthermore, we prove that for an artin algebra R satisfying the Auslander condition, R is Gorenstein if and only if the subcategory consisting of finitely generated modules satisfying the Auslander condition is contravariantly finite. As applications, we get some equivalent characterizations of Auslander–Gorenstein rings and Auslander-regular rings.

2020 Mathematics Subject Classification: Primary 16E65; Secondary 16E10, 18G25, 16G10.
Keywords: Auslander-type conditions, flat dimension, injective dimension, minimal flat resolutions, minimal injective coresolutions, Gorenstein algebras, contravariantly finite subcategories.

§1. Introduction

It is well known that commutative Gorenstein rings are fundamental and important research objects in commutative algebra and algebraic geometry. Bass proved in [B2] that a commutative Noetherian ring R is a Gorenstein ring (that is, the self-injective dimension of R is finite) if and only if the flat dimension of the ith term in a minimal injective coresolution of R as an R-module is at most $i-1$ for any $i \geq 1$. In the non-commutative case, Auslander proved that this condition is left–right symmetric ([FGR, Thm. 3.7]); in this case, R is said to satisfy the *Auslander condition*. Motivated by this philosophy, Huang and Iyama introduced the notion of Auslander-type conditions of rings as follows. For any $m, n \geq 0$, a left and right Noetherian ring is said to be $G_n(m)$ if the flat dimension of the ith term in a minimal injective coresolution of R_R is at most $m+i-1$ for any

Communicated by S. Mochizuki. Received September 8, 2020. Revised November 29, 2020.

Z. Y. Huang: Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P.R. China;
e-mail: huangzy@nju.edu.cn

$1 \leq i \leq n$. Auslander-type conditions are non-commutative analogs of commutative Gorenstein rings. Such conditions play a crucial role in homological algebra, representation theory of algebras and non-commutative algebraic geometry ([AR3, AR4, Bj, EHIS, FGR, H1, HI, IS, I1, I2, I3, I4, M, Ro, S, W] and so on). In particular, by constructing an injective coresolution of the last term in an exact sequence of finite length from that of the other terms, Miyachi obtained in [M] an equivalent characterization of the Auslander condition in terms of the relation between the flat dimensions of any module and its injective envelope. Then he got some properties of Auslander–Gorenstein rings and Auslander-regular rings.

Note that a commutative Noetherian ring satisfies the Auslander condition if and only if it is Gorenstein ([B2]). Auslander and Reiten conjectured in [AR3] that an artin algebra satisfying the Auslander condition is Gorenstein. This conjecture is situated between the well-known Nakayama conjecture and the finitistic dimension conjecture. For an artin algebra R, the Nakayama conjecture states that R is self-injective if all terms in a minimal injective coresolution of ${}_RR$ are projective, and the finitistic dimension conjecture states that the supremum of the projective dimensions of all finitely generated left R-modules with finite projective dimension is finite. All of these conjectures remain open.

Based on the above-mentioned details, in this paper we will introduce modules satisfying Auslander-type conditions and study the homological properties of such modules. By using the obtained properties we get some equivalent characterizations of rings satisfying the Auslander condition, Auslander–Gorenstein rings and Auslander-regular rings respectively. Then we study when an artin algebra satisfying the Auslander condition is Gorenstein.

Throughout this paper, R is an associative ring with identity, $\operatorname{Mod} R$ is the category of left R-modules and $\operatorname{mod} R$ is the category of finitely generated left R-modules. This paper is organized as follows.

In Section 2 we give some terminology and some preliminary results.

Let $M \in \operatorname{Mod} R$. We use $\operatorname{fd}_R M$, $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the flat, projective and injective dimensions of M, respectively. Bican, El Bashir and Enochs [BEE, Thm. 3] proved that every R-module has a flat cover. For an R-module M, we call an exact sequence

$$\cdots \longrightarrow F_i \xrightarrow{\pi_i} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

a *proper flat resolution* of M if $\pi_i \colon F_i \to \operatorname{Im} \pi_i$ is a flat precover of $\operatorname{Im} \pi_i$ for any $i \geq 0$. Furthermore, we call the exact sequence

$$\cdots \longrightarrow F_i(M) \xrightarrow{\pi_i(M)} \cdots \xrightarrow{\pi_2(M)} F_1(M) \xrightarrow{\pi_1(M)} F_0(M) \xrightarrow{\pi_0(M)} M \longrightarrow 0$$

a *minimal flat resolution* of M, where $\pi_i(M)\colon F_i(M) \to \operatorname{Im}\pi_i(M)$ is a flat cover of $\operatorname{Im}\pi_i(M)$ for any $i \geq 0$. It is easy to verify that $\operatorname{fd}_R M \leq n$ if and only if $F_{n+1}(M) = 0$. In addition, we use

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to \cdots$$

to denote a minimal injective coresolution of M.

In Section 3, by using some techniques of direct limits and transfinite induction, we prove the following theorem.

Theorem 1.1 (Theorem 3.1). *Let R be a left Noetherian ring and $n, k \geq 0$, and let $\{M_i\}_{i\in I}$ be a family of left R-modules and $M = \varinjlim_{i\in I} M_i$, where I is a directed index set. If $\operatorname{fd}_R E^n(M_i) \leq k$ for any $i \in I$, then $\operatorname{fd}_R E^n(M) \leq k$.*

For any $m, n \geq 0$, we introduce in Section 4 the notion of modules satisfying the Auslander-type conditions $G_n(m)$; in particular, a module M in $\operatorname{Mod} R$ is said to satisfy the *Auslander condition* if $\operatorname{fd}_R E^{i-1}(M) \leq i-1$ for any $i \geq 1$. By using Theorem 1.1 and the constructions of (co)proper (co)resolutions of modules in [H2] we will investigate the homological behavior of modules satisfying Auslander-type conditions in terms of the relation between the flat (resp. injective) dimensions of modules and their injective envelopes (resp. flat covers). We prove the following theorem.

Theorem 1.2 (Theorem 4.9). *Let R be a left and right Noetherian ring. Then the following statements are equivalent:*

(1) *${}_RR$ satisfies the Auslander condition.*
(2) *Every flat left R-module satisfies the Auslander condition.*
(3) $\operatorname{fd}_R E^i(M) \leq \operatorname{fd}_R M + i$ *for any* $M \in \operatorname{Mod} R$ *and* $i \geq 0$.
(4) $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M$ *for any* $M \in \operatorname{Mod} R$.
(5) $\operatorname{id}_R F_i(Q) \leq i$ *for any injective left R-module Q and* $i \geq 0$.
(6) $\operatorname{id}_R F_i(M) \leq \operatorname{id}_R M + i$ *for any* $M \in \operatorname{Mod} R$ *and* $i \geq 0$.
(7) $\operatorname{id}_R F_0(M) \leq \operatorname{id}_R M$ *for any* $M \in \operatorname{Mod} R$.
$(\mathrm{i})^{\mathrm{op}}$ *The opposite version of* (i) *($1 \leq \mathrm{i} \leq 7$).*

As applications of this theorem, we obtain some equivalent characterizations of Auslander–Gorenstein rings and Auslander-regular rings, respectively (Theorems 4.15 and 4.18).

In Section 5 we first obtain the approximation presentations of a given module relative to the subcategory of modules satisfying the Auslander condition and that of modules with finite injective dimension respectively. Then we establish the connection between the Auslander and Reiten conjecture mentioned above with the contravariant finiteness of some certain subcategories as follows.

Theorem 1.3 (Theorem 5.8). *Let R be an artin algebra satisfying the Auslander condition. Then the following statements are equivalent:*

(1) *R is Gorenstein.*

(2) *The subcategory consisting of finitely generated modules satisfying the Auslander condition is contravariantly finite.*

(3) *The subcategory consisting of finitely generated modules which are n-syzygy for any $n \geq 1$ is contravariantly finite.*

(4) *The subcategory consisting of finitely generated modules which are n-torsion-free for any $n \geq 1$ is contravariantly finite.*

As a consequence, we get that an artin algebra is Auslander-regular if and only if the subcategory consisting of projective modules and that consisting of modules satisfying the Auslander condition coincide (Theorem 5.9).

§2. Preliminaries

In this section we give some terminology and some preliminary results.

Definition 2.1 ([E]). Let $\mathscr{C} \subseteq \mathscr{D}$ be full subcategories of $\operatorname{Mod} R$. A homomorphism $f\colon C \to D$ in $\operatorname{Mod} R$ with $C \in \mathscr{C}$ and $D \in \mathscr{D}$ is said to be a *$\mathscr{C}$-precover* of D if for any homomorphism $g\colon C' \to D$ in $\operatorname{Mod} R$ with $C' \in \mathscr{C}$, there exists a homomorphism $h\colon C' \to C$ such that the following diagram commutes:

$$\begin{tikzcd} & C' \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ C \arrow[r, "f"] & D. \end{tikzcd}$$

The homomorphism $f\colon C \to D$ is said to be *right minimal* if an endomorphism $h\colon C \to C$ is an automorphism whenever $f = fh$. A $\mathscr{C}$-precover $f\colon C \to D$ is called a *$\mathscr{C}$-cover* if f is right minimal. Dually, the notions of a *$\mathscr{C}$-preenvelope*, a *left minimal homomorphism* and *a $\mathscr{C}$-envelope* are defined. Following Auslander and Reiten's terminology in [AR1], for a module over an artin algebra, a $\mathscr{C}$-(pre)cover and a $\mathscr{C}$-(pre)envelope are called a (*minimal*) *right $\mathscr{C}$-approximation* and a (*minimal*) *left $\mathscr{C}$-approximation*, respectively. If each module in $\mathscr{D}$ has a right (resp. left) $\mathscr{C}$-approximation, then $\mathscr{C}$ is called *contravariantly finite* (resp. *covariantly finite*) in $\mathscr{D}$.

We use $\mathscr{F}^0(\operatorname{Mod} R)$ and $\mathscr{I}^0(\operatorname{Mod} R)$ to denote the subcategories of $\operatorname{Mod} R$ consisting of flat modules and injective modules, respectively. Recall that an $\mathscr{F}^0(\operatorname{Mod} R)$-(pre)cover and an $\mathscr{I}^0(\operatorname{Mod} R)$-(pre)envelope are called a *flat (pre) cover* and an *injective (pre)envelope*, respectively.

Lemma 2.2 ([X, Thm. 1.2.9]). *Let $\mathscr{C}$ be a full subcategory of $\operatorname{Mod} R$ closed under direct products. If $f_i\colon C_i \to M_i$ is a $\mathscr{C}$-precover of M_i in $\operatorname{Mod} R$ for any $i \in I$, where I is an index set, then $\prod_{i\in I} f_i\colon \prod_{i\in I} C_i \to \prod_{i\in I} M_i$ is a $\mathscr{C}$-precover of $\prod_{i\in I} M_i$.*

We write $(-)^+ := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Lemma 2.3 ([EH, Thm. 3.7]). *The following statements are equivalent:*

(1) *R is a left Noetherian ring.*

(2) *A monomorphism $f\colon A \rightarrowtail E$ in $\operatorname{Mod} R$ is an injective preenvelope of A if and only if $f^+\colon E^+ \twoheadrightarrow A^+$ is a flat precover of A^+ in $\operatorname{Mod} R^{\mathrm{op}}$.*

Lemma 2.4.

(1) *([F, Thm. 2.1]) For any $M \in \operatorname{Mod} R$, $\operatorname{fd}_R M = \operatorname{id}_{R^{\mathrm{op}}} M^+$.*

(2) *([F, Thm. 2.2]) If R is a right Noetherian ring, then $\operatorname{fd}_R N^+ = \operatorname{id}_{R^{\mathrm{op}}} N$ for any $N \in \operatorname{Mod} R^{\mathrm{op}}$.*

Recall that $\operatorname{Fin.dim} R = \sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \operatorname{pd}_R M < \infty\}$. Observe that the first assertion in the following result was proved by Bass in [B1, Cor. 5.5] when R is a commutative Noetherian ring.

Lemma 2.5.

(1) *For a left Noetherian ring R, we have*

$$\operatorname{id}_R R \geq \sup\{\operatorname{fd}_R M \mid M \in \operatorname{Mod} R \textit{ with } \operatorname{fd}_R M < \infty\}.$$

(2) *For a left and right Noetherian ring R, we have*

$$\operatorname{id}_R R \geq \sup\{\operatorname{id}_{R^{\mathrm{op}}} N \mid N \in \operatorname{Mod} R^{\mathrm{op}} \textit{with } \operatorname{id}_{R^{\mathrm{op}}} N < \infty\}.$$

Proof. (1) Let $\operatorname{id}_R R = n$ $(< \infty)$. Then $\operatorname{Fin.dim} R \leq n$ by [B1, Prop. 4.3]. It follows from [J1, Prop. 6] that the projective dimension of any flat left R-module is finite. So, if $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M < \infty$, then $\operatorname{pd}_R M < \infty$ and $\operatorname{pd}_R M \leq n$. Thus we have $\operatorname{fd}_R M$ $(\leq \operatorname{pd}_R M) \leq n$.

(2) By [B1, Prop. 4.1] we have $\sup\{\operatorname{fd}_R M \mid M \in \operatorname{Mod} R \text{ with } \operatorname{fd}_R M < \infty\} = \sup\{\operatorname{id}_{R^{\mathrm{op}}} N \mid N \in \operatorname{Mod} R^{\mathrm{op}} \text{ with } \operatorname{id}_{R^{\mathrm{op}}} N < \infty\}$. So the assertion follows from (1). $\square$

§3. Flat dimension of E^n of direct limits

In this section, R is a left Noetherian ring. The aim of this section is to prove the following theorem.

Theorem 3.1. *Let $n, k \geq 0$ and let $\{M_i\}_{i \in I}$ be a family of left R-modules, where I is a directed index set. If $M = \varinjlim_{i \in I} M_i$ and $\operatorname{fd}_R E^n(M_i) \leq k$ for any $i \in I$, then $\operatorname{fd}_R E^n(M) \leq k$.*

By [R, Thm. 5.40], every flat left R-module is a direct limit (over a directed index set) of finitely generated free left R-modules. So by Theorem 3.1 we have the following corollary.

Corollary 3.2. *We have $\operatorname{fd}_R E^n({}_R R) = \sup\{\operatorname{fd}_R E^n(F) \mid F \in \operatorname{Mod} R$ is flat$\}$ for any $n \geq 0$.*

Before giving the proof of Theorem 3.1 we need some preliminaries.

Definition 3.3 ([J2]). Let β be an ordinal number. A set S is called a *continuous union* of a family of subsets indexed by ordinals α with $\alpha < \beta$ if for each such α we have a subset $S_\alpha \subset S$ such that if $\alpha \leq \alpha'$ then $S_\alpha \subset S_{\alpha'}$, and such that if $\gamma < \beta$ is a limit ordinal then $S_\gamma = \bigcup_{\alpha < \gamma} S_\alpha$.

A main tool in our proof is the next result.

Lemma 3.4 ([J2, Lem. 1.4]). *If I is an infinite directed index set, then for some ordinal β, I can be written as a continuous union $I = \bigcup_{\alpha < \beta} I_\alpha$, where each I_α is a directed index set with the order induced by that of I and where $|I_\alpha| < |I|$ for each $\alpha < \beta$.*

This result will be useful since it will allow us to rewrite a direct limit as a well-ordered direct limit. So if $M = \varinjlim_{i \in I} M_i$ with I infinite, then write $I = \bigcup_{\alpha < \beta} I_\alpha$ as above, and put $M_\alpha = \varinjlim_{i \in I_\alpha} M_i$. Hence if $\alpha \leq \alpha' < \beta$, since $I_\alpha \subset I_{\alpha'}$ we have an obvious map $M_\alpha \to M_{\alpha'}$. These maps then give us a direct system $\{M_\alpha\}_{\alpha < \beta}$. Clearly then $\varinjlim_{\alpha < \beta} M_\alpha = \varinjlim_{i \in I} M_i$.

Proposition 3.5. *Let κ be an ordinal number and $\{M_\alpha, f_{\alpha\beta} \colon M_\alpha \to M_\beta \mid \alpha \leq \beta < \kappa\}$ a direct system of left R-modules. If*

$$\zeta_\alpha := 0 \to M_\alpha \to E^0(M_\alpha) \to E^1(M_\alpha) \to \cdots$$

is a minimal injective coresolution of M_α in $\operatorname{Mod} R$ for each α, then these exact sequences ζ_α are the members of a direct system indexed by $\alpha < \kappa$ in such a way that if $\alpha \leq \beta < \kappa$, the map from the sequence indexed by α into that indexed by β agrees with the original map $f_{\alpha\beta} \colon M_\alpha \to M_\beta$.

Proof. We only need to construct a direct system $\{\zeta_\alpha, F_{\alpha\beta}\colon \zeta_\alpha \to \zeta_\beta \mid \alpha \leq \beta < \kappa\}$ indexed by κ, consisting of complexes ζ_α of minimal injective coresolution of M_α and system maps $F_{\alpha\beta}\colon \zeta_\alpha \to \zeta_\beta$, where $F_{\alpha\beta}$ is a sequence of maps $(f_{\alpha\beta}, f^0_{\alpha\beta}, f^1_{\alpha\beta}, \ldots)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & M_\alpha \arrow[r] \arrow[d, "f_{\alpha\beta}"] & E^0(M_\alpha) \arrow[r] \arrow[d, dashed, "f^0_{\alpha\beta}"] & E^1(M_\alpha) \arrow[r] \arrow[d, dashed, "f^1_{\alpha\beta}"] & \cdots \\ 0 \arrow[r] & M_\beta \arrow[r] & E^0(M_\beta) \arrow[r] & E^1(M_\beta) \arrow[r] & \cdots \end{tikzcd} \tag{3.1}$$

is commutative, and the original map in $F_{\alpha\beta}$ is $f_{\alpha\beta}\colon M_\alpha \to M_\beta$.

Next we will give the construction of $F_{\alpha\beta}\colon \zeta_\alpha \to \zeta_\beta$, $\alpha \leq \beta < \kappa$ in (3.1) by transfinite induction on $\beta < \kappa$.

(1) For the successional case, let $\beta + 1 < \kappa$. We can form a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_\beta \arrow[r] \arrow[d, "f_{\beta,\beta+1}"] & E^0(M_\beta) \arrow[r] \arrow[d, dashed, "f^0_{\beta,\beta+1}"] & E^1(M_\beta) \arrow[r] \arrow[d, dashed, "f^1_{\beta,\beta+1}"] & \cdots \\ 0 \arrow[r] & M_{\beta+1} \arrow[r] & E^0(M_{\beta+1}) \arrow[r] & E^1(M_{\beta+1}) \arrow[r] & \cdots . \end{tikzcd} \tag{3.2}$$

Let $F_{\beta,\beta+1} = (f_{\beta,\beta+1}, f^0_{\beta,\beta+1}, f^1_{\beta,\beta+1}, \ldots)\colon \zeta_\beta \to \zeta_{\beta+1}$. Therefore, $F_{\alpha,\beta+1} = F_{\beta,\beta+1}F_{\alpha\beta}$, $\alpha < \beta$, are the desired maps in (3.1).

(2) For the limit case, let $\beta < \kappa$ be a limit ordinal. By induction, assume $\{\zeta_\alpha, F_{\alpha\gamma}\colon \zeta_\alpha \to \zeta_\gamma \mid \alpha \leq \gamma < \beta\}$ is the desired direct subsystem in (3.1). Taking the direct limit, we get the following commutative diagram:

$$\begin{tikzcd} \zeta_\alpha : \arrow[d, "F_\alpha"] & 0 \arrow[r] & M_\alpha \arrow[r] \arrow[d] & E^0(M_\alpha) \arrow[r] \arrow[d, dashed] & E^1(M_\alpha) \arrow[r] \arrow[d, dashed] & \cdots \\ \varinjlim_{\alpha<\beta} \zeta_\alpha : & 0 \arrow[r] & \varinjlim_{\alpha<\beta} M_\alpha \arrow[r] & \varinjlim_{\alpha<\beta} E^0(M_\alpha) \arrow[r] & \varinjlim_{\alpha<\beta} E^1(M_\alpha) \arrow[r] & \cdots , \end{tikzcd}$$

where F_α is the limit map such that $F_\alpha = F_\gamma F_{\alpha\gamma}$ for any $\alpha \leq \gamma < \beta$. Since R is left Noetherian, any direct limit of injective left R-modules is injective by [B1, Thm. 1.1]. So $\varinjlim_{\alpha<\beta} \zeta_\alpha$ is in fact an injective coresolution of $\varinjlim_{\alpha<\beta} M_\alpha$. We have a map $f_\beta\colon \varinjlim_{\alpha<\beta} M_\alpha \to M_\beta$ given by the maps $f_{\alpha\beta}\colon M_\alpha \to M_\beta$. As the construction in (3.2), we have a map $F_\beta\colon \varinjlim \zeta_\alpha \to \zeta_\beta$ such that the original in F_β is the map f_β. So, composing F_β with F_α, we get maps $F_{\alpha\beta} = F_\beta F_\alpha\colon \zeta_\alpha \to \zeta_\beta$. It follows that $F_{\alpha\beta} = F_{\gamma\beta}F_{\alpha\gamma}$ for any $\alpha \leq \gamma < \beta$. By transfinite induction, this completes the construction. $\square$

Note that this result gives that if ζ is an injective coresolution of M, then $\zeta \cong \varinjlim_{\alpha<\beta} \zeta_\alpha$. In particular, this gives that $E^n(M) \cong \varinjlim_{\alpha<\beta} E^n(M_\alpha)$. This then gives that if $\mathrm{fd}_R\, E^n(M_\alpha) \leq k$ for each α then $\mathrm{fd}_R\, E^n(M) \leq k$. In other words, Theorem 3.1 holds true when our direct system is over the well-ordered index set of $\alpha < \beta$ for some ordinal β.

Proof of Theorem 3.1. We proceed by transfinite induction on $|I|$. So to begin the induction we suppose that $|I| = \aleph_0$ (the first infinite cardinal number). Then I is countable, so we suppose $I = \{i_n \mid n \in \mathbb{N}\}$ with $\mathbb{N}$ the set of non-negative integers. We construct a sequence $j_0, j_1, j_2, \ldots$ of elements in I by letting $j_0 = i_0$. Then we choose j_1 so that $j_1 \geq j_0, i_1$. So in general we choose j_n so that $j_n \geq j_{n-1}, i_n$. Then let $J = \{j_n \mid n \in \mathbb{N}\}$. We have that J is well ordered and is clearly a cofinal subset of I. Hence $M = \varinjlim_{i\in I} M_i = \varinjlim_{j\in J} M_j$. Since J is well ordered, we have $E^n(M) = \varinjlim_{j\in J} E^n(M_j)$. So the assumption that $\mathrm{fd}_R\, E^n(M_j) \leq k$ for each j gives $\mathrm{fd}_R\, E^n(M) \leq k$.

Now we make the induction hypothesis and assume $|I| > \aleph_0$. We appeal to Lemma 3.4 and write $I = \bigcup_{\alpha<\beta} I_\alpha$ as in that lemma. Then $M = \varinjlim_{\alpha<\beta} M_\alpha$. We have that M_α is the limit over I_α. But $|I_\alpha| < |I|$, so the assertion holds true for direct limits over I_α by the induction hypothesis. This means that we have $\mathrm{fd}_R\, M_\alpha \leq k$ for each α. Because the system $\{M_\alpha\}_{\alpha<\beta}$ is over a well-ordered index set of indices, we get that $\mathrm{fd}_R\, E^n(M_\alpha) \leq k$ for each α, which gives the assertion that $\mathrm{fd}_R\, E^n(M) \leq k$. $\square$

Remark 3.6. The same techniques show that, for a given $n \geq 0$, if

$$0 \to M_\alpha \to E^0(M_\alpha) \to E^1(M_\alpha) \to \cdots \to E^{n-1}(M_\alpha) \to C^n(M_\alpha) \to 0$$

is a partial minimal injective coresolution of M_α with $\mathrm{fd}_R\, C^n(M_\alpha) \leq k$ for each α, then we get $\mathrm{fd}_R\, C^n(M) \leq k$, where

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^{n-1}(M) \to C^n(M) \to 0$$

is a partial minimal injective coresolution of M.

§4. Modules satisfying the Auslander-type conditions

As a generalization of rings satisfying the Auslander condition, Huang and Iyama introduced in [HI] the notion of rings satisfying Auslander-type conditions. Now we introduce the notion of modules satisfying the Auslander-type conditions as follows.

Definition 4.1. Let $M \in \operatorname{Mod} R$ and let $m, n \geq 0$. Then M is said to be $G_n(m)$ if $\operatorname{fd}_R E^i(M) \leq m+i$ for any $0 \leq i \leq n-1$, and M is said to be $G_\infty(m)$ if it is $G_n(m)$ for all n. In particular, M is said to satisfy the *Auslander condition* if it is $G_\infty(0)$.

Recall from [FGR] that a left and right Noetherian ring R is called *Auslander's n-Gorenstein* if $\operatorname{fd}_R E^i({}_RR) \leq i$ for any $0 \leq i \leq n-1$, and R is said to satisfy the *Auslander condition* if it is Auslander's n-Gorenstein for all n.

Example 4.2. Let R be a left and right Noetherian ring. Then we have the following:

(1) ${}_RR$ is $G_n(m)$ if and only if R is $G_n(m)^{\mathrm{op}}$ in the sense of Huang and Iyama in [HI].

(2) ${}_RR$ is $G_n(0)$ if and only if R is Auslander's n-Gorenstein. Note that the notion of Auslander's n-Gorenstein rings (and hence that of the Auslander condition) is left–right symmetric ([FGR, Thm. 3.7]). So R satisfies the Auslander condition if and only if both ${}_RR$ and R_R satisfy the Auslander condition. However, in general, the notion of R being $G_n(m)$ is not left–right symmetric when $m \geq 1$ ([AR4, HI]).

(3) Let $\operatorname{id}_{R^{\mathrm{op}}} R = m$ $(< \infty)$. Then $\operatorname{fd}_R E \leq m$ for any injective left R-module E by [I, Prop. 1]. So any module in $\operatorname{Mod} R$ is $G_\infty(m)$.

(4) Let K be an algebraically closed field, and let Q be the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n+1$$

and $R = KQ/J^2$, where J is the Jacobson radical of KQ. Then $\operatorname{gl.dim} R = n$, $E^j(R)$ is projective for any $0 \leq j \leq n-1$ and $\operatorname{pd}_R E^n(R) = n$. The Auslander–Reiten quiver of $\operatorname{mod} R$ is

$$\begin{tikzcd}[column sep=small]
& P(2) \arrow[dr] & & P(3) \arrow[dr] & & \cdots \arrow[dr] & & P(n) \arrow[dr] & & P(n+1) \arrow[dr] & \\
S(1)\,(=P(1)) \arrow[ur] \arrow[rr, dotted, no head] & & S(2) \arrow[ur] \arrow[rr, dotted, no head] & & S(3) \arrow[ur] & & S(n-1) \arrow[ur] \arrow[rr, dotted, no head] & & S(n) \arrow[ur] \arrow[rr, dotted, no head] & & S(n+1),
\end{tikzcd}$$

where $P(i)$ and $S(i)$ are the projective and simple modules corresponding to the vertex i respectively for any $1 \leq i \leq n+1$. By [HZ, Thm. 4.8 and Cor. 4.9] we have $\operatorname{pd}_R S(i) + \operatorname{id}_R S(i) = n$ for any $1 \leq i \leq n+1$. In the minimal injective coresolution

$$0 \to S(i) \to E^0(S(i)) \to E^1(S(i)) \to \cdots \to E^{n-i+1}(S(i)) \to 0$$

of $S(i)$ in $\operatorname{mod} R$, we have that $E^j(S(i))$ is projective and $\operatorname{pd}_R E^{n-i+1}(S(i)) = n$ for any $1 \leq i \leq n+1$ and $0 \leq j \leq n-i$. So $S(1)$ is $G_{n+1}(0)$ and hence $G_\infty(0)$, and $S(i)$ is both $G_{n-i+1}(0)$ and $G_\infty(i-1)$ for any $2 \leq i \leq n+1$.

(5) Let K be an algebraically closed field, and let Q be the quiver

$$\begin{tikzcd}
5 \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r] & 3 \arrow[r] \arrow[d] & 2 \arrow[r, shift left] & 1 \arrow[l, shift left] \\
& & 6 \arrow[d] & & \\
& & 7 \arrow[d] & & \\
& & \vdots \arrow[d] & & \\
& & k+5 \arrow[d] & & \\
& & k+6 & &
\end{tikzcd}$$

and $R = KQ/J^2$ with $n \geq 1$. We use $P(i)$, $I(i)$ and $S(i)$ to denote the projective, injective and simple modules corresponding to the vertex i respectively for any $1 \leq i \leq k+6$. Then we have

(5.1) For any $1 \leq i \leq 5$,

$$0 \to P(1) \to I(2) \to I(3) \to I(4) \to I(5) \to I(4) \to \cdots,$$
$$0 \to P(2) \to I(1) \to 0,$$
$$0 \to P(3) \to I(2) \oplus I(6) \to I(3) \to I(4) \to I(5) \to I(4) \to I(5) \to \cdots,$$
$$0 \to P(4) \to I(3) \oplus I(5) \to I(4) \to I(5) \to I(4) \to \cdots,$$
$$0 \to P(5) \to I(4) \to 0$$

are minimal injective coresolutions of $P(i)$ respectively. So $\operatorname{id}_R R = \infty$. Because $\operatorname{pd}_R I(2) = \infty$, $\operatorname{pd}_R I(3) = \infty$ and $\operatorname{pd}_R I(5) = \infty$, we have that none of $P(1)$, $P(3)$, $P(4)$ and R is $G_n(m)$ for any $n, m \geq 0$.

(5.2) For any $7 \leq i \leq k+6$,

$$\begin{aligned} 0 \to S(i) \to I(i) \to I(i-1) \to \cdots \to I(7) \to I(6) \\ \to I(3) \to I(4) \to I(5) \to I(4) \to \cdots \end{aligned}$$

is a minimal injective coresolution of $S(i)$, where all of $I(i), I(i-1), \ldots, I(7)$ are projective and $\mathrm{pd}_R I(6) = \infty$. Thus $S(i)$ is $G_{i-6}(0)$ but not $G_{i-5}(0)$, and $S(6)$ is not $G_n(m)$ for any $n, m \geq 0$.

(6) Let R and S be finite-dimensional algebras over a field K, and let $M \in \mathrm{mod}\, R$ be $G_n(m)$ for some $n, m \geq 0$. Because

$$0 \to M \otimes_K S \to E^0(M) \otimes_K S \to E^1(M) \otimes_K S \to \cdots$$

is a minimal injective coresolution of $M \otimes_K S$ in $\mathrm{mod}\, R \otimes_K S$, by [CE, Thm. XI.3.2] we have that $M \otimes_K S$ is $G_n(m)$ in $\mathrm{mod}\, R \otimes_K S$.

The aim of this section is to study the homological behavior of modules (especially ${}_RR$) satisfying certain Auslander-type conditions. The following proposition plays an important role in proving the main result of this section.

Proposition 4.3. *For a left Noetherian ring R, $\mathrm{id}_{R^{\mathrm{op}}} F_i(E) \leq \mathrm{fd}_R E^i({}_RR)$ for any injective right R-module E and $i \geq 0$.*

Proof. By Lemma 2.3 we have that

$$\cdots \longrightarrow [E^i({}_RR)]^+ \xrightarrow{\pi_i} \cdots \xrightarrow{\pi_2} [E^1({}_RR)]^+ \xrightarrow{\pi_1} [E^0({}_RR)]^+ \xrightarrow{\pi_0} ({}_RR)^+ \longrightarrow 0$$

is a proper flat resolution of $({}_RR)^+$ in $\mathrm{Mod}\, R^{\mathrm{op}}$.

Let E be an injective right R-module. Because $({}_RR)^+$ is an injective cogenerator for $\mathrm{Mod}\, R^{\mathrm{op}}$, we have that E is isomorphic to a direct summand of $[({}_RR)^+]^I$ for some index set I. Because the subcategory of $\mathrm{Mod}\, R^{\mathrm{op}}$ consisting of flat modules is closed under direct products by [C, Thm. 2.1], we have that ${\pi_i}^I \colon ([E^i({}_RR)]^+)^I \to (\mathrm{Im}\, \pi_i)^I$ is a flat precover of $(\mathrm{Im}\, \pi_i)^I$ for any $i \geq 0$ by Lemma 2.2. Note that $F_i(E)$ is isomorphic to a direct summand of $([E^i({}_RR)]^+)^I$ for any $i \geq 0$. So by Lemma 2.4(1), we have

$$\mathrm{id}_{R^{\mathrm{op}}} F_i(E) \leq \mathrm{id}_{R^{\mathrm{op}}} ([E^i({}_RR)]^+)^I = \mathrm{id}_{R^{\mathrm{op}}} [E^i({}_RR)]^+ = \mathrm{fd}_R E^i({}_RR)$$

for any $i \geq 0$. □

We also have the following result.

Proposition 4.4. *For any $m \geq 0$, $\mathrm{id}_{R^{\mathrm{op}}} F_i(E) \leq m + i$ for any injective right R-module E and $i \geq 0$ if and only if $\mathrm{id}_{R^{\mathrm{op}}} F_i(N) \leq \mathrm{id}_{R^{\mathrm{op}}} N + m + i$ for any $N \in \mathrm{Mod}\, R^{\mathrm{op}}$ and $i \geq 0$.*

Proof. The sufficiency is trivial. We next prove the necessity. Let $N \in \mathrm{Mod}\, R^{\mathrm{op}}$ with $\mathrm{id}_{R^{\mathrm{op}}} N = s < \infty$. We will proceed by induction on s. If $s = 0$, then the assertion follows from assumption. Now suppose $s \geq 1$. Then we have an exact sequence

$$0 \to N \to E^0(N) \to N^1 \to 0$$

in $\operatorname{Mod} R^{\mathrm{op}}$ with $\operatorname{id}_{R^{\mathrm{op}}} N^1 = s-1$. By the induction hypothesis we have $\operatorname{id}_{R^{\mathrm{op}}} F_i(N^1) \leq (s-1)+m+i$ and $\operatorname{id}_{R^{\mathrm{op}}} F_i(E^0(N)) \leq m+i$ for any $i \geq 0$.

By [H2, Cor. 3.3] we have that

$$\cdots \to F_{i+1}(N^1) \bigoplus F_i(E^0(N)) \to \cdots \to F_2(N^1) \bigoplus F_1(E^0(N)) \to F_0 \to N \to 0$$

is a proper flat resolution of N and

$$0 \to F_0 \to F_1(N^1) \bigoplus F_0(E^0(N)) \to F_0(N^1) \to 0$$

is exact. So $\operatorname{id}_{R^{\mathrm{op}}} F_0 \leq s+m$ and $\operatorname{id}_{R^{\mathrm{op}}} F_{i+1}(N^1) \bigoplus F_i(E^0(N)) \leq s+m+i$ for any $i \geq 1$. Notice that $F_0(N)$ is isomorphic to a direct summand of F_0 and $F_i(N)$ is isomorphic to a direct summand of $F_{i+1}(N^1) \bigoplus F_i(E^0(N))$ for any $i \geq 1$, so we have $\operatorname{id}_{R^{\mathrm{op}}} F_i(N) \leq s+m+i$ for any $i \geq 0$. □

As a consequence of Propositions 4.3 and 4.4, we get the following corollary.

Corollary 4.5. *Let R be a left Noetherian ring. If ${}_RR$ is $G_\infty(m)$ with $m \geq 0$, then $\operatorname{id}_{R^{\mathrm{op}}} F_i(N) \leq \operatorname{id}_{R^{\mathrm{op}}} N + m + i$ for any $N \in \operatorname{Mod} R^{\mathrm{op}}$ and $i \geq 0$.*

Proof. If ${}_RR$ is $G_\infty(m)$, then $\operatorname{fd}_R E^i({}_RR) \leq m+i$ for any $i \geq 0$. By Proposition 4.3 we have $\operatorname{id}_{R^{\mathrm{op}}} F_i(E) \leq m+i$ for any injective right R-module E and $i \geq 0$. Now the assertion follows from Proposition 4.4. □

The following result can be regarded as a dual version of Proposition 4.4.

Proposition 4.6. *For any $m \geq 0$, any flat left R-module is $G_\infty(m)$ if and only if $\operatorname{fd}_R E^i(M) \leq \operatorname{fd}_R M + m + i$ for any $M \in \operatorname{Mod} R$ and $i \geq 0$.*

Proof. The sufficiency is trivial. We next prove the necessity. Let $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M = s < \infty$. We will proceed by induction on s. If $s = 0$, then the assertion follows from assumption. Now suppose $s \geq 1$. Then we have an exact sequence

$$0 \to M_1 \to F_0(M) \to M \to 0$$

in $\operatorname{Mod} R$ with $\operatorname{fd}_R M_1 = s-1$. By the induction hypothesis we have $\operatorname{fd}_R E^i(M_1) \leq (s-1)+m+i$ and $\operatorname{fd}_R E^i(F_0(M)) \leq m+i$ for any $i \geq 0$.

By [M, Cor. 1.3] (cf. [H2, Cor. 3.5]) we have that

$$\begin{aligned} 0 \to M \to I^0 \to E^1(F_0(M)) \bigoplus E^2(M_1) \to \cdots \\ \to E^i(F_0(M)) \bigoplus E^{i+1}(M_1) \to \cdots \end{aligned}$$

is an injective coresolution of M and

$$0 \to E^0(M_1) \to E^0(F_0(M)) \bigoplus E^1(M_1) \to I^0 \to 0$$

is exact and split. So $\mathrm{fd}_R\, I^0 \leq s+m$ and $\mathrm{fd}_R\, E^i(F_0(M)) \bigoplus E^{i+1}(M_1) \leq s+m+i$ for any $i \geq 1$. Notice that $E^0(M)$ is isomorphic to a direct summand of I^0 and $E^i(M)$ is isomorphic to a direct summand of $E^i(F_0(M)) \bigoplus E^{i+1}(M_1)$ for any $i \geq 1$, so we have $\mathrm{fd}_R\, E^i(M) \leq s+m+i$ for any $i \geq 0$. □

By the dimension shifting we get the following lemma.

Lemma 4.7.

(1) $\mathrm{fd}_R\, E^0(M) \leq \mathrm{fd}_R\, M$ *for any* $M \in \mathrm{Mod}\, R$ *if and only if* $\mathrm{fd}_R\, E^i(M) \leq \mathrm{fd}_R\, M+i$ *for any* $M \in \mathrm{Mod}\, R$ *and* $i \geq 0$.

(2) $\mathrm{id}_{R^{\mathrm{op}}}\, F_0(N) \leq \mathrm{id}_{R^{\mathrm{op}}}\, N$ *for any* $N \in \mathrm{Mod}\, R^{\mathrm{op}}$ *if and only if* $\mathrm{id}_{R^{\mathrm{op}}}\, F_i(N) \leq \mathrm{id}_{R^{\mathrm{op}}}\, N+i$ *for any* $N \in \mathrm{Mod}\, R^{\mathrm{op}}$ *and* $i \geq 0$.

We also need the following lemma.

Lemma 4.8. *Let* $M \in \mathrm{Mod}\, R$ *and* $n \geq 0$.

(1) *If* R *is a right Noetherian ring and* $\mathrm{id}_{R^{\mathrm{op}}}\, F_0(M^+) \leq \mathrm{id}_{R^{\mathrm{op}}}\, M^+ + n$, *then* $\mathrm{fd}_R\, E^0(M) \leq \mathrm{fd}_R\, M + n$.

(2) *If* R *is a left Noetherian ring and* $\mathrm{id}_{R^{\mathrm{op}}}\, M^+ \leq \mathrm{id}_{R^{\mathrm{op}}}\, F_0(M^+)+n$, *then* $\mathrm{fd}_R\, M \leq \mathrm{fd}_R\, E^0(M) + n$.

Proof. (1) Let $\mathrm{fd}_R\, M = s < \infty$. Then $\mathrm{id}_{R^{\mathrm{op}}}\, M^+ = s$ by Lemma 2.4(1). So $\mathrm{id}_{R^{\mathrm{op}}}\, F_0(M^+) \leq \mathrm{id}_{R^{\mathrm{op}}}\, M^+ = s+n$ by assumption, and hence we get an injective preenvelope $0 \to M^{++} \to [F_0(M^+)]^+$ of M^{++} with $\mathrm{fd}_R[F_0(M^+)]^+ = \mathrm{id}_{R^{\mathrm{op}}}\, F_0(M^+) \leq s+n$ by Lemma 2.4(2). Notice that there exists an embedding $M \hookrightarrow M^{++}$ by [St, p. 48, Exe. 41], thus $E^0(M)$ is isomorphic to a direct summand of $[F_0(M^+)]^+$ and therefore $\mathrm{fd}_R\, E^0(M) \leq s+n$.

(2) Let $\mathrm{fd}_R\, E^0(M) = s < \infty$. By Lemmas 2.3 and 2.4(1), $[E^0(M)]^+ \twoheadrightarrow M^+$ is a flat precover of M^+ in $\mathrm{Mod}\, R^{\mathrm{op}}$ with $\mathrm{id}_{R^{\mathrm{op}}}[E^0(M)]^+ = s$. So $F_0(M^+)$ is isomorphic to a direct summand of $[E^0(M)]^+$ and $\mathrm{id}_{R^{\mathrm{op}}}\, F_0(M^+) \leq s$. Then by assumption, we have

$$\mathrm{id}_{R^{\mathrm{op}}}\, M^+ \leq \mathrm{id}_{R^{\mathrm{op}}}\, F_0(M^+) + n \leq s+n.$$

It follows from Lemma 2.4(1) that $\mathrm{fd}_R\, M \leq s+n$. □

We are now in a position to state the main result in this section, which is more general than Theorem 1.2.

Theorem 4.9. *For a left Noetherian ring R, consider the following conditions:*

(1) *${}_RR$ satisfies the Auslander condition.*

(2) *Any flat left R-module satisfies the Auslander condition.*

(3) $\operatorname{fd}_R E^i(M) \leq \operatorname{fd}_R M + i$ *for any left R-module M and $i \geq 0$.*

(4) $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M$ *for any left R-module M.*

(5) $\operatorname{id}_{R^{\mathrm{op}}} F_i(E) \leq i$ *for any injective right R-module E and $i \geq 0$.*

(6) $\operatorname{id}_{R^{\mathrm{op}}} F_i(N) \leq \operatorname{id}_{R^{\mathrm{op}}} N + i$ *for any right R-module N and $i \geq 0$.*

(7) $\operatorname{id}_{R^{\mathrm{op}}} F_0(N) \leq \operatorname{id}_{R^{\mathrm{op}}} N$ *for any right R-module N.*

We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7)*. If R is also right Noetherian, then all of the above and below conditions are equivalent:*

(i)$^{\mathrm{op}}$ *The opposite version of* (i) *(*$1 \leq \mathrm{i} \leq 7$*).*

Proof. (2) $\Rightarrow$ (1) is trivial, and (1) $\Rightarrow$ (2) follows from Corollary 3.2. The assertions (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Proposition 4.6 and Lemma 4.7(1), and (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) follow from Proposition 4.4 and Lemma 4.7(2). By Corollary 4.5 we have (1) $\Rightarrow$ (5).

Assume that R is a left and right Noetherian ring. Then (1) $\Leftrightarrow$ (1)$^{\mathrm{op}}$ follows from [FGR, Thm. 3.7], and (7) $\Rightarrow$ (4) follows from Lemma 4.8(1). □

Observe that Miyachi proved in [M, Thm. 4.1] that if R is a right coherent and left Noetherian projective K-algebra over a commutative ring K, then R satisfies the Auslander condition (that is, ${}_RR$ is $G_\infty(0)$) if and only if $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M$ for any $M \in \operatorname{Mod} R$. Theorem 4.9 extends this result. Moreover, by Theorem 4.9, we immediately have the following corollary.

Corollary 4.10. *Let R be a left Noetherian ring such that ${}_RR$ satisfies the Auslander condition. If $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M \leq s$ $(< \infty)$, then M is $G_\infty(s)$.*

Remark 4.11. By the dimension shifting it is easy to verify that the converse of Corollary 4.10 holds true when $\operatorname{id}_R M < \infty$, even without the assumption that R is a left Noetherian ring satisfying the Auslander condition. However, this converse does not hold true in general. For example, let R be a quasi-Frobenius ring with the global dimension $\operatorname{gl.dim} R$ of R infinite. Then R is a left and right artin ring satisfying the Auslander condition and every module in $\operatorname{Mod} R$ is $G_\infty(0)$, but there exists a module in $\operatorname{Mod} R$ which is not flat because $\operatorname{gl.dim} R$ is infinite.

For any $n, k \geq 0$, we use $\mathscr{G}_n(k)$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of modules being $G_n(k)$, and write $\mathscr{G}_\infty(k) := \bigcap_{n \geq 0} \mathscr{G}_n(k)$. By [H2, Cor. 3.9] it is easy to get the following proposition.

Proposition 4.12. *Let*

$$0 \to X \to X^0 \to X^1 \to 0$$

be an exact sequence in $\operatorname{Mod} R$*, and let* $s \geq 0$ *and* $n \geq 1$*. If* $X^0 \in \mathscr{G}_n(s)$ *and* $X^1 \in \mathscr{G}_{n-1}(s+1)$*, then* $X \in \mathscr{G}_n(s)$*.*

For any $n \geq 0$, we use $\mathscr{F}^n(\operatorname{Mod} R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of modules with flat dimension at most n.

Corollary 4.13. *Let* R *be a left Noetherian ring. Then we have*

(1) $\mathscr{G}_\infty(0) = \mathscr{F}^0(\operatorname{Mod} R)$ *if and only if* $\mathscr{G}_\infty(s) = \mathscr{F}^s(\operatorname{Mod} R)$ *for any* $s \geq 0$*;*

(2) $\mathscr{G}_\infty(0) \cap \operatorname{mod} R = \mathscr{F}^0(\operatorname{mod} R)$ *if and only if* $\mathscr{G}_\infty(s) \cap \operatorname{mod} R = \mathscr{F}^s(\operatorname{mod} R)$ *for any* $s \geq 0$*.*

Proof. (1) The sufficiency is trivial, so it suffices to prove the necessity. By Corollary 4.10 we have $\mathscr{F}^s(\operatorname{Mod} R) \subseteq \mathscr{G}_\infty(s)$ for any $s \geq 0$. In the following we will prove the converse inclusion by induction on s. The case for $s = 0$ follows from assumption. Now suppose $s \geq 1$ and $M \in \mathscr{G}_\infty(s)$. Let

$$0 \to K \to F_0(M) \to M \to 0$$

be an exact sequence in $\operatorname{Mod} R$. By assumption $F_0(M) \in \mathscr{G}_\infty(0)$. So $K \in \mathscr{G}_\infty(s-1)$ by Proposition 4.12, and hence $\operatorname{fd}_R K \leq s-1$ by the induction hypothesis. It follows that $\operatorname{fd}_R M \leq s$ and $M \in \mathscr{F}^s(\operatorname{Mod} R)$, which implies $\mathscr{G}_\infty(s) \subseteq \mathscr{F}^s(\operatorname{Mod} R)$.

(2) It is an immediate consequence of (1). □

As applications of the results obtained above, in the rest of this section we will study the properties of rings satisfying the Auslander condition with finite certain homological dimension. In particular, we will get some equivalent characterizations of Auslander–Gorenstein rings and Auslander-regular rings.

For a module $M \in \operatorname{Mod} R$ and $t \geq 0$, we use $\Omega^t(M)$ to denote the tth syzygy of M (note: $\Omega^0(M) = M$). It is known that $\Omega^t(M)$ is unique up to projective equivalence for a given module M.

Lemma 4.14. *Let* R *be a left Noetherian ring, and let* $t \geq 1$ *and* $n \geq 0$*. For a module* $M \in \operatorname{Mod} R$*, if* $\operatorname{fd}_R \Omega^t(M) \leq \operatorname{fd}_R E^0(\Omega^t(M)) + n$*, then* $\operatorname{fd}_R M \leq \operatorname{fd}_R E^0({}_RR) + n + t$*.*

Proof. Let $M \in \operatorname{Mod} R$. Then there exist index sets $J_0, \ldots, J_{t-1}$ such that we have the following exact sequence:

$$0 \to \Omega^t(M) \to R^{(J_{t-1})} \to \cdots \to R^{(J_0)} \to M \to 0$$

in $\operatorname{Mod} R$. Because $E^0(R^{(J_{t-1})}) = [E^0({}_RR)]^{(J_{t-1})}$ by [B1, Thm. 1.1] and [AF, Prop. 18.12(4)], we have $\operatorname{fd}_R E^0(R^{(J_{t-1})}) = \operatorname{fd}_R E^0({}_RR)$. Notice that $E^0(\Omega^t(M))$ is isomorphic to a direct summand of $E^0(R^{(J_{t-1})})$, so $\operatorname{fd}_R E^0(\Omega^t(M)) \leq \operatorname{fd}_R E^0({}_RR)$. It follows from assumption that

$$\operatorname{fd}_R \Omega^t(M) \leq \operatorname{fd}_R E^0(\Omega^t(M)) + n \leq \operatorname{fd}_R E^0({}_RR) + n$$

and $\operatorname{fd}_R M \leq \operatorname{fd}_R E^0({}_RR) + n + t$. □

Recall from [Bj] that a left and right Noetherian ring R is called *Auslander–Gorenstein* (resp. *Auslander-regular*) if R satisfies the Auslander condition and $\operatorname{id}_R R = \operatorname{id}_{R^{\mathrm{op}}} R$ (resp. $\operatorname{gl.dim} R$) $< \infty$. Also recall that $\operatorname{fin.dim} R = \sup\{\operatorname{pd}_R M \mid M \in \operatorname{mod} R \text{ with } \operatorname{pd}_R M < \infty\}$.

As an application of Theorem 4.9 we get some equivalent characterizations of rings satisfying the Auslander condition with finite left self-injective dimension as follows, which generalizes [M, Prop. 4.4].

Theorem 4.15. *For a left and right Noetherian ring R and $n \geq 1$, the following statements are equivalent:*

(1) *R satisfies the Auslander condition with $\operatorname{id}_R R \leq n$.*

(2) $\operatorname{id}_{R^{\mathrm{op}}} F_0(N) \leq \operatorname{id}_{R^{\mathrm{op}}} N \leq \operatorname{id}_{R^{\mathrm{op}}} F_0(N) + n - 1$ *for any $N \in \operatorname{Mod} R^{\mathrm{op}}$ with finite injective dimension.*

(3) $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M \leq \operatorname{fd}_R E^0(M) + n - 1$ *for any $M \in \operatorname{Mod} R$ with finite flat dimension.*

Proof. (1) ⇒ (2). When $N \in \operatorname{Mod} R^{\mathrm{op}}$ is flat, it is trivial that assertion (2) holds true. Now let $N \in \operatorname{Mod} R^{\mathrm{op}}$ be non-flat with finite injective dimension. By Theorem 4.9 we have $\operatorname{id}_{R^{\mathrm{op}}} F_0(N) \leq \operatorname{id}_{R^{\mathrm{op}}} N$. So we only need to prove the latter inequality. Because $\operatorname{id}_R R \leq n$, we have $\operatorname{id}_{R^{\mathrm{op}}} N \leq n$ by Lemma 2.5(2). So if $\operatorname{id}_{R^{\mathrm{op}}} F_0(N) \geq 1$, then the assertion holds true. Suppose that $F_0(N)$ is injective. We have an exact sequence

$$0 \to B \to F_0(N) \to N \to 0$$

in $\operatorname{Mod} R^{\mathrm{op}}$ with $\operatorname{id}_{R^{\mathrm{op}}} B < \infty$. If $\operatorname{id}_{R^{\mathrm{op}}} N = n$, then $\operatorname{id}_{R^{\mathrm{op}}} B = n+1$. It follows from Lemma 2.5(2) that $\operatorname{id}_R R \geq n+1$, which is a contradiction. Thus we have $\operatorname{id}_{R^{\mathrm{op}}} N \leq n-1$.

(2) ⇒ (3). Let $M \in \operatorname{Mod} R$ with finite flat dimension. Then $M^+ \in \operatorname{Mod} R^{\mathrm{op}}$ with finite injective dimension by Lemma 2.4(1). Now the assertion follows from Lemma 4.8.

(3) ⇒ (1). By (3) and Theorem 4.9, R satisfies the Auslander condition. Let $M \in \operatorname{mod} R$ with $\operatorname{pd}_R M$ $(= \operatorname{fd}_R M) < \infty$. Then $\operatorname{fd}_R \Omega^1(M) < \infty$. By (3) we have

$\mathrm{fd}_R\,\Omega^1(M) \leq \mathrm{fd}_R\,E^0(\Omega^1(M)) + n - 1$. So

$$\mathrm{pd}_R\,M = \mathrm{fd}_R\,M \leq \mathrm{fd}_R\,E^0({}_RR) + n = n$$

by Lemma 4.14. Thus we have $\mathrm{fin.dim}\,R \leq n$. It follows from [HI, Cor. 5.3] that $\mathrm{id}_R\,R \leq n$. □

In view of Theorem 4.15 it would be interesting to ask the following question.

Question 4.16. Let R be a left and right Noetherian ring satisfying the Auslander condition with $\mathrm{id}_R\,R < \infty$. Then, is $\mathrm{id}_{R^{\mathrm{op}}}\,R < \infty$? That is, is R Auslander–Gorenstein?

By [H1, Prop. 4.6] the answer to Question 4.16 is positive if R is a left and right artin ring. It is a generalization of [AR3, Cor. 5.5(b)].

Putting $n = 1$ in Theorem 4.15 we have the following corollary.

Corollary 4.17. *For a left and right Noetherian ring R, the following statements are equivalent:*

(1) *R satisfies the Auslander condition with $\mathrm{id}_R\,R \leq 1$.*
(2) *$\mathrm{id}_{R^{\mathrm{op}}}\,F_0(N) = \mathrm{id}_{R^{\mathrm{op}}}\,N$ for any $N \in \mathrm{Mod}\,R^{\mathrm{op}}$ with finite injective dimension.*
(3) *$\mathrm{fd}_R\,E^0(M) = \mathrm{fd}_R\,M$ for any $M \in \mathrm{Mod}\,R$ with finite flat dimension.*

As another application of Theorem 4.9 we get some equivalent characterizations of Auslander-regular rings as follows, which generalizes [M, Cor. 4.5].

Theorem 4.18. *For a left and right Noetherian ring R and $n \geq 1$, the following statements are equivalent:*

(1) *R is an Auslander-regular ring with $\mathrm{gl.dim}\,R \leq n$.*
(2) *$\mathrm{id}_{R^{\mathrm{op}}}\,F_0(N) \leq \mathrm{id}_{R^{\mathrm{op}}}\,N \leq \mathrm{id}_{R^{\mathrm{op}}}\,F_0(N) + n - 1$ for any $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.*
(3) *$\mathrm{fd}_R\,E^0(M) \leq \mathrm{fd}_R\,M \leq \mathrm{fd}_R\,E^0(M) + n - 1$ for any $M \in \mathrm{Mod}\,R$.*

Proof. By Theorem 4.15 and Lemma 4.8 we have (1) ⇒ (2) ⇒ (3).

(3) ⇒ (1). By (3) and Theorem 4.9, R satisfies the Auslander condition. Let $M \in \mathrm{mod}\,R$. By (3) we have $\mathrm{fd}_R\,\Omega^1(M) \leq \mathrm{fd}_R\,E^0(\Omega^1(M)) + n - 1$. So

$$\mathrm{pd}_R\,M = \mathrm{fd}_R\,M \leq \mathrm{fd}_R\,E^0({}_RR) + n = n$$

by Lemma 4.14, and hence $\mathrm{gl.dim}\,R \leq n$. □

Putting $n = 1$ in Theorem 4.18 we have the following corollary.

Corollary 4.19. *For a left and right Noetherian ring R, the following statements are equivalent:*

(1) *R is an Auslander-regular ring with* $\operatorname{gl.dim} R \leq 1$.
(2) $\operatorname{id}_{R^{\mathrm{op}}} F_0(N) = \operatorname{id}_{R^{\mathrm{op}}} N$ *for any* $N \in \operatorname{Mod} R^{\mathrm{op}}$.
(3) $\operatorname{fd}_R E^0(M) = \operatorname{fd}_R M$ *for any* $M \in \operatorname{Mod} R$.

§5. Approximation presentations and Gorenstein algebras

In this section, R is an artin algebra. We will establish the connection between Auslander and Reiten's conjecture mentioned in the introduction and the contravariant finiteness of the full subcategory of $\operatorname{mod} R$ consisting of modules satisfying the Auslander condition.

For $n \geq 0$, we use $\mathscr{I}^n(\operatorname{Mod} R)$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of modules with injective dimension at most n. For a module $M \in \operatorname{Mod} R$, we denote by $\Omega^{-n}(M)$ the nth cosyzygy of M. The following approximation theorem plays a crucial role in the rest of this section.

Theorem 5.1. *Let ${}_RR \in \mathscr{G}_n(k)$ and $R_R \in \mathscr{G}_n(k)^{\mathrm{op}}$ with $n, k \geq 0$. Then for any $M \in \operatorname{Mod} R$ and $1 \leq i \leq n-1$, there exist the following commutative diagrams with exact rows:*

(5.1)
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & I_{i+1}(M) \arrow[r] \arrow[d] & G_{i+1}(M) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & I_i(M) \arrow[r] & G_i(M) \arrow[r] & 0 \end{tikzcd}$$

and

(5.2)
$$\begin{tikzcd} 0 \arrow[r] & I^{i+1}(M) \arrow[r] \arrow[d] & G^{i+1}(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I^i(M) \arrow[r] & G^i(M) \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

in $\operatorname{Mod} R$ *with $G_j(M), G^j(M) \in \mathscr{G}_j(k)$, and $I_j(M), I^j(M) \in \mathscr{I}^{j+k}(\operatorname{Mod} R)$ for $j = i, i+1$. Furthermore, if M is in* $\operatorname{mod} R$, *then all modules in the above two commutative diagrams are also in* $\operatorname{mod} R$.

Proof. By [H2, Cor. 3.7 and Lem. 3.1(1)] we have the following commutative diagrams with exact columns and rows:

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & M \arrow[r, dashed] \arrow[d] & I_i(M) \arrow[r, dashed] \arrow[d, dashed] & G_i(M) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & E^0(M) \arrow[r, dashed] \arrow[d] & E^0(M)\bigoplus(\bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M))) \arrow[r, dashed] \arrow[d, dashed] & \bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & E^1(M) \arrow[r, dashed] \arrow[d] & E^1(M)\bigoplus(\bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M))) \arrow[r, dashed] \arrow[d, dashed] & \bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r, dashed] & E^{i-2}(M) \arrow[r, dashed] \arrow[d] & E^{i-2}(M)\bigoplus(P_1(E^i(M))\bigoplus P_0(E^{i-1}(M))) \arrow[r, dashed] \arrow[d, dashed] & P_1(E^i(M))\bigoplus P_0(E^{i-1}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & E^{i-1}(M) \arrow[r, dashed] \arrow[d] & E^{i-1}(M)\bigoplus P_0(E^i(M)) \arrow[r, dashed] \arrow[d, dashed] & P_0(E^i(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{-i}(M) \arrow[r] \arrow[d] & E^i(M) \arrow[r] \arrow[d, dashed] & \Omega^{-(i+1)}(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where, for any $i \geq 1$,

$$I_i(M) = \operatorname{Ker}\left(E^0(M)\bigoplus\left(\bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M))\right) \to E^1(M)\bigoplus\left(\bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M))\right)\right),$$

$$G_i(M) = \operatorname{Ker}\left(\bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M)) \to \bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M))\right).$$

Consider the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & \Omega^1(E^{i+1}(M)) \arrow[r, dashed] \arrow[d, equal] & X_{i+1} \arrow[r, dashed] \arrow[d, dashed] & \Omega^{-(i+1)}(M) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1(E^{i+1}(M)) \arrow[r] & P_0(E^{i+1}(M)) \arrow[r] \arrow[d, dashed] & E^{i+1}(M) \arrow[r] \arrow[d] & 0 \\
 & & \Omega^{-(i+2)}(M) \arrow[r, equal] \arrow[d, dashed] & \Omega^{-(i+2)}(M) \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By [H2, Cor. 3.7 and Lem. 3.1(1)] again, for any $i \geq 1$ we have the following commutative and exact columns and rows:

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & \Omega^{i+1}(E^{i+1}(M)) \arrow[r, dashed] \arrow[d] & G_{i+1}(M) \arrow[r, dashed] \arrow[d, dashed] & G_i(M) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & P_i(E^{i+1}(M)) \arrow[r, dashed] \arrow[d] & \bigoplus_{j=0}^{i} P_j(E^{j+1}(M)) \arrow[r, dashed] \arrow[d, dashed] & \bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & P_{i-1}(E^{i+1}(M)) \arrow[r, dashed] \arrow[d] & \bigoplus_{j=0}^{i-1} P_j(E^{j+2}(M)) \arrow[r, dashed] \arrow[d, dashed] & \bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r, dashed] & P_2(E^{i+1}(M)) \arrow[r, dashed] \arrow[d] & P_2(E^{i+1}(M)) \bigoplus (P_1(E^{i}(M)) \bigoplus P_0(E^{i-1}(M))) \arrow[r, dashed] \arrow[d, dashed] & P_1(E^{i}(M)) \bigoplus P_0(E^{i-1}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & P_1(E^{i+1}(M)) \arrow[r, dashed] \arrow[d] & P_1(E^{i+1}(M)) \bigoplus P_0(E^{i}(M)) \arrow[r, dashed] \arrow[d, dashed] & P_0(E^{i}(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{1}(E^{i+1}(M)) \arrow[r] \arrow[d] & X_{i+1} \arrow[r] \arrow[d, dashed] & \Omega^{-(i+1)}(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Then we get the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
 & & \Omega^{i+1}(E^{i+1}(M)) \arrow[r, equal] \arrow[d, dashed] & \Omega^{i+1}(E^{i+1}(M)) \arrow[d] & \\
0 \arrow[r, dashed] & M \arrow[r, dashed] \arrow[d, equal] & I_{i+1}(M) \arrow[r, dashed] \arrow[d, dashed] & G_{i+1}(M) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & I_i(M) \arrow[r] \arrow[d, dashed] & G_i(M) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Because $R_R \in \mathscr{G}_n(k)^{\mathrm{op}}$, we have $\mathrm{id}_R\, P_j(E^t(M)) \leq j+k$ for any $0 \leq j \leq n-1$ and $t \geq 0$ by Proposition 4.3. So from the middle column in the first diagram we get

$\mathrm{id}_R\, I_i(M) \le i+k$ for any $1 \le i \le n$. Because ${}_RR \in \mathscr{G}_n(k)$, any projective module in $\mathrm{Mod}\, R$ is also in $\mathscr{G}_n(k)$. So by [H2, Cor. 3.9] and the exactness of the rightmost column in the first diagram, we have $G_i(M) \in \mathscr{G}_i(k)$ for any $1 \le i \le n$. Thus the above diagram is (5.1).

Put $I^i(M) = I_i(\Omega^1(M))$. Then we have the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & & \\
0 \arrow[r] & \Omega^1(M) \arrow[r] \arrow[d] & P_0(M) \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dashed] & I^i(M) \arrow[r, dashed] \arrow[d] & G^i(M) \arrow[r, dashed] \arrow[d, dashed] & M \arrow[r, dashed] & 0 \\
 & G_i(\Omega^1(M)) \arrow[r, equal] \arrow[d] & G_i(\Omega^1(M)) \arrow[d, dashed] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Note that $P_0(M) \in \mathscr{G}_n(k)$. For any $1 \le i \le n$, because $G_i(\Omega^1(M)) \in \mathscr{G}_i(k)$ by the above argument, we have that $G^i(M)$ is also in $\mathscr{G}_i(k)$ by the horseshoe lemma and the exactness of the middle column in the above diagram. By the above argument we have the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
 & & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) \arrow[r, equal] \arrow[d, dashed] & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) \arrow[d] & \\
0 \arrow[r, dashed] & P_0(M) \arrow[r, dashed] \arrow[d, equal] & G^{i+1}(M) \arrow[r, dashed] \arrow[d, dashed] & G_{i+1}(\Omega^1(M)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & P_0(M) \arrow[r] & G^i(M) \arrow[r] \arrow[d, dashed] & G_i(\Omega^1(M)) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Then the pull-back diagram

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d] & & \\
 & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) \arrow[r, equal, dashed] \arrow[d, dashed] & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) \arrow[d] & & \\
0 \arrow[r, dashed] & I^{i+1}(M) \arrow[r, dashed] \arrow[d, dashed] & G^{i+1}(M) \arrow[r, dashed] \arrow[d] & M \arrow[r, dashed] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^i(M) \arrow[r] \arrow[d, dashed] & G^i(M) \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

is (5.2). □

Let $G \in \mathscr{G}_i(0)$ and $I \in \mathscr{I}^i(\operatorname{Mod} R)$ with $i \geq 1$. Then applying the functor $\operatorname{Hom}_R(-, I)$ to the minimal injective coresolution of G, we get $\operatorname{Ext}_R^1(G, I) = 0$ by the dimension shifting. So, if R satisfies the Auslander condition, then the exact sequences

$$0 \to M \to I_i(M) \to G_i(M) \to 0$$

and

$$0 \to I^i(M) \to G^i(M) \to M \to 0$$

in Theorem 5.1 are a left $\mathscr{I}^i(\operatorname{Mod} R)$-approximation and a right $\mathscr{G}_i(0)$-approximation of M respectively for any $1 \leq i \leq n$.

Lemma 5.2. *Let $X \in \operatorname{mod} R$ and $\{M_i\}_{i \in I}$ be a family of left R-modules, where I is a directed index set. Then for any $n \geq 0$ we have*

$$\operatorname{Ext}_R^n\Big(\varinjlim_{i \in I} M_i, X\Big) \cong \varprojlim_{i \in I} \operatorname{Ext}_R^n(M_i, X).$$

Proof. Because R is an artin algebra, any module in $\operatorname{mod} R$ is pure-injective by [GT, Cor. 1.2.22]. Then the assertion follows from [GT, Lem. 3.3.4]. □

Let $M \in \operatorname{Mod} R$ and $n, k \geq 0$, and let

$$\cdots \to P_i(M) \to \cdots \to P_1(M) \to P_0(M) \to M \to 0$$

be a minimal projective resolution of M. We use $\operatorname{Co}\mathscr{G}_n(k)$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of the modules M satisfying $\operatorname{id}_R P_i(M) \leq i + k$

for any $0 \leq i \leq n-1$, and denote $\mathrm{Co}\mathscr{G}_\infty(k) = \bigcap_{n\geq 0} \mathrm{Co}\mathscr{G}_n(k)$. We use $\mathscr{P}^n(\operatorname{mod} R)$ (resp. $\mathscr{I}^n(\operatorname{mod} R)$) to denote the full subcategory of $\operatorname{mod} R$ consisting of modules with projective (resp. injective) dimension at most n. We use $\mathbb{D}$ to denote the ordinary duality between $\operatorname{mod} R$ and $\operatorname{mod} R^{\mathrm{op}}$. As a consequence of Theorem 5.1 we get the following proposition.

Proposition 5.3. *Let R satisfy the Auslander condition and $M \in \operatorname{mod} R$. Then we have the following:*

(1) *There exists a countably generated left R-module $N \in \mathrm{Co}\mathscr{G}_\infty(0)$ and a monomorphism $\beta\colon M \rightarrowtail N$ in $\operatorname{Mod} R$ such that $\operatorname{Hom}_R(\beta, T)$ is epic for any $T \in \mathrm{Co}\mathscr{G}_\infty(0) \cap \operatorname{mod} R$.*

(2) *There exists a countably generated right R-module $N' \in \mathrm{Co}\mathscr{G}_\infty(0)^{\mathrm{op}}$ and an epimorphism $\alpha\colon \mathbb{D}N' \twoheadrightarrow M$ in $\operatorname{Mod} R$ such that $\mathbb{D}N' \in \mathscr{G}_\infty(0)$ and $\operatorname{Hom}_R(T', \alpha)$ is epic for any $T' \in \mathscr{G}_\infty(0) \cap \operatorname{mod} R$.*

Proof. (1) Let R satisfy the Auslander condition. By Theorem 5.1, for any $M \in \operatorname{mod} R$ and $n \geq 1$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}(\mathbb{D}M) \arrow[r] \arrow[d] & G^{n+1}(\mathbb{D}M) \arrow[r] \arrow[d] & \mathbb{D}M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^{n}(\mathbb{D}M) \arrow[r] & G^{n}(\mathbb{D}M) \arrow[r] & \mathbb{D}M \arrow[r] & 0
\end{tikzcd}$$

with $G^i(\mathbb{D}M) \in \mathscr{G}_i(0)^{\mathrm{op}} \cap \operatorname{mod} R^{\mathrm{op}}$ and $I^i(\mathbb{D}M) \in \mathscr{I}^i(\operatorname{mod} R^{\mathrm{op}})$ for $i = n, n+1$. Then we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\beta_n"] \arrow[d, equal] & \mathbb{D}G^{n}(\mathbb{D}M) \arrow[r] \arrow[d] & \mathbb{D}I^{n}(\mathbb{D}M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\beta_{n+1}"] & \mathbb{D}G^{n+1}(\mathbb{D}M) \arrow[r] & \mathbb{D}I^{n+1}(\mathbb{D}M) \arrow[r] & 0
\end{tikzcd}$$

with $\mathbb{D}G^i(\mathbb{D}M) \in \mathrm{Co}\mathscr{G}_i(0) \cap \operatorname{mod} R$ and $\mathbb{D}I^i(\mathbb{D}M) \in \mathscr{P}^i(\operatorname{mod} R)$ for $i = n, n+1$. Put $N_n := \mathbb{D}G^n(\mathbb{D}M)$ and $K_n := \mathbb{D}I^n(\mathbb{D}M)$ for any $n \geq 1$. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
P_k(N_n) \arrow[r] \arrow[d, dashed, "g^k_{n+1,n}"] & P_{k-1}(N_n) \arrow[r] \arrow[d, dashed, "g^{k-1}_{n+1,n}"] & \cdots \arrow[r] & P_1(N_n) \arrow[r] \arrow[d, dashed, "g^1_{n+1,n}"] & P_0(N_n) \arrow[r] \arrow[d, dashed, "g^0_{n+1,n}"] & N_n \arrow[r] \arrow[d, "g_{n+1,n}"] & 0 \\
P_k(N_{n+1}) \arrow[r] & P_{k-1}(N_{n+1}) \arrow[r] & \cdots \arrow[r] & P_1(N_{n+1}) \arrow[r] & P_0(N_{n+1}) \arrow[r] & N_{n+1} \arrow[r] & 0.
\end{tikzcd}$$

If $n > m$, then put

$$g_{n,m} := g_{n,n-1} g_{n-1,n-2} \cdots g_{m+1,m}$$

and

$$g^k_{n,m} := g^k_{n,n-1} g^k_{n-1,n-2} \cdots g^k_{m+1,m}.$$

In this way, for any $k \geq 0$ we get direct systems $\{N_n, g_{n,m}\}_{n\in\mathbb{Z}^+}$ and $\{P_k(N_n), g^k_{n,m}\}_{n\in\mathbb{Z}^+}$, where $\mathbb{Z}^+$ is the set of positive integers. Because each $g_{n,m}\colon N_m \to N_n$ is monic, we can identify $\varinjlim_{n\geq 1} N_n$ with the direct union. It follows that $\varinjlim_{n\geq 1} N_n = \varinjlim_{n\geq t} N_n$ for any $1 \leq t \leq n$. Put $N := \varinjlim_{n\geq 1} N_n$. Then N is countably generated.

Because $N_t \in \mathrm{Co}\mathscr{G}_t(0) \cap \operatorname{mod} R$, we have $\mathrm{id}_R\, P_k(N_t) \leq k$ for any $0 \leq k \leq t$. So $\varinjlim_{n\geq t} P_k(N_n)$ is projective and $\mathrm{id}_R \varinjlim_{n\geq t} P_k(N_n) \leq k$ for any $0 \leq k \leq t$ by [B1, Thm. 1.1]. On the other hand, we have an exact sequence

$$\cdots \to \varinjlim_{n\geq t} P_t(N_n) \to \varinjlim_{n\geq t} P_{t-1}(N_n) \to \cdots \to \varinjlim_{n\geq t} P_0(N_n) \to \varinjlim_{n\geq t} N_n \, (= N) \to 0.$$

So $N \in \mathrm{Co}\mathscr{G}_\infty(0)$. Put $K := \varinjlim_{n\geq t} K_n$ and $\beta := \varinjlim_{n\geq t} \beta_n$. Then we get the following exact sequence:

$$0 \to M \xrightarrow{\beta} N \to K \to 0$$

in $\operatorname{Mod} R$. Note that $K_n \in \mathscr{P}^n(\operatorname{mod} R)$ for any $n \geq 1$. So by Lemma 5.2 and the dimension shifting, for any $T \in \mathrm{Co}\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ we have

$$\begin{aligned}\mathrm{Ext}^1_R(K,T) \cong \mathrm{Ext}^1_R\Big(\varinjlim_{n\geq t} K_n, T\Big) &\cong \varprojlim_{n\geq t} \mathrm{Ext}^1_R(K_n,T)\\ &\cong \varprojlim_{n\geq t} \mathrm{Ext}^{n+1}_R(K_n, \Omega^n(T)) = 0,\end{aligned}$$

which implies that $\mathrm{Hom}_R(\beta,T)$ is epic.

(2) Let $M \in \operatorname{mod} R$ and $T' \in \mathscr{G}_\infty(0) \cap \operatorname{mod} R$. Then $\mathbb{D}M \in \operatorname{mod} R^{\mathrm{op}}$ and $\mathbb{D}T' \in \mathrm{Co}\mathscr{G}_\infty(0)^{\mathrm{op}} \cap \operatorname{mod} R^{\mathrm{op}}$. By (1), there exists a monomorphism $\beta\colon \mathbb{D}M \rightarrowtail N'$ in $\operatorname{Mod} R^{\mathrm{op}}$ with N' countably generated and $N' \in \mathrm{Co}\mathscr{G}_\infty(0)^{\mathrm{op}}$ such that $\mathrm{Hom}_{R^{\mathrm{op}}}(\beta, \mathbb{D}T')$ is epic. Then $\mathbb{D}\beta\colon \mathbb{D}N' \twoheadrightarrow M \ (\cong \mathbb{D}\mathbb{D}M)$ is epic in $\operatorname{Mod} R$ such that $\mathrm{Hom}_R(T', \mathbb{D}\beta)$ $(\cong \mathrm{Hom}_R(\mathbb{D}\mathbb{D}T', \mathbb{D}\beta))$ is also epic. Because $N' \in \mathrm{Co}\mathscr{G}_\infty(0)^{\mathrm{op}}$, we have that $\mathrm{id}_{R^{\mathrm{op}}}\, P_i(N') \leq i$ for any $i \geq 0$. Note that $P_i(N') = \bigoplus_j P^i_j$ with all P^i_j projective in $\operatorname{mod} R$ for any $i \geq 0$. So we get an exact sequence

$$0 \to \mathbb{D}N' \to \prod_j \mathbb{D}P^0_j \to \prod_j \mathbb{D}P^1_j \to \cdots \to \prod_j \mathbb{D}P^i_j \to \cdots$$

in $\operatorname{Mod} R$ with $\prod_j \mathbb{D}P^i_j$ injective and $\mathrm{pd}_R \prod_j \mathbb{D}P^i_j \leq i$ (by [C, Thm. 3.3]) for any $i \geq 0$. It implies $\mathbb{D}N' \in \mathscr{G}_\infty(0)$. $\square$

Following [AR2], for a full subcategory $\mathscr{X}$ of $\operatorname{mod} R$ we write

$$\begin{aligned}\operatorname{Rapp}(\mathscr{X}) &:= \{M \in \operatorname{mod} R \mid \text{there exists a right } \mathscr{X}\text{-approximation of } M\},\\ \operatorname{Lapp}(\mathscr{X}) &:= \{M \in \operatorname{mod} R \mid \text{there exists a left } \mathscr{X}\text{-approximation of } M\}.\end{aligned}$$

We use $\mathscr{P}^{\infty}(\operatorname{mod} R)$ (resp. $\mathscr{I}^{\infty}(\operatorname{mod} R)$) to denote the full subcategory of $\operatorname{mod} R$ consisting of modules with finite projective (resp. injective) dimension.

Proposition 5.4. *Let R satisfy the Auslander condition. Then we have*

(1) $\operatorname{Lapp}(\operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R)$
$$= \{M \in \operatorname{mod} R \mid \textit{there exists an exact sequence } 0 \to M \to X \to Y \to 0 \textit{ in } \operatorname{mod} R \textit{ with } X \in \operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R \textit{ and } Y \in \mathscr{P}^{\infty}(\operatorname{mod} R)\}.$$

(2) $\operatorname{Rapp}(\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R)$
$$= \{M \in \operatorname{mod} R \mid \textit{there exists an exact sequence } 0 \to Y \to X \to M \to 0 \textit{ in } \operatorname{mod} R \textit{ with } X \in \mathscr{G}_{\infty}(0) \cap \operatorname{mod} R \textit{ and } Y \in \mathscr{I}^{\infty}(\operatorname{mod} R)\}.$$

Proof. It is easy to see that $\operatorname{Lapp}(\operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R) \supseteq \{M \in \operatorname{mod} R \mid$ there exists an exact sequence $0 \to M \to X \to Y \to 0$ in $\operatorname{mod} R$ with $X \in \operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R$ and $Y \in \mathscr{P}^{\infty}(\operatorname{mod} R)\}$ and $\operatorname{Rapp}(\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R) \supseteq \{M \in \operatorname{mod} R \mid$ there exists an exact sequence $0 \to Y \to X \to M \to 0$ in $\operatorname{mod} R$ with $X \in \mathscr{G}_{\infty}(0) \cap \operatorname{mod} R$ and $Y \in \mathscr{I}^{\infty}(\operatorname{mod} R)\}$. So it suffices to prove the converse inclusions.

(1) Let $M \in \operatorname{Lapp}(\operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R)$. Because R satisfies the Auslander condition, the injective cogenerator $\mathbb{D}(R_R)$ for $\operatorname{Mod} R$ is in $\operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R$. So we may assume that

$$0 \to M \xrightarrow{f} X^M \to Y^M \to 0$$

is exact in $\operatorname{mod} R$ such that f is a minimal left $\operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R$-approximation of M.

By the proof of Proposition 5.3(1) we have an exact sequence

$$0 \to M \xrightarrow{\beta} N \to K \to 0$$

in $\operatorname{Mod} R$ satisfying the following properties:

(a) $N \in \operatorname{Co}\mathscr{G}_{\infty}(0)$ and $N = \varinjlim_{n\geq 1} N_n$ $(= \bigcup_{n\geq 1} N_n)$ with all $N_n \in \operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R$.

(b) $K = \varinjlim_{n\geq 1} K_n$ $(= \bigcup_{n\geq 1} K_n)$ with $\operatorname{pd}_R K_n \leq n$ for any $n \geq 1$.

(c) $0 \to M \xrightarrow{\beta_n} N_n \to K_n \to 0$ is exact for any $n \geq 1$ and $\beta = \varinjlim_{n\geq 1} \beta_n$.

(d) $\operatorname{Hom}_R(\beta, T)$ is epic for any $T \in \operatorname{Co}\mathscr{G}_{\infty}(0) \cap \operatorname{mod} R$.

Then there exist $u \in \mathrm{Hom}_R(N, X^M)$ and $v_n \in \mathrm{Hom}_R(X^M, N_n)$ such that $f = u\beta$ and $\beta_n = v_n f$ for any $n \geq 1$. It induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, equal] & X^M \arrow[r] \arrow[d, "v"] & Y^M \arrow[r] \arrow[d, dashed, "v'"] & 0 \\
0 \arrow[r] & M \arrow[r, "\beta"] \arrow[d, equal] & N \arrow[r] \arrow[d, "u"] & K \arrow[r] \arrow[d, dashed, "u'"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"] & X^M \arrow[r] & Y^M \arrow[r] & 0,
\end{tikzcd}$$

where $v = \varinjlim_{n \geq 1} v_n$, v' and u' are induced homomorphisms. By the minimality of f we have that uv is an isomorphism and so is $u'v'$. It implies that $v' \colon Y^M \to K$ $(= \varinjlim_{n\geq 1} K_n = \bigcup_{n\geq 1} K_n)$ is a split monomorphism. Because Y^M is finitely generated, we have $\mathrm{Im}\, v' \subseteq K_n$ for some n. So Y^M is isomorphic to a direct summand of K_n, and hence $\mathrm{pd}_R Y^M \leq n$.

(2) Let $M \in \mathrm{Rapp}(\mathscr{G}_\infty(0) \cap \mathrm{mod}\, R)$. Then $\mathbb{D}M \in \mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0)^{\mathrm{op}} \cap \mathrm{mod}\, R^{\mathrm{op}})$. By (1) there exists an exact sequence

$$0 \to \mathbb{D}M \to X \to Y \to 0$$

in $\mathrm{mod}\, R^{\mathrm{op}}$ with $X \in \mathrm{Co}\mathscr{G}_\infty(0)^{\mathrm{op}} \cap \mathrm{mod}\, R^{\mathrm{op}}$ and $Y \in \mathscr{P}^\infty(\mathrm{mod}\, R^{\mathrm{op}})$. So we get an exact sequence

$$0 \to \mathbb{D}Y \to \mathbb{D}X \to M \to 0$$

in $\mathrm{mod}\, R$ with $\mathbb{D}X \in \mathscr{G}_\infty(0) \cap \mathrm{mod}\, R$ and $\mathbb{D}Y \in \mathscr{I}^\infty(\mathrm{mod}\, R)$. $\square$

As a consequence of Proposition 5.4, we get the following proposition.

Proposition 5.5. *Let R satisfy the Auslander condition. Then we have*

(1) $\mathrm{Rapp}(\mathscr{G}_\infty(0) \cap \mathrm{mod}\, R) = \{M \in \mathrm{mod}\, R \mid$ *there exists* $n \geq 1$ *such that* $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \cap \mathrm{mod}\, R\}$.

(2) $\mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0) \cap \mathrm{mod}\, R) = \{M \in \mathrm{mod}\, R \mid$ *there exists* $n \geq 1$ *such that* $\Omega^n(M) \in \mathrm{Co}\mathscr{G}_\infty(n) \cap \mathrm{mod}\, R\}$.

Proof. (1) Let $M \in \mathrm{Rapp}(\mathscr{G}_\infty(0) \cap \mathrm{mod}\, R)$. Then by Proposition 5.4(2), there exists an exact sequence

$$0 \to Y \to X \to M \to 0$$

in $\mathrm{mod}\, R$ with $X \in \mathscr{G}_\infty(0) \cap \mathrm{mod}\, R$ and $Y \in \mathscr{I}^\infty(\mathrm{mod}\, R)$. Suppose $\mathrm{id}_R Y = k$ $(< \infty)$. Then for any $n > k$ we have

$$\mathrm{Ext}^1_R(-, \Omega^{-n+1}(X)) \cong \mathrm{Ext}^n_R(-, X) \cong \mathrm{Ext}^n_R(-, M) \cong \mathrm{Ext}^1_R(-, \Omega^{-n+1}(M)),$$

which implies that $\Omega^{-n+1}(X)$ and $\Omega^{-n+1}(M)$ are injectively equivalent. Because $X \in \mathscr{G}_\infty(0)$, we have $\Omega^{-n+1}(X) \in \mathscr{G}_\infty(n-1)$. So $\Omega^{-n+1}(M) \in \mathscr{G}_\infty(n-1)$ and $\Omega^{-n}(M) \in \mathscr{G}_\infty(n)$.

Conversely, let $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \cap \operatorname{mod} R$. We have the following commutative diagrams with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & I \arrow[r, dashed] \arrow[d, dashed] & G \arrow[r, dashed] \arrow[d, dashed] & M \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d, dashed] & P_0(E^0(M)) \arrow[r] \arrow[d, dashed] & E^0(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d, dashed] & P_0(E^1(M)) \arrow[r] \arrow[d] & E^1(M) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r] & K_{n-2} \arrow[r] \arrow[d, dashed] & P_0(E^{n-2}(M)) \arrow[r] \arrow[d, dashed] & E^{n-2}(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{n-1} \arrow[r] \arrow[d, dashed] & P_0(E^{n-1}(M)) \arrow[r] \arrow[d, dashed] & E^{n-1}(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & \Omega^{-n}(M) \arrow[r, equal] \arrow[d, dashed] & \Omega^{-n}(M) \arrow[d] & \\
 & & 0 & 0, &
\end{tikzcd}$$

where $K_i = \operatorname{Ker}(P_0(E^i(M)) \to E^i(M))$ for any $0 \leq i \leq n-1$, $G = \operatorname{Ker}(P_0(E^0(M)) \to P_0(E^1(M)))$ and $I = \operatorname{Ker}(K_0 \to K_1)$. Because R satisfies the Auslander condition, we have that $P_0(E^i(M))$ is injective and satisfies the Auslander condition for any $0 \leq i \leq n-1$ by Theorem 4.9. So $\operatorname{id}_R K_i \leq 1$ for any $0 \leq i \leq n-1$, and hence $\operatorname{id}_R I \leq n$ by the exactness of the leftmost column in the above diagram. On the other hand, by [H2, Cor. 3.9] and the exactness of the middle column in the above diagram, we have $G \in \mathscr{G}_\infty(0) \cap \operatorname{mod} R$. Thus the exact sequence

$$0 \to I \to G \to M \to 0$$

in $\operatorname{mod} R$ is a right $\mathscr{G}_\infty(0) \cap \operatorname{mod} R$-approximation of M and $M \in \operatorname{Rapp}(\mathscr{G}_\infty(0) \cap \operatorname{mod} R)$.

(2) It is dual to the proof of (1), so we omit it. □

Corollary 5.6. *Let R satisfy the Auslander condition. Then we have*

(1) *$\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ is contravariantly finite in $\operatorname{mod} R$ if and only if there exists $n \geq 1$ such that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \cap \operatorname{mod} R$ for any $M \in \operatorname{mod} R$;*

(2) *$\operatorname{Co}\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ is covariantly finite in $\operatorname{mod} R$ if and only if there exists $n \geq 1$ such that $\Omega^{n}(M) \in \operatorname{Co}\mathscr{G}_\infty(n) \cap \operatorname{mod} R$ for any $M \in \operatorname{mod} R$.*

Proof. (1) The sufficiency follows from Proposition 5.5(1).

Conversely, let $\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ be contravariantly finite in $\operatorname{mod} R$ and $\{S_1, S_2, \dots, S_t\}$ a complete set of non-isomorphic simple modules in $\operatorname{mod} R$. By Proposition 5.5(1), there exists $n_i \geq 1$ such that $\Omega^{-n_i}(S_i) \in \mathscr{G}_\infty(n_i)$ for any $1 \leq i \leq t$. Put $n := \max\{n_1, n_2, \dots, n_t\}$. Then $\Omega^{-n}(S_i) \in \mathscr{G}_\infty(n)$ for any $1 \leq i \leq t$.

We will prove that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n)$ for any $M \in \operatorname{mod} R$ by induction on $\operatorname{length}(M)$ (the length of M). If $\operatorname{length}(M) = 1$, then $M \cong S_i$ for some $1 \leq i \leq t$ and the assertion follows. Now suppose $\operatorname{length}(M) \geq 2$. Then there exists an exact sequence

$$0 \to S \to M \to M/S \to 0$$

in $\operatorname{mod} R$ with S simple and $\operatorname{length}(M/S) < \operatorname{length}(M)$. By the induction hypothesis, both S and M/S are in $\mathscr{G}_\infty(n)$. Then M is also in $\mathscr{G}_\infty(n)$ by the horseshoe lemma.

(2) It is dual to the proof of (1), so we omit it. □

Let $M \in \operatorname{mod} R$ and let

$$P_1(M) \to P_0(M) \to M \to 0$$

be a minimal projective presentation of $M \in \operatorname{mod} R$. For any $n \geq 1$, recall from [AB] that M is called *n-torsion-free* if $\operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) = 0$ for any $1 \leq i \leq n$, where $\operatorname{Tr} M = \operatorname{Coker}(P_0(M)^* \to P_1(M)^*)$ is the *transpose* of M and $(-)^* = \operatorname{Hom}_R(-, R)$. We use $\Omega^n(\operatorname{mod} R)$ (resp. $\mathscr{T}_n(\operatorname{mod} R)$) to denote the full subcategory of $\operatorname{mod} R$ consisting of n-syzygy (resp. n-torsion-free) modules. Put

$$\Omega^\infty(\operatorname{mod} R) := \bigcap_{n \geq 1} \Omega^n(\operatorname{mod} R) \quad \text{and} \quad \mathscr{T}_\infty(\operatorname{mod} R) := \bigcap_{n \geq 1} \mathscr{T}_n(\operatorname{mod} R).$$

In general, we have $\Omega^n(\operatorname{mod} R) \supseteq \mathscr{T}_n(\operatorname{mod} R)$ for any $n \geq 1$ (cf. [AB, Thm. 2.17]).

Lemma 5.7. *If $R \in \mathscr{G}_n(0)$ with $n \geq 1$, then*

$$\mathscr{G}_n(0) \cap \operatorname{mod} R = \Omega^n(\operatorname{mod} R) = \mathscr{T}_n(\operatorname{mod} R);$$

in particular, if R satisfies the Auslander condition, then

$$\mathscr{G}_\infty(0) \cap \operatorname{mod} R = \Omega^\infty(\operatorname{mod} R) = \mathscr{T}_\infty(\operatorname{mod} R).$$

Proof. We have $\mathscr{G}_n(0) \cap \operatorname{mod} R = \Omega^n(\operatorname{mod} R)$ by [AR3, Prop. 5.1] and $\Omega^n(\operatorname{mod} R) = \mathscr{T}_n(\operatorname{mod} R)$ by [AR4, Prop. 1.6 and Thm. 4.7]. □

For a full subcategory $\mathscr{C}$ of $\operatorname{mod} R$, we write

$$\mathscr{C}^{\perp_1} := \{M \in \operatorname{mod} R \mid \operatorname{Ext}^1_R(C, M) = 0 \text{ for any } C \in \mathscr{C}\}.$$

Auslander and Reiten conjectured in [AR3] that R is Gorenstein (that is, $\operatorname{id}_R R = \operatorname{id}_{R^{\mathrm{op}}} R < \infty$) if R satisfies the Auslander condition. It remains open. Now we are in a position to establish the connection between this conjecture and the contravariant finiteness of $\mathscr{G}_\infty(0) \cap \operatorname{mod} R$, $\Omega^\infty(\operatorname{mod} R)$ and $\mathscr{T}_\infty(\operatorname{mod} R)$ as follows.

Theorem 5.8. *Let R satisfy the Auslander condition. Then the following statements are equivalent:*

(1) *R is Gorenstein.*
(2) *$\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ is contravariantly finite in $\operatorname{mod} R$.*
(3) *$\operatorname{Co}\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ is covariantly finite in $\operatorname{mod} R$.*
(4) *$\Omega^\infty(\operatorname{mod} R)$ is contravariantly finite in $\operatorname{mod} R$.*
(5) *$\mathscr{T}_\infty(\operatorname{mod} R)$ is contravariantly finite in $\operatorname{mod} R$.*

Proof. Because R satisfies the Auslander condition if and only if R^{op} does, we get (2) $\Leftrightarrow$ (3). By Lemma 5.7 we have (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (2). Assume that R is Gorenstein with $\operatorname{id}_R R = \operatorname{id}_{R^{\mathrm{op}}} R = n$. By [I, Prop. 1], we have $\operatorname{pd}_R E \leq n$ for any injective left R-module E. So $\mathscr{G}_\infty(0) \cap \operatorname{mod} R = \mathscr{G}_n(0) \cap \operatorname{mod} R$, and hence $\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ is contravariantly finite in $\operatorname{mod} R$ by Theorem 5.1.

(2) $\Rightarrow$ (1). Assume that $\mathscr{G}_\infty(0) \cap \operatorname{mod} R$ is contravariantly finite in $\operatorname{mod} R$. Then there exists $n \geq 1$ such that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \cap \operatorname{mod} R$ for any $M \in \operatorname{mod} R$ by Corollary 5.6, which implies $\mathscr{G}_\infty(0) \cap \operatorname{mod} R = \mathscr{G}_n(0) \cap \operatorname{mod} R$. Because $\mathscr{G}_n(0) \cap \operatorname{mod} R = \mathscr{T}_n(\operatorname{mod} R)$ by Lemma 5.7, we have

$$(\mathscr{G}_\infty(0) \cap \operatorname{mod} R)^{\perp_1} = (\mathscr{G}_n(0) \cap \operatorname{mod} R)^{\perp_1} = \mathscr{T}_n(\operatorname{mod} R)^{\perp_1} = \mathscr{I}^n(\operatorname{mod} R)$$

by [HI, Thm. 1.3]. On the other hand, it is easy to see that $\mathscr{I}^\infty(\operatorname{mod} R) \subseteq (\mathscr{G}_\infty(0) \cap \operatorname{mod} R)^{\perp_1}$. So $\mathscr{I}^\infty(\operatorname{mod} R) = \mathscr{I}^n(\operatorname{mod} R)$, and hence $\mathscr{P}^\infty(\operatorname{mod} R^{\mathrm{op}}) = \mathscr{P}^n(\operatorname{mod} R^{\mathrm{op}})$. Thus $\operatorname{id}_{R^{\mathrm{op}}} R \le n$ by [HI, Cor. 5.3], which implies that R is Gorenstein by [AR3, Cor. 5.5(b)]. □

We remark that the equivalence between (1) and (5) in Theorem 5.8 has been known for a commutative ring under some mild assumption (see [T, Cor. 3.15]).

As an application of Theorem 5.8, we obtain in the following result some equivalent characterizations of Auslander-regular algebras. Note that the converse of Corollary 4.10 does not hold true in general by Remark 4.11. The following result also shows when this converse holds true.

Theorem 5.9. *The following statements are equivalent:*

(1) *R is Auslander-regular.*
(2) $\mathscr{G}_\infty(0) = \mathscr{P}^0(\operatorname{Mod} R)$.
(3) $\mathscr{G}_\infty(0) \cap \operatorname{mod} R = \mathscr{P}^0(\operatorname{mod} R)$.
(4) $\mathscr{G}_\infty(s) = \mathscr{P}^s(\operatorname{Mod} R)$ *for any* $s \ge 0$.
(5) $\mathscr{G}_\infty(s) \cap \operatorname{mod} R = \mathscr{P}^s(\operatorname{mod} R)$ *for any* $s \ge 0$.

Proof. Both (2) ⇒ (3) and (4) ⇒ (5) are trivial. By Corollary 4.13 we have (2) ⇔ (4) and (3) ⇔ (5).

(1) ⇒ (2). By (1) and Corollary 4.10 we have $\mathscr{P}^0(\operatorname{Mod} R) \subseteq \mathscr{G}_\infty(0)$.

Let $\operatorname{gl.dim} R = n$ $(< \infty)$ and $M \in \mathscr{G}_\infty(0)$. Then in a minimal injective coresolution

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^n(M) \to 0$$

of M in $\operatorname{Mod} R$, we have $\operatorname{pd}_R E^i(M) \le i$ for any $0 \le i \le n$. By the dimension shifting we have that M is projective, which implies $\mathscr{G}_\infty(0) \subseteq \mathscr{P}^0(\operatorname{Mod} R)$.

(5) ⇒ (1). By (5), R satisfies the Auslander condition and $\mathscr{G}_\infty(0) \cap \operatorname{mod} R = \mathscr{P}^0(\operatorname{mod} R)$ is contravariantly finite in $\operatorname{mod} R$. So R is Gorenstein by Theorem 5.8. Suppose $\operatorname{id}_{R^{\mathrm{op}}} R = \operatorname{id}_R R = n$ $(< \infty)$. Then $\operatorname{pd}_R E \le n$ for any injective left R-module E by [I, Prop. 1]. So for any $M \in \operatorname{mod} R$, we have $M \in \mathscr{G}_\infty(n) \cap \operatorname{mod} R$, and hence $\operatorname{pd}_R M \le n$ by (5). It follows that $\operatorname{gl.dim} R \le n$. □

Acknowledgements

This research was partially supported by NSFC (grant nos. 11971225, 12171207). The author thanks Edgar E. Enochs, Xiaojin Zhang and Guocheng Dai for helpful discussions, and also thanks the referee for very useful suggestions.

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