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Relative Left Derived Functors of Tensor Product Functors

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Abstract We introduce and study the relative left derived functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$ in the module category, which unifies several related left derived functors. Then we give some criteria for computing the $\mathscr{F}$-resolution dimensions of modules in terms of the properties of $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$. We also construct a complete and hereditary cotorsion pair relative to balanced pairs. Some known results are obtained as corollaries.

Keywords Tensor product functors, relative left derived functors, balanced pairs, cotorsion pairs, (co)resolution dimension

MR(2010) Subject Classification 18G10, 18G25, 16E10, 16E30

1 Introduction

The derived functor is a powerful tool in studying homological properties of rings and modules in classical homological algebra, see [2, 3, 5, 6, 8, 9], and so on. It is well known that the classical left derived functor $\mathrm{Tor}(-,-)$ induced by $-\otimes-$ simultaneously measures unflatness of the first variable and the second variable, and that one has the adjoint isomorphism theorem with the classical right derived functor $\mathrm{Ext}(-,-)$ induced by $\mathrm{Hom}(-,-)$. In fact, these theories are based on a standard balanced pair $(\mathcal{P}_0, \mathcal{I}_0)$, where $\mathcal{P}_0$ and $\mathcal{I}_0$ are subcategories of R-modules consisting of projective modules and injective modules respectively.

The relative homological algebra, especially Gorenstein homological algebra, as a generalization of the classical one, was introduced by Enochs and Jenda in 1970s. It has been developed to an advanced level in recent years, see [3, 4, 6, 7, 10, 12] and the references therein. In particular, in [3], Enochs and Jenda introduced and studied the Gorenstein left (resp. right) derived functor $\mathrm{Gtor}(-,-)$ (resp. $\mathrm{Gext}(-,-)$). Then Holm provided in [7] a sufficient condition for the

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functor $-\otimes-$ being balanced with respect to Gorenstein flat modules, and he investigated the relations between the Gorenstein left derived functor $\mathrm{Gtor}(-,-)$ and the classical left derived functor $\mathrm{Tor}(-,-)$. Recently, we introduced a more general right derived functors $\mathcal{E}\mathrm{xt}^i_{\mathscr{A}}(-,-)$ induced by $\mathrm{Hom}(-,-)$ relative to a given balanced pair in [11], where $\mathscr{A}$ is an abelian category. The aim of this paper is to introduce and study the relative left derived functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$ induced by $-\otimes-$.

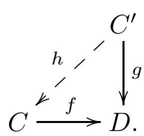
This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results. In Section 3, we write $(-)^+ := \mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. Let $(\mathscr{C},\mathscr{D})$ be a balanced pair in Mod-R and $(\mathscr{C}',\mathscr{D}')$ a balanced pair in R-Mod, and let $\mathscr{F}$ be a precovering subcategory of Mod-R containing $\mathscr{C}$ and $\mathscr{F}'$ a precovering subcategory of R-Mod containing $\mathscr{C}'$, such that ${\mathscr{F}'}^+\subseteq\mathscr{D}$ and $\mathscr{F}^+\subseteq\mathscr{D}'$. Then we may define the relative left derived functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$ of $-\otimes_R-$, which unifies several related left derived functors. We get some criteria for computing the $\mathscr{F}$-resolution dimensions of modules in terms of the properties of the relative left derived functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$. Furthermore, by using the properties of $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$, we construct in Section 4 a complete and hereditary cotorsion pair relative to $(\mathscr{C},\mathscr{D})$. Some known results are obtained as corollaries.

2 Definitions and Notations

In this section, $\mathscr{A}$ is an abelian category and a subcategory of $\mathscr{A}$ means a full and additive subcategory closed under isomorphisms and direct summands.

Definition 2.1 ([2]) *Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. A morphism $f: C\to D$ in $\mathscr{A}$ with $C\in\mathscr{C}$ is called a* $\mathscr{C}$-precover *of D if for any morphism $g: C'\to D$ in $\mathscr{A}$ with $C'\in\mathscr{C}$, there exists a morphism $h: C'\to C$ such that the following diagram commutes:*



The morphism $f: C\to D$ is called right minimal *if an endomorphism $h: C\to C$ is an automorphism whenever $f=fh$. A $\mathscr{C}$-precover is called a* $\mathscr{C}$-cover *if it is right minimal; $\mathscr{C}$ is called a* (pre)covering subcategory *of $\mathscr{A}$ if every object in $\mathscr{A}$ has a $\mathscr{C}$-(pre)cover; $\mathscr{C}$ is called a* surjective (pre)covering subcategory *of $\mathscr{A}$ if every object in $\mathscr{A}$ has a surjective $\mathscr{C}$-(pre)cover. Dually, the notions of a* $\mathscr{C}$-(pre)envelope, *a* (pre)enveloping subcategory *and a* monic (pre)enveloping subcategory *are defined.*

The following result is useful.

Lemma 2.2 ([8, Theorem 2.5]) *Let $\mathscr{T}$ be any class of modules which is closed under pure quotient modules. Then the following statements are equivalent:*

(1) *$\mathscr{T}$ is closed under direct sums.*

(2) *$\mathscr{T}$ is precovering.*

(3) *$\mathscr{T}$ is covering.*

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. Recall that a sequence in $\mathscr{A}$ is called $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-*exact* if it is exact after applying the functor $\mathrm{Hom}_{\mathscr{A}}(C,-)$ for any object $C\in\mathscr{C}$. Let $M\in\mathscr{A}$. An exact sequence (of finite or infinite length):

$$\cdots \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \to 0$$

in $\mathscr{A}$ with all $C_i \in \mathscr{C}$ is called a *$\mathscr{C}$-resolution* of M if it is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact, that is, each f_i is an epic $\mathscr{C}$-precover of $\mathrm{Im}\, f_i$. We denote sometimes the $\mathscr{C}$-resolution of M by $\mathscr{C}_\bullet \to M$, where

$$\mathscr{C}_\bullet := \cdots \to C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \to 0$$

is the *deleted $\mathscr{C}$-resolution* of M. Note that by a version of the comparison theorem, the $\mathscr{C}$-resolution is unique up to homotopy (see [3, p. 169]). The *$\mathscr{C}$-resolution dimension* $\mathscr{C}$-res.dim M of M is defined to be the minimal integer $n \geq 0$ such that there is a $\mathscr{C}$-resolution

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to M \to 0$$

of M. If there exists no such an integer, we set $\mathscr{C}$-res.dim $M = \infty$. Dually, the notions of a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact sequence, a *$\mathscr{C}$-coresolution* and the *$\mathscr{C}$-coresolution dimension* $\mathscr{C}$-cores.dim M of M are defined.

Definition 2.3 ([1]) *A pair $(\mathscr{C}, \mathscr{D})$ of additive subcategories in $\mathscr{A}$ is called a* balanced pair *if the following conditions are satisfied*:

(1) *$\mathscr{C}$ is surjective precovering and $\mathscr{D}$ is monic preenveloping.*

(2) *For any $M \in \mathscr{A}$, there is a $\mathscr{C}$-resolution $\mathscr{C}_\bullet \to M$ such that it is* $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{D})$*-exact.*

(3) *For any $N \in \mathscr{A}$, there is a $\mathscr{D}$-coresolution $N \to \mathscr{D}_\bullet$ such that it is* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$*-exact.*

Let $(\mathscr{C}, \mathscr{D})$ be a balanced pair in $\mathscr{A}$. For any $M, N \in \mathscr{A}$, there exist a $\mathscr{C}$-resolution $\mathscr{C}_\bullet \to M$ of M and a $\mathscr{D}$-coresolution $N \to \mathscr{D}_\bullet$ of N. We write

$$\mathrm{Ext}^n_{(\mathscr{C},\mathscr{D})}(M, N) := \mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}_\bullet, N)) = \mathrm{H}^n(\mathrm{Hom}_{\mathscr{A}}(M, \mathscr{D}_\bullet)).$$

Definition 2.4 ([10]) *Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. The* Gorenstein subcategory $\mathcal{G}(\mathscr{C})$ *of $\mathscr{A}$ is defined as $\mathcal{G}(\mathscr{C}) := \{M \in \mathscr{A} \mid$ there exists an exact sequence $\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$ in $\mathscr{A}$ with all terms in $\mathscr{C}$, which is both* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$*-exact and* $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$*-exact, such that $M \cong \mathrm{Im}(C_0 \to C^0)\}$.*

Let R be a ring. If $\mathscr{C}$ is the category of projective (resp. injective) R-modules, then $\mathcal{G}(\mathscr{C})$ is exactly the category of Gorenstein projective (resp. injective) R-modules, which is denoted by GProj (resp. GInj). We use $\mathrm{Gpd}\, M$ (resp. $\mathrm{Gid}\, M$) to denote the Gorenstein projective (resp. injective) dimension of M.

Definition 2.5 ([3]) *Let R be a ring. A right R-module M is called* Gorenstein flat *if there exists an exact sequence of right R-modules*

$$\mathbb{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

with all terms flat, such that $M = \mathrm{Im}(F_0 \to F^0)$ and the sequence $\mathbb{F} \otimes_R I$ is exact for any injective left R-module I. The category of Gorenstein flat R-modules is denoted by GFlat*, and we use* $\mathrm{Gfd}\, M$ *to denote the Gorenstein flat dimension of M.*

Let R be a ring. Recall that a short exact sequence ξ: $0 \to A \to B \to C \to 0$ of R-modules is called *pure exact* if the induced sequence $\mathrm{Hom}_R(F, \xi)$ is exact for every finitely presented R-module F. In this case A is called a *pure submodule* of B and C is called a *pure quotient* *module* of B. In addition, modules that are projective (resp. injective) with respect to pure exact sequences are called *pure projective* (resp. *pure injective*). The class of all pure projective (resp. pure injective) R-modules is denoted by $\mathcal{PP}$ (resp. $\mathcal{PI}$).

3 Relative Left Derived Functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$

From now on, R is an associative ring with identity, Mod-R is the category of right R-modules and R-Mod is the category of left R-modules. We use $\mathcal{P}_0$ (resp. $\mathcal{I}_0$, $\mathcal{F}_0$) to denote the subcategory of Mod-R consisting of projective (resp. injective, flat) modules, and use $\mathrm{pd}\,M$ (resp. $\mathrm{id}\,M$, $\mathrm{fd}\,M$) to denote the projective (resp. injective, flat) dimension of M for any R-module M. For a subcategory of $\mathscr{X}$ of Mod-R, $\mathscr{X}^{\mathrm{op}}$ denotes the corresponding opposite subcategory of R-Mod.

We start with the following theorem which is crucial to define the relative left derived functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$.

Theorem 3.1 *Let $(\mathscr{C},\mathscr{D})$ be a balanced pair in* Mod-R *and $(\mathscr{C}',\mathscr{D}')$ a balanced pair in* R-Mod, *and let $\mathscr{F}$ be a surjective precovering subcategory of* Mod-R *containing $\mathscr{C}$ and $\mathscr{F}'$ a surjective precovering subcategory of* R-Mod *containing $\mathscr{C}'$, such that ${\mathscr{F}'}^{+}\subseteq\mathscr{D}$ and $\mathscr{F}^{+}\subseteq\mathscr{D}'$. Then we have*

$$\mathrm{H}_n(\mathscr{F}_\bullet\otimes_R N)\cong\mathrm{H}_n(M\otimes_R\mathscr{F}'_\bullet)$$

for any $M\in$ Mod-R, *$N\in$* R-Mod *and $n\geq 0$, where $\mathscr{F}_\bullet$ (resp. $\mathscr{F}'_\bullet$) is the deleted resolution of M (resp. N).*

Proof Note that $\mathscr{F}$ and $\mathscr{F}'$ are both surjective precovering by assumption. Take an $\mathscr{F}$-resolution

$$\cdots\xrightarrow{f_{n+1}}F_n\xrightarrow{f_n}\cdots\xrightarrow{f_2}F_1\xrightarrow{f_1}F_0\xrightarrow{f_0}M\to 0$$

of M in Mod-R. We claim that

$$\cdots\xrightarrow{f_{n+1}\otimes 1_{F'}}F_n\otimes_R F'\xrightarrow{f_n\otimes 1_{F'}}\cdots\xrightarrow{f_2\otimes 1_{F'}}F_1\otimes_R F'\xrightarrow{f_1\otimes 1_{F'}}F_0\otimes_R F'\xrightarrow{f_0\otimes 1_{F'}}M\otimes_R F'\to 0$$

is exact for any $F'\in\mathscr{F}'$. It suffices to show that

$$0\to\mathrm{Ker}\,f_i\otimes_R F'\to F_i\otimes_R F'\to\mathrm{Ker}\,f_{i-1}\otimes_R F'\to 0 \tag{3.1}$$

is exact for any $i\geq 0$ (where $\mathrm{Ker}\,f_{-1}=M$). Because ${\mathscr{F}'}^{+}\subseteq\mathscr{D}$, we have $\mathrm{Ext}^1_{(\mathscr{C},\mathscr{D})}(\mathrm{Ker}\,f_{i-1},{F'}^{+})=0$ for any $i\geq 0$. Then by [3, Theorem 8.2.3(2)], we get the following commutative diagram with bottom row exact:

$$\begin{array}{ccccccccc}
0 & \to & (\mathrm{Ker}\,f_{i-1}\otimes_R F')^{+} & \to & (F_i\otimes_R F')^{+} & \to & (\mathrm{Ker}\,f_i\otimes_R F')^{+} & \to & 0\\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \to & \mathrm{Hom}_R(\mathrm{Ker}\,f_{i-1},{F'}^{+}) & \to & \mathrm{Hom}_R(F_i,{F'}^{+}) & \to & \mathrm{Hom}_R(\mathrm{Ker}\,f_i,{F'}^{+}) & \to & \mathrm{Ext}^1_{(\mathscr{C},\mathscr{D})}(\mathrm{Ker}\,f_{i-1},{F'}^{+})=0.
\end{array}$$

The vertical isomorphisms follow from the adjoint isomorphism theorem. So the upper row is also exact, which induces the exactness of the sequence (3.1). The claim is proved.

Similarly, take an $\mathscr{F}'$-resolution

$$\cdots\xrightarrow{f'_{n+1}}F'_n\xrightarrow{f'_n}\cdots\xrightarrow{f'_2}F'_1\xrightarrow{f'_1}F'_0\xrightarrow{f'_0}N\to 0$$

of N in R-Mod. Then we have a long exact sequence

$$\cdots \xrightarrow{1_F\otimes f_{n+1}} F\otimes_R F'_n \xrightarrow{1_F\otimes f_n} \cdots \xrightarrow{1_F\otimes f_2} F\otimes_R F'_1 \xrightarrow{1_F\otimes f_1} F\otimes_R F'_0 \xrightarrow{1_F\otimes f'_0} F\otimes_R N \to 0$$

for any $F \in \mathscr{F}$.

By [3, Definition 8.2.13], $-\otimes-$ is left balanced relative to $\mathscr{F}\times\mathscr{F}'$. So $\mathrm{H}_n(\mathscr{F}_\bullet\otimes_R N)\cong \mathrm{H}_n(M\otimes_R \mathscr{F}'_\bullet)$ for any $M\in$ Mod-R and $N\in R$-Mod. □

In the rest of this paper, the assumptions in Theorem 3.1 *are satisfied.* Now we can give the following definition.

Definition 3.2 *For any $M\in$ Mod-R and $N\in R$-Mod, we define*

$$\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N) := \mathrm{H}_n(\mathscr{F}_\bullet\otimes_R N)\cong \mathrm{H}_n(M\otimes_R \mathscr{F}'_\bullet).$$

By Theorem 3.1, we have that, for any $n\geq 0$, $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)$ is independent of the choices of $\mathscr{F}$-resolutions of M and $\mathscr{F}'$-resolutions of N. So we have $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)\cong \mathrm{Tor}_n^{(\mathscr{C},\mathscr{C}')}(M,N)$.

The following example shows that the above definition unifies several related notions.

Example 3.3 (1) Note that $(\mathcal{P}_0,\mathcal{I}_0)$ is the standard balanced pair in Mod-R. Let $\mathscr{C}=\mathcal{P}_0$, $\mathscr{D}=\mathcal{I}_0$ and $\mathscr{F}=\mathcal{F}_0$, and let $\mathscr{C}'={\mathcal{P}_0}^{\mathrm{op}}$, $\mathscr{D}'={\mathcal{I}_0}^{\mathrm{op}}$ and $\mathscr{F}'={\mathcal{F}_0}^{\mathrm{op}}$. Then $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)$ is exactly the standard homology $\mathrm{Tor}_n^R(M,N)$.

(2) By [3, Example 8.3.2], $(\mathcal{PP},\mathcal{PI})$ is a balanced pair in Mod-R. Let $\mathscr{C}=\mathcal{PP}$ and $\mathscr{D}=\mathcal{PI}$, and let $\mathscr{C}'=\mathcal{PP}^{\mathrm{op}}$ and $\mathscr{D}'=\mathcal{PI}^{\mathrm{op}}$. In this case, we write $\mathrm{Ptor}_n^R(M,N) := \mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)$.

(3) Let R be a Gorenstein ring (that is, R is a left and right and noetherian ring with finite left and right self-injective dimensions). Then by [3, Theorem 12.1.4], we have that (GProj, GInj) is a balanced pair in Mod-R. Now let $\mathscr{C}=$ GProj, $\mathscr{D}=$ GInj and $\mathscr{F}=$ GFlat, and let $\mathscr{C}'=\mathrm{GProj}^{\mathrm{op}}$, $\mathscr{D}'=\mathrm{GInj}^{\mathrm{op}}$ and $\mathscr{F}'=\mathrm{GFlat}^{\mathrm{op}}$. Then $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)$ coincides with $\mathrm{Gtor}_n^R(M,N)$ defined in [3, p. 299].

(4) Let R be an FC ring (that is, R is a left and right and coherent ring with finite left and right FP-self-injective dimensions). Then by [3, Theorem 12.1.4], we have that $(\mathcal{SGF},\mathcal{GFI})$ is a balanced pair in Mod-R, where $\mathcal{SGF}$ and $\mathcal{GFI}$ are the subcategories of Mod-R consisting of strongly Gorenstein flat modules and Gorenstein FP-injective modules, respectively. Now let $\mathscr{C}=\mathscr{F}=\mathcal{SGF}$ and $\mathscr{D}=\mathcal{GFI}$, and let $\mathscr{C}'=\mathscr{F}'=\mathcal{SGF}^{\mathrm{op}}$ and $\mathscr{D}'=\mathcal{GFI}^{\mathrm{op}}$. Then $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)$ coincides with $\mathrm{Dtor}_n^R(M,N)$ defined in [12].

Lemma 3.4 (1) $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)=0$ *for any* $n<0$.

(2) $\mathrm{Tor}_0^{(\mathscr{F},\mathscr{F}')}(M,N)$ *is naturally isomorphic to* $M\otimes_R N$.

(3) $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M,N)=0$ *for any* $n\geq 1$ *if either* $M\in\mathscr{F}$ *or* $N\in\mathscr{F}'$.

(4) *Let*

$$0\to A\to B\to C\to 0$$

be a $\mathrm{Hom}_R(\mathscr{C},-)$*-exact* (*equivalently* $\mathrm{Hom}_R(-,\mathscr{D})$*-exact*) *sequence in* Mod-R. *Then for any $N\in R$-Mod, we have the following long exact sequences*:

$$\cdots\to \mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(C,N)\to \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(A,N)\to \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(B,N)\to \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(C,N)\to$$

$$\cdots\to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(C,N)\to A\otimes_R N\to B\otimes_R N\to C\otimes_R N\to 0.$$

(5) *Let*

$$0 \to A' \to B' \to C' \to 0$$

be a $\mathrm{Hom}_R(\mathscr{C}', -)$*-exact* (*equivalently* $\mathrm{Hom}_R(-, \mathscr{D}')$*-exact*) *sequence in* R-Mod. *Then for any* $M \in$ Mod-R, *we have the following long exact sequences*:

$$\cdots \to \mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M, C') \to \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(M, A') \to \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(M, B') \to \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(M, C') \to$$

$$\cdots \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M, C') \to M \otimes_R A' \to M \otimes_R B' \to M \otimes_R C' \to 0.$$

Proof The assertions (1), (2) and (3) are trivial. Both (4) and (5) follow from [3, Theorem 8.2.3(1)]. □

As an immediate consequence of Lemma 3.4, we have the following

Proposition 3.5 (1) *Let*

$$0 \to A \to B \to C \to 0$$

be a $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact* (*equivalently* $\mathrm{Hom}_R(-, \mathscr{D})$*-exact*) *sequence in* Mod-R. *If* $B \in \mathscr{F}$, *then* $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(C, N) \cong \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(A, N)$ *for any* $N \in R$-Mod *and* $n \geq 2$.

(2) *Let*

$$0 \to A' \to B' \to C' \to 0$$

be a $\mathrm{Hom}_R(\mathscr{C}', -)$*-exact* (*equivalently* $\mathrm{Hom}_R(-, \mathscr{D}')$*-exact*) *sequence in* R-Mod. *If* $B' \in \mathscr{F}'$, *then* $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(M, C') \cong \mathrm{Tor}_{n-1}^{(\mathscr{F},\mathscr{F}')}(M, A')$ *for any* $M \in$ Mod-R *and* $n \geq 2$.

Proposition 3.6 *Let* $N \in R$-Mod. *Then*

(1) *the canonical map* $\mathrm{Ext}^1_{(\mathscr{C}',\mathscr{D}')}(M, N) \to \mathrm{Ext}^1_R(M, N)$ *is an injection for any* $M \in R$-Mod, *and*

(2) *the canonical map* $\mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M, N)$ *is a surjection for any* $M \in$ Mod-R.

Proof (1) By [1, Proposition 2.2], take a $\mathrm{Hom}_R(\mathscr{C}', -)$-exact exact sequence

$$0 \to N \to D \to L \to 0$$

in R-Mod with $D \in \mathscr{D}'$. Then by [3, Theorem 8.2.5(1)], we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(M, D) & \longrightarrow & \mathrm{Hom}_R(M, L) & \longrightarrow & \mathrm{Ext}^1_{(\mathscr{C}',\mathscr{D}')}(M, N) & \longrightarrow & \mathrm{Ext}^1_{(\mathscr{C}',\mathscr{D}')}(M, D) = 0 \\
\| & & \| & & \downarrow & & \\
\mathrm{Hom}_R(M, D) & \longrightarrow & \mathrm{Hom}_R(M, L) & \longrightarrow & \mathrm{Ext}^1_R(M, N), & &
\end{array}$$

and the assertion (1) follows.

(2) The argument is dual to that of (1). By the assumption, take a $\mathrm{Hom}_R(\mathscr{C}', -)$-exact exact sequence

$$0 \to K \to F \to N \to 0$$

in R-Mod with $F \in \mathscr{F}'$. Then by Lemma 3.4(5), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Tor}_1^R(M,N) & \longrightarrow & M \otimes_R K & \longrightarrow & M \otimes_R F \\ & & \downarrow & & \| & & \| \\ 0 = \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M,F) & \longrightarrow & \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M,N) & \longrightarrow & M \otimes_R K & \longrightarrow & M \otimes_R F, \end{array}$$

and the assertion (2) follows. □

Proposition 3.7 *For any* $M \in \mathrm{Mod}\text{-}R$, $N \in R\text{-}\mathrm{Mod}$ *and* $i \geq 0$, *we have*

$$\mathrm{Ext}^i_{(\mathscr{C},\mathscr{D})}(M,N^+) \cong \mathrm{Tor}_i^{(\mathscr{F},\mathscr{F}')}(M,N)^+ \cong \mathrm{Ext}^i_{(\mathscr{C}',\mathscr{D}')}(N,M^+).$$

Proof If $i=0$, then $\mathrm{Hom}_R(M,N^+) \cong (M \otimes_R N)^+ \cong \mathrm{Hom}_{R^{\mathrm{op}}}(N,M^+)$ by [9, Theorems 2.75 and 2.76]. If $i \geq 1$, then

$$\begin{aligned}\mathrm{Ext}^i_{(\mathscr{C},\mathscr{D})}(M,N^+) &\cong \mathrm{H}^i(\mathrm{Hom}_R(\mathscr{C}_\bullet,N^+)) \cong \mathrm{H}^i((\mathscr{C}_\bullet \otimes_R N)^+) \\ &\cong (\mathrm{H}_i(\mathscr{C}_\bullet \otimes_R N))^+ \cong \mathrm{Tor}_i^{(\mathscr{F},\mathscr{F}')}(M,N)^+.\end{aligned}$$

Similarly, we have $\mathrm{Ext}^i_{(\mathscr{C}',\mathscr{D}')}(N,M^+) \cong \mathrm{Tor}_i^{(\mathscr{F},\mathscr{F}')}(M,N)^+$. □

Let R be a Gorenstein ring. Then the right derived functors of $\mathrm{Hom}_R(M,N)$ using a Gorenstein projective resolution of M or a Gorenstein injective coresolution of N are denoted by $\mathrm{Gext}^i_R(M,N)$ (see [3, p. 296]).

Corollary 3.8 *Let R be a Gorenstein ring. Then for any* $M \in \mathrm{Mod}\text{-}R$, $N \in R\text{-}\mathrm{Mod}$ *and* $i \geq 0$, *we have*

$$\mathrm{Gext}^i_R(M,N^+) \cong \mathrm{Gtor}_i^R(M,N)^+ \cong \mathrm{Gext}^i_{R^{\mathrm{op}}}(N,M^+).$$

Proof The assertion follows from Proposition 3.7 and Example 3.3 (3). □

Corollary 3.9 *For any* $M \in \mathrm{Mod}\text{-}R$, $N \in R\text{-}\mathrm{Mod}$ *and* $n \geq 1$, *we have* $\mathrm{Ptor}_n^R(M,N) = 0$.

Proof Put $(\mathscr{C},\mathscr{D}) = (\mathcal{PP}, \mathcal{PI})$. Notice that N^+ is pure injective right R-module for any $N \in R$-Mod by [3, Proposition 5.3.7], so the assertion follows from Proposition 3.7 and Example 3.3(2). □

In the following, we will give some criteria for computing the $\mathscr{F}$-resolution dimensions of modules in terms of the properties of the relative left derived functor $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$.

Theorem 3.10 *Consider the following conditions for any* $M \in \mathrm{Mod}\text{-}R$ *and* $n \geq 0$.

(1) $\mathscr{F}$*-res.dim* $M \leq n$.

(2) $\mathrm{Tor}_{n+i}^{(\mathscr{F},\mathscr{F}')}(M,N) = 0$ *for any* $N \in R\text{-}\mathrm{Mod}$ *and* $i \geq 1$.

(3) $\mathrm{Tor}_{n+1}^{(\mathscr{F},\mathscr{F}')}(M,N) = 0$ *for any* $N \in R\text{-}\mathrm{Mod}$.

(4) $\mathrm{Tor}_{n+1}^{(\mathscr{F},\mathscr{F}')}(M,N) = 0$ *for any finitely presented left R-module N.*

(5) *For any $\mathscr{F}$-resolution*

$$\cdots \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

of M, we have $\mathrm{Ker}\, f_{n-1} \in \mathscr{F}$ (*where* $\mathrm{Ker}\, f_{-1} = M$).

(6) *There exists an* $\mathscr{F}$*-resolution*

$$\cdots \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0,$$

of M*, such that* $\operatorname{Ker} f_{n-1} \in \mathscr{F}$ *(where* $\operatorname{Ker} f_{-1} = M$*).*

Then we have (5) ⇒ (6) ⇒ (1) ⇒ (2) ⇒ (3) ⇒ (4)*. If* $\mathscr{F}$ *is closed under pure quotient modules, then* (4) ⇒ (5)*, and thus all statements are equivalent.*

Proof The implications (5) ⇒ (6) ⇒ (1) and (2) ⇒ (3) ⇒ (4) are trivial.

(1) ⇒ (2) Let $\mathscr{F}$-res.dim $M \leq n$ and

$$0 \longrightarrow F_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

be an $\mathscr{F}$-resolution of M. By Lemma 3.4 and Proposition 3.5(1), we have

$$\operatorname{Tor}_{n+i}^{(\mathscr{F},\mathscr{F}')}(M,N) \cong \operatorname{Tor}_i^{(\mathscr{F},\mathscr{F}')}(F_n,N) = 0$$

for any $N \in R$-Mod and $i \geq 1$.

Now suppose that $\mathscr{F}$ is closed under pure quotient modules. We will prove (4) ⇒ (5)

Let $K = \operatorname{Ker} f_{n-1}$. Then $\operatorname{Tor}_1^{(\mathscr{F},\mathscr{F}')}(K,N) \cong \operatorname{Tor}_{n+1}^{(\mathscr{F},\mathscr{F}')}(M,N) = 0$ for any finitely presented left R-module N by (4). Take a $\operatorname{Hom}_R(\mathscr{C},-)$-exact exact sequence

$$0 \to L \to F \to K \to 0 \tag{$\star$}$$

with $F \in \mathscr{F}$. Then we have

$$0 = \operatorname{Tor}_1^{(\mathscr{F},\mathscr{F}')}(K,N) \to L \otimes_R N \to F \otimes_R N \to K \otimes_R N \to 0,$$

and so ($\star$) is a pure exact sequence. Now the assertion follows from the assumption. □

In the following, we will give some applications of Theorem 3.10.

Corollary 3.11 ([9, Proposition 8.17]) *The following are equivalent for any* $M \in \operatorname{Mod} R$ *and* $n \geq 0$:

(1) $\operatorname{fd} M \leq n$.

(2) $\operatorname{Tor}_{n+i}^R(M,N) = 0$ *for any* $N \in R$-Mod *and* $i \geq 1$.

(3) $\operatorname{Tor}_{n+1}^R(M,N) = 0$ *for any* $N \in R$-Mod.

(4) $\operatorname{Tor}_{n+1}^R(M,N) = 0$ *for any finitely presented left* R*-module* N.

(5) *For any flat resolution*

$$\cdots \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

of M*, we have* $\operatorname{Ker} f_{n-1}$ *is flat.*

(6) *There exists a flat resolution*

$$\cdots \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

of M*, such that* $\operatorname{Ker} f_{n-1}$ *is flat.*

Proof The assertions follow Theorem 3.10 and Example 3.3(1). □

Corollary 3.12 *Let* R *be a Gorenstein ring. Then the following are equivalent for any* $M \in$ Mod-R *and* $n \geq 0$:

(1) $\operatorname{Gfd} M \leq n$.

(2) $\mathrm{Gtor}_{n+i}^R(M,N)=0$ *for any* $N\in R$-Mod *and* $i\geq 1$.

(3) $\mathrm{Gtor}_{n+1}^R(M,N)=0$ *for any* $N\in R$-Mod.

(4) $\mathrm{Gtor}_{n+1}^R(M,N)=0$ *for any finitely presented left* R*-module* N.

(5) *For any Gorenstein flat resolution*

$$\cdots \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

of M*, we have* $\mathrm{Ker}\, f_{n-1}$ *is Gorenstein flat.*

(6) *There exists a Gorenstein flat resolution*

$$\cdots \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

of M*, such that* $\mathrm{Ker}\, f_{n-1}$ *is Gorenstein flat.*

Proof If R is a Gorenstein ring, then by Example 3.3(3), we take a pure exact sequence

$$0\to A\to B\to C\to 0$$

with B is Gorenstein flat. So

$$0\to C^+\to B^+\to A^+\to 0$$

splits by [3, Proposition 5.3.8], and hence C is Gorenstein flat by [3, Theorem 10.3.8]. Now the assertion follows from Theorem 3.10. □

Corollary 3.13 *Let* R *be an* n*-Gorenstein ring. Then* $\mathrm{Gtor}_{n+i}^R(M,N)=0$ *for any* $M\in$ Mod-R*,* $N\in R$-Mod *and* $i\geq 1$.

Proof Since $\mathrm{Gfd}\, M\leq \mathrm{Gpd}\, M\leq n$ for any $M\in$ Mod-R over an n-Gorenstein ring, the assertion follows from Corollary 3.12. □

Theorem 3.14 (1) *If* $\mathscr{F}$ *is closed under pure quotient modules, then* $\mathscr{F}$-res.dim $M=\mathscr{D}'$-cores.dim M^+ *for any* $M\in$ Mod-R.

(2) *If* $\mathscr{F}'$ *is closed under pure quotient modules, then* $\mathscr{F}'$-res.dim $N=\mathscr{D}$-cores.dim N^+ *for any* $N\in R$-Mod.

Proof (1) Let $\mathscr{F}$-res.dim $M=n$. Because $\mathscr{F}$ is closed under pure quotient modules, we have $\mathrm{Tor}_{n+1}^{(\mathscr{F},\mathscr{F}')}(M,N)=0$ for any $N\in R$-Mod by Theorem 3.10. So $\mathrm{Ext}^{n+1}_{(\mathscr{C}',\mathscr{D}')}(N,M^+)=0$ by Proposition 3.7, and hence $\mathscr{D}'$-cores.dim $M^+\leq n$, that is, $\mathscr{D}'$-cores.dim $M^+\leq \mathscr{F}$-res.dim M. Dually, we have $\mathscr{F}$-res.dim $M\leq \mathscr{D}'$-cores.dim M^+.

(2) It follows from Proposition 3.7 and the opposite version of Theorem 3.10. □

Immediately, we have the following

Corollary 3.15 (1) ([3, Theorem 3.2.19]) $\mathrm{fd}\, M=\mathrm{id}\, M^+$ *for any* $M\in$ Mod-R.

(2) ([6, Proposition 3.11]) *Let* R *be a Gorenstein ring. Then* $\mathrm{Gfd}\, M=\mathrm{Gid}\, M^+$ *for any* $M\in$ Mod-R.

4 Cotorsion Pairs

In [11], we introduced cotorsion pairs relative to a given balanced pair $(\mathscr{C},\mathscr{D})$ in an abelian category. In this section, we study cotorsion pairs induced by $\mathrm{Tor}_n^{(\mathscr{F},\mathscr{F}')}(-,-)$ relative to $(\mathscr{C},\mathscr{D})$ in Mod-R and give some applications.

Let $\mathscr{X} \subseteq \text{Mod-}R$ and $\mathscr{T} \subseteq R\text{-Mod}$. We write

$$\mathscr{X}^{\perp_*} := \{N \in \text{Mod } R \mid \text{Ext}^1_{(\mathscr{C},\mathscr{D})}(M,N) = 0 \text{ for any } M \in \mathscr{X}\},$$

$$^{\perp_*}\mathscr{X} := \{M \in \text{Mod } R \mid \text{Ext}^1_{(\mathscr{C},\mathscr{D})}(M,N) = 0 \text{ for any } N \in \mathscr{X}\},$$

$$^{\top_*}\mathscr{T} := \{M \in \text{Mod } R \mid \text{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M,N) = 0 \text{ for any } N \in \mathscr{T}\}.$$

Proposition 4.1 *Let $\mathscr{C} \subseteq \mathcal{PP}$ or $\mathscr{D} \subseteq \mathcal{PI}$. Then we have*

(1) *$^{\top_*}\mathscr{T}$ is covering and closed under direct limits.*

(2) *If $\mathscr{X} \subseteq \{N^+ \mid N \in R\text{-Mod}\}$, then $^{\perp_*}\mathscr{X}$ is covering.*

Proof (1) Let $0 \to A \to B \to C \to 0$ be a pure exact sequence in Mod-R with $B \in {}^{\top_*}\mathscr{T}$. Since $\mathscr{C} \subseteq \mathcal{PP}$ or $\mathscr{D} \subseteq \mathcal{PI}$, the above sequence is $\text{Hom}_R(\mathscr{C},-)$-exact by [1, Proposition 2.2]. So we have $C \in {}^{\top_*}\mathscr{T}$. Notice that $^{\top_*}\mathscr{T}$ is closed under direct sums, so $^{\top_*}\mathscr{T}$ is covering by Lemma 2.2, and it is closed under direct limits by [5, Corollary 2.9].

(2) By Proposition 3.7, $M \in {}^{\perp_*}(N^+) \iff M \in {}^{\top_*}N$ for any right R-module M. So the assertion follows from (1). □

In order to give the main result in this section, we need some definitions.

Definition 4.2 *Let $\mathscr{E}$ be a full subcategory of* Mod-R.

(1) *$\mathscr{E}$ is said to be* closed under $\mathscr{C}$-extensions *if $0 \to A \to B \to C \to 0$ is $\text{Hom}_R(\mathscr{C},-)$-exact exact in* Mod-$R$ *with A, C in $\mathscr{E}$, then B is also in $\mathscr{E}$.*

(2) *$\mathscr{E}$ is said to be* closed under kernels of $\mathscr{C}$-epimorphisms *if $0 \to A \to B \to C \to 0$ is $\text{Hom}_R(\mathscr{C},-)$-exact exact in* Mod-$R$ *with B, C in $\mathscr{E}$, then A is also in $\mathscr{E}$.*

(3) *$\mathscr{E}$ is called* $\mathscr{C}$-resolving *if $\mathscr{C} \subseteq \mathscr{E}$ and $\mathscr{E}$ is closed under direct summands, $\mathscr{C}$-extensions and kernels of $\mathscr{C}$-epimorphisms.*

Definition 4.3 *If there exists an exact sequence*

$$0 \to A \to C \to M \to 0$$

in R-Mod *with $C \in \mathscr{C}'$, then A is called a $\mathscr{C}'$-syzygy module of M, and denoted by $\Omega^1_{\mathscr{C}'}(M)$. A subcategory $\mathscr{T}$ of* R-Mod *is called* $\mathscr{C}'$-syzygy closed *if $\Omega^1_{\mathscr{C}'}(M) \in \mathscr{T}$ for any $M \in \mathscr{T}$; in particular, if $\mathscr{C}'$ is the category of projective left R-modules, then $\mathscr{T}$ is called* syzygy closed.

Definition 4.4 (1) *Let $\mathscr{T}$ be a full subcategory of* Mod-R *and $M \in$* Mod-R. *A $\mathscr{T}$-(pre)cover $f: C \to M$ of M is called* special *relative to $(\mathscr{C},\mathscr{D})$ if it is epic and* $\text{Ker} f \in \mathscr{T}^{\perp_*}$. *Dually, a $\mathscr{T}$-(pre)envelope $f: M \to C$ of M is called* special *relative to $(\mathscr{C},\mathscr{D})$ if f is monic and* $\text{Coker} f \in {}^{\perp_*}\mathscr{T}$. *If each module in* Mod-R *has a special $\mathscr{T}$-(pre)cover (resp. special $\mathscr{T}$-(pre)envelope) relative to $(\mathscr{C},\mathscr{D})$, then $\mathscr{C}$ is called* special (pre)covering (*resp.* special (pre)enveloping) *relative to $(\mathscr{C},\mathscr{D})$.*

(2) *A pair $(\mathscr{X},\mathscr{Y})$ is called a* cotorsion pair *relative to $(\mathscr{C},\mathscr{D})$ in* Mod-R *if $\mathscr{X} = {}^{\perp_*}\mathscr{Y}$ and $\mathscr{Y} = \mathscr{X}^{\perp_*}$. The cotorsion pair $(\mathscr{X},\mathscr{Y})$ is called* complete *relative to $(\mathscr{C},\mathscr{D})$ if $\mathscr{X}$ is special precovering and $\mathscr{Y}$ is special preenveloping relative to $(\mathscr{C},\mathscr{D})$; it is called* perfect *relative to $(\mathscr{C},\mathscr{D})$ if $\mathscr{X}$ is special covering and $\mathscr{Y}$ is special enveloping relative to $(\mathscr{C},\mathscr{D})$; and it is called* hereditary *relative to $(\mathscr{C},\mathscr{D})$ if $\mathscr{X}$ is $\mathscr{C}$-resolving and $\mathscr{Y}$ is $\mathscr{D}$-coresolving.*

Putting $(\mathscr{C},\mathscr{D}) = (\mathcal{P}_0,\mathcal{I}_0)$, then those in Definition 4.4(2) are exactly the so-called (classical) *complete*, *perfect* and *hereditary cotorsion pairs*, respectively (see [3]).

The main result in this section is the following

Theorem 4.5 *Let $\mathscr{T}$ be $\mathscr{C}'$-syzygy closed. If ${}^{\top_*}\mathscr{T}$ is closed under pure quotient modules, then $({}^{\top_*}\mathscr{T}, ({}^{\top_*}\mathscr{T})^{\perp_*})$ is a complete and hereditary cotorsion pair relative to $(\mathscr{C}, \mathscr{D})$.*

Proof Note that ${}^{\top_*}\mathscr{T}$ is closed under direct sums. So ${}^{\top_*}\mathscr{T}$ is covering by Lemma 2.2. Since $\mathscr{C} \subseteq {}^{\top_*}\mathscr{T}$, we have that ${}^{\top_*}\mathscr{T}$ is surjective covering.

Next we show that ${}^{\top_*}\mathscr{T}$ is $\mathscr{C}$-resolving. Obviously, we have that $\mathscr{C} \subseteq {}^{\top_*}\mathscr{T}$ and ${}^{\top_*}\mathscr{T}$ is closed under direct summands and $\mathscr{C}$-extensions. Now let $0 \to M \to N \to L \to 0$ be a $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact sequence in Mod-R with $N, L \in {}^{\top_*}\mathscr{T}$. By Lemma 3.4(4), for any $T \in \mathscr{T}$, we get the following exact sequence:

$$\mathrm{Tor}_2^{(\mathscr{F},\mathscr{F}')}(L,T) \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M,T) \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(N,T) \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(L,T).$$

In order to prove that ${}^{\top_*}\mathscr{T}$ is closed under kernels of $\mathscr{C}$-epimorphisms, it suffices to show that $\mathrm{Tor}_2^{(\mathscr{F},\mathscr{F}')}(L,T) = 0$. Take a $\mathrm{Hom}_R(\mathscr{C}', -)$-exact exact sequence $0 \to K \to C \to T \to 0$ in R-Mod with $C \in \mathscr{C}'$. Then $K \in \mathscr{T}$ since $\mathscr{T}$ is $\mathscr{C}'$-syzygy closed. By Lemma 3.4(5), we get the following exact sequence:

$$0 = \mathrm{Tor}_2^{(\mathscr{F},\mathscr{F}')}(L,C) \to \mathrm{Tor}_2^{(\mathscr{F},\mathscr{F}')}(L,T) \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(L,K) \to \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(L,C) = 0.$$

Since $L \in {}^{\top_*}\mathscr{T}$, we have $\mathrm{Tor}_2^{(\mathscr{F},\mathscr{F}')}(L,T) = 0$. So we conclude that ${}^{\top_*}\mathscr{T}$ is $\mathscr{C}$-resolving. Then it follows from [11, Theorems 3.1 and 3.2] that $({}^{\top_*}\mathscr{T}, ({}^{\top_*}\mathscr{T})^{\perp_*})$ is a complete hereditary cotorsion pair relative to $(\mathscr{C}, \mathscr{D})$. □

In the rest of this section, we will give some applications of Theorem 4.5.

Corollary 4.6 (1) *If $\mathscr{F}$ is closed under pure quotient modules, then $(\mathscr{F}, \mathscr{F}^{\perp_*})$ is a complete and hereditary cotorsion pair relative to $(\mathscr{C}, \mathscr{D})$.*

(2) *If ${}^{\top_*}\mathcal{G}(\mathscr{C}')$ is closed under pure quotient modules, then $({}^{\top_*}\mathcal{G}(\mathscr{C}'), ({}^{\top_*}\mathcal{G}(\mathscr{C}'))^{\perp_*})$ is a complete and hereditary cotorsion pair relative to $(\mathscr{C}, \mathscr{D})$.*

(3) *Let $\mathscr{C} \subseteq \mathcal{PP}$ or $\mathscr{D} \subseteq \mathcal{PI}$. If $\mathscr{T}$ is $\mathscr{C}'$-syzygy closed, then $({}^{\top_*}\mathscr{T}, ({}^{\top_*}\mathscr{T})^{\perp_*})$ is a complete and hereditary cotorsion pair relative to $(\mathscr{C}, \mathscr{D})$.*

Proof (1) Observe that $M \in \mathscr{F} \iff \mathrm{Tor}_1^{(\mathscr{F},\mathscr{F}')}(M,N) = 0$ for any $N \in R$-Mod by Theorem 3.10. Now the assertion follows from Theorem 4.5 by taking $\mathscr{T} = R$-Mod.

(2) It follows from Theorem 4.5 and the fact that $\mathcal{G}(\mathscr{C}')$ is $\mathscr{C}'$-syzygy closed.

(3) ${}^{\top_*}\mathscr{T}$ is closed under pure quotient modules by the proof of Proposition 4.1(1). So the assertion follows from Theorem 4.5. □

Let $\mathscr{X} \subseteq \mathrm{Mod}\, R$ and $\mathscr{T} \subseteq R$-Mod. We write

$$\mathscr{X}^{\perp_G} := \{N \in \mathrm{Mod}\ R \mid \mathrm{Gext}_R^1(M,N) = 0 \text{ for any } M \in \mathscr{X}\},$$
$${}^{\top_G}\mathscr{T} := \{M \in \mathrm{Mod}\ R \mid \mathrm{Gtor}_1^R(M,T) = 0 \text{ for any } T \in \mathscr{T}\},$$
$${}^{\top_1}\mathscr{T} := \{M \in \mathrm{Mod}\ R \mid \mathrm{Tor}_1^R(M,T) = 0 \text{ for any } T \in \mathscr{T}\}.$$

Corollary 4.7 (1) ([4, Theorems 2.11 and 2.12]) *If R is left coherent, then* $(\mathrm{GFlat}, \mathrm{GFlat}^{\perp_1})$ *is a perfect and hereditary cotorsion pair in* Mod-R; *in particular,* $\mathrm{GFlat}^{\perp_1}$ *is enveloping.*

(2) *If R is Gorenstein, then* $(\mathrm{GFlat}, \mathrm{GFlat}^{\perp_G})$ *is a complete and hereditary cotorsion pair relative to* $(\mathrm{GProj}, \mathrm{GInj})$ *in* Mod-R; *in particular,* $\mathrm{GFlat}^{\perp_G}$ *is preenveloping.*

Proof (1) Because GFlat is precovering by [13, Theorem *A*], and we have that GFlat is $\mathcal{P}_0$-resolving and closed under direct limits by [6, Theorem 3.7]. Thus GFlat is surjective covering, and therefore $(\mathrm{GFlat}, \mathrm{GFlat}^{\perp_1})$ is a perfect and hereditary cotorsion pair by [11, Theorems 3.1 and 3.2] and [3, Theorem 7.2.6].

(2) By Corollary 3.12, we have that GFlat $= {}^{\top_G}(R\text{-Mod})$ and GFlat is closed under pure quotient modules. So the assertion follows from Theorem 4.5. □

Corollary 4.8 *Let* $\mathscr{T} \subseteq R$-Mod *such that* $\mathscr{T}$ *is syzygy closed. Then* $({}^{\top_1}\mathscr{T}, ({}^{\top_1}\mathscr{T})^{\perp_1})$ *is a perfect and hereditary cotorsion pair in* Mod-R*.*

Proof Put $(\mathscr{C}, \mathscr{D}) = (\mathcal{P}_0, \mathcal{I}_0)$ in Mod-R and $(\mathscr{C}', \mathscr{D}') = (\mathcal{P}_0, \mathcal{I}_0)$ in R-Mod. By Corollary 4.6(3), we have that $({}^{\top_1}\mathscr{T}, ({}^{\top_1}\mathscr{T})^{\perp_1})$ is a complete and hereditary classical cotorsion pair in Mod-R. Now the assertion follows from Proposition 4.1(1) and [3, Theorem 7.2.6]. □

Recall that a module C in Mod-R is called *cotorsion* if $\mathrm{Ext}_R^1(F, C) = 0$ for any flat right R-module F. We use $\mathcal{C}_0$ to denote the subcategory of Mod-R consisting of cotorsion modules.

Corollary 4.9 (1) ([5, Theorem 8.1]) $(\mathcal{F}_0, \mathcal{C}_0)$ *is a perfect and hereditary cotorsion pair in* Mod-R*.*

(2) $({}^{\top_1}\mathrm{GProj}, ({}^{\top_1}\mathrm{GProj})^{\perp_1})$ *is a perfect and hereditary cotorsion pair in* Mod-R*.*

(3) *Let R be a right coherent ring. Then* $({}^{\top_1}\mathrm{GFlat}, ({}^{\top_1}\mathrm{GFlat})^{\perp_1})$ *is a perfect and hereditary cotorsion pair in* Mod-R*.*

Proof (1) and (2) are trivial by Corollary 4.8.

(3) Since GFlat is syzygy closed by [6, Theorem 3.7], the assertion follows from Corollary 4.8. □

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