Vol. 46 No. 5 SCIENCE IN CHINA (Series A) September 2003

Extension closure of relative syzygy modules

HUANG Zhaoyong (黄兆泳)

Department of Mathematics, Nanjing University, Nanjing 210093, China (email: huangzy@nju.edu.cn)

Received May 21, 2002; revised October 29, 2002

Abstract In this paper we introduce the notion of relative syzygy modules. We then study the extension closure of the category of modules consisting of relative syzygy modules (resp. relative k-torsionfree modules).

Keywords: extension closed, ω-k-syzygy modules, ω-k-torsionfree modules.

DOI: 10.1360/02ys0169

1 Introduction

Throughout this paper Λ is a left noetherian ring and Γ is a right noetherian ring, mod Λ (resp. mod Γ^{op}) is the category of finitely generated left Λ-modules (resp. right Γ-modules). All modules considered are finitely generated.

Let ${}_\Lambda\omega_\Gamma$ be a (Λ,Γ)-bimodule with ${}_\Lambda\omega$ in mod Λ and ω_Γ in mod Γ^{op}.

Definition 1.1. Let $A\in$ mod Λ (resp. mod Γ^{op}) and i a non-negative integer. We say that the grade of A with respect to ω, written grade$_\omega A$, is greater than or equal to i if $\mathrm{Ext}^j_\Lambda(A,\omega)=0$ (resp. $\mathrm{Ext}^j_\Gamma(A,\omega)=0$) for any $0\leqslant j<i$. We say that the strong grade of A with respect to ω, written s.grade$_\omega A$, is greater than or equal to i if grade$_\omega B\geqslant i$ for all submodules B of A.

Definition 1.2. Let $A\in$ mod Λ (resp. mod Γ^{op}) and k a positive integer. We call A a ω-k-syzygy module if there is an exact sequence $0\to A\to X_0\to X_1\to\cdots\xrightarrow{f_{k-1}}X_{k-1}$ with all X_i in add$_\Lambda\omega$ (resp. addω_Γ), where add$_\Lambda\omega$ (resp. addω_Γ) denotes the full subcategory of mod Λ (resp. mod Γ^{op}) consisting of all modules isomorphic to the direct summands of finite direct sums of copies of ${}_\Lambda\omega$ (resp. ω_Γ). We further call Cokerf_{k-1} a ω-k-cosyzygy module. We use $\Omega^k_\omega(\Lambda)$ (resp. $\Omega^k_\omega(\Gamma^{op})$) and $\Omega^{-k}_\omega(\Lambda)$ (resp. $\Omega^{-k}_\omega(\Gamma^{op})$) to denote the full subcategory of mod Λ (resp. mod Γ^{op}) consisting of ω-k-syzygy modules and ω-k-cosyzygy modules, respectively.

For any $A\in$ mod Λ, there is an exact sequence $P_1\xrightarrow{f}P_0\to A\to 0$ in mod Λ with P_0 and P_1 projective. Then we have an exact sequence $0\to A^\omega\to P_0^\omega\xrightarrow{f^\omega}P_1^\omega\to X\to 0$, where $()^\omega=$Hom$(,\omega)$ and $X=$Cokerf^ω.

Definition 1.3[1]**.** Suppose that the natural maps $\Lambda\to$End(ω_Γ) and $\Gamma^{op}\to$End$({}_\Lambda\omega)$ are isomorphisms and $\mathrm{Ext}^i_\Gamma(\omega,\omega)=0$ for any $1\leqslant i\leqslant k$. Let A and X be as above. A is called a ω-k-torsionfree module if $\mathrm{Ext}^i_\Gamma(X,\omega)=0$ for any $1\leqslant i\leqslant k$. We use $\mathcal{T}^k_\omega(\Lambda)$ to denote the full subcategory of mod Λ consisting of ω-k-torsionfree modules.

Remarks. (1) If Λ is a two-sided noetherian ring and ${}_\Lambda\omega_\Gamma={}_\Lambda\Lambda_\Lambda$, then the notions in Definitions 1.1—1.3 are just the grade, the strong grade, k-syzygy modules and k-torsionfree modules in the usual sense[2,3] respectively. Particularly, in this case $\Omega_\omega^{-k}(\Lambda)$ =mod Λ for any $k \geqslant 1$.

(2) The definition of ω-k-torsionfree modules above is well-defined[1].

(3) Let $\sigma_A : A \to A^{\omega\omega}$ be the canonical evaluation homomorphism. A is called a ω-torsionless module if σ_A is a monomorphism; and A is called a ω-reflexive module if σ_A is an isomorphism. By Lemma 4 of ref. [1], A is ω-torsionless (resp. ω-reflexive) if and only if A is ω-1-torsionfree (resp. ω-2-torsionfree).

A full subcategory $\mathcal{X}$ of mod Λ is said to be extension closed if the middle term B of any short exact sequence $0 \to A \to B \to C \to 0$ is in $\mathcal{X}$ provided that the end terms A and C are in $\mathcal{X}$. For any positive integer k, we showed in ref. [1] that a ω-k-torsionfree module is a ω-k-syzygy module and so $\mathcal{T}_\omega^k(\Lambda) \subset \Omega_\omega^k(\Lambda)$. In this paper, we mainly discuss the extension closure of $\Omega_\omega^k(\Lambda)$ and $\mathcal{T}_\omega^k(\Lambda)$. This paper is mainly motivated by the work of Auslander and Reiten[3].

In sec. 2 we give some lemmas which will be used later. In sec. 3 we show that $\text{grade}_\omega \text{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ for any $M \in \Omega_\omega^{-(i+1)}(\Lambda)$ and $1 \leqslant i \leqslant k-1$ if and only if $\Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$ for any $1 \leqslant i \leqslant k$ (Theorem 3.1), which is applied to showing that $\text{s.grade}_\omega \text{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ for any $M \in \Omega_\omega^{-i}(\Lambda)$ and $1 \leqslant i \leqslant k$ if and only if $\Omega_\omega^i(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$ (Theorem 3.3). These are generalizations of Proposition 2.26 in ref. [2] and Theorem 1.7 in ref. [3], respectively. In sec. 4 we deal with the extension closure of ω-torsionless modules and ω-reflexive modules. If $k \leqslant 2$, then $\mathcal{T}_\omega^i(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$ if and only if $\text{grade}_\omega \text{Ext}_\Gamma^i(C,\omega) \geqslant i$ for any $C \in \text{mod } \Gamma^{op}$ (or $\Omega_\omega^{-i}(\Gamma^{op})$) and $1 \leqslant i \leqslant k$ (Theorem 4.1).

In the following, k is a positive integer, ${}_\Lambda\omega_\Gamma$ is a faithfully balanced bimodule, that is, the natural maps $\Lambda \to \text{End}(\omega_\Gamma)$ and $\Gamma^{op} \to \text{End}({}_\Lambda\omega)$ are isomorphisms, satisfying $\text{Ext}_\Lambda^i(\omega,\omega) = 0 = \text{Ext}_\Gamma^i(\omega,\omega)$ for any $1 \leqslant i \leqslant k$. Under the assumption of ${}_\Lambda\omega_\Gamma$ being faithfully balanced, it is easy to see that any projective module in mod Λ (resp. mod Γ^{op}) and any module in $\text{add}_\Lambda\omega$ (resp. $\text{add}\omega_\Gamma$) are ω-reflexive.

2 Some lemmas

In this section we give some lemmas which will be used later.

Lemma 2.1. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to X \xrightarrow{\alpha} Y$ be two exact sequences in mod Λ. If $\text{Hom}_\Lambda(g,Y)$ is an isomorphism, then $\text{Hom}_\Lambda(g,X)$ is also an isomorphism.

Proof. Since $0 \to \text{Hom}_\Lambda(C,Y) \xrightarrow{\text{Hom}_\Lambda(g,Y)} \text{Hom}_\Lambda(B,Y) \xrightarrow{\text{Hom}_\Lambda(f,Y)} \text{Hom}_\Lambda(A,Y)$ is exact and $\text{Hom}_\Lambda(g,Y)$ is an isomorphism, $\text{Hom}_\Lambda(f,Y)$ is a zero homomorphism.

Consider the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_\Lambda(C,X) & \xrightarrow{\text{Hom}_\Lambda(g,X)} & \text{Hom}_\Lambda(B,X) & \xrightarrow{\text{Hom}_\Lambda(f,X)} & \text{Hom}_\Lambda(A,X) \\
 & & \downarrow{\scriptstyle \text{Hom}_\Lambda(C,\alpha)} & & \downarrow{\scriptstyle \text{Hom}_\Lambda(B,\alpha)} & & \downarrow{\scriptstyle \text{Hom}_\Lambda(A,\alpha)} \\
0 & \longrightarrow & \text{Hom}_\Lambda(C,Y) & \xrightarrow{\text{Hom}_\Lambda(g,Y)} & \text{Hom}_\Lambda(B,Y) & \xrightarrow{\text{Hom}_\Lambda(f,Y)} & \text{Hom}_\Lambda(A,Y)
\end{array}$$

Since $\mathrm{Hom}_\Lambda(f,Y)$ is a zero homomorphism and $\mathrm{Hom}_\Lambda(A,\alpha)$ is a monomorphism, $\mathrm{Hom}_\Lambda(f,X)$ is also a zero homomorphism. By the exactness of the upper row in the above diagram, $\mathrm{Hom}_\Lambda(g,X)$ is an epimorphism and hence an isomorphism.

Lemma 2.2. Let A and B be in mod Λ. If B is ω-torsionless and $A^\omega = 0$, then $\mathrm{Hom}_\Lambda(A,B) = 0$.

Proof. Since B is ω-torsionless, there is an embedding $0 \to B \to \omega^n$ with n a positive integer, and an exact sequence $0 \to \mathrm{Hom}_\Lambda(A,B) \to [\mathrm{Hom}_\Lambda(A,\omega)]^n = (A^\omega)^n$. Hence we are done.

Let $A \in$ mod Λ (resp. mod Γ^{op}) and $P_1 \xrightarrow{f} P_0 \to A \to 0$ an exact sequence in mod Λ (resp. mod Γ^{op}) with P_0 and P_1 projective (or P_0 and P_1 in $\mathrm{add}_\Lambda\omega$ (resp. $\mathrm{add}\omega_\Gamma$)). Then we get an exact sequence

$$0 \to A^\omega \to P_0^\omega \xrightarrow{f^\omega} P_1^\omega \to X \to 0$$

in mod Γ^{op} (resp. mod Λ), where $X = \mathrm{Coker} f^\omega$.

Lemma 2.3 (Lemma 2.1 in ref. [4]). Let A and X be as above. Then we have the following exact sequences:

$$0 \to \mathrm{Ext}^1_\Gamma(X,\omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}^2_\Gamma(X,\omega) \to 0,$$
$$0 \to \mathrm{Ext}^1_\Lambda(A,\omega) \to X \xrightarrow{\sigma_X} X^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(A,\omega) \to 0.$$

The following Lemmas 2.4 and 2.5 have analogous proofs to Lemmas 2.6 and 2.12 of ref. [15] respectively.

Lemma 2.4. Let $0 \to A \to H \xrightarrow{f} B$ be an exact sequence in mod Λ (resp. mod Γ^{op}) with H ω-reflexive and B ω-torsionless. Then $A \cong (\mathrm{Coker} f^\omega)^\omega$.

Lemma 2.5. For any $A \in$ mod Λ (resp. mod Γ^{op}), the following statements are equivalent.

(1) A^ω is ω-reflexive;

(2) $A^{\omega\omega}$ is ω-reflexive;

(3) $(\mathrm{Coker}\sigma_A)^\omega = 0$.

Lemma 2.6. A module $M \in \Omega^2_\omega(\Lambda)$ if and only if there is a module $N \in$ mod Γ^{op} such that $M \cong N^\omega$.

Proof. Suppose $M \cong N^\omega$ with $N \in$ mod Γ^{op}. Because there is an exact sequence $Q_1 \to Q_0 \to N \to 0$ in mod Γ^{op} with Q_0 and Q_1 projective, we have an exact sequence $0 \to N^\omega \to Q_0^\omega \to Q_1^\omega$ with $Q_0^\omega, Q_1^\omega \in \mathrm{add}_\Lambda\omega$ and $M(\cong N^\omega) \in \Omega^2_\omega(\Lambda)$. The converse follows from Lemma 2.4.

Lemma 2.7. The following statements are equivalent.

(1) A^ω is ω-reflexive for any $A \in$ mod Λ.

$(1)^{op}$ B^ω is ω-reflexive for any $B \in$ mod Γ^{op}.

(2) $[\mathrm{Ext}^2_\Lambda(A,\omega)]^\omega = 0$ for any $A \in$ mod Λ.

$(2)^{op}$ $[\mathrm{Ext}^2_\Gamma(B,\omega)]^\omega = 0$ for any $B \in$ mod Γ^{op}.

(3) $[\mathrm{Ext}^2_\Lambda(A,\omega)]^\omega = 0$ for any $A \in \Omega^{-2}_\omega(\Lambda)$.

$(3)^{op}$ $[\mathrm{Ext}^2_\Gamma(B,\omega)]^\omega = 0$ for any $B \in \Omega^{-2}_\omega(\Gamma^{op})$.

(4) Every module in $\Omega^2_\omega(\Lambda)$ is ω-reflexive.

$(4)^{op}$ Every module in $\Omega^2_\omega(\Gamma^{op})$ is ω-reflexive.

Proof. We will prove (1) $\Leftrightarrow$ $(1)^{op}$ $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ $(1)^{op}$ and $(1)^{op}$ $\Leftrightarrow$ (4). Then by symmetry, we are done.

(1) $\Leftrightarrow$ $(1)^{op}$ The argument for Lemma 2.13 in ref. [5] remains valid here, so we omit it.

$(1)^{op}$ $\Rightarrow$ (2) By Lemma 2.3, for any $A \in \text{mod}\ \Lambda$ there is an exact sequence $X \xrightarrow{\sigma_X} X^{\omega\omega} \to \text{Ext}^2_\Lambda(A,\omega) \to 0$ with $X \in \text{mod}\ \Gamma^{op}$, and then $0 \to [\text{Ext}^2_\Lambda(A,\omega)]^\omega \to X^{\omega\omega\omega} \xrightarrow{\sigma^\omega_X} X^\omega$ is exact. By Proposition 20.14 in ref. [6], $\sigma^\omega_X \sigma_{X^\omega} = 1_{X^\omega}$, so σ^ω_X is a split epimorphism and hence $X^{\omega\omega\omega} \cong X^\omega \bigoplus [\text{Ext}^2_\Lambda(A,\omega)]^\omega$. By $(1)^{op}$, X^ω is ω-reflexive, so we have $[\text{Ext}^2_\Lambda(A,\omega)]^\omega = 0$.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ $(1)^{op}$ For any $B \in \text{mod}\ \Gamma^{op}$, there is an exact sequence $P_1 \xrightarrow{f} P_0 \to B \to 0$ in mod Γ^{op} with P_0 and P_1 projective. Then we have an exact sequence $0 \to B^\omega \to P_0^\omega \xrightarrow{f^\omega} P_1^\omega \to X \to 0$ in mod Λ with P_0^ω, P_1^ω in $\text{add}_\Lambda\omega$ and $X \in \Omega^{-2}_\omega(\Lambda)$, where X =Cokerf^ω. By Lemma 2.3, we have exact sequences $B \xrightarrow{\sigma_B} B^{\omega\omega} \to \text{Ext}^2_\Gamma(X,\omega) \to 0$ and $0 \to [\text{Ext}^2_\Gamma(X,\omega)]^\omega \to B^{\omega\omega\omega} \xrightarrow{\sigma^\omega_B} B^\omega$. Similar to the above argument we have $B^{\omega\omega\omega} \cong B^\omega \bigoplus [\text{Ext}^2_\Gamma(X,\omega)]^\omega$. $[\text{Ext}^2_\Gamma(X,\omega)]^\omega = 0$ by (3), so $B^{\omega\omega\omega} \cong B^\omega$ and hence B^ω is ω-reflexive.

$(1)^{op}$ $\Leftrightarrow$ (4) It follows from Lemma 2.6.

Lemma 2.8 (Lemma 4 in ref. [1]). A module in mod Λ is ω-torsionless (resp. ω-reflexive) if and only if it is ω-1-torsionfree (resp. ω-2-torsionfree).

Lemma 2.9. Let $k \geqslant 3$. Then a ω-reflexive module A in mod Λ is ω-k-torsionfree if and only if $\text{Ext}^i_\Gamma(A^\omega,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

Proof. Let $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a projective resolution of A in mod Λ. Then

$$0 \to A^\omega \to P_0^\omega \xrightarrow{f^\omega} P_1^\omega \to X \to 0 \qquad (2.9.1)$$

is exact in mod Γ^{op} with P_0^ω and P_1^ω in $\text{add}\omega_\Gamma$, where X =Cokerf^ω. By Lemma 2.3, A is ω-reflexive if and only if $\text{Ext}^1_\Gamma(X,\omega) = 0 = \text{Ext}^2_\Gamma(X,\omega)$. On the other hand, from the exactness of the sequence (2.9.1) we get that $\text{Ext}^{i-2}_\Gamma(A^\omega,\omega) \cong \text{Ext}^i_\Gamma(X,\omega)$ for any $1 \leqslant i \leqslant k$. Now our conclusion follows easily.

3 Extension closure of $\Omega^k_\omega(\Lambda)$

In this section we discuss the extension closure of $\Omega^k_\omega(\Lambda)$.

Theorem 3.1. The following statements are equivalent.

(1) $\text{grade}_\omega \text{Ext}^{i+1}_\Lambda(M,\omega) \geqslant i$ for any $M \in \Omega^{-(i+1)}_\omega(\Lambda)$ and $1 \leqslant i \leqslant k-1$.

(2) $\Omega^i_\omega(\Lambda) = \mathcal{T}^i_\omega(\Lambda)$ for any $1 \leqslant i \leqslant k$.

Proof. Proceed by induction on k. It is not difficult to verify that a module in mod Λ is ω-torsionless if and only if it is in $\Omega^1_\omega(\Lambda)$. Then by Lemma 2.8 we have $\Omega^1_\omega(\Lambda) = \mathcal{T}^1_\omega(\Lambda)$. On the other hand, when $k=1$ the assumption of (1) is empty. So the case for $k=1$ is trivial. The case for $k=2$ follows from Lemma 2.7. Now suppose $k \geqslant 3$.

(1) $\Rightarrow$ (2) By Theorem 1 in ref. [1], $\mathcal{T}^k_\omega(\Lambda) \subset \Omega^k_\omega(\Lambda)$. So we only need to prove $\mathcal{T}^k_\omega(\Lambda) \supset \Omega^k_\omega(\Lambda)$.

Let $L \in \Omega^k_\omega(\Lambda)$. Then there is an exact sequence $0 \to L \to X_{k-1} \xrightarrow{f} X_{k-2} \to \cdots \to X_0 \to M \to 0$ in mod Λ with all $X_i \in \text{add}_\Lambda\omega$. Since we have assumed (1) at level k, we also know (1) at level $k-1$, so by induction assumption we have $\Omega^i_\omega(\Lambda) = \mathcal{T}^i_\omega(\Lambda)$ for any $1 \leqslant i \leqslant k-1$. Hence $L \in \mathcal{T}^{k-1}_\omega(\Lambda)$.

Let $P_1 \xrightarrow{g} P_0 \to L \to 0$ be an exact sequence in mod Λ with P_0 and P_1 projective. Then we have an exact sequence $0 \to L^{\omega} \to P_0^{\omega} \xrightarrow{g^{\omega}} P_1^{\omega} \to X \to 0$ in mod Γ^{op} with P_0^{ω} and P_1^{ω} in addω_{Γ}, where $X =$Cokerg^{ω}. We will show that L is ω-k-torsionfree.

Notice that $L \in \mathcal{T}_{\omega}^{k-1}(\Lambda)$ and $k \geqslant 3$, so L is ω-reflexive and hence it suffices to show that $\mathrm{Ext}_{\Gamma}^{i}(L^{\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$ by Lemma 2.9.

Put $N =$Cokerf^{ω}. Then, by Lemma 2.4, $L \cong N^{\omega}$ and $L^{\omega} \cong N^{\omega\omega}$. We claim that $\mathrm{Ext}_{\Gamma}^{i}(N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$. If $k = 3$, then Coker$f$ is a submodule of X_0. But X_0 is in add$_{\Lambda}\omega$, so X_0 is ω-reflexive and Coker$f$ is ω-torsionless. By Lemma 2.3, $\mathrm{Ext}_{\Gamma}^{1}(N, \omega) \cong$Ker$\sigma_{\mathrm{Coker}f} = 0$. If $k = 4$, then Coker$f \in \Omega_{\omega}^{2}(\Lambda)$ $(= \mathcal{T}_{\omega}^{2}(\Lambda))$ and Coker$f$ is ω-reflexive. Thus $\mathrm{Ext}_{\Gamma}^{1}(N, \omega) \cong$Ker$\sigma_{\mathrm{Coker}f} = 0$ and $\mathrm{Ext}_{\Gamma}^{2}(N, \omega) \cong$Coker$\sigma_{\mathrm{Coker}f} = 0$ and the case for $k = 4$ follows. If $k \geqslant 5$, then Coker$f \in \Omega_{\omega}^{k-2}(\Lambda)$ and Coker$f \in \mathcal{T}_{\omega}^{k-2}(\Lambda)$. Thus $\mathrm{Ext}_{\Gamma}^{i}((\mathrm{Coker}f)^{\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-4$ by Lemma 2.9. It follows from the exact sequence $0 \to (\mathrm{Coker}f)^{\omega} \to X_{k-2}^{\omega} \xrightarrow{f^{\omega}} X_{k-1}^{\omega} \to N \to 0$ with X_{k-2}^{ω} and X_{k-1}^{ω} projective that $\mathrm{Ext}_{\Gamma}^{i}(N, \omega) = 0$ for any $3 \leqslant i \leqslant k-2$. So $\mathrm{Ext}_{\Gamma}^{i}(N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

By Lemma 2.3, we have an exact sequence

$$0 \to \mathrm{Ext}_{\Lambda}^{1}(\mathrm{Coker}f, \omega) \to N \xrightarrow{\sigma_N} N^{\omega\omega} \to \mathrm{Ext}_{\Lambda}^{2}(\mathrm{Coker}f, \omega) \to 0.$$

Then Ker$\sigma_N \cong \mathrm{Ext}_{\Lambda}^{1}(\mathrm{Coker}f, \omega) \cong \mathrm{Ext}_{\Lambda}^{k-1}(M, \omega)$ and Coker$\sigma_N \cong \mathrm{Ext}_{\Lambda}^{2}(\mathrm{Coker}f, \omega) \cong \mathrm{Ext}_{\Lambda}^{k}(M, \omega)$. So we get the following exact sequences:

$$0 \to \mathrm{Ext}_{\Lambda}^{k-1}(M, \omega) \to N \xrightarrow{\pi} \mathrm{Im}\sigma_N \to 0, \tag{3.1.1}$$

$$0 \to \mathrm{Im}\sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_{\Lambda}^{k}(M, \omega) \to 0, \tag{3.1.2}$$

where $\sigma_N = \mu\pi$. Since $\mathrm{Ext}_{\Gamma}^{i}(N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$ and grade$_{\omega}\mathrm{Ext}_{\Lambda}^{k-1}(M, \omega) \geqslant k-2$, from the exact sequence (3.1.1) we have $\mathrm{Ext}_{\Gamma}^{i}(\mathrm{Im}\sigma_N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$. Moreover, since grade$_{\omega}\mathrm{Ext}_{\Lambda}^{k}(M, \omega) \geqslant k-1$, from the exact sequence (3.1.2) we get that $\mathrm{Ext}_{\Gamma}^{i}(N^{\omega\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$, which yields $\mathrm{Ext}_{\Gamma}^{i}(L^{\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

(2) $\Rightarrow$ (1) Let $M \in \Omega_{\omega}^{-k}(\Lambda)$. Then there is an exact sequence $0 \to L \to X_{k-1} \xrightarrow{f} X_{k-2} \to \cdots \to X_0 \to M \to 0$ in mod Λ with all $X_i \in$ add$_{\Lambda}\omega$. By (2), $L \in \mathcal{T}_{\omega}^{k}(\Lambda)$. By induction assumption, grade$_{\omega}\mathrm{Ext}_{\Lambda}^{i+1}(M, \omega) \geqslant i$ for any $1 \leqslant i \leqslant k-2$. So it remains to show that grade$_{\omega}\mathrm{Ext}_{\Lambda}^{k}(M, \omega) \geqslant k-1$. Put $N =$Cokerf^{ω}. From the proof of (1) $\Rightarrow$ (2), we have the following facts:

(i) there is exact sequences $0 \to \mathrm{Ext}_{\Lambda}^{k-1}(M, \omega) \to N \xrightarrow{\pi} \mathrm{Im}\sigma_N \to 0$ and $0 \to \mathrm{Im}\sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_{\Lambda}^{k}(M, \omega) \to 0$, where $\sigma_N = \mu\pi$;

(ii) $L \cong N^{\omega}$;

(iii) $\mathrm{Ext}_{\Gamma}^{i}(N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$;

(iv) $\mathrm{Ext}_{\Gamma}^{i}(\mathrm{Im}\sigma_N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

Since $L \in \mathcal{T}_{\omega}^{k}(\Lambda)$ and $L \cong N^{\omega}$, N^{ω} is ω-reflexive and $\mathrm{Ext}_{\Gamma}^{i}(N^{\omega\omega}, \omega) \cong \mathrm{Ext}_{\Gamma}^{i}(L^{\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$ by Lemma 2.9. Since $\mathrm{Ext}_{\Gamma}^{i}(\mathrm{Im}\sigma_N, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$ and we have the exact sequence $0 \to \mathrm{Im}\sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_{\Lambda}^{k}(M, \omega) \to 0$, $\mathrm{Ext}_{\Gamma}^{i}(\mathrm{Ext}_{\Lambda}^{k}(M, \omega), \omega) = 0$ for any $2 \leqslant i \leqslant k-2$. On the other hand, N^{ω} is ω-reflexive, so $\pi^{\omega}\mu^{\omega} = \sigma_N^{\omega}$ is an isomorphism by Proposition 20.14 in ref. [6], and it follows easily that π^{ω} and μ^{ω} are isomorphisms. Moreover, we have a long exact sequence:

$$0 \to [\mathrm{Ext}_\Lambda^k(M,\omega)]^\omega \to N^{\omega\omega\omega} \xrightarrow{\mu^\omega} (\mathrm{Im}\sigma_N)^\omega \to \mathrm{Ext}_\Gamma^1(\mathrm{Ext}_\Lambda^k(M,\omega),\omega) \to \mathrm{Ext}_\Gamma^1(N^{\omega\omega},\omega) = 0.$$

So $[\mathrm{Ext}_\Lambda^k(M,\omega)]^\omega \cong \mathrm{Ker}\mu^\omega = 0$ and $\mathrm{Ext}_\Gamma^1(\mathrm{Ext}_\Lambda^k(M,\omega),\omega) \cong \mathrm{Coker}\mu^\omega = 0$. Therefore we conclude that $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^k(M,\omega) \geqslant k-1$.

If ${}_\Lambda\omega$ is a generator in mod Λ (for example, when ${}_\Lambda\omega = {}_\Lambda \Lambda$), then $\Omega_\omega^{-k}(\Lambda) =$mod Λ for any $k \geqslant 1$ and we have the following:

Corollary 3.1. If ${}_\Lambda\omega$ is a generator in mod Λ, then the following statements are equivalent.

(1) $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ for any $M \in$ mod Λ and $1 \leqslant i \leqslant k-1$.

(2) $\Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$ for any $1 \leqslant i \leqslant k$.

The following theorem is analogous to the result of Theorem 1.1 in ref. [3]. Since the proof here is similar to that given in ref. [3], we omit it.

Theorem 3.2. Let $N \in \mathcal{T}_\omega^k(\Lambda)$. The following statements are equivalent.

(1) $\mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^1(N,\omega) \geqslant k$.

(2) If $0 \to L \to M \to N \to 0$ is exact in mod Λ with L in $\mathcal{T}_\omega^k(\Lambda)$, then $M \in \mathcal{T}_\omega^k(\Lambda)$.

(3) If $0 \to \omega^n \to E \to N \to 0$ is exact in mod Λ with n a positive integer, then $E \in \mathcal{T}_\omega^k(\Lambda)$.

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.2. The following statements are equivalent.

(1) $\mathcal{T}_\omega^i(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$.

(2) $\mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^1(N,\omega) \geqslant i$ for any $N \in \mathcal{T}_\omega^i(\Lambda)$ and $1 \leqslant i \leqslant k$.

Proposition 3.1. If $\Omega_\omega^i(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$, then $\Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$ for any $1 \leqslant i \leqslant k$.

Proof. Proceed by induction on k. There is nothing to do for the case $k = 1$.

Now suppose $k \geqslant 2$. Then, by induction assumption, $\Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$, which is extension closed for any $1 \leqslant i \leqslant k-1$. For any $M \in \Omega_\omega^{-(i+1)}(\Lambda)$ $(1 \leqslant i \leqslant k-1)$, there is an exact sequence $X_i \xrightarrow{f_i} \cdots \to X_0 \to M \to 0$ with all X_j in $\mathrm{add}_\Lambda\omega$. Then $\mathrm{Im}f_i \in \Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$ $(1 \leqslant i \leqslant k-1)$ and $\mathrm{Ext}_\Lambda^{i+1}(M,\omega) \cong \mathrm{Ext}_\Lambda^1(\mathrm{Im}f_i,\omega)$. So we have $\mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) = \mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^1(\mathrm{Im}f_i,\omega) \geqslant i$ for any $1 \leqslant i \leqslant k-1$ by Corollary 3.2. Then by Theorem 3.1 we have that $\Omega_\omega^k(\Lambda) = \mathcal{T}_\omega^k(\Lambda)$, which finishes the proof.

The main result in this section is the following, which is a generalization of Theorem 1.7 in ref. [3].

Theorem 3.3. The following statements are equivalent.

(1) $\mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ for any $M \in \Omega_\omega^{-i}(\Lambda)$ and $1 \leqslant i \leqslant k$.

(2) $\Omega_\omega^i(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$.

(3) $\Omega_\omega^i(\Lambda)$ is extension closed and $\Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$ for any $1 \leqslant i \leqslant k$.

Proof. (1) $\Rightarrow$ (2) By (1) and Theorem 3.1 we have $\Omega_\omega^i(\Lambda) = \mathcal{T}_\omega^i(\Lambda)$ for any $1 \leqslant i \leqslant k$. Let $N \in \mathcal{T}_\omega^i(\Lambda)$ $(1 \leqslant i \leqslant k)$, then $N \in \Omega_\omega^i(\Lambda)$ and there is an exact sequence $0 \to N \to X_{i-1} \to \cdots \to X_0 \to M \to 0$ with all X_j in $\mathrm{add}_\Lambda\omega$. Then $\mathrm{Ext}_\Lambda^1(N,\omega) \cong \mathrm{Ext}_\Lambda^{i+1}(M,\omega)$ and $M \in \Omega_\omega^{-i}(\Lambda)$. So $\mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^1(N,\omega) = \mathrm{s.grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ $(1 \leqslant i \leqslant k)$ by (1) and hence $\mathcal{T}_\omega^i(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$ by Corollary 3.2. Therefore we conclude that $\Omega_\omega^i(\Lambda)$ is also extension closed for any $1 \leqslant i \leqslant k$.

(2) $\Rightarrow$ (3) By Proposition 3.1.

(3) $\Rightarrow$ (1) By (3) and Corollary 3.2, $\text{s.grade}_\omega \text{Ext}^1_\Lambda(N,\omega) \geqslant i$ for any $N \in \mathcal{T}^i_\omega(\Lambda) = \Omega^i_\omega(\Lambda)$ and $1 \leqslant i \leqslant k$. So $\text{s.grade}_\omega \text{Ext}^{i+1}_\Lambda(M,\omega) \geqslant i$ for any $M \in \Omega^{-i}_\omega(\Lambda)$ and $1 \leqslant i \leqslant k$.

Corollary 3.3. If ${}_\Lambda\omega$ is a generator in mod Λ, then the following statements are equivalent.

(1) $\text{s.grade}_\omega \text{Ext}^{i+1}_\Lambda(M,\omega) \geqslant i$ for any $M \in \text{mod } \Lambda$ and $1 \leqslant i \leqslant k$.

(2) $\Omega^i_\omega(\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$.

(3) $\Omega^i_\omega(\Lambda)$ is extension closed and $\Omega^i_\omega(\Lambda) = \mathcal{T}^i_\omega(\Lambda)$ for any $1 \leqslant i \leqslant k$.

4 Extension closure of $\mathcal{T}^k_\omega(\Lambda)$

In this section we deal with the extension closure of $\mathcal{T}^k_\omega(\Lambda)$, especially, of $\mathcal{T}^1_\omega(\Lambda)$ and $\mathcal{T}^2_\omega(\Lambda)$. We use $\text{l.id}_\Lambda(\omega)$ to denote the left injective dimension of ω as a left Λ-module.

Proposition 4.1. If $\text{l.id}_\Lambda(\omega) \leqslant k$, then $\mathcal{T}^k_\omega(\Lambda)$ is extension closed.

Proof. Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an exact sequence in mod Λ with A and C ω-k-torsionfree. Consider the following exact commutative diagram with last two rows splitting:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & A & \xrightarrow{f} & B & \to & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F_0 & \to & F_0 \bigoplus G_0 & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F_1 & \to & F_1 \bigoplus G_1 & \longrightarrow & G_1 & \longrightarrow & 0
\end{array}$$

where all F_i and G_i are projective. Then we get the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^\omega & \to & B^\omega & \xrightarrow{f^\omega} & A^\omega & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_0^\omega & \to & G_0^\omega \bigoplus F_0^\omega & \longrightarrow & F_0^\omega & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_1^\omega & \to & G_1^\omega \bigoplus F_1^\omega & \longrightarrow & F_1^\omega & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Z & & Y & & X & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It follows from the snake lemma that there is an exact sequence $0 \to C^\omega \to B^\omega \xrightarrow{f^\omega} A^\omega \to Z \to Y \to X \to 0$. Because C is ω-k-torsionfree, $C \in \Omega^k_\omega(\Lambda)$ by Theorem 1 in ref. [1]. On the other hand, $\text{l.id}_\Lambda(\omega) \leqslant k$, so $\text{Ext}^1_\Lambda(C,\omega) \cong \text{Ext}^{k+1}_\Lambda(\Omega^{-k}_\omega(C),\omega) = 0$ and hence f^ω is epic, which induces an exact sequence $0 \to Z \to Y \to X \to 0$. Since A and C are ω-k-torsionfree, $\text{Ext}^i_\Gamma(X,\omega) = 0 = \text{Ext}^i_\Gamma(Z,\omega)$ for any $1 \leqslant i \leqslant k$. So $\text{Ext}^i_\Gamma(Y,\omega) = 0$ for any $1 \leqslant i \leqslant k$ and hence B is ω-k-torsionfree.

${}_\Lambda\omega_\Gamma$ is called a cotilting bimodule if $\text{l.id}_\Lambda(\omega) < \infty$ and $\text{r.id}_\Gamma(\omega) < \infty$[7].

Corollary 4.1. If ${}_{\Lambda}\omega_{\Gamma}$ is a cotilting bimodule with $\mathrm{l.id}_{\Lambda}(\omega) \leqslant k$, then $\mathcal{T}_{\omega}^{k}(\Lambda)$ is extension closed.

Proposition 4.2. The following statements are equivalent.

(1) $\mathcal{T}_{\omega}^{1}(\Lambda)$ is extension closed.

(2) $\mathrm{grade}_{\omega}\mathrm{Ext}_{\Gamma}^{1}(C,\omega) \geqslant 1$ for any $C \in \mathrm{mod}\ \Gamma^{op}$.

(3) $\mathrm{grade}_{\omega}\mathrm{Ext}_{\Gamma}^{1}(C,\omega) \geqslant 1$ for any $C \in \Omega_{\omega}^{-1}(\Gamma^{op})$.

Proof. (1) $\Rightarrow$ (2) Let $C \in \mathrm{mod}\ \Gamma^{op}$ and $P_1 \xrightarrow{f} P_0 \to C \to 0$ a projective resolution of C in $\mathrm{mod}\ \Gamma^{op}$. By Lemma 2.3, we have an exact sequence:

$$0 \to \mathrm{Ext}_{\Gamma}^{1}(C,\omega) \to X \xrightarrow{\sigma_X} X^{\omega\omega} \to \mathrm{Ext}_{\Gamma}^{2}(C,\omega) \to 0,$$

where $X = \mathrm{Coker} f^{\omega}$.

Put $Y = \mathrm{Im}\sigma_X$ and assume that $\sigma_X = \mu\pi$, where $\pi : X \to Y$ is an epimorphism and $\mu : Y \to X^{\omega\omega}$ is a monomorphism. Since $\pi^{\omega}\mu^{\omega} = \sigma_X^{\omega}$ is an epimorphism by Proposition 20.14 in ref. [6], π^{ω} is also an epimorphism and hence an isomorphism. So, by applying $()^{\omega}$ to the exact sequence $0 \to \mathrm{Ext}_{\Gamma}^{1}(C,\omega) \to X \xrightarrow{\pi} Y \to 0$, we have $\mathrm{Ker}\mathrm{Ext}_{\Lambda}^{1}(\pi,\omega) \cong [\mathrm{Ext}_{\Gamma}^{1}(C,\omega)]^{\omega}$.

Suppose

$$\eta : 0 \to \omega \to K \xrightarrow{\gamma} Y \to 0$$

is an element in $\mathrm{Ker}\mathrm{Ext}_{\Lambda}^{1}(\pi,\omega)$, that is, $\mathrm{Ext}_{\Lambda}^{1}(\pi,\omega)(\eta) = 0$. Then we have the following pull-back diagram with the first row splitting:

$$\begin{array}{ccccccccc}
 & 0 & \to & \omega & \to & N & \xrightarrow{u} & X & \longrightarrow 0 \\
 & & & \| & & \downarrow v & & \downarrow \pi & \\
\eta: & 0 & \to & \omega & \to & K & \xrightarrow{\gamma} & Y & \longrightarrow 0
\end{array}$$

So there is a homomorphism $u' : X \to N$ such that $uu' = 1_X$ and hence $\pi = \gamma(vu')$. Notice that Y is ω-torsionless since Y is a submodule of a ω-torsionless module $X^{\omega\omega}$. Since ω is ω-torsionless, K is also ω-torsionless by (1). So we have an embedding $0 \to K \to \omega^n$ with n a positive integer. Since π^{ω} is an isomorphism, $\mathrm{Hom}_{\Lambda}(\pi,\omega^n)$ is also an isomorphism. It follows from Lemma 2.1 that $\mathrm{Hom}_{\Lambda}(\pi,K)$ is an isomorphism. Then there is a homomorphism $h : Y \to K$ such that $vu' = h\pi$ and so $\pi = \gamma(vu') = \gamma h\pi$. But π is an epimorphism which implies $1_Y = \gamma h$. So we conclude that the exact sequence η splits, which implies that $\mathrm{Ker}\mathrm{Ext}_{\Lambda}^{1}(\pi,\omega) = 0$ and $[\mathrm{Ext}_{\Gamma}^{1}(C,\omega)]^{\omega} = 0$.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Let $0 \to K \xrightarrow{\beta} L \xrightarrow{\alpha} M \to 0$ be an exact sequence in $\mathrm{mod}\ \Lambda$ with K and M ω-torsionless.

Suppose $P_1 \xrightarrow{f} P_0 \to L \to 0$ is a projective resolution of L in $\mathrm{mod}\ \Lambda$. Put $N = \mathrm{Coker} f^{\omega}$. By Lemma 2.3, $\mathrm{Ker}\sigma_L \cong \mathrm{Ext}_{\Gamma}^{1}(N,\omega)$. Since $N \in \Omega_{\omega}^{-1}(\Gamma^{op})$, $[\mathrm{Ext}_{\Gamma}^{1}(N,\omega)]^{\omega} = 0$ by (3) and thus $(\mathrm{Ker}\sigma_L)^{\omega} = 0$. Notice that K is ω-torsionless, so $\mathrm{Hom}_{\Lambda}(\mathrm{Ker}\sigma_L, K) = 0$ by Lemma 2.2. Moreover, σ_M is a monomorphism and $\alpha^{\omega\omega}\sigma_L = \sigma_M\alpha$, so $\mathrm{Ker}\sigma_L \subset \mathrm{Ker}\alpha \cong K$ and hence $\mathrm{Ker}\sigma_L = 0$, which implies that L is ω-torsionless.

Corollary 4.2. If $\mathcal{T}_{\omega}^{1}(\Lambda)$ is extension closed, then M^{ω} is ω-reflexive for any $M \in \mathrm{mod}\ \Lambda$.

Proof. Let $M \in \mathrm{mod}\ \Lambda$ and $P_1 \xrightarrow{f} P_0 \to M \to 0$ a projective resolution of M in $\mathrm{mod}\ \Lambda$. Put $N = \mathrm{Coker} f^{\omega}$ and $L = \mathrm{Im} f^{\omega}$. By Lemma 2.3, $\mathrm{Coker}\sigma_M \cong \mathrm{Ext}_{\Gamma}^{2}(N,\omega)$. Since $0 \to L \to P_1^{\omega} \to$

$N \to 0$ is exact with $P_1^\omega \in \mathrm{add}\omega_\Gamma$, $[\mathrm{Ext}_\Gamma^2(N,\omega)]^\omega \cong [\mathrm{Ext}_\Gamma^1(L,\omega)]^\omega = 0$ by Proposition 4.2. Thus $(\mathrm{Coker}\sigma_M)^\omega = 0$ and therefore M^ω is ω-reflexive by Lemma 2.5.

Proposition 4.3. The following statements are equivalent.

(1) $\mathcal{T}_\omega^i(\Lambda)$ is extension closed for $1 \leqslant i \leqslant 2$.

(2) $\mathrm{grade}_\omega \mathrm{Ext}_\Gamma^i(C,\omega) \geqslant i$ for any $C \in \mathrm{mod}\ \Gamma^{op}$ and $1 \leqslant i \leqslant 2$.

(3) $\mathrm{grade}_\omega \mathrm{Ext}_\Gamma^i(C,\omega) \geqslant i$ for any $C \in \Omega_\omega^{-i}(\Gamma^{op})$ and $1 \leqslant i \leqslant 2$.

(4) $\mathrm{Ext}_\Lambda^{i-1}(\mathrm{Ext}_\Gamma^i(C,\omega),\omega) = 0$ for any $C \in \mathrm{mod}\ \Gamma^{op}$ and $1 \leqslant i \leqslant 2$.

(5) $\mathrm{Ext}_\Lambda^{i-1}(\mathrm{Ext}_\Gamma^i(C,\omega),\omega) = 0$ for any $C \in \Omega_\omega^{-i}(\Gamma^{op})$ and $1 \leqslant i \leqslant 2$.

Proof. (1) $\Rightarrow$ (2) Let $C \in \mathrm{mod}\ \Gamma^{op}$ and $P_1 \xrightarrow{f} P_0 \to C \to 0$ a projective resolution of C in mod Γ^{op}. Put $B = \mathrm{Coker} f^\omega$. By Lemma 2.3, $\mathrm{Ker}\sigma_B \cong \mathrm{Ext}_\Gamma^1(C,\omega)$ and $\mathrm{Coker}\sigma_B \cong \mathrm{Ext}_\Gamma^2(C,\omega)$. Put $L = \mathrm{Im}\sigma_B$ and let $\sigma_B = \mu\pi$, where $\pi : B \to L$ is an epimorphism and $\mu : L \to B^{\omega\omega}$ is a monomorphism. By Proposition 4.2, $\mathrm{grade}_\omega \mathrm{Ext}_\Gamma^1(C,\omega) \geqslant 1$ and $[\mathrm{Ext}_\Gamma^1(C,\omega)]^\omega = 0$. By applying $()^\omega$ to the exact sequence $0 \to \mathrm{Ext}_\Gamma^1(C,\omega) \to B \xrightarrow{\pi} L \to 0$, we know that $\mathrm{Ext}_\Lambda^1(\pi,\omega)$ is a monomorphism.

Since $\sigma_B^\omega = \pi^\omega\mu^\omega$ and π^ω is an isomorphism (see the proof of (1) $\Rightarrow$ (2) in Proposition 4.2), and since σ_B^ω is an epimorphism by Proposition 20.14 in ref. [6], μ^ω is also an epimorphism. On the other hand, $\mathrm{Ext}_\Lambda^1(\sigma_B,\omega) = \mathrm{Ext}_\Lambda^1(\mu\pi,\omega) = \mathrm{Ext}_\Lambda^1(\pi,\omega)\mathrm{Ext}_\Lambda^1(\mu,\omega)$. By applying $()^\omega$ to the exact sequence $0 \to L \xrightarrow{\mu} B^{\omega\omega} \xrightarrow{\nu} \mathrm{Ext}_\Gamma^2(C,\omega) \to 0$, $\mathrm{Ker}\mathrm{Ext}_\Lambda^1(\sigma_B,\omega) \cong \mathrm{Ker}\mathrm{Ext}_\Lambda^1(\mu,\omega) \cong \mathrm{Ext}_\Lambda^1$ $(\mathrm{Ext}_\Gamma^2(C,\omega),\omega)$.

Suppose

$$\zeta : 0 \to \omega \to M \xrightarrow{\alpha} B^{\omega\omega} \to 0$$

is an element in $\mathrm{Ker}\mathrm{Ext}_\Lambda^1(\sigma_B,\omega)$, that is, $\mathrm{Ext}_\Lambda^1(\sigma_B,\omega)(\zeta) = 0$. Then we have the following pull-back diagram with the first row splitting:

$$\begin{array}{ccccccccc} & 0 & \to & \omega & \to & N & \xrightarrow{\beta} & B & \longrightarrow 0 \\ & & & \| & & \downarrow \gamma & & \downarrow \sigma_B & \\ \zeta : & 0 & \to & \omega & \to & M & \xrightarrow{\alpha} & B^{\omega\omega} & \longrightarrow 0 \end{array}$$

So there is a homomorphism $\beta' : B \to N$ such that $\beta\beta' = 1_B$ and hence $\sigma_B = \alpha(\gamma\beta')$. By Corollary 4.2, B^ω is ω-reflexive. It follows from Lemma 2.5 that $B^{\omega\omega}$ is ω-reflexive. Since ω is ω-reflexive, M is also ω-reflexive by (1). Since $\sigma_M(\gamma\beta') = (\gamma\beta')^{\omega\omega}\sigma_B$, $\sigma_B = \alpha(\gamma\beta') = \alpha\sigma_M^{-1}(\gamma\beta')^{\omega\omega}\sigma_B$. So $(1_{B^{\omega\omega}} - \alpha\sigma_M^{-1}(\gamma\beta')^{\omega\omega})\sigma_B = 0$ and hence $\mathrm{Ker}\nu = \mathrm{Im}\sigma_B \subset \mathrm{Ker}(1_{B^{\omega\omega}} - \alpha\sigma_M^{-1}(\gamma\beta')^{\omega\omega})$. Then by Theorem 3.6 in ref. [6] there is a homomorphism $\delta : \mathrm{Ext}_\Gamma^2(C,\omega) \to B^{\omega\omega}$ such that $1_{B^{\omega\omega}} - \alpha\sigma_M^{-1}(\gamma\beta')^{\omega\omega} = \delta\nu$. In addition, $[\mathrm{Ext}_\Gamma^2(C,\omega)]^\omega \cong (\mathrm{Coker}\sigma_B)^\omega = 0$ by Lemma 2.5. Then by Lemma 2.2 $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^2(C,\omega), B^{\omega\omega}) = 0$ since $B^{\omega\omega}$ is ω-reflexive. So $\delta = 0$ and hence $1_{B^{\omega\omega}} = \alpha\sigma_M^{-1}(\gamma\beta')^{\omega\omega}$, which implies that the exact sequence ζ splits. Thus $\mathrm{Ker}\mathrm{Ext}_\Lambda^1(\sigma_B,\omega) = 0$ and $\mathrm{Ext}_\Lambda^1(\mathrm{Ext}_\Gamma^2(C,\omega),\omega) = 0$. So we conclude that $\mathrm{grade}_\omega \mathrm{Ext}_\Gamma^2(C,\omega) \geqslant 2$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1) By Proposition 4.2, $\mathcal{T}_\omega^1(\Lambda)$ is extension closed. Let $0 \to K \to L \xrightarrow{\alpha} M \to 0$ be an exact sequence in mod Λ with K and M ω-reflexive. Then L is ω-torsionless by Proposition 4.2.

Let $P_1 \xrightarrow{f} P_0 \to L \to 0$ be a projective resolution of L in mod Λ. By Lemma 2.3, $\mathrm{Coker}\sigma_L \cong \mathrm{Ext}^2_\Gamma(N,\omega)$, where $N = \mathrm{Coker} f^\omega$ $(\in \Omega_\omega^{-2}(\Lambda))$. Consider the following exact commutative digram:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & L & \xrightarrow{\alpha} & M & \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \sigma_L & & \downarrow \sigma_M & \\ 0 & \to & \mathrm{Ker}\alpha^{\omega\omega} & \to & L^{\omega\omega} & \xrightarrow{\alpha^{\omega\omega}} & M^{\omega\omega} & \longrightarrow 0 \end{array}$$

where σ_M is an isomorphism, σ_L is a monomorphism and β is an induced homomorphism. By the snake lemma, we get an exact sequence $0 \to K \xrightarrow{\beta} \mathrm{Ker}\alpha^{\omega\omega} \to \mathrm{Ext}^2_\Gamma(N,\omega) \to 0$. By Corollary 4.2, L^ω is ω-reflexive. It follows from Lemma 2.5 that $[\mathrm{Ext}^2_\Gamma(N,\omega)]^\omega \cong (\mathrm{Coker}\sigma_L)^\omega = 0$. By (5), $\mathrm{Ext}^1_\Lambda(\mathrm{Ext}^2_\Gamma(N,\omega),\omega) = 0$. Thus, by applying $()^\omega$ to the last exact sequence, we know that β^ω is an isomorphism and then $\beta^{\omega\omega}$ is also an isomorphism. On the other hand, $\beta^{\omega\omega}\sigma_K = \sigma_{\mathrm{Ker}\alpha^{\omega\omega}}\beta$ and σ_K is an isomorphism, so $\sigma_{\mathrm{Ker}\alpha^{\omega\omega}}\beta$ is an isomorphism which implies that $\sigma_{\mathrm{Ker}\alpha^{\omega\omega}}$ is an epimorphism. Then $\sigma_{\mathrm{Ker}\alpha^{\omega\omega}}$ is an isomorphism since $\mathrm{Ker}\alpha^{\omega\omega}$ is clearly ω-torsionless. So we conclude that β is also an isomorphism, which implies that $\mathrm{Ext}^2_\Gamma(N,\omega) = 0$ and $\mathrm{Coker}\sigma_L = 0$. Thus L is ω-reflexive.

We are now in a position to state the main result in this section.

Theorem 4.1. Let $k \leqslant 2$. The following statements are equivalent.

(1) $\mathcal{T}^i_\omega(\Lambda)$ is extension closed for $1 \leqslant i \leqslant k$.

(2) $\mathrm{grade}_\omega \mathrm{Ext}^i_\Gamma(C,\omega) \geqslant i$ for any $C \in \mathrm{mod}\ \Gamma^{op}$ and $1 \leqslant i \leqslant k$.

(3) $\mathrm{grade}_\omega \mathrm{Ext}^i_\Gamma(C,\omega) \geqslant i$ for any $C \in \Omega_\omega^{-i}(\Gamma^{op})$ and $1 \leqslant i \leqslant k$.

(4) $\mathrm{Ext}^{i-1}_\Lambda(\mathrm{Ext}^i_\Gamma(C,\omega),\omega) = 0$ for any $C \in \mathrm{mod}\ \Gamma^{op}$ and $1 \leqslant i \leqslant k$.

(5) $\mathrm{Ext}^{i-1}_\Lambda(\mathrm{Ext}^i_\Gamma(C,\omega),\omega) = 0$ for any $C \in \Omega_\omega^{-i}(\Gamma^{op})$ and $1 \leqslant i \leqslant k$.

Proof. Use Propositions 4.2 and 4.3.

Acknowledgements The author was partially supported by the National Natural Science Foundation of China (Grant No. 10001017), Scientific Research Foundation for Returned Overseas Chinese Scholars (State Education Ministry) and Nanjing University Talent Development Foundation.

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