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Homological dimensions of gentle algebras via geometric models

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Abstract Let $A = kQ/I$ be a finite-dimensional basic algebra over an algebraically closed field k, which is a gentle algebra with the marked ribbon surface $(\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$. It is known that $\mathcal{S}_A$ can be divided into some elementary polygons $\{\Delta_i \mid 1 \leqslant i \leqslant d\}$ by Γ_A, which has exactly one side in the boundary of $\mathcal{S}_A$. Let $\mathfrak{C}(\Delta_i)$ be the number of sides of Δ_i belonging to Γ_A if the unmarked boundary component of $\mathcal{S}_A$ is not a side of Δ_i; otherwise, $\mathfrak{C}(\Delta_i) = \infty$, and let $\mathsf{f}\text{-}\Delta$ be the set of all the non-∞-elementary polygons and $\mathcal{F}_A$ (resp. $\mathsf{f}\text{-}\mathcal{F}_A$) be the set of all the forbidden threads (resp. of finite length). Then we have

(1) the global dimension of A is $\max_{1\leqslant i\leqslant d} \mathfrak{C}(\Delta_i) - 1 = \max_{\Pi\in\mathcal{F}_A} l(\Pi)$, where $l(\Pi)$ is the length of Π;

(2) the left and right self-injective dimensions of A are

$$\begin{cases} 0, & \text{if } Q \text{ is either a point or an oriented cycle with full relations,} \\ \max\limits_{\Delta_i\in\mathsf{f}\text{-}\Delta} \{1, \mathfrak{C}(\Delta_i) - 1\} = \max\limits_{\Pi\in\mathsf{f}\text{-}\mathcal{F}_A} l(\Pi), & \text{otherwise.} \end{cases}$$

As a consequence, we get that the finiteness of the global dimension of gentle algebras is invariant under Avella-Geiss (AG)-equivalence. In addition, we get that the number of indecomposable non-projective Gorenstein projective modules over gentle algebras is also invariant under AG-equivalence.

Keywords global dimension, self-injective dimension, gentle algebras, marked ribbon surfaces, geometric models, AG-equivalence

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1 Introduction

Gentle algebras were introduced by Assem and Skowroński [3] as appropriate context for the study of algebras derived equivalently to hereditary algebras of type $\widetilde{\mathbb{A}}_n$. Note that every gentle algebra is a special biserial algebra and all the indecomposable modules over a special biserial algebra were described by Butler and Ringel [9, Section 3] and Wald and Waschbäsch [28, Proposition 2.3]. Thus, we can portray all the indecomposable modules over a gentle algebra. To be precise, each indecomposable module over

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a gentle algebra is either a string module or a band module. Moreover, every gentle algebra is also a string algebra and all the irreducible morphisms between string and band modules over a string algebra were studied by Butler and Ringel [9], Crawley-Boevey [11] and Krause [21]. In addition, by proving that the left and right self-injective dimensions of a gentle algebra are equal to the maximal length of certain paths starting with a gentle arrow and bounded by one, if there is no such arrow, Geiss and Reiten [14] obtained that every gentle algebra is Gorenstein, i.e., its left self-injective dimension $\text{inj.dim}_A A$ and right self-injective dimension $\text{inj.dim} A_A$ are finite.

The geometric models of gentle algebras were introduced in [6,15,25] and have been extensively studied in [1,17,26] based on the works in [2,7,13,27]. They originated in triangulated surfaces which are used in the study of cluster algebras and cluster categories, such as [8,22,23] and so on. Opper et al. [25] introduced ribbon graphs and marked ribbon surfaces for gentle algebras. One can calculate all the objects, morphisms and AG-invariants (a derived invariant defined in [5]) in the derived category of a gentle algebra by marked ribbon surfaces (see [25, Theorems 2.5, 3.3 and 6.1]). Furthermore, Baur and Coelho-Simões [6] introduced permissible curves (see Definition 2.5 below) to describe indecomposable modules, and constructed the geometric model of the category of finitely generated modules over a gentle algebra; they also provided the depictions of the Auslander-Reiten translate τ and Auslander-Reiten sequences through the rotations of permissible curves in marked ribbon surfaces. Recently, He et al. [17] studied the category of finitely generated modules over skew-gentle algebras (a generalization of gentle algebras) by punctured marked surfaces (a generalization of marked ribbon surfaces). They provided a dimension formula for calculating morphisms between the indecomposable modules M and τM by equivalence classes of tagged permissible curves, tagged intersections and intersection numbers. In addition, by using the Koszul dual, Opper et al. [25, Section 1 and Subsection 1.7] proved that the global dimension $\text{gl.dim} A$ of a gentle algebra A is infinite if and only if its quiver has at least one oriented cycle with full relations.

The aim of this paper is to give some definite formulae for calculating the global and self-injective dimensions of a given gentle algebra by marked ribbon surfaces.

Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Then $\mathcal{S}_A$ is divided into some elementary polygons $\{\Delta_i \mid 1 \leqslant i \leqslant d\}$ by Γ_A which has exactly one side $\subseteq \partial\mathcal{S}_A$ (the boundary of $\mathcal{S}_A$). We use $\mathfrak{C}(\Delta_i)$ to denote the number of sides of Δ_i belonging to Γ_A if the unmarked boundary component of $\mathcal{S}_A$ is not a side of Δ_i; otherwise, $\mathfrak{C}(\Delta_i) = \infty$. We use $\mathsf{f}\text{-}\Delta$ to denote the set of all the non-∞-elementary polygons (see Remark 5.8), and use $\mathcal{F}_A$ (resp. $\mathsf{f}\text{-}\mathcal{F}_A$) to denote the set of all the forbidden threads (resp. of finite length) (see Definition 5.9). For any $\mathit{\Pi} \in \mathcal{F}_A$, we use $l(\mathit{\Pi})$ to denote its length. Our main result is as follows.

Theorem 1.1 (= Theorems 5.10 and 6.9). *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Then we have*

(1) $\text{gl.dim} A = \max_{1\leqslant i\leqslant d} \mathfrak{C}(\Delta_i) - 1 = \max_{\mathit{\Pi}\in\mathcal{F}_A} l(\mathit{\Pi})$;

(2)

$$\text{inj.dim}_A A = \text{inj.dim} A_A = \begin{cases} 0, & \textit{if } Q \textit{ is either a point or an oriented cycle with full relations,} \\ \max\limits_{\Delta_i\in\mathsf{f}\text{-}\Delta}\{1, \mathfrak{C}(\Delta_i) - 1\} = \max\limits_{\mathit{\Pi}\in\mathsf{f}\text{-}\mathcal{F}_A} l(\mathit{\Pi}), & \textit{otherwise.} \end{cases}$$

The rest of this paper is organized as follows. In Section 2, we recall some terminologies and some preliminary results needed in this paper. In particular, we give the definition of gentle algebras and some related notions related to their geometric models. In Section 3, we give the descriptions of some short exact sequences by geometric models, which will be used frequently in the sequel.

Theorem 1.2 (= Theorem 3.12). *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface.*

(1) *If c, c' and c'' are permissible curves such that the positional relationship of them is given by Case* I *in Figure* 1*, then there exists an exact sequence*

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0.$$

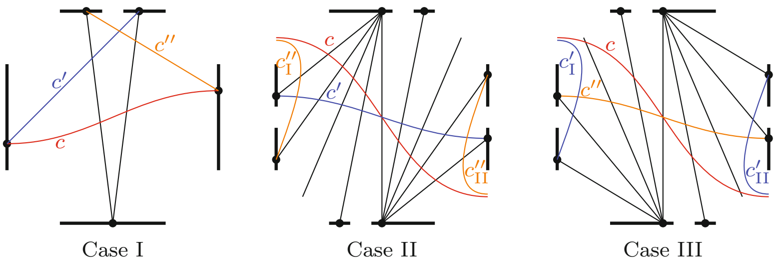


Figure 1 (Color online) For each case, the module M' corresponding to blue permissible curve(s) is a submodule of M corresponding to the red permissible curve, and the module M'' corresponding to orange permissible curve(s) is isomorphic to the quotient M/M'

(2) *If c, c', c''_{I} and c''_{II} are permissible curves such that the positional relationship of them is given by Case* II *in Figure* 1*, then there exists an exact sequence*

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c''_{\mathrm{I}} \oplus c''_{\mathrm{II}}) \longrightarrow 0.$$

(3) *If c, c'_{I}, c'_{II} and c'' are permissible curves such that the positional relationship of them is given by Case* III *in Figure* 1*, then there exists an exact sequence*

$$0 \longrightarrow M(c'_{\mathrm{I}} \oplus c'_{\mathrm{II}}) \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0.$$

In Section 4, we describe the projective covers and injective envelopes of indecomposable modules over gentle algebras by marked ribbon surfaces (see Theorems 4.6 and 4.9). Then in Section 5, we give the proof of Theorem 1.1(1).

In Section 6, by describing all the Gorenstein projective modules in geometric models, we give a proof of the following equalities:

$$\text{inj.dim}_A A = \text{inj.dim} A_A = \begin{cases} 0, & \text{if } Q \text{ is either a point or an oriented cycle with full relations,} \\ \max\limits_{\Delta_i \in \text{f-}\Delta} \{1, \mathfrak{C}(\Delta_i) - 1\}, & \text{otherwise.} \end{cases}$$

Then by describing minimal projective resolutions of all the injective modules, we obtain a formula for calculating the left and right self-injective dimensions of A by marked ribbon surfaces (see Proposition 6.8). As a consequence, we give the proof of Theorem 1.1(2) including another proof of the above equalities. In addition, we prove that the number of indecomposable non-projective Gorenstein projective modules over gentle algebras is invariant under AG-equivalence (see Proposition 6.12). In Section 7, we give some examples to illustrate the obtained results, i.e., we apply Theorem 1.1 to calculate the global and self-injective dimensions of some gentle algebras.

2 The geometric models of gentle algebras

In this paper, assume that $A = kQ/I$ is a finite-dimensional basic algebra over an algebraically closed field k, where I is an admissible ideal of kQ and $Q = (Q_0, Q_1)$ is a finite quiver with Q_0 and Q_1 the sets of all the vertices and arrows, respectively. We use s and t to denote two functions from Q_1 to Q_0 which send each arrow to its source and target, respectively. The multiplication $\alpha_1\alpha_2$ of two arrows α_1 and α_2 in Q_1 is defined by the concatenation if $t(\alpha_1) = s(\alpha_2)$ or zero if $t(\alpha_1) \neq s(\alpha_2)$. For a set X, the number of elements of X is denoted by $\sharp X$. We use $\text{mod}A$ to denote the category of finitely generated right A-modules, and use $\text{gl.dim}A$ to denote the global dimension of A. For a module $M \in \text{mod}A$, we use $\text{proj.dim}M$ and $\text{inj.dim}M$ to denote the projective and injective dimensions of M, respectively, and use $\text{top}M$, $\text{soc}M$ and $P(M)$ to denote the top, socle and projective cover of M, respectively. We define $N \leqslant M$ (resp. $N \leqslant_{\oplus} M$) if N is a submodule (resp. direct summand) of M.

2.1 Marked ribbon surfaces

In gentle algebras, Opper et al. [25] introduced the notion of marked ribbon surfaces by defining ribbon graphs first. Marked ribbon surfaces are often referred to as marked surfaces, such as [1, 6, 17]. In this paper, we still use the terminology from [25] (but we do not need the definition of ribbon graphs).

Definition 2.1 (Marked ribbon surfaces [1,6,25]). A *marked ribbon surface* is a triple $(\mathcal{S}, \mathcal{M}, \Gamma)$, where

- $\mathcal{S}$ is an oriented connected surface with non-empty boundary, and we use $\partial\mathcal{S}$ to denote its boundary;
- $\mathcal{M}$ is a finite set of points on $\partial\mathcal{S}$, and each element in $\mathcal{M}$ is called a *marked point* and denoted by the symbol $\bullet$;
- Γ is called a *full formal* $\bullet$*-arc system* of $\mathcal{S}$ such that the following hold:

– Each element in Γ, called a $\bullet$*-arc* (or an arc), is a curve whose ending points belong to $\mathcal{M}$.

– For any two $\bullet$-arcs γ and γ', there is no intersection in the inner of $\mathcal{S}$.

– $\mathcal{S}$ is divided into some parts $\{\Delta_i \mid 1 \leqslant i \leqslant d\}$ by Γ such that each part is a polygon which has exactly a side, say a *single boundary arc*, not belonging to Γ. The set of all the polygons $\Delta = \{\Delta_i \mid 1 \leqslant i \leqslant d\}$ is called an *original* $\bullet$*-dissection* (or *original dissection*) of $\mathcal{S}$ (see [1]), and each polygon Δ_i above is said to be *elementary.* We denote by $\mathfrak{S}(\Delta_i)$, say the *arc set*, the set of all the sides of Δ_i belong to Γ.

We give some supplements to Definition 2.1 as follows.

(1) Each arc in Γ can be seen as a pair (γ, γ^-), where γ is a continuous function $\gamma : (0, 1) \to \mathcal{S}\backslash\partial\mathcal{S}$ such that $\gamma(0^+) = \lim_{x\to 0^+} \gamma(x)$ and $\gamma(1^-) = \lim_{x\to 1^-} \gamma(x)$ lie in $\mathcal{M}$, and γ^- is the formal inverse of γ, i.e., $\gamma^-(x) = \gamma(1-x)$. Based on this, $\gamma = \gamma(x)$ and $\gamma^- = \gamma(1-x)$ correspond to the same arc in Γ. For the sake of simplicity, we suppose that γ and γ^- both are elements in Γ and $\gamma \simeq \gamma^- \simeq (\gamma, \gamma^-)$. This hypothesis can be used to simplify some descriptions, such as Theorem 5.6 and Example 7.1.

(2) If there is some i with $1 \leqslant i \leqslant d$ such that Δ_i is a two-gons, then Δ_i is one of two forms (i) and (ii) shown in Figure 2. If Δ_i is of the form (i), then adding a new point (will be denoted by the symbol $\circ$ and called an *extra point*) on the side not belonging to Γ. The set of all the extra points is denoted by E.

(3) The positive direction of $\partial\mathcal{S}$ is defined as follows: the interior of $\mathcal{S}$ is on the left while walking along the boundary.

Remark 2.2. (1) In some cases, we do not know whether a point in the surface is a marked point or an extra point, we use "$\times$" to express it.

(2) We say that arcs $\gamma_1, \ldots, \gamma_n$ with a common endpoint p *surround* p *in the clockwise* (*resp. counterclockwise*) *order* if γ_{i+1} is right (resp. left) to γ_i $(1 \leqslant i \leqslant n-1)$ at the point p (see Figure 3).

(3) In [1], Amiot et al. defined the $\circ$-points and $\circ$-dissection for marked ribbon surfaces. The $\circ$*-point* is either a point belonging to $\partial\mathcal{S}$ which lies between two adjacent marked points or a boundary component of $\mathcal{S}$ with no marked point. The $\circ$*-dissection* $\Delta_\circ$ of $\mathcal{S}$, whose elements are called $\circ$*-arcs*, is dual of $\bullet$-dissection $\Delta = \Delta_\bullet$, i.e., each $\circ$-arc is a curve whose endpoints are $\circ$-points, any two $\circ$-arcs have no intersection point in $\mathcal{S}\backslash\partial\mathcal{S}$, and for every $\bullet$-arc (resp. $\circ$-arc), there exists a unique $\circ$-arc (resp. $\bullet$-arc) such that they have only one intersection point in $\mathcal{S}\backslash\partial\mathcal{S}$. Indeed, $\Delta_\circ$ is uniquely determined by $\Delta_\bullet$, and it is said to be *lamination* of $\mathcal{S}$ by Opper et al. [25], although their definitions seem different. In this paper, we do not need the definitions of $\circ$-point and $\circ$-dissection, but we need to define the extra points which are special $\circ$-points. They are used in the description of some indecomposable modules.

Definition 2.3 (k-algebras of marked ribbon surfaces [1, 25]). Let $\mathcal{S} = (\mathcal{S}, \mathcal{M}, \Gamma)$ be a marked ribbon surface. We associate a quiver $Q_{\mathcal{S}} = (Q_0, Q_1)$ and a relation $I_{\mathcal{S}} = \langle R\rangle$ to $\mathcal{S}$ as follows:

(1) The vertexes in Q_0 correspond to the $\bullet$-arcs in Γ and this corresponding is one-to-one, denoted by $\mathfrak{v} : Q_0 \to \Gamma$, i.e., each $\bullet$-arc can be seen as a vertex of $Q_{\mathcal{S}}$.

(2) For every elementary polygon Δ_i with two sides $u, v \in \Gamma$, if v follows u in the counterclockwise order, then there is an arrow $u \to v$ in Q_1.

(3) R is the set of all the compositions $\alpha\beta$, where $\alpha : u \to v$ and $\beta : v \to w$ satisfy that u, v and w are sides of the same elementary polygon.

Then *the* k*-algebra* $A_{\mathcal{S}}$ *of the marked ribbon surfaces* $\mathcal{S}$ is defined by $A_{\mathcal{S}} := kQ_{\mathcal{S}}/I_{\mathcal{S}}$.

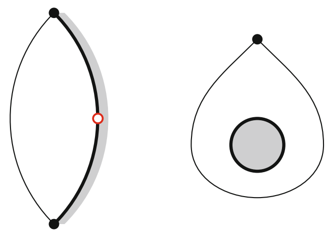


Figure 2 (Color online) (i) and (ii) are 2-gons. In (i), we add an extra point on the side not belonging to Γ

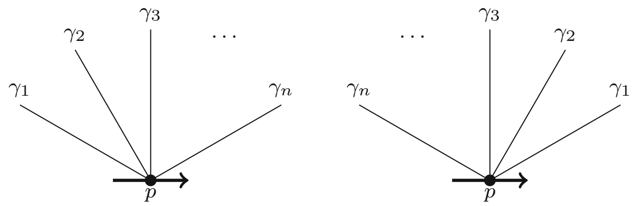


Figure 3 $\gamma_1, \ldots, \gamma_n$ with the common endpoint p are surround p in the clockwise (resp. counterclockwise) order

Definition 2.4 (Gentle algebras [1,3,5,25]). A k-algebra $A = kQ/I$ is said to be *special biserial* if the following conditions are satisfied:

(1) For each vertex $v \in Q_0$, $\sharp\{\alpha \in Q_1 \mid s(\alpha) = v\} \leqslant 2$ and $\sharp\{\alpha \in Q_1 \mid t(\alpha) = v\} \leqslant 2$.

(2) For each arrow $\beta \in Q_1$, $\sharp\{\alpha \in Q_1 \mid s(\beta) = t(\alpha), \alpha\beta \notin I\} \leqslant 1$ and $\sharp\{\alpha \in Q_1 \mid s(\alpha) = t(\beta), \beta\alpha \notin I\} \leqslant 1$.

(3) For each arrow $\beta \in Q_1$, there exists a bound $n \in \mathbb{N}^+$ such that

- for each path $p = \beta_1 \cdots \beta_{n-1}$ where $t(p) = s(\beta)$, the path $p\beta$ contains a subpath in I;
- for each path $p' = \beta'_1 \cdots \beta'_{n-1}$ where $t(\beta) = s(p')$, the path $\beta p'$ contains a subpath in I.

Moreover, $A = kQ/I$ is said to be *gentle* if it is a special biserial algebra such that the following hold:

(4) Q is a bound quiver.

(5) All the relations in I are paths of length 2.

(6) For each arrow $\beta \in Q_1$, $\sharp\{\alpha \in Q_1 \mid s(\beta) = t(\alpha), \alpha\beta \in I\} \leqslant 1$ and $\sharp\{\alpha \in Q_1 \mid s(\alpha) = t(\beta), \beta\alpha \in I\} \leqslant 1$.

Notice that a special biserial algebra is finite-dimensional if and only if Q is finite, and thus a gentle algebra is always finite-dimensional. Moreover, Opper et al. [25, Proposition 1.21] proved that $A_{\mathcal{S}} := kQ_{\mathcal{S}}/I_{\mathcal{S}}$ is gentle, and there is a bijection between isomorphism classes of gentle algebras and homotopy classes of marked ribbon surfaces. It should be pointed out that there are different definitions of gentle algebras. For example, the definition of gentle algebras in [3,5] does not require the fourth condition, so there are gentle algebras whose quivers are not bound; that of gentle algebras in [1,25] does not require the third condition, so there are gentle algebras whose quivers have at least one oriented cycle with no relations.

2.2 Permissible curves and permissible closed curves

Each curve c in a marked ribbon surface $\mathcal{S} = (\mathcal{S}, \mathcal{M}, \Gamma)$ can be defined as a function $c : [0,1] \to \mathcal{S}$, where $c(0)$ and $c(1)$ are its endpoints. We say that $c(0)$ and $c(1)$ are its *starting point* and *ending point*, respectively. In this paper, we only consider such curves whose points lie in the interior of $\mathcal{S}$ except endpoints, i.e., $\{c(x) \mid 0 < x < 1\} \subseteq \mathcal{S}\backslash\partial\mathcal{S}$, and whose endpoints are elements belong to $\mathcal{M} \cup E$. Thus each non-closed curve can be regarded as a function $c : (0,1) \to \mathcal{S}\backslash\partial\mathcal{S}$ where $c(0^+)$ and $c(1^-)$ are its endpoints and each closed curve can be regarded as a function $c : [0,1] \to \mathcal{S}\backslash\partial\mathcal{S}$ where $c(0) = c(1) \in \mathcal{S}\backslash\partial\mathcal{S}$. In this paper, for each curve c with the endpoints $c(0^+)$ and $c(1^-)$ on the marked ribbon surface, we always suppose that the number of intersections between c and Γ is minimal up to homotopy.

Definition 2.5 (Permissible curves and permissible closed curves [6, Definition 3.1])**.** Let $\mathcal{S} = (\mathcal{S}, \mathcal{M}, \Gamma)$ be a marked ribbon surface.

(1) A curve $c : (0,1) \to \mathcal{S}\backslash\partial\mathcal{S}$ (or a closed curve $c : [0,1] \to \mathcal{S}\backslash\partial\mathcal{S}$) in $\mathcal{S}$ is said to *consecutively cross* $u, v \in \Gamma$ if $p_1 = c \cap u$ and $p_2 = c \cap v$ belong to $\mathcal{S}\backslash\partial\mathcal{S}$, and the segment of c between the points p_1 and p_2 does not cross any other arc in Γ.

Then each non-closed curve c consecutively crossing $\gamma_1, \ldots, \gamma_r \in \Gamma$ can be written as $c = c_1c_2\cdots c_rc_{r+1}$, where each c_i $(2 \leqslant i \leqslant r)$ is a segment between γ_{i-1} and γ_i which does not cross any other arc in Γ, and c_1 (resp. c_{r+1}) is a segment between $c(0^+)$ and γ_1 (resp. γ_r and $c(1^-)$) which does not cross any other arc in Γ. The segment c_i $(2 \leqslant i \leqslant r-1)$ is called *middle*, and the segments c_1 and c_r are called *end*. Similarly, each closed curve c consecutively crossing $\gamma_1, \ldots, \gamma_r \in \Gamma$ can be written as $c = c_1c_2\cdots c_r$, where each c_i $(2 \leqslant i \leqslant r)$ is a segment between γ_{i-1} and γ_i which does not cross any other arc in Γ, and c_1 is a segment between γ_r and γ_1.

(2) Let B be a boundary component with no marked point of $\mathcal{S}$, called an *unmarked boundary component* of $\mathcal{S}$, and let $c : (0,1) \to \mathcal{S}\backslash\partial\mathcal{S}$ be a curve in $\mathcal{S}$. Write $c = c_1c'c_2$, where c_1 (resp. c_2) is the end segment between $c(0^+)$ (resp. $c(1^-)$) and the first (resp. last) crossing point in Γ. We define the *winding number* of c_i $(i = 1, 2)$ around B as the minimum number of times travels $\bar{c}$ around B in either direction, where $\bar{c}$ lies in the homotopy class of c_i.

(3) A curve c is called *permissible* if the following conditions are satisfied:

(a) The winding number of c around an unmarked boundary component is either 0 or 1.

(b) If c consecutively crosses two arcs u and v in Γ, then u and v have a common endpoint p, denoted by $p(u, v, c)$, lying in $\mathcal{M}$, and locally we have a triangle with p a vertex (as shown in Figure 4(I), the curves c_1, c, c' and c'' are permissible).

(4) A *permissible closed curve* is a closed curve c satisfying the condition (3)(b) (as shown in Figure 4(II), the algebra of the marked ribbon surface is a 2-Kronecker algebra, and the curve c is a permissible closed curve).

Definition 2.6 (Equivalence class of permissible curves [6])**.** Let $\mathcal{S} = (\mathcal{S}, \mathcal{M}, \Gamma)$ be a marked ribbon surface. Two permissible curves $c, c' : (0,1) \to \mathcal{S}\backslash\partial\mathcal{S}$ in $\mathcal{S}$ are called *equivalent*, denoted by $c \simeq c'$, if one of the following conditions is satisfied:

(1) c' is the *reverse curve* c^- of c, i.e., $c(x) = c'(1-x)$ for any $x \in (0,1)$.

(2) There is a sequence of consecutive arcs $(u_i : (0,1) \to \mathcal{S}\backslash\partial\mathcal{S})_{1\leqslant i\leqslant n}$ in the arc system Γ such that

- all u_i $(1 \leqslant i \leqslant n)$ are sides of the same elementary polygon Δ_j;
- $c(0^+)$ (resp. $c(1^-)$) is a marked point $\bullet$ or an extra point $\circ$, $c(1^-) = u_1(0^+)$ (resp. $c(0^+) = u_1(0^+)$), $u_{i-1}(1^-) = u_i(0^+)$ for any $1 \leqslant i \leqslant n$;
- c is homotopic to the concatenation of c' and $(u_i)_{1\leqslant i\leqslant n}$;

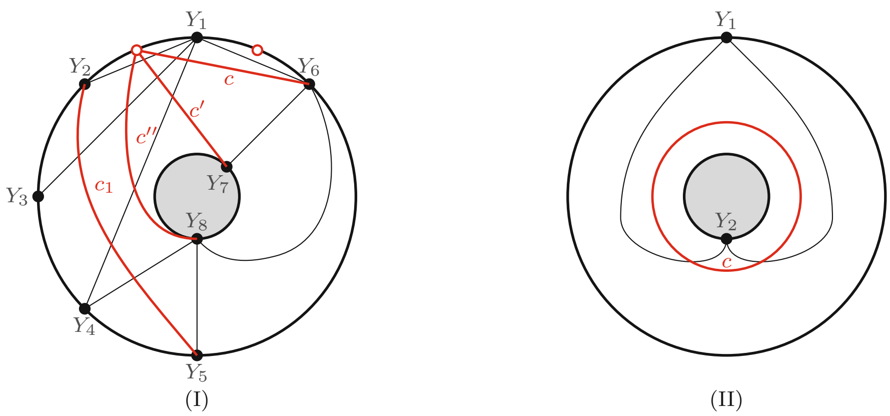


Figure 4 (Color online) Some examples of permissible (closed) curves in marked ribbon surfaces

• c' starts at an endpoint of u_n, and the first crossing with Γ of c and c' are with the same side of Δ_j.

(3) $c(0^+) = c'(0^+)$ and $c(1^-) = c'(1^-)$ (resp. $c(0^+) = c'(1^-)$ and $c(1^-) = c'(0^+)$) are marked points or extra points, arcs consecutively crossed by c and c' (c^- and c') are the same, and each segment of c cut by two arcs is homotopic to that of c'.

(4) c (resp. c') is either a curve satisfying $c \cap \Gamma = \emptyset$ (resp. $c' \cap \Gamma = \emptyset$) or an arc; in this case, we call c and c' are *trivial*. Thus each arc can be regarded as a trivial permissible curve.

Example 2.7. In Figure 4(I), the curves c, c' and c'' are permissible and $c \simeq c' \simeq c''$, but $c \not\simeq c_1$.

3 Finitely generated module categories

3.1 The geometric models of string and band modules

Let $A = kQ/I$ be a gentle algebra. For each arrow $\alpha \in Q_1$, the *formal inverse* of α, denoted by α^-, is an arrow with $s(\alpha^-) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. Obviously, $(\alpha^-)^- = \alpha$. We use Q_1^- to denote the set of all the formal inverses of arrows in Q_1. Also, we can define that the formal inverse of any path $p = \alpha_1 \cdots \alpha_n$ is $p^- = \alpha_n^- \cdots \alpha_1^-$. In this subsection, we recall the definitions of string modules and band modules over a gentle algebra A (see [9]), and review the description of indecomposable modules in marked ribbon surfaces (see [6]).

A *string* s is a reduced walk in the quiver Q with no relations, i.e., $s = a_1 \cdots a_n$ with $a_i \in Q_1 \cup Q_1^-$ has neither subwalks of the form aa^- and a^-a nor subwalks of the form ab such that $ab \in I$ or $b^-a^- \in I$; particularly, $s = 1_v$ is a path of length 0 corresponding to a vertex $v \in Q_0$, and it is called a *simple string* (note that $1_v^- = 1_v$ for each $v \in Q_0$). Trivially, we define the *trivial string* $s = 0$.

Each string $s = a_1 \cdots a_n$ in A corresponds to a module $M(s) \in \operatorname{mod} A$ as follows:

• replacing each vertex of s by a copy of the field k, i.e., for each $v \in Q_0$, $\dim_k M(s)1_v = 1$ if 1_v is a trivial subwalk of s or zero otherwise, and

• the action of an arrow $\alpha \in Q_1$ on $M(s)$ is the identity morphism if α is an arrow of s or zero otherwise.

Obviously, $s = 0$ yields $M(s) = 0$; s being simple yields that $M(s)$ is simple; $M(s) \cong M(s')$ if and only if $s' = s^-$ or $s' = s$.

A *band* $b = a_1 \cdots a_n$ is a cyclic string (i.e., $t(a_n) = s(a_1)$) such that each power b^n is a string but b is not a proper power of any string.

Each band $b = a_1 \cdots a_n$ in A corresponds to a family of modules $M(b, \varphi) \in \operatorname{mod} A$ as follows:

• φ is an indecomposable k-linear automorphism of k^n with $n \geqslant 0$ (i.e., $\varphi \in \operatorname{Aut}(k^n)$ is a Jordan block),

• replacing each vertex of s by a copy of the vector space k^n, and

• the action of an arrow $\alpha \in Q_1$ on $M(b, \varphi)$ is the identity morphism if $\alpha = a_i$ for $1 \leqslant i \leqslant n-1$ or φ if $\alpha = a_n$ (thus $M(b, \varphi)\alpha = 0$ if α is not an arrow of b).

Remark 3.1. By [9], we know that each indecomposable A-module is either a string module or a band module. Baur and Coelho Simões [6] showed that there is a bijection between the equivalence classes of non-trivial permissible curves in $\mathcal{S}_A$ and non-zero strings of A, and there is a bijection between the homotopy classes of permissible closed curves c in $\mathcal{S}_A$ with $|I_{\Gamma_A}(c)| \geqslant 2$ and the powers of bands of A, where $|I_{\Gamma_A}(c)|$ is the number of intersection points of c and Γ_A (see [6]). To be precise, we have the following:

• If $[c]$ is an equivalence class of non-trivial permissible curve, then $c = c_1 \cdots c_r$, where each c_i is a segment cut by two arcs γ_{i-1} and γ_i (we suppose that c consecutively crosses $\gamma_1, \ldots, \gamma_{r-1}$), and c corresponds to such string $s = s_1 \cdots s_r$ that s_i is one of

– the arrow $\gamma_{i-1} \to \gamma_i$, if c counterclockwise crosses γ_{i-1} and γ_i around their common endpoint $p(\gamma_{i-1}, \gamma_i, c)$;

– the formal inversion of the arrow $\gamma_i \to \gamma_{i-1}$, if c clockwise crosses γ_{i-1} and γ_i around their common endpoint $p(\gamma_{i-1}, \gamma_i, c)$.

(Note that each arc in Γ can be regarded as a vertex of quiver Q by Definition 2.3.)

Thus c can correspond to the string module of s.

• If c is a homotopy class of permissible closed curves, then c with an indecomposable invertible linear transformation $\varphi : k^n \to k^n$ corresponds to a band which induces a band module $M(c,\varphi)$. Band modules lie in homogeneous tubes of the Auslander-Reiten quiver, and thus the Auslander-Reiten translate τ acts on them as the identity morphism.

Remark 3.2 (See [6, Theorem 3.8 and Proposition 3.9])**.** Let $[c]$ (resp. $\mathrm{htp}(c)$) be the equivalence class of the permissible curve (resp. the homotopy class of permissible closed curve) c. Define the following three sets:

• $\text{Eq-cls}(\mathcal{S}_A) = \{[c] \mid c \text{ is a permissible curve}\}$.

• $\text{Htp-cls}(\mathcal{S}_A) = \{(\mathrm{htp}(c),\varphi) \mid c \text{ is a permissible closed curve with } |I_{\Gamma_A}(c)| \geqslant 2 \text{ and } \varphi \text{ a Jordan block}\}$.

• $\mathfrak{P}(\mathcal{S}_A) = \text{Eq-cls}(\mathcal{S}_A) \cup \text{Htp-cls}(\mathcal{S}_A)$.

Then we have a bijection

$$M : \mathfrak{P}(\mathcal{S}_A) \to \{\text{string modules}\} \cup \{\text{band modules}\} = \mathrm{ind}(\mathrm{mod}A), \quad x \mapsto M(x).$$

Obviously, each trivial permissible curve corresponds to zero by M.

Definition 3.3 (Formal direct sums)**.** Let $\mathcal{S}_A$ be the marked ribbon surface of a gentle algebra A. A *formal direct sum* of $c_1, \ldots, c_n$, denoted by $\bigoplus_{i=1}^n c^i$, naturally corresponds to the direct sum of modules $\bigoplus_{i=1}^n M(c^i)$ by the corresponding M in Remark 3.2.

3.2 The geometric models of some short exact sequences

In this subsection, we depict certain short exact sequences by marked ribbon surfaces for gentle algebras, which are used to describe the minimal projective and injective resolutions of simple modules in Section 5, and describe the minimal projective resolutions of injective modules in Section 6. First at all, we fix some terms and notations.

We always assume that each non-trivial permissible curve $c : (0,1) \to \mathcal{S}\backslash\partial\mathcal{S}$ in $\mathcal{S} = (\mathcal{S},\mathcal{M},\Gamma)$ consecutively crossing arcs $\gamma_1, \ldots, \gamma_{r-1}$ satisfies that γ_1 is the first arc crossed by c, i.e., there is a sequence $0 = x_0 < x_1 < x_2 < \cdots < x_r = 1$ such that $c = c_1 \cup c_2 \cup \cdots \cup c_r$ (we write $c = c_1c_2\cdots c_r$ for simplicity), where each $c_i = \{c(x) \mid x_{i-1} \leqslant x \leqslant x_i\}$ is a segment of c satisfying $c_i \cap \gamma_i = c(x_i) = c_{i+1} \cap \gamma_i$ if $2 \leqslant i \leqslant r-1$; $c_1 = \{c(x) \mid 0 < x \leqslant x_1\}$ and $c_r = \{c(x) \mid x_{r-1} \leqslant x < 1\}$ are the end segments of c satisfying $c_1 \cap \gamma_1 = c(x_1) = c_2 \cap \gamma_1$ and $c_r \cap \gamma_{r-1} = c(x_{r-1}) = c_{r-1} \cap \gamma_{r-1}$, respectively.

Let $c = c_1c_2\cdots c_r$ be a permissible curve consecutively crossing $\gamma_1, \ldots, \gamma_{r-1}$. Then it induces a new permissible curve $c' = c_{i-1}^{\mathfrak{T}} c_i \cdots c_j c_{j+1}^{\mathfrak{T}}$ for any $2 \leqslant i \leqslant j \leqslant r-1$ such that

(1) $c_{i-1}^{\mathfrak{T}}$ and c_{i-1} lie in the inner of the same elementary polygon of $\mathcal{S}$;

(2) one endpoint of $c_{i-1}^{\mathfrak{T}}$ is $c_{i-1} \cap \gamma_{i-1}$ and the other is an endpoint of γ_{i-2};

(3) the dual of the condition (1), i.e., $c_{j+1}^{\mathfrak{T}}$ and c_{j+1} lie in the inner of the same elementary polygon;

(4) the dual of the condition (2), i.e., one endpoint of $c_{j+1}^{\mathfrak{T}}$ is $c_{j+1} \cap \gamma_j$ and the other is an endpoint of γ_{j+1}.

Note that it is a trivial case for $c_1^{\mathfrak{T}} c_2 \cdots c_{r-1} c_r^{\mathfrak{T}} = c_1c_2\cdots c_{r-1}c_r$, i.e., $c_1^{\mathfrak{T}} = c_1$, and $c_r^{\mathfrak{T}} = c_r$. We provide an example in Figure 5, in which $c = \cdots c_{i-1}c_ic_{i+1}\cdots c_{j-1}c_jc_{j+1}$ and $c' = c_i^{\mathfrak{T}} c_{i+1} \cdots c_{j-1} c_j^{\mathfrak{T}}$ are the red and the blue permissible curves, respectively. In this example, c corresponds to the string

$$\gamma_1 \longrightarrow \cdots \longrightarrow \gamma_{i-1} \longrightarrow \gamma_i \longrightarrow \cdots \longrightarrow \gamma_j \longrightarrow \gamma_{j+1} \longrightarrow \cdots \longleftarrow \gamma_{r-1},$$

and c' corresponds to the string

$$\gamma_i \longrightarrow \cdots \longrightarrow \gamma_j.$$

A string module which corresponds to the permissible curve c (resp. its equivalence class $[c]$) is denoted by $M(c)$ (resp. $M([c])$); a band module with a Jordan block $\varphi \in \mathrm{Aut}(k^n)$ which corresponds to the permissible closed curve c (resp. its homotopic class $\mathrm{htp}(c)$) is denoted by $M(c,\varphi)$ (resp. $M(\mathrm{htp}(c),\varphi)$).

For the convenience of describing rotations, we suppose that the positive direction of the boundary $\partial\mathcal{S}$ of the surface $\mathcal{S}$ is the following walking direction: walking along $\partial\mathcal{S}$, we see that the inner of $\mathcal{S}$ is on the left.

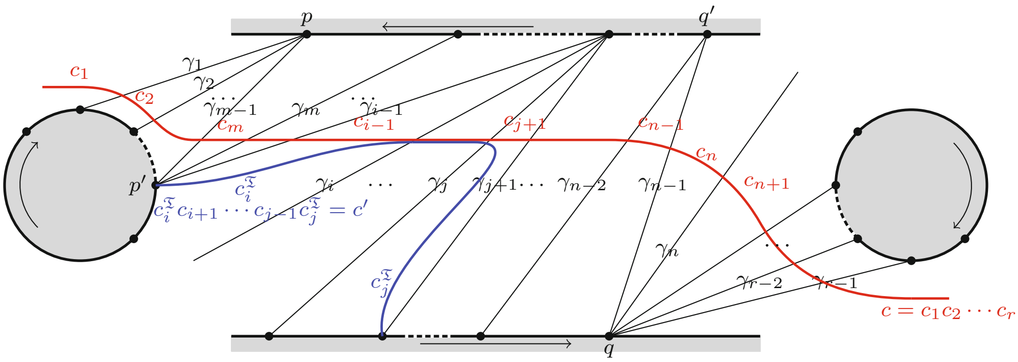


Figure 5 (Color online) Two examples of permissible curves corresponding to strings

Definition 3.4 (Rotations of Type 1). Let $c^1 = c_1 \cdots c_r$ be a permissible curve consecutively crossing $\gamma_1, \ldots, \gamma_{r-1}$, and let c^2 be equivalent to $c' c_{i+1} \cdots c_r$ (in this case $c^1(1^-) = c^2(1^-)$), where c' is either $c_i^{\mathfrak{T}}$ or a curve with an endpoint being an extra point. We say that the permissible curve c^3, denoted by $\mathrm{prot}_{c^1}(c^2)$, is obtained by the *positive rotating* c^2 with respect to c^1 if it is given as follows:

Step 1. Move the endpoint of c^2 which is the common endpoint of c^2 and c^1 to the other of c^1.

Step 2. Following the positive direction of the boundary, we move the other endpoint of c^2 to the vertex p of Δ_j, where

– Δ_j is the elementary polygon such that c_i lies in the inner of Δ_j (in this case, $\gamma_{i-1}, \gamma_i \in \mathfrak{S}(\Delta_j)$), and

– p is a vertex such that $c^3 = \mathrm{prot}_{c^1}(c^2)$ crosses γ_{i-1}.

Dually, we can define $\mathrm{nrot}_{c^1}(c^2)$ obtained by the *negative rotating* c^2 with respect to c^1 (see Figure 6).

Definition 3.5 (Rotations of Type 2). Let $c^1 = c_1 \cdots c_r$ be a permissible curve consecutively crossing $\gamma_1, \ldots, \gamma_{r-1}$, and c^2 be equivalent to either $c_m^{\mathfrak{T}} c_{m+1} \cdots c_n c_{n+1}^{\mathfrak{T}}$ ($2 \leqslant m \leqslant n+1 \leqslant r-2$) such that c^1 and c^2 have an intersection, say x, in $\mathcal{S} \backslash \partial \mathcal{S}$. Then c^2 is divided into two parts c^2_{I} and c^2_{II}, where $c^2_{\mathrm{I}}(1^-) = x = c^2_{\mathrm{II}}(0^+)$. We say that the permissible curve c^3_{I} (resp. c^3_{II}), denoted by $\mathrm{prot}_{c^1}(c^2_{\mathrm{I}})$ (resp. $\mathrm{prot}_{c^1}(c^2_{\mathrm{II}})$), is obtained by the *positive rotating* c^2_{I} (resp. c^2_{II}) with respect to c^1 if it is given as follows:

Step 1. Move the endpoint x of c^2_{I} (resp. c^2_{II}) to the endpoint $c^1(0^+)$ (resp. $c^1(1^-)$).

Step 2. Following the positive direction of boundary, we move the endpoint $c^2_{\mathrm{I}}(0^+)$ of c^2_{I} (resp. the endpoint $c^2_{\mathrm{II}}(1^-)$ of c^2_{II}) to the next vertex of Δ_j, where Δ_j is an elementary polygon such that c_m (resp. c_{n+1}) lies in the inner of Δ_j (in this case, we have $\gamma_{m-1}, \gamma_m \in \mathfrak{S}(\Delta_j)$ (resp., $\gamma_n, \gamma_{n+1} \in \mathfrak{S}(\Delta_j)$)).

Dually, we can define $\mathrm{nrot}_{c^1}(c^2_{\mathrm{I}})$ and $\mathrm{nrot}_{c^1}(c^2_{\mathrm{II}})$ obtained by the *negative rotating* c^2_{I} *and* c^2_{II} with respect to c^1, respectively (see Figure 7).

Proposition 3.6. *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. If there are three permissible curves c, c' and c'' such that the following conditions are satisfied* (*see Figure* 8):

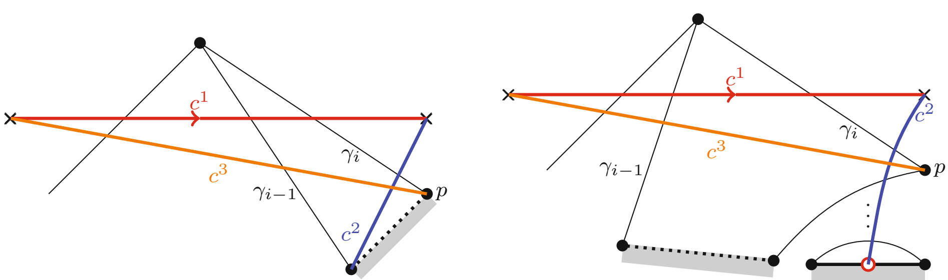


Figure 6 (Color online) $c^3 = \mathrm{nrot}_{c^1}(c^2)$, where the point "×" is either a marked point or an extra point

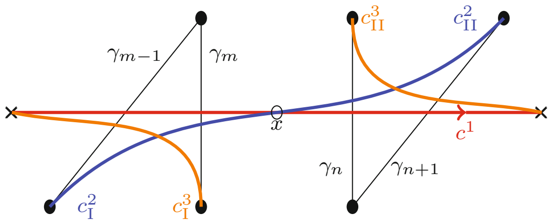


Figure 7 (Color online) $c^3_{\mathrm{I}} \cup c^3_{\mathrm{II}} = \mathrm{nrot}_{c^1}(c^2)$, where the point "×" is either a marked point or an extra point

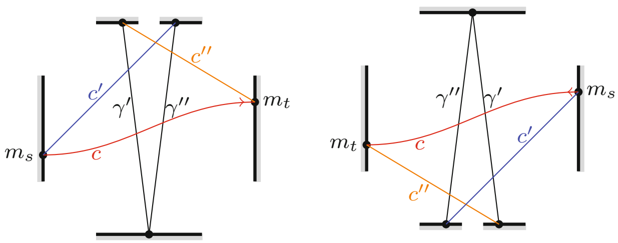


Figure 8 (Color online) The short exact sequence is described in the marked ribbon surface

(1) *c' and c have a common endpoint $m_s = c(0^+)$, and c'' and c have another common endpoint $m_t = c(1^-)$.*

(2) *c consecutively crosses arcs γ' and γ'' such that $p(\gamma', \gamma'', c)$ is on the right of c; c' crosses γ' such that c' and γ'' have a common endpoint which is not m_t; c'' crosses γ'' such that c'' and γ' have a common endpoint which is not m_s.*

(3) *$c'' \simeq \mathrm{prot}_c(c')$ (equivalently, $c' \simeq \mathrm{nrot}_c(c'')$) (see Definition 3.4).*

Then we have a short exact sequence

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0.$$

Proof. Notice that c corresponds to such a string which is of the form $s_c := \cdots — \gamma' \longleftarrow \gamma'' — \cdots$. Then c' and c'' correspond to $s_{c'} := \cdots — \gamma'$ and $s_{c''} := \gamma'' — \cdots$, respectively. Furthermore, $M(c)$, $M(c')$ and $M(c'')$ are string modules corresponding to s_c, $s_{c'}$ and $s_{c''}$, respectively, and hence

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0$$

is exact. □

Lemma 3.7. *Let A be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let $c = c_1 \cdots c_r : (0,1) \to \mathcal{S}_A \backslash \partial\mathcal{S}_A$ be a permissible curve consecutively crossing $\gamma_1, \ldots, \gamma_{r-1}$ such that the endpoints p and q of γ_i $(1 < i < r-1)$ are on the left and right of c, respectively (see Figure 9). Then for any permissible curve c' consecutively crossing $\gamma_m, \ldots, \gamma_n$ $(1 \leqslant m < i < n \leqslant r-1)$ such that*

- *$\gamma_m, \ldots, \gamma_i$ have a common endpoint $p = p(\gamma_m, \gamma_{m+1}, c) = \cdots = p(\gamma_{i-1}, \gamma_i, c)$, and*
- *$\gamma_i, \ldots, \gamma_n$ have a common endpoint $q = p(\gamma_i, \gamma_{i+1}, c) = \cdots = p(\gamma_{n-1}, \gamma_n, c)$,*

we have $M(c') \leqslant M(c)$.

Proof. The strings s_c and $s_{c'}$ corresponding to c and c' are

$$\cdots \longrightarrow \gamma_m \longrightarrow \gamma_{m+1} \longrightarrow \cdots \longrightarrow \gamma_i \longleftarrow \cdots \longleftarrow \gamma_{n-1} \longleftarrow \gamma_n \longleftarrow \cdots$$

and

$$\gamma_m \longrightarrow \gamma_{m+1} \longrightarrow \cdots \longrightarrow \gamma_i \longleftarrow \cdots \longleftarrow \gamma_{n-1} \longleftarrow \gamma_n,$$

respectively. Then the string module corresponding to $s_{c'}$ is a submodule of the module corresponding to s_c, and thus $M(c') \leqslant M(c)$. □

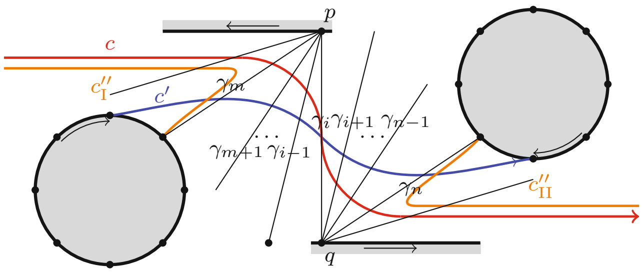


Figure 9 (Color online) c and c' are given in Lemma 3.7 corresponding to two modules $M(c)$ and $M(c')$, and we have $M(c') \leqslant M(c)$. The formal direct sum $c''_{\rm I} \oplus c''_{\rm II}$ corresponds to the module $M(c''_{\rm I} \oplus c''_{\rm II})$ which is isomorphic to the quotient $M(c)/M(c')$

Proposition 3.8. *Keeping the notations in Lemma* 3.7*, we have that*

$$c' = c_m^{\mathfrak{T}} c_{m+1} \cdots c_n c_{n+1}^{\mathfrak{T}} : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$$

crosses γ_i and that c and c' have an intersection. Assume that

- *$c''_{\rm I}$ is a permissible curve which is given as follows:*
- *c' is divided into two parts by c, where denote the part with endpoints $c'(0^+)$ and $c' \cap \gamma_i$ by $c'_{\rm I}$;*
- *$c''_{\rm I} \simeq \mathrm{prot}_c(c'_{\rm I})$ (see Definition* 3.5*);*
- *$c''_{\rm II}$ is a permissible curve which is given as follows:*
- *c' is divided into two parts by c, where denote the part with endpoints $c' \cap \gamma_i$ and $c'(1^-)$ by $c'_{\rm II}$;*
- *$c''_{\rm II} \simeq \mathrm{prot}_c(c'_{\rm II})$ (see Definition* 3.5*).*

Then the quotient module $M(c)/M(c')$ is isomorphic to $M(c''_{\rm I} \oplus c''_{\rm II})$ (see Figure 10(1)*).*

Proof. Similar to the proof of Lemma 3.7, we have that $M(c)/M(c')$ corresponds to two permissible curves

$$\tilde{c}''_{\rm I} = c_1 \cdots c_{m-1} c_m^{\mathfrak{T}} \quad \text{and} \quad \tilde{c}''_{\rm II} = c_{n+1}^{\mathfrak{T}} c_{n+2} \cdots c_r.$$

We can see that $\tilde{c}''_{\rm I}$ consecutively crosses arcs $\gamma_1, \ldots, \gamma_{m-1}$ and $\tilde{c}''_{\rm II}$ consecutively crosses arcs $\gamma_{n+1}, \ldots, \gamma_{r-1}$ $(1 \leqslant m \leqslant i \leqslant n \leqslant r-1)$ (see Figure 9). Thus $\tilde{c}''_{\rm I}$ and $\tilde{c}''_{\rm II}$ lie in the equivalence class $[c''_{\rm I}]$ and $[c''_{\rm II}]$, respectively, and therefore $M(c)/M(c') \cong M(\tilde{c}''_{\rm I} \oplus \tilde{c}''_{\rm II}) \cong M(c''_{\rm I} \oplus c''_{\rm II})$. $\square$

Remark 3.9. If $\gamma_1 = \gamma_m = \gamma_i$ (resp. $\gamma_i = \gamma_n = \gamma_r$), then $c' = c_1^{\mathfrak{T}} c_2 \cdots c_n c_{n+1}^{\mathfrak{T}} \simeq c_1 c_2 \cdots c_n c_{n+1}^{\mathfrak{T}}$ (resp. $c' = c_m^{\mathfrak{T}} c_{m+1} \cdots c_{r-1} c_r^{\mathfrak{T}} \simeq c_m^{\mathfrak{T}} c_{m+1} \cdots c_{r-1} c_r$), and thus $c''_{\rm I}$ (resp. $c''_{\rm II}$) is zero.

The following two results are dual to Lemma 3.7 and Proposition 3.8, respectively.

Lemma 3.10. *Let A be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let $c = c_1 \cdots c_r : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$ be a permissible curve consecutively crossing $\gamma_1, \ldots, \gamma_{r-1}$ such that the endpoints p and q of γ_i $(1 \leqslant i \leqslant r-1)$ are on the right and left of c, respectively. Then for any permissible curve c'' consecutively crossing $\gamma_m, \ldots, \gamma_n$ $(1 \leqslant m < i < n \leqslant r-1)$ such that*

- *$\gamma_m, \ldots, \gamma_i$ have a common endpoint $p = p(\gamma_m, \gamma_{m+1}, c) = \cdots = p(\gamma_{i-1}, \gamma_i, c)$, and*
- *$\gamma_i, \ldots, \gamma_n$ have a common endpoint $q = p(\gamma_i, \gamma_{i+1}, c) = \cdots = p(\gamma_{n-1}, \gamma_n, c)$,*

we have that $M(c'')$ is a quotient module of $M(c)$.

Proposition 3.11. *Keeping the notations in Lemma* 3.10*, we have that*

$$c'' = c_m^{\mathfrak{T}} c_{m+1} \cdots c_n c_{n+1}^{\mathfrak{T}} : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$$

crosses γ_i and that c and c'' have an intersection. Assume that

- *$c'_{\rm I}$ is a permissible curve which is given as follows:*
- *c'' is divided into two parts, where denote the part with endpoints $c''(0^+)$ and $c'' \cap \gamma_i$ by $c''_{\rm I}$;*
- *$c'_{\rm I} \simeq \mathrm{nrot}_c(c''_{\rm I})$ (see Definition* 3.5*);*
- *$c'_{\rm II}$ is a permissible curve which is given as follows:*
- *c'' is divided into two parts, where denote the part with endpoints $c'' \cap \gamma_i$ and $c''(1^-)$ by $c''_{\rm II}$;*
- *$c'_{\rm II} \simeq \mathrm{nrot}_c(c''_{\rm II})$ (see Definition* 3.5*).*

Then $M(c'')$ is isomorphic to $M(c)/M(c'_{\rm I} \oplus c'_{\rm II})$ (see Figure 10(2)*).*

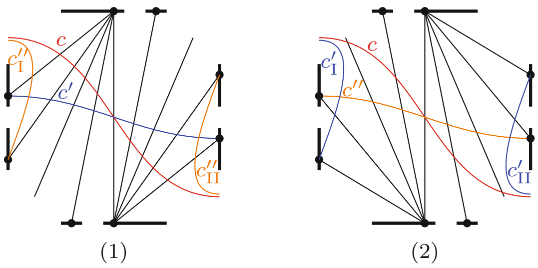


Figure 10 (Color online) The descriptions of short exact sequences given in Propositions 3.8 and 3.11

By Propositions 3.6, 3.8 and 3.11, we get the following theorem.

Theorem 3.12. *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface.*

(1) *If c, c' and c'' are permissible curves satisfying the conditions in Proposition* 3.6, *then there is a short exact sequence*

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0.$$

(2) *If c and c' are permissible curves satisfy the conditions in Lemma* 3.7, *and c''_{I} and c''_{II} are permissible curves satisfying the conditions in Proposition* 3.8, *then there is a short exact sequence*

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c''_{\mathrm{I}} \oplus c''_{\mathrm{II}}) \longrightarrow 0.$$

(3) (The dual of (2)) *If c and c'' are permissible curves satisfying the conditions in Lemma* 3.10, *and c'_{I} and c'_{II} are permissible curves satisfying the conditions in Proposition* 3.11, *then there is a short exact sequence*

$$0 \longrightarrow M(c'_{\mathrm{I}} \oplus c'_{\mathrm{II}}) \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0.$$

4 Special modules

4.1 Simple modules and top (resp. socle) of indecomposable modules

A permissible curve in $\mathcal{S}_A$ corresponding to a simple A-module is a permissible curve crossing a unique arc in Γ_A, and we call it *simple*. The top (resp. socle) of an A-module is semi-simple, and thus we can use a family of simple permissible curves to describe the top (resp. socle) of any A-module. Moreover, for a permissible curve c consecutively crossing arcs γ and γ', recall that $p(\gamma, \gamma', c)$ is the marked point lying in $\gamma \cap \gamma'$ such that it is the vertex of the triangle decided by γ, γ' and the segment given by γ and γ' cutting c. We will use this notation frequently in the sequel.

Lemma 4.1. *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let $c = c_1c_2\cdots c_r$ be a non-trivial permissible* (*non-closed*) *curve in $\mathcal{S}_A$ consecutively crossing arcs $\gamma_1, \gamma_2, \ldots, \gamma_{r-1}$* (*note*: *$c_i$ is the segment between γ_{i-1} and γ_i, c_1 is the segment between $\partial\mathcal{S}_A$ and γ_1, and c_r is the segment between γ_{r-1} and $\partial\mathcal{S}_A$*)*. If there exists some i with $1 \leqslant i \leqslant r$ such that one of the following conditions is satisfied* (*see Figure* 11)*:*

Case 1 (resp. Case 1′). *$p(\gamma_{i-1}, \gamma_i, c)$ is on the right* (*resp. left*) *of c, and $p(\gamma_i, \gamma_{i+1}, c)$ is on the left* (*resp. right*) of c ($2 \leqslant i \leqslant r-2$);

Case 2 (resp. Case 2′). *$c_i = c_1$ is an end segment such that $p(\gamma_1, \gamma_2, c)$ is on the left* (*resp. right*) *of c*;

Case 3 (resp. Case 3′). *$c_i = c_r$ is an end segment such that $p(\gamma_{r-2}, \gamma_{r-1}, c)$ is on the right* (*resp. left*) *of c,*

then $M(c^{\gamma_i}_{\mathrm{simp}}) \leqslant_{\oplus} \mathrm{top}M(c)$ (*resp. $M(c^{\gamma_i}_{\mathrm{simp}}) \leqslant_{\oplus} \mathrm{soc}M(c)$*)*, where $c^{\gamma_i}_{\mathrm{simp}}$ is the simple permissible curve crossing γ_i and $M : \mathfrak{P}(\mathcal{S}_A) \to \mathrm{ind}\,\mathrm{mod}\,A$ is the bijection given in Remark* 3.2.

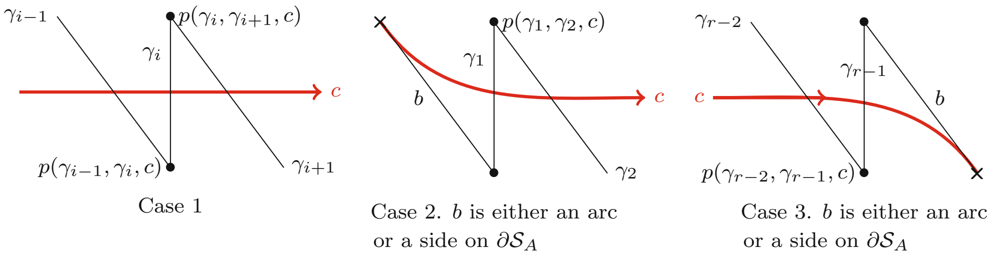


Figure 11 (Color online) $M(c_{\text{simp}}^{\gamma_i})$ is a direct summand of top$M(c)$ (the point "×" is either a marked point or an extra point), where $i = 1$ in Case 2 and $i = r-1$ in Case 3. Cases 1′, 2′ and 3′ are dual

Proof. We only prove the case for top, and the case for socle is dual. For any string module M, the source, denoted by i, of the string s corresponding to M is one of the following three types (we denote by γ_i the arc corresponding to i):

$$\text{Type 1: } \cdots \longleftarrow i \longrightarrow \cdots, \quad \text{Type 2: } i \longrightarrow \cdots, \quad \text{Type 3: } \cdots \longleftarrow i.$$

Then the module $S(i)$, the simple module corresponding to the vertex i, is a direct summand of top$M$, i.e., $S(i) \leqslant_{\oplus} \text{top}M$. On the other hand, the permissible curve $c = c_1 \cdots c_r$ corresponding to s satisfies one of the conditions in Cases 1–3 accordingly (in Cases 2 and 3, we have $i = 1$ and $i = r-1$, respectively). Thus $S(i)$ corresponds to the permissible curve $c_{\text{simp}}^{\gamma_i}$ and $M(c_{\text{simp}}^{\gamma_i}) \leqslant_{\oplus} \text{top}M \cong \text{top}M(c)$. □

Lemma 4.2. *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let $c = c_1c_2\cdots c_r$ be a non-trivial permissible closed curve in $\mathcal{S}_A$ consecutively crossing arcs $\gamma_1, \gamma_2, \ldots, \gamma_{r-1}, \gamma_r = \gamma_0$ (c_i is the segment between $\gamma_{\overline{i-1}}$ and $\gamma_{\overline{i}}$, where $\overline{i}$ is i modulo r), and let $\varphi \in \mathrm{Aut}(k^m)$ with $m \geqslant 1$ be a Jordan block. If there exists some i with $1 \leqslant i \leqslant r$ such that the following condition is satisfied*:

Case RL (resp. Case LR). *$p(\gamma_{\overline{i-1}}, \gamma_{\overline{i}}, c)$ is on the right (resp. left) of c and $p(\gamma_{\overline{i}}, \gamma_{\overline{i+1}}, c)$ is on the left (resp. right) of c,*

then $M(c_{\text{simp}}^{\gamma_i})^{\oplus m} \leqslant_{\oplus} \text{top}(M(c,\varphi))$ (resp. $M(c_{\text{simp}}^{\gamma_i})^{\oplus m} \leqslant_{\oplus} \text{soc}(M(c,\varphi))$).

Proof. For any band module M, a source, denoted by i, of the band corresponding to M must be Type 1 in the proof of Lemma 4.1. Note that $M(c,\varphi)e_i \cong k^m$ is k-linearly isomorphic to $M(c_{\text{simp}}^{\gamma_i})^{\oplus m}$, where e_i is the trivial path corresponding to $i \in Q_0$. Now, similar to the proof of Lemma 4.1, we have $M(c_{\text{simp}}^{\gamma_i})^{\oplus m} \leqslant_{\oplus} \text{top}M(c,\varphi)$. The case for socle is dual. □

As a consequence of Lemmas 4.1 and 4.2, we get the following proposition.

Proposition 4.3. (1) *Under the assumptions in Lemma 4.1, let*

$$\mathfrak{I} = \{1 \leqslant i \leqslant r-1 \mid i \textit{ satisfies one of Cases } 1, 2 \textit{ and } 3 \textit{ in Lemma } 4.1\}$$
$$(\textit{resp. } \mathfrak{I} = \{1 \leqslant i \leqslant r-1 \mid i \textit{ satisfies one of Cases } 1', 2' \textit{ and } 3' \textit{ in Lemma } 4.1\}).$$

Then $\text{top}M(c) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\text{simp}}^{\gamma_i})$ (*resp.* $\text{soc}M(c) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\text{simp}}^{\gamma_i})$).

(2) *Under the assumptions in Lemma 4.2, let*

$$\mathfrak{I} = \{1 \leqslant i \leqslant r-1 \mid i \textit{ satisfies Case } \mathrm{RL} \textit{ in Lemma } 4.2\},$$
$$(\textit{resp. } \mathfrak{I} = \{1 \leqslant i \leqslant r-1 \mid i \textit{ satisfies Case } \mathrm{LR} \textit{ in Lemma } 4.2\}).$$

Then $\text{top}M(c,\varphi) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\text{simp}}^{\gamma_i})^{\oplus m}$ (*resp.* $\text{soc}M(c,\varphi) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\text{simp}}^{\gamma_i})^{\oplus m}$).

Proof. (1) Let $M \in \text{mod}\,A$ be string and

$$\mathfrak{I}' = \{1 \leqslant i \leqslant r-1 \mid i \text{ is a source which is one of Types 1–3 in the proof of Lemma 4.1}\}.$$

Then

$$\mathrm{top}M \cong \mathrm{top}M(c) \cong \bigoplus_{i\in\mathfrak{I}'} S(i) \cong M\left(\bigoplus_{i\in\mathfrak{I}} c_{\mathrm{simp}}^{\gamma_i}\right) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{simp}}^{\gamma_i})$$

($\bigoplus_{i\in\mathfrak{I}} c_{\mathrm{simp}}^{\gamma_i}$ is the formal direct sum of $c_{\mathrm{simp}}^{\gamma_i}$, see Definition 3.3).

(2) It is similar to (1). □

4.2 Projective modules and projective covers

Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface of A. If $s = \alpha_1\alpha_2\cdots\alpha_n$ ($\alpha_i \in Q_0 \cup Q_0^-$) is a string corresponding to a projective module, then s is one of the following cases:

- s is *left-maximal inverse*, i.e., s is inverse and $\alpha^- s \in I$ for any $\alpha \in Q_0$ with $t(\alpha_1^-) = s(\alpha)$.
- s is *right-maximal direct*, i.e., s is direct and $s\alpha \in I$ for any $\alpha \in Q_0$ with $t(\alpha_n) = s(\alpha)$.
- There exists a unique i with $1 \leqslant i \leqslant n-1$ such that $\alpha_1\cdots\alpha_i$ is left-maximal inverse and $\alpha_{i+1}\cdots\alpha_n$ is right-maximal direct.

Therefore, a permissible curve $c : (0,1) \to \mathcal{S}_A\backslash\partial\mathcal{S}_A$ (which consecutively crosses arcs $\gamma_1, \ldots, \gamma_{r-1}$) corresponds to an indecomposable projective module by $M : \mathfrak{P}(\mathcal{S}_A) \to \mathrm{ind}(\bmod A)$ defined in Remark 3.2 if and only if c is a permissible curve such that the following conditions are satisfied:

- (PQC) (Projective quotient condition) There exits an i with $1 \leqslant i \leqslant r-1$ such that
– (PQC1) $m := p(\gamma_1, \gamma_2, c) = \cdots = p(\gamma_{i-1}, \gamma_i, c)$ which is on the right of c, and
– (PQC2) $m' := p(\gamma_i, \gamma_{i+1}, c) = \cdots = p(\gamma_{r-2}, \gamma_{r-1}, c)$ which is on the left of c.
- (MC) (Maximal conditions) There are no arcs γ and γ' such that
– (LMC) $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_i$ have a common endpoint m and they surround m in the clockwise order, and
– (RMC) $\gamma_i, \gamma_{i+1}, \ldots, \gamma_n, \gamma'$ have a common endpoint m' and they surround m' in the counterclockwise order.

Figure 12 provides an example of a permissible curve corresponding to a projective module.

Definition 4.4 (Projective permissible curves)**.** Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. For an arc $\gamma \in \Gamma_A$, a permissible curve c consecutively crossing arcs $\gamma_1, \ldots, \gamma_{r-1}$ is called a *projective permissible curve* of γ, denoted by $c_{\mathrm{proj}}^{\gamma}$, if c is decided by the pair (c_1, c_2) of two curves $c_1 : (0,1] \to \mathcal{S}_A\backslash\partial\mathcal{S}_A$ and $c_2 : [0,1) \to \mathcal{S}_A\backslash\partial\mathcal{S}_A$, where

- there is a unique i with $1 \leqslant i \leqslant r-1$ such that $\gamma = \gamma_i$,
- $c(0^+) = c_1(0^+)$, $c(1^-) = c_2(1^-)$, $c_1(1) = c_2(0) = \gamma \cap c$,
- c is obtained by connecting c_1 and c_2, i.e., $c = c_1 \cup c_2$, and
- c_1 and c_2 consecutively cross arcs $\gamma_1, \ldots, \gamma_i$ and $\gamma_i, \ldots, \gamma_{r-1}$ such that (PQC1), (LMC) and (PQC2), (RMC) are satisfied, respectively.

We use $\mathfrak{P}_{\mathrm{proj}}(\mathcal{S}_A)$ to denote the set of all the equivalence classes of projective permissible curves. Obviously, we have $\mathfrak{P}_{\mathrm{proj}}(\mathcal{S}_A) \subseteq \text{Eq-cls}(\mathcal{S}_A)$.

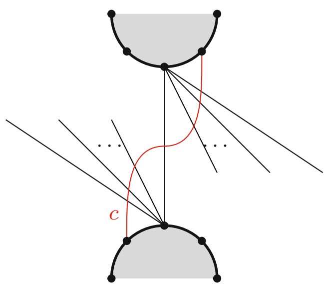


Figure 12 (Color online) Each point • is either a marked point or an extra point. The red curve c is a permissible curve satisfying (PQC) and (MC), and thus the string module corresponding to c is a projective module

Lemma 4.5 (Projective modules). *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Then there exists a one-to-one mapping between the equivalence classes of projective permissible curves and the isomorphic classes of indecomposable projective modules which are given by*

$$M\,|_{\mathfrak{P}_{\mathrm{proj}}(\mathcal{S}_A)} : \mathfrak{P}_{\mathrm{proj}}(\mathcal{S}_A) \to \mathrm{ind}(\mathrm{proj}A), \quad c \mapsto M(c).$$

Proof. A permissible c lies in $\mathfrak{P}_{\mathrm{proj}}(\mathcal{S}_A)$ if and only if it corresponds to a string s which is the connection of a left-maximal (trivial) inverse string s_1 and a right-maximal (trivial) direct string s_2, i.e., $s = \alpha_1\alpha_2\cdots\alpha_n$ is of the form

$$\cdot \xleftarrow{\alpha_1^-} \cdot \xleftarrow{\alpha_2^-} \cdots \xleftarrow{\alpha_i^-} \cdot \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_{n-1}} \cdot \xrightarrow{\alpha_n} \cdot\,.$$

Thus the string module corresponding to s is projective. (If $\alpha_1\cdots\alpha_i$ (resp. $\alpha_{i+1}\cdots\alpha_n$) is trivial, then the module corresponding to s is $P(v) \cong e_vA$, where $v \in Q_0$ is the source of α_{i+1} (resp. α_i) and e_v is the primitive idempotent corresponding to the vertex v.) □

Now we consider the projective cover of a module by marked ribbon surfaces of gentle algebras.

Theorem 4.6 (Projective covers of indecomposable modules). *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let $M = M(c)$ (resp. $M = M(c, \varphi)$) be a string module (resp. band module) which corresponds to the permissible curve c (resp. the permissible closed curve c with a Jordan block over a k-linear space V). Then*

$$P(M) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{proj}}^{\gamma_i}) \ \left(\textit{resp. } P(M) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{proj}}^{\gamma_i})^{\oplus \dim_k V}\right),$$

where $\{\gamma_i \mid i \in \mathfrak{I}\}$ and $\mathfrak{I}$ are the same as that in Lemma 4.1 and Proposition 4.3(2) (resp. Lemma 4.2 and Proposition 4.3(2)), respectively.

Proof. For each indecomposable module M, its projective cover $P(M)$ is isomorphic to the projective cover $P(\mathrm{top}M)$ of its top. Thus we only need to consider the projective cover of each simple module. Note that for each simple permissible curve $c_{\mathrm{simp}}^{\gamma}$, we have $P(M(c_{\mathrm{simp}}^{\gamma})) = M(c_{\mathrm{proj}}^{\gamma})$. Now suppose $\mathrm{top}M \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{simp}}^{\gamma_i})$ (when M is string, each γ_i corresponds the vertex of Q such that one of three cases in Lemma 4.1 holds; when M is band, each γ_i corresponds to the vertex of Q such that the condition in Lemma 4.2 holds). If M is a string module, then

$$P(M) \cong P(\mathrm{top}M) \cong P\left(\bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{simp}}^{\gamma_i})\right) \cong \bigoplus_{i\in\mathfrak{I}} P(M(c_{\mathrm{simp}}^{\gamma_i})) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{proj}}^{\gamma_i}).$$

If M is a band module, then

$$P(M) \cong P(\mathrm{top}M) \cong P\left(\bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{simp}}^{\gamma_i})^{\oplus \dim_k V}\right) \cong \bigoplus_{i\in\mathfrak{I}} P(M(c_{\mathrm{simp}}^{\gamma_i})^{\oplus \dim_k V}) \cong \bigoplus_{i\in\mathfrak{I}} M(c_{\mathrm{proj}}^{\gamma_i})^{\oplus \dim_k V}.$$

This completes the proof. □

4.3 Injective modules and injective envelopes

If a permissible curve $c : (0,1) \to \mathcal{S}_A \backslash \partial\mathcal{S}_A$ consecutively crosses arcs $\gamma_1, \ldots, \gamma_{r-1}$ and corresponds to an injective module by $M : \text{Eq-cls}(\mathcal{S}_A) \cup \text{Htp-cls}(\mathcal{S}_A) \to \mathrm{ind}(\mathrm{mod}A)$, then there is a unique i with $1 \leqslant i \leqslant n-1$ such that the following conditions are satisfied:

- (ISC) (Injective submodule condition) There exits an i with $1 \leqslant i \leqslant r-1$ such that
  - (ISC1) $m := p(\gamma_1, \gamma_2, c) = \cdots = p(\gamma_{i-1}, \gamma_i, c)$ which is on the left of c, and
  - (ISC2) $m' := p(\gamma_i, \gamma_{i+1}, c) = \cdots = p(\gamma_{r-2}, \gamma_{r-1}, c)$ which is on the right of c.
- (MC′) There are no arcs γ and γ' such that

– (LMC′) $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_i$ have a common endpoint m and they surround m in the counterclockwise order, and

– (RMC′) $\gamma_i, \gamma_{i+1}, \ldots, \gamma_n, \gamma'$ have a common endpoint m' and they surround m' in the clockwise order.

We define injective permissible curves and $\mathfrak{P}_{\text{inj}}(\mathcal{S}_A)$ as follows.

Definition 4.7 (Injective permissible curves). Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. For an arc $\gamma \in \Gamma_A$, a permissible curve c consecutively crossing arcs $\gamma_1, \ldots, \gamma_{r-1}$ is called an *injective permissible curve* of γ, denoted by c^{γ}_{inj}, if c is decided by the pair (c_1, c_2) of two curves $c_1 : (0, 1] \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$ and $c_2 : [0, 1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$, where

- there is a unique i with $1 \leqslant i \leqslant r-1$ such that $\gamma = \gamma_i$,
- $c(0^+) = c_1(0^+)$, $c(1^-) = c_2(1^-)$, $c_1(1) = c_2(0) = \gamma \cap c$,
- c is obtained by connecting c_1 and c_2, i.e., $c = c_1 \cup c_2$, and
- c_1 and c_2 consecutively crosses arcs $\gamma_1, \ldots, \gamma_i$ and $\gamma_i, \ldots, \gamma_{r-1}$ such that (ISC1), (LMC′) and (ISC2), (RMC′) are satisfied, respectively.

We use $\mathfrak{P}_{\text{inj}}(\mathcal{S}_A)$ to denote the set of all the equivalence classes of injective permissible curves. Obviously, we have

$$\mathfrak{P}_{\text{inj}}(\mathcal{S}_A) \subseteq \text{Eq-cls}(\mathcal{S}_A).$$

The following results are dual to Lemma 4.5 and Theorem 4.6.

Lemma 4.8 (Injective modules). *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Then there exists a one-to-one mapping between the equivalence classes of injective permissible curves and the isomorphic classes of indecomposable injective modules which is given by*

$$\mathfrak{P}\,|_{\text{inj}}(\mathcal{S}_A) : \mathfrak{P}_{\text{inj}}(\mathcal{S}_A) \to \text{ind}(\text{inj}A), \quad c \mapsto M(c).$$

Theorem 4.9 (Injective envelopes of indecomposable modules). *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let $M = M(c)$ (resp. $M = M(c, \varphi)$) be a string module (resp. band module) which corresponds to the permissible curve c (resp. the permissible closed curve c with a Jordan block over a k-linear space V). Then*

$$E(M) \cong \bigoplus_{i \in \mathfrak{I}} M(c^{\gamma_i}_{\text{inj}}) \left(\textit{resp. } E(M) \cong \bigoplus_{i \in \mathfrak{I}} M(c^{\gamma_i}_{\text{inj}})^{\oplus \dim_k V} \right),$$

where $\{\gamma_i \mid i \in \mathfrak{I}\}$ and $\mathfrak{I}$ are the same as that in Lemma 4.1 *and Proposition* 4.3(2) (*resp. Lemma* 4.2 *and Proposition* 4.3(2)), *respectively.*

5 Global dimension

5.1 The descriptions of projective and injective resolutions of simple modules in geometric models

In this subsection, we consider the minimal projective and injective resolutions of simple modules through marked ribbon surfaces.

Definition 5.1 (P-condition and I-condition). Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let γ be an arc in Γ_A with two endpoints p and q. We say that γ satisfies the P-*condition at* p (resp. I-*condition at* p) if the following conditions are satisfied:

- there are arcs $\gamma_1, \ldots, \gamma_r$ such that p is the common endpoint of γ and γ_i (for any $1 \leqslant i \leqslant r$), and
- $\gamma, \gamma_1, \ldots, \gamma_r$ (resp. $\gamma_1, \ldots, \gamma_r, \gamma$) surround p in the counterclockwise order.

Furthermore, we say that γ satisfies the P-*condition* (resp. I-*condition*) if the P-conditions (resp. I-conditions) at p and q are satisfied.

Remark 5.2. If an arc γ with endpoints p and q satisfies the P-condition at p or q, then $M(c^{\gamma}_{\text{proj}})$ is not simple.

We provide an example in Figure 13. In this figure, γ satisfies the P-condition.

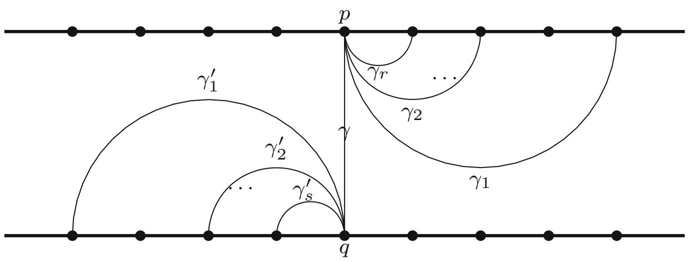


Figure 13 The arc γ satisfying the P-condition

Lemma 5.3. *Under the conditions in Definition* 5.1*, let* $c = c^{\gamma}_{\mathrm{simp}}$ *be a simple permissible curve, where* γ *is an arc in* Γ_A *with endpoints* p *and* q *such that the P-condition is satisfied at* p *(resp.* q*) but not at* q *(resp.* p*). Then the kernel of the projective cover* $p(M(c^{\gamma}_{\mathrm{simp}})) : M(c^{\gamma}_{\mathrm{proj}}) \twoheadrightarrow M(c^{\gamma}_{\mathrm{simp}})$ *is the module* $M(c')$*, where* c' *is the permissible curve consecutively crossing* $\gamma_1, \ldots, \gamma_r$ *(resp.* $\gamma_1', \ldots, \gamma_s'$*).*

Proof. We only prove the case for γ satisfying the P-condition at p but not at q, and the argument for the other case is similar. As shown in Figure 14, let $\gamma = \tilde{\gamma}_1$, and $\gamma_j = \tilde{\gamma}_{j+1}$ $(1 \leqslant j \leqslant r)$. Then we may suppose $c = c^{\gamma}_{\mathrm{proj}} = c^{\tilde{\gamma}_1}_{\mathrm{proj}} \simeq c_1c_2 \cdots c_{r+2}$, where each c_t $(2 \leqslant t \leqslant r+1)$ is a segment between $\tilde{\gamma}_{t-1}$ and $\tilde{\gamma}_t$, and c_1 and c_{r+2} are end segments of c.

Let $c'' = c^{\gamma}_{\mathrm{simp}} = c^{\tilde{\gamma}_1}_{\mathrm{simp}}$ and c' be the permissible curve consecutively crossing $\gamma_1, \ldots, \gamma_r$. Then $c'' \simeq c_1c_2^{\mathfrak{T}}$ and $c' \simeq \mathrm{nrot}_c(c'')$ (by Definition 3.4) and the positional relationship of c, c' and c'' satisfies the conditions in Proposition 3.6 (i.e., c consecutively crosses $\tilde{\gamma}_2$ and $\tilde{\gamma}_1$, c' crosses only $\tilde{\gamma}_2$, and c'' crosses $\tilde{\gamma}_1$, where $\tilde{\gamma}_2$ and $\tilde{\gamma}_1$ are viewed as arcs γ' and γ'' in Proposition 3.6, respectively). Thus we get the following exact sequence:

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0$$

by Theorem 3.12(1). $\square$

Lemma 5.4. *Under the conditions in Definition* 5.1*, let* $c = c^{\gamma}_{\mathrm{simp}}$ *be a simple permissible curve, where* γ *is an arc in* Γ_A *with endpoints* p *and* q *such that the* P*-condition is satisfied. Then the kernel of the projective cover* $p(M(c^{\gamma}_{\mathrm{simp}})) : M(c^{\gamma}_{\mathrm{proj}}) \twoheadrightarrow M(c^{\gamma}_{\mathrm{simp}})$ *is the module* $M(c'_{\mathrm{I}} \oplus c'_{\mathrm{II}})$*, where* c'_{I} *consecutively crosses* $\gamma_1', \ldots, \gamma_s'$ *and* c_2' *consecutively crosses* $\gamma_1, \ldots, \gamma_r$*.*

Proof. Let $\gamma_i' = \tilde{\gamma}_{s-i+1}$ $(1 \leqslant i \leqslant s)$, $\gamma = \tilde{\gamma}_{s+1}$ and $\gamma_j = \tilde{\gamma}_{s+j+1}$ $(1 \leqslant j \leqslant r)$. Then we may suppose $c = c^{\gamma}_{\mathrm{proj}} = c^{\tilde{\gamma}_{s+1}}_{\mathrm{proj}} \simeq c_1c_2 \cdots c_{s+r+2}$, where each c_t $(2 \leqslant t \leqslant s+r+1)$ is a segment between $\tilde{\gamma}_{t-1}$ and $\tilde{\gamma}_t$, and c_1 and c_{s+r+2} are end segments of c. Let $c'' = c^{\gamma}_{\mathrm{simp}}$, and x be the intersection of c and c'' (note that $\sharp(c \cap c'') = 1$ in this case) such that c''_{I} and c''_{II} are the curves obtained by x dividing c''. Then the positional relationship of c, $\mathrm{prot}_c(c''_{\mathrm{I}}), \mathrm{prot}_c(c''_{\mathrm{II}})$ and c'' satisfies the conditions in Proposition 3.11.

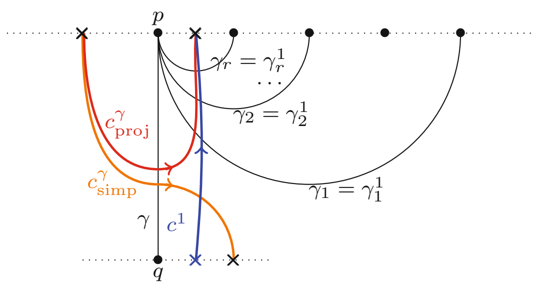


Figure 14 (Color online) The projective cover $p(M(c^{\gamma}_{\mathrm{simp}})) : M(c^{\gamma}_{\mathrm{proj}}) \to M(c^{\gamma}_{\mathrm{simp}})$ of $M(c^{\gamma}_{\mathrm{simp}})$ and its kernel $\mathrm{Ker}\, p(M(c^{\gamma}_{\mathrm{simp}})) \cong M(c^1)$, where $c^1 \simeq \mathrm{nrot}_{c^{\gamma}_{\mathrm{proj}}}(c^{\gamma}_{\mathrm{simp}})$ by Proposition 3.11

We get the following exact sequence:

$$0 \longrightarrow M(\mathrm{prot}_c(c''_{\mathrm{I}}) \oplus \mathrm{prot}_c(c''_{\mathrm{II}})) \longrightarrow M(c) \xrightarrow{p(M(c^{\gamma}_{\mathrm{simp}}))} M(c'') \longrightarrow 0$$

by Theorem 3.12(3). Notice that $\mathrm{prot}_c(c''_{\mathrm{I}}) \simeq c'_{\mathrm{I}}$ and $\mathrm{prot}_c(c''_{\mathrm{II}}) \simeq c'_{\mathrm{II}}$, and thus

$$\mathrm{Ker}p(M(c^{\gamma}_{\mathrm{simp}})) = M(\mathrm{prot}_c(c''_{\mathrm{I}}) \oplus \mathrm{prot}_c(c''_{\mathrm{II}})) \cong M(c'_{\mathrm{I}} \oplus c'_{\mathrm{II}}).$$

This completes the proof. □

Lemma 5.5. *Under the conditions Definition* 5.1 *and Lemma* 5.3*, let c^1 be the permissible curve c' corresponding to* $\mathrm{Ker}p(M(c^{\gamma}_{\mathrm{simp}}))$*, where $p(M(c^{\gamma}_{\mathrm{simp}}))$ is given by Lemma* 5.3*. Then $P(M(c^1)) = P(M(c^{\gamma_1}_{\mathrm{proj}}))$ is indecomposable, and the kernel of $P(M(c^1)) \twoheadrightarrow M(c^1)$ is $M(\mathrm{nrot}_{c^{\gamma_1}_{\mathrm{proj}}}(c^1))$. To be precise, if $\gamma^2_1, \gamma^2_2, \ldots, \gamma^2_s$ are arcs with the endpoint of γ_1 which is not p such that γ, γ_1 and γ^2_1 are sides of the same elementary polygon of $\mathcal{S}_A$, then $\mathrm{nrot}_{c^{\gamma_1}_{\mathrm{proj}}}(c^1) \simeq c^2$, where c^2 is the permissible curve consecutively crosses $\gamma^2_1, \gamma^2_2, \ldots, \gamma^2_s$.*

Proof. Let $\gamma_{r-i} = \tilde{\gamma}_{i+1}$ $(0 \leqslant i \leqslant r-1)$ and $\gamma^2_j = \tilde{\gamma}_{r+j}$ $(1 \leqslant j \leqslant s)$. Then $c'' := c^1 = c''_1 c''_2 \cdots c''_{r+1}$, where each c''_i $(2 \leqslant i \leqslant r)$ is a segment between $\tilde{\gamma}_{i-1}$ and $\tilde{\gamma}_i$, and c''_1 and c''_{r+1} are its end segments; $c := c^{\gamma_1}_{\mathrm{proj}} = c^{\tilde{\gamma}_r}_{\mathrm{proj}} = c_1 c_2 \cdots c_{r+s+1}$, where each c_j $(2 \leqslant j \leqslant r+s)$ is a segment between $\tilde{\gamma}_{j-1}$ and $\tilde{\gamma}_j$, and c_1 and c_j are its end segments. Thus $c'' = c''_1 c''_2 \cdots c''_{r+1} \simeq c_1 c_2 \cdots c^{\mathfrak{T}}_{r+1}$. We have the permissible curve $c' = c^{\mathfrak{T}}_{r+1} c_{r+2} \cdots c_{r+s+1}$ which consecutively crosses $\tilde{\gamma}_{r+1} = \gamma^2_1$, $\tilde{\gamma}_{r+2} = \gamma^2_2, \ldots, \tilde{\gamma}_{r+s} = \gamma^2_s$, and the positional relationship of c, c' and c'' satisfies the conditions in Proposition 3.6. So by Theorem 3.12(1), we get the following short exact sequence:

$$0 \longrightarrow M(c') \longrightarrow M(c) \longrightarrow M(c'') \longrightarrow 0,$$

where $M(c'') = M(c^1)$, $M(c) = M(c^{\gamma_1}_{\mathrm{proj}})$ and $M(c') = M(c^{\mathfrak{T}}_{r+1} c_{r+2} \cdots c_{r+s+1}) \cong M(\mathrm{nrot}_{c^{\gamma_1}_{\mathrm{proj}}}(c^1))$ (note that $c' \simeq \mathrm{nrot}_{c^{\gamma_1}_{\mathrm{proj}}}(c^1)$ since γ and γ_1 are sides of the same elementary polygon). □

Keep the notations in Lemma 5.5 and denote by γ^1_i the permissible curve γ_i for any $1 \leqslant i \leqslant r$. In Figure 15, $p(M(c^{\gamma}_{\mathrm{simp}})) : M(c^{\gamma}_{\mathrm{proj}}) \twoheadrightarrow M(c^{\gamma}_{\mathrm{simp}})$ is the projective cover of $M(c^{\gamma}_{\mathrm{simp}})$ which is simple, and the module $M(c^1)$ corresponding to the orange permissible curve c^1 is the kernel of $p(M(c^{\gamma}_{\mathrm{simp}}))$. In addition, $c^{\gamma^1_1}_{\mathrm{proj}}$, the red permissible curve in this figure, provides the projective cover of $M(c^1)$, i.e.,

$$p(M(c^1)) : M(c^{\gamma^1_1}_{\mathrm{proj}}) \twoheadrightarrow M(c^1).$$

Then $\mathrm{nrot}_{c^{\gamma^1_1}_{\mathrm{proj}}}(c^1)$ is the kernel of $p(M(c^1))$. Furthermore, let

$$\mathrm{nrot}_{c^{\gamma^1_1}_{\mathrm{proj}}}(c^1) = c^2.$$

If there are arcs $\gamma^3_1, \ldots, \gamma^3_t$ having a common endpoint which is the endpoint of γ^2_1 but not the common endpoint of $\gamma^2_1, \ldots, \gamma^2_s$ such that γ, γ^1_1, γ^2_1 and γ^3_1 are sides of the same elementary polygon of $\mathcal{S}_A$, then the kernel of $M(c^{\gamma^2_1}_{\mathrm{proj}}) \twoheadrightarrow M(c^2)$ is isomorphic to $M(c^3)$, where

$$c^3 \simeq \mathrm{nrot}_{c^{\gamma^2_1}_{\mathrm{proj}}}(c^2)$$

is a permissible curve consecutively crossing $\gamma^3_1, \ldots, \gamma^3_t$. Repeating this process, we get the minimal projective resolution of $M(c^{\gamma}_{\mathrm{simp}})$ as follows:

$$\cdots \longrightarrow M(c^{\gamma^2_1}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma^1_1}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma}_{\mathrm{simp}}) \longrightarrow 0.$$

Thus we obtain a one-to-one mapping $M(c^{\gamma^t_1}_{\mathrm{proj}}) \mapsto \gamma^t_1$, where $\gamma^0_1 = \gamma$ and $\gamma^0_1, \gamma^1_1, \ldots$ are sides of the same elementary polygon of $\mathcal{S}_A$ (see Figure 16).

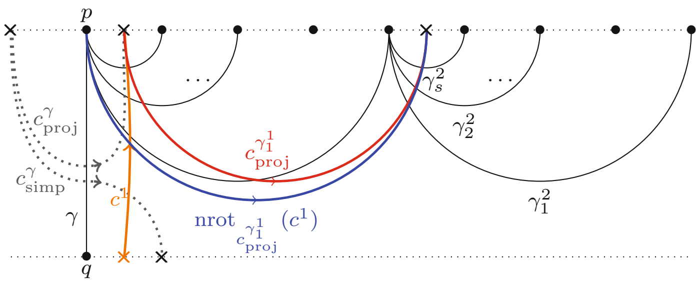


Figure 15 (Color online) The projective cover $p(M(c^{\gamma}_{\mathrm{simp}})) : M(c^{\gamma_1^1}_{\mathrm{proj}}) \to M(c^{\gamma}_{\mathrm{simp}})$ of $M(c^{\gamma}_{\mathrm{simp}})$ and its kernel $\mathrm{Ker}\, p(M(c^{\gamma}_{\mathrm{simp}})) \cong M(c^2)$, where $c^2 \simeq \mathrm{nrot}_{c^{\gamma_1}_{\mathrm{proj}}}(c^1)$ by Proposition 3.11

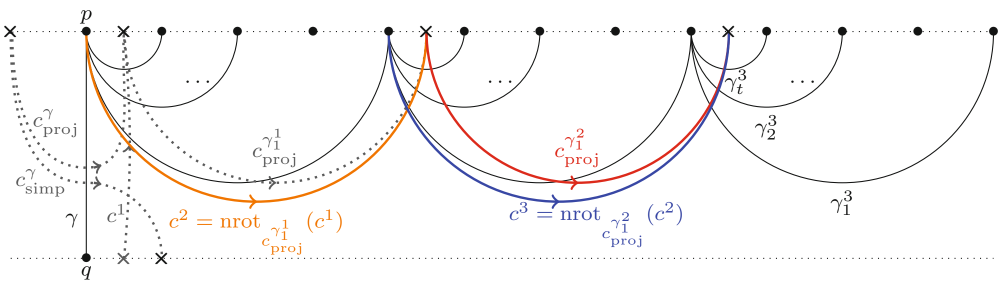


Figure 16 (Color online) Keep the notations in Figure 15. Then the positional relationship of c^2, $c^{\gamma_1^2}_{\mathrm{proj}}$ and c^3 satisfies the conditions in Proposition 3.11

In the case where γ satisfies the P-condition at q but not at p, we get dually the minimal projective resolution of $M(c^{\gamma}_{\mathrm{simp}})$ as follows:

$$\cdots \longrightarrow M(c^{\gamma_1'^2}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma_1'^1}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma}_{\mathrm{simp}}) \longrightarrow 0.$$

Moreover, if γ satisfies the P-condition, then the minimal projective resolution of $M(c^{\gamma}_{\mathrm{simp}})$ is as follows:

$$\cdots \longrightarrow M(c^{\gamma_1^2}_{\mathrm{proj}}) \oplus M(c^{\gamma_1'^2}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma_1^1}_{\mathrm{proj}}) \oplus M(c^{\gamma_1'^1}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma}_{\mathrm{simp}}) \longrightarrow 0.$$

Proposition 5.6. *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Let γ^0 be an arc $\gamma^0 : (0,1) \to \mathcal{S}_A \backslash \partial\mathcal{S}_A$ whose endpoints are $p := \gamma^0(0^+)$ and $q := \gamma^0(1^-)$. As shown in Figure 17, assume that*

- (PD) (*resp.* (ID)) *there are arcs $\gamma^1, \ldots, \gamma^n : (0,1) \to \mathcal{S}_A \backslash \partial\mathcal{S}_A$ such that*

– (PD1) = (ID1) *$\gamma^0, \gamma^1, \ldots, \gamma^n$ are sides of the same elementary polygon of $\mathcal{S}_A$,*

– (PD2) = (ID2) *$\gamma^1(0^+) = q$ and $\gamma^{i-1}(1^-) = \gamma^i(0^+)$ for any $2 \leqslant i \leqslant n$,*

– (PD3) (*resp.* (ID3)) *in the clockwise* (*resp. counterclockwise*) *order at the common endpoint of γ^{i-1} and γ^i, γ^i appears after γ^{i-1} for any $1 \leqslant i \leqslant n$, and*

– (PD4) = (ID4) *for any $\gamma \in \Gamma_A$, none of the arcs $\gamma^0, \gamma^1, \ldots, \gamma^n, \gamma$ satisfies one of the above three conditions*;

assume that

- (PD$'$) (*resp.* (ID$'$)) *there are arcs $\gamma'^1, \ldots, \gamma'^m : (0,1) \to \mathcal{S}_A \backslash \partial\mathcal{S}_A$ such that*

– (PD1$'$) = (ID1$'$) *$\gamma'^0 := \gamma^0, \gamma'^1, \ldots, \gamma'^m$ are sides of the same elementary polygon of $\mathcal{S}_A$,*

– (PD2$'$) = (ID2$'$) *$\gamma'^1(0^+) = p$ and $\gamma'^{j-1}(1^-) = \gamma'^j(0^+)$ for any $2 \leqslant j \leqslant m$,*

– (PD3$'$) (*resp.* (ID3$'$)) *in the counterclockwise* (*resp. clockwise*) *order at the common endpoint of γ'^{j-1} and γ'^j, γ'^j appears after γ^{j-1} for any $1 \leqslant j \leqslant m$, and*

– (PD4$'$) = (ID4$'$) *for any $\gamma' \in \Gamma_A$, none of the arcs $\gamma'^0, \gamma'^1, \ldots, \gamma'^m, \gamma'$ satisfies one of the above three conditions.*

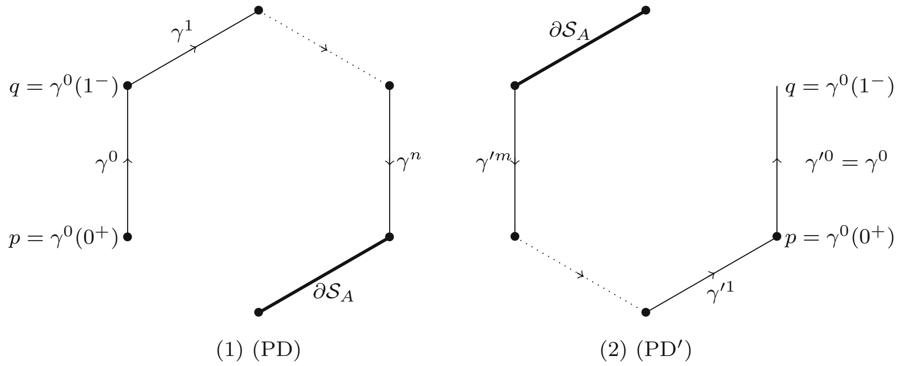


Figure 17 $\gamma^0, \gamma^1, \ldots, \gamma^n$ are arcs satisfying (PD), and $\gamma'^0, \gamma'^1, \ldots, \gamma'^m$ are arcs satisfying (PD′). The figures of (ID) and (ID′) are dual

Then the projective dimension (resp. the injective dimension) of $M(c_{\mathrm{simp}}^{\gamma^0})$ *is* $\max\{n, m\}$*.*

Proof. Notice that $M(c_{\mathrm{proj}}^{\gamma}) \mapsto \gamma$ for any $\gamma \in \{\gamma^i \mid 1 \leqslant i \leqslant n\} \cup \{\gamma^j \mid 1 \leqslant j \leqslant m\} \cup \{\gamma^0\}$ by Lemmas 5.3–5.5, and thus the minimal projective resolution of $M(c_{\mathrm{simp}}^{\gamma^0})$ is as follows:

$$\begin{aligned} 0 &\longrightarrow M(c_{\mathrm{proj}}^{\gamma^{\max\{n,m\}}}) \oplus M(c_{\mathrm{proj}}^{\gamma'^{\max\{n,m\}}}) \longrightarrow \cdots \\ &\longrightarrow M(c_{\mathrm{proj}}^{\gamma^2}) \oplus M(c_{\mathrm{proj}}^{\gamma'^2}) \longrightarrow M(c_{\mathrm{proj}}^{\gamma^1}) \oplus M(c_{\mathrm{proj}}^{\gamma'^1}) \longrightarrow M(c_{\mathrm{proj}}^{\gamma^0}) \longrightarrow M(c_{\mathrm{simp}}^{\gamma^0}) \longrightarrow 0. \end{aligned} \tag{5.1}$$

If $m \geqslant n$, then $M(c_{\mathrm{proj}}^{\gamma^i}) = 0$ for any $i > n$, so $\mathrm{proj.dim} M(c_{\mathrm{simp}}^{\gamma^0}) = m$. If $n \geqslant m$, then $\mathrm{proj.dim} M(c_{\mathrm{simp}}^{\gamma^0}) = n$ similarly. Dually, we can prove the case for the injective dimension. □

The exact sequence (5.1) is a description of the minimal projective resolution of a simple module in marked ribbon surfaces. We will give two examples in Section 7 to illustrate it (see Examples 7.1 and 7.2).

5.2 The descriptions of the global dimension in geometric models

Definition 5.7 (Consecutive arcs numbers)**.** Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Recall from Definition 2.1 that Γ_A divides $\mathcal{S}_A$ to some elementary polygons $\{\Delta_i \mid 1 \leqslant i \leqslant d\}$ whose sides are arcs in Γ_A except one side which is a boundary arc. We define $\mathfrak{C}(\Delta_i)$, the *consecutive arcs numbers* of Δ_i, as the number of sides belonging to Γ_A, if the unmarked boundary component of $\mathcal{S}_A$ is not a side of Δ_i; otherwise, $\mathfrak{C}(\Delta_i) = \infty$.

Remark 5.8. An elementary polygon Δ_i is called an ∞-*elementary polygon* if Δ_i has a side which is an unmarked boundary component b of $\mathcal{S}_A$. Obviously, $\mathfrak{C}(\Delta_i) = \infty$ if and only if Δ_i is ∞-elementary. In this case, for any side belonging to Γ_A of Δ_i, say γ^0, there are arcs $\gamma^0, \gamma^1, \ldots, \gamma^n$ satisfying the conditions (PD1), (PD2) and (PD3) in Theorem 5.6. Let $\gamma^t = \gamma^{\bar{t}}$ for all $t > n$ where $\bar{t}$ equals t modulo $n+1$. Then for any $t \geqslant 0$, $\gamma^0, \gamma^1, \ldots, \gamma^t$ are arcs such that (PD1), (PD2) and (PD3) hold. Thus the minimal projective resolution of $M(c_{\mathrm{simp}}^{\gamma^0})$ is as follows:

$$\cdots \longrightarrow P^i \longrightarrow \cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow M(c_{\mathrm{proj}}^{\gamma^0}) \longrightarrow M(c_{\mathrm{simp}}^{\gamma^0}) \longrightarrow 0,$$

where each $M(c_{\mathrm{proj}}^{\gamma^{\bar{i}}})$ is a quotient of P^i. Therefore, $\mathrm{proj.dim} M(c_{\mathrm{simp}}^{\gamma^0}) = \infty$.

We recall some notions from [5, Section 2].

Definition 5.9. Let $A = kQ/I$ be a gentle algebra. A *non-trivial permitted* (resp. *forbidden*) *path* is either a path of length 1, or a path $\mathit{\Pi} = \alpha_1 \cdots \alpha_m$ ($m \geqslant 2$) such that $\alpha_i\alpha_{i+1} \notin I$ (resp. $\alpha_i\alpha_{i+1} \in I$) for any $1 \leqslant i \leqslant m-1$. A *non-trivial permitted* (resp. *forbidden*) *thread* is a maximal non-trivial permitted (resp. forbidden) path, i.e., for each $\beta \in Q_1$ satisfying $t(\beta) = s(\alpha_1)$ we have $\beta\alpha_1 \in I$ (resp. $\beta\alpha_1 \notin I$), and for each $\beta' \in Q_1$ satisfying $t(\alpha_m) = s(\beta')$ we have $\alpha_m\beta' \in I$ (resp. $\alpha_m\beta' \notin I$). A *trivial permitted* (resp. *forbidden*) *thread* is a trivial path $v \in Q_0$ such that one of the following holds: (i) v is a source point and $\sharp\{\beta \in Q_1 \mid t(\beta) = v\} = 0$. (ii) v is a sink point and $\sharp\{\beta' \in Q_1 \mid s(\beta') = v\} = 0$. (iii) $\sharp\{\beta \in Q_1 \mid t(\beta) = v\} = 1 = \sharp\{\beta' \in Q_1 \mid s(\beta') = v\}$ and $\beta\beta' \notin I$ (resp. $\beta\beta' \in I$).

We use $\mathcal{P}_A$ (resp. $\mathcal{F}_A$) to denote the set of all the permitted (resp. forbidden) threads. By [25, Section 1], we have a bijection $\mathfrak{p} : \mathcal{P}_A \to \mathcal{M}_A$ (resp. $\mathfrak{f} : \mathcal{F}_A \to \{\Delta_i \mid 1 \leqslant i \leqslant d\}$), where $\mathfrak{p}(H)$ is the marked point p such that every arc with endpoint $p \in \mathcal{M}_A$ corresponds to a vertex of $H \in \mathcal{P}_A$ (resp. $\mathfrak{f}(\mathit{\Pi})$ is the elementary polygons whose sides belonging to Γ_A correspond to the vertices of $\mathit{\Pi} \in \mathcal{F}_A$).

For any path $\wp$, we use $l(\wp)$ to denote the length of $\wp$. In particular, for an *oriented cycle with the full relation* of the quiver (Q, I) of a gentle algebra A, i.e., an oriented cycle $\alpha_1\alpha_2\cdots\alpha_\ell$ $(t(\alpha_\ell) = s(\alpha_1))$ such that $\alpha_{\bar{i}}\alpha_{\overline{i+1}} \in I$ where $\bar{i}$ is equal to i modulo ℓ, it deduces a forbidden threads of infinite length $\cdots\alpha_\ell\alpha_1\alpha_2\cdots\alpha_\ell\alpha_1\alpha_2\cdots$.

Theorem 5.10. *Let $A = kQ/I$ be gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface, and let $\{\Delta_i \mid 1 \leqslant i \leqslant d\}$ be the set of all the elementary polygons of $\mathcal{S}_A$. Then*

$$\mathrm{gl.dim}A = \max_{1\leqslant i\leqslant d} \mathfrak{C}(\Delta_i) - 1 = \max_{\mathit{\Pi}\in\mathcal{F}_A} l(\mathit{\Pi}).$$

Proof. If there exists an elementary polygons Δ_i which is of the form given in Remark 5.8, then $\mathfrak{C}(\Delta_i) = \infty$. In this case, for each side γ of Δ_i, we have $\mathrm{proj.dim}M(c^{\gamma}_{\mathrm{simp}}) = \infty$ by Remark 5.8, and the assertion follows.

Otherwise, for each elementary polygons Δ_i whose sides are denoted by $\gamma^0, \gamma^1, \ldots, \gamma^{m_i}$ such that (PD) holds, then $m_i + 1 = \mathfrak{C}(\Delta_i)$. For the arc γ^t $(0 \leqslant t \leqslant m_i - 1)$, there is an elementary polygon Δ'_i such that γ^t is the unique common side of Δ_i and Δ'_i. We can find a sequence of arcs, say $\gamma'^0(= \gamma^t), \gamma'^1, \ldots, \gamma'^{n_{it}}$ such that (PD) holds, and thus the minimal projective resolution of $M(c^{\gamma^t}_{\mathrm{simp}})$ is as follows:

$$\cdots \longrightarrow M(c^{\gamma^{t+2}}_{\mathrm{proj}}) \oplus M(c^{\gamma'^2}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma^{t+1}}_{\mathrm{proj}}) \oplus M(c^{\gamma'^1}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma^t}_{\mathrm{proj}}) \longrightarrow M(c^{\gamma^t}_{\mathrm{simp}}) \longrightarrow 0.$$

It stops from the $\max\{m_i - t, n_{it}\}$-th projective module. Notice that $n_{it} \leqslant \mathfrak{C}(\Delta'_i) - 1$, so

$$\mathrm{proj.dim}M(c^{\gamma^t}_{\mathrm{simp}}) = \max\{m_i, n_{it}\} \leqslant \max\{\mathfrak{C}(\Delta_i) - 1, \mathfrak{C}(\Delta'_i) - 1\},$$

and hence

$$\mathrm{gl.dim}A \leqslant \max_{1\leqslant i\leqslant d}\{\max\{\mathfrak{C}(\Delta_i) - 1, \mathfrak{C}(\Delta'_i) - 1\}\} = \max_{1\leqslant i\leqslant d} \mathfrak{C}(\Delta_i) - 1.$$

Let Δ_ℓ be the elementary polygon such that

$$\max_{1\leqslant i\leqslant d} \mathfrak{C}(\Delta_i) = \mathfrak{C}(\Delta_\ell) \xlongequal{\text{denoted by}} m_\ell,$$

and let $\gamma^0, \gamma^1, \ldots, \gamma^{m_\ell} : (0, 1) \to \mathcal{S}_A \backslash \partial\mathcal{S}_A$ be its sides such that $\gamma_i(1^-) = p_i = \gamma_{i+1}(0^+)$ and γ_{i+1} follows γ_i around p_i in the counterclockwise order $(0 \leqslant i \leqslant m_\ell - 1)$. Then $\gamma^0, \gamma^1, \ldots, \gamma^{m_\ell}$ satisfy the condition (PD), and thus

$$\mathrm{gl.dim}A \geqslant \mathrm{proj.dim}M(c^{\gamma^0}_{\mathrm{simp}}) = m_\ell = \mathfrak{C}(\Delta_\ell) - 1 = \max_{1\leqslant j\leqslant d} \mathfrak{C}(\Delta_j) - 1$$

by Proposition 5.6. It follows that

$$\mathrm{gl.dim}A = \max_{1\leqslant i\leqslant d} \mathfrak{C}(\Delta_i) - 1.$$

Since the map $\mathfrak{f}$ above is bijective, the length $l(\mathit{\Pi})$ of the forbidden thread $\mathit{\Pi}$ is equal to $\mathfrak{C}(\mathfrak{f}(\mathit{\Pi})) - 1$ and $\mathrm{gl.dim}A = \max_{\mathit{\Pi}\in\mathcal{F}_A} l(\mathit{\Pi})$. □

5.3 AG-invariants and AG-equivalence

The *AG-invariant* of a gentle algebra $A = kQ/I$ introduced by Avella-Alaminos and Geiß [5] is a function $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ with $\mathbb{N}$ the set of natural numbers mapping $\phi[(m_t, n_t)]$ to the number of pairs (m_t, n_t) in the sequence $(m_t, n_t)_{1 \leqslant t \leqslant r}$, where $(m_i, n_i)_{1 \leqslant i \leqslant r}$ is obtained as follows:

Step 1. Let $H_0 = \alpha_1^0 \cdots \alpha_{u_0}^0$ ($u_0 \in \mathbb{N}$) be a (trivial) permitted thread of A.

Step 2. If H_0 is non-trivial and there exists a non-trivial forbidden thread $F = \beta_1^0 \cdots \beta_{v_0}^0$ ($v_0 \in \mathbb{N}$) such that $t(\beta_{v_0}^0) = t(\alpha_{u_0}^0)$ and $\alpha_{u_0}^0 \neq \beta_{v_0}^0$, consider $\mathit{\Pi}_0 = F$; otherwise, consider the (trivial) forbidden thread $\mathit{\Pi}_0$ satisfying $t(\mathit{\Pi}_0) = t(H_0)$.

Step 3. If $\mathit{\Pi}_0$ is non-trivial and there exists a non-trivial permissible thread $P = \alpha_1^1 \cdots \alpha_{u_1}^1$ ($u_1 \in \mathbb{N}$) such that $s(\alpha_1^1) = s(\beta_1^0)$ and $\alpha_1^1 \neq \beta_1^0$, consider $H_1 = P$; otherwise, consider the (trivial) permitted thread H_1 satisfying $s(H_1) = s(\mathit{\Pi}_0)$.

Step 4. If we obtain a permitted thread H_i (resp. forbidden thread $\mathit{\Pi}_i$), then just as in Step 2 (resp. Step 3), determine the forbidden thread $\mathit{\Pi}_i$ (resp. permitted thread H_{i+1}). Repeat this step until $H_{m_1} = H_0$ first appears, and we induce a pair (m_1, n_1) where $n_1 = \sum_{i=0}^{m_1-1} l(\mathit{\Pi}_i)$.

Step 5. Until all the permitted threads are considered in Steps 1–4, we obtain a sequence of pairs $(m_1, n_1), \ldots, (m_{r'}, n_{r'})$ with $1 \leqslant r' \leqslant r$.

Step 6. For every oriented cycle of length ℓ having full relations, we add a pair $(0, \ell)$ in the sequence obtained in Step 5 and obtain the sequence $(m_t, n_t)_{1 \leqslant t \leqslant r}$, where $m_t = 0$ if and only if $t > r'$.

The bijection $\mathfrak{f} : \mathcal{F}_A \to \{\Delta_i \mid 1 \leqslant i \leqslant d\}$ provides a method for calculating AG-invariants by marked ribbon surfaces. Indeed, each elementary polygon, say $\Delta_j^t \in \{\Delta_i \mid 1 \leqslant i \leqslant d\}$, with a unique single boundary arc on the boundary component b^t of $\mathcal{S}_A$ corresponds to a forbidden thread $\mathfrak{f}^{-1}(\Delta_j^t)$, and $\sharp\mathfrak{S}(\Delta_j^t)$ is equal to

- either $l(\mathfrak{f}^{-1}(\Delta_j^t)) + 1$, if Δ_j^t is not ∞-elementary,
- or the number of vertices of Δ_j^t, otherwise.

Thus $(m_t, n_t)_{1 \leqslant t \leqslant r}$ can be obtained by

$$m_t = \text{ the number of marked points on the boundary component } b^t,$$
$$n_t = \sum_{j \in J_t} (\sharp\mathfrak{S}(\Delta_j^t) - 1) = \sum_{j \in J_t} \sharp\mathfrak{S}(\Delta_j^t) - m_t,$$

where $J_t \subseteq \{1, 2, \ldots, d\}$ is a set of all the elementary polygons which have a side is on the boundary component b^t (in this case, we have $\bigcup_{1 \leqslant t \leqslant r} = \{1, 2, \ldots, d\}$ and $J_t \cap J_{t'} = \emptyset$ holds for all $1 \leqslant t \neq t' \leqslant r$) (see [24, Theorem 3.3]).

Remark 5.11. The AG-invariant of gentle algebras is an invariant up to derived equivalence, i.e., if two gentle algebras A and B are derived equivalent, then the number of cycles of A and that of B are identical and $\phi_A = \phi_B$. Furthermore, if A and B are gentle one-cycle, i.e., the quivers of A and B both have at most one cycle, then $\phi_A = \phi_B$ yields that A is derived equivalent to B (see [5, Theorems A and C and Proposition B]).

We say that two gentle algebras A and A' are *AG-equivalent* if $\phi_A = \phi_{A'}$. By [5, Proposition B], derived equivalence yields AG-equivalence. We know that the finiteness of the global dimension is invariant under derived equivalence. In the following, we show that it is also invariant under AG-equivalence. We need the following observation.

Proposition 5.12. *For a gentle algebra $A = kQ/I$, we have* gl.dim$A = \infty$ *if and only if there is an $\ell \geqslant 1$ such that $\phi_A(0, \ell) \neq 0$.*

Proof. For a gentle algebra A, if there is an $\ell \geqslant 1$ such that $\phi_A(0, \ell) = x \geqslant 1$, then its marked ribbon surface $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ has x unmarked boundary component(s), say $b^1, b^2, \ldots, b^x$. So $\mathcal{S}_A$ has x ∞-elementary polygon(s) $\Delta^1, \Delta^2, \ldots, \Delta^x$. Thus, gl.dim$A = \infty$ by Theorem 5.10.

Conversely, if $\phi_A(0, \ell) = 0$ for any $\ell \geqslant 1$, then each boundary component b^t of $\mathcal{S}_A$ has at least one marked point and each elementary polygon Δ_j^t ($j \in J$ with J some finite set) whose single boundary arc included in b^t is not a unmarked boundary component of $\mathcal{S}_A$. Then $\mathfrak{C}(\Delta_j^t) < \infty$, so gl.dim$A$ $= \max_{1\leqslant t\leqslant r, j\in J} \mathfrak{C}(\Delta_j^t) - 1 < \infty$ by Theorem 5.10. □

As a consequence, we get the following corollary, which was proved by Opper et al. [25] by using the Koszul dual of gentle algebras.

Corollary 5.13 (See [25, Section 1 and Subsection 1.7]). *Let $A = kQ/I$ be a gentle algebra. Then* gl.dim$A = \infty$ *if and only if its quiver (Q, I) has at least one oriented cycle with full relations.*

Proof. If gl.dim$A = \infty$, then its marked ribbon surface has at least one ∞-elementary polygon Δ_j by Theorem 5.10. Suppose $\mathfrak{S}(\Delta_j) = \{\gamma_0, \ldots, \gamma_{\ell-1}\}$, where $\gamma_{\overline{i}}(1^-) = \gamma_{\overline{i+1}}(0^+)$ for any $i \geqslant 0$ and $\overline{i}$ is equal to i modulo ℓ. Then (Q, I) has an oriented cycle with full relations of length ℓ whose vertices are $\{\mathfrak{v}^{-1}(\gamma_i) \mid 0 \leqslant i \leqslant \ell - 1\}$.

Conversely, if (Q, I) has at least one oriented cycle of length ℓ ($\geqslant 1$) with full relations, then $\phi_A(0, \ell) > 0$, and thus gl.dim$A = \infty$ by Proposition 5.12. □

The following result shows that the finiteness of the global dimension of gentle algebras is invariant under AG-equivalence.

Corollary 5.14. *If two gentle algebras A and A' are AG-equivalent, then* gl.dim$A < \infty$ *if and only if* gl.dim$A' < \infty$.

Proof. By Proposition 5.12, we have gl.dim$A < \infty$ if and only if $\phi_A(0, \ell) = 0$ for any $\ell \geqslant 1$. Suppose that A and A' are AG-equivalent and gl.dim$A < \infty$. Then $\phi_A = \phi_{A'}$ and $\phi_{A'}(0, \ell) = \phi_A(0, \ell) = 0$ for any $\ell \geqslant 1$. Thus, gl.dim$A' < \infty$ by Proposition 5.12. □

6 Self-injective dimension

A module $G \in$ mod$A$ is called *Gorenstein projective* if there exists an exact sequence of projective modules

$$\cdots \longrightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots,$$

which remains still exact after applying the functor $\mathrm{Hom}_A(-, A)$ such that $G \cong \mathrm{Im}d^{-1}$ (see [4, 12]). Obviously, every projective module is Gorenstein projective. For each module M, its *Gorenstein projective dimension* G-proj.dim$M$, is defined as

$$\begin{aligned}\inf\{n \mid &\text{ there exists an exact sequence } 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \\ &\text{ in mod } A \text{ with all } G_i \text{ Gorenstein projective}\}.\end{aligned}$$

6.1 The descriptions of Gorenstein projective modules

For a gentle algebra $A = kQ/I$, the following theorem shows that all the indecomposable Gorenstein projective modules can be determined by its quiver (Q, I).

Theorem 6.1 (See [20]). *Let $A = kQ/I$ be a gentle algebra. An A-module G is Gorenstein projective if and only if G is isomorphic to a projective module or an αA where $\alpha \in Q_1$ is an arrow on some oriented cycle with full relations of A (note that αA is a direct summand of* $\mathrm{rad}P(s(\alpha)) \cong \mathrm{rad}(e_{s(\alpha)}A)$).

In this subsection, we depict all the indecomposable Gorenstein projective modules in marked ribbon surfaces.

Lemma 6.2. *Let $A = kQ/I$ be a gentle algebra having an oriented cycle with the full relation, $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface, and $\gamma \in \Gamma_A$ be an arc.*

(1) *If γ is a side of an ∞-elementary polygon Δ_i of $\mathcal{S}_A$, then there is an arc γ' such that*

– γ' and γ are two adjacent sides of Δ_i, the common endpoint of them is denoted by p, and

– γ' and γ surround p in the counterclockwise order.

(2) *Moreover, suppose that $c_{\mathrm{proj}}^{\gamma'} = c_1 \cdots c_r$ is the projective permissible curve of γ', where c_j $(1 \leqslant j \leqslant r)$ is the segment from γ' to γ. Then the module $M(c_j^{\mathfrak{T}} c_{j+1} \cdots c_r)$ is a non-projective Gorenstein projective module (see Figure 18).*

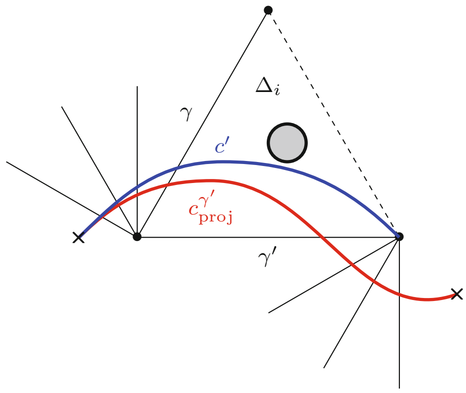


Figure 18 (Color online) $c' \simeq c_j^{\mathfrak{T}} c_{j+1} \cdots c_r$ is the permissible curve corresponded by the non-projective Gorenstein-projective module αA, where $\alpha : \gamma' \to \gamma$ is the arrow on the oriented cycle with full relations corresponded by Δ_i

Proof. (1) Since the quiver (Q, I) of A has an oriented cycle with the full relation, we have $\mathrm{gl.dim}A = \infty$ by Corollary 5.13, and moreover, each oriented cycle of length ℓ with the full relation corresponds to a pair $(0, \ell)$ of the AG-invariant ϕ_A and an ∞-elementary ℓ-polygon of the marked ribbon surface $\mathcal{S}_A$. Let q be another endpoint of γ. Obviously, in the case where $p = q$, we have $\gamma = \gamma'$. If $p \neq q$, then Δ_i corresponds to some oriented cycle with the full relation

$$0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} 2 \longrightarrow \cdots \longrightarrow \ell - 1 \xrightarrow{\alpha_\ell} 0$$

such that the side $\gamma = \mathfrak{v}^{-1}(x)$ of Δ_i corresponds to some vertex $x \in \{0, 1, \ldots, \ell - 1\}$ through the map $\mathfrak{v} : Q_0 \to \Gamma_A$ which is defined in Definition 2.3. Then $\gamma' = \mathfrak{v}(\overline{x-1})$, where $\overline{x-1}$ is equal to $x - 1$ modulo ℓ.

(2) Following Definition 2.3(1), for simplicity, we use γ to denote the vertex $\mathfrak{v}^{-1}(\gamma) \in Q_0$. We have that $c^{\gamma'}_{\mathrm{proj}}$ corresponds to the string

$$\cdots \xleftarrow{\alpha(c_{j-1})} \gamma' \xrightarrow{\alpha(c_j)} \gamma \xrightarrow{\alpha(c_{j+1})} \gamma'' \xrightarrow{\alpha(c_{j+2})} \cdots,$$

where each arrow $\alpha(c_j) \in Q_1$ is induced by the segment c_j for any $1 \leqslant j \leqslant r - 1$ (see Definition 2.3(2)). Then $c_j^{\mathfrak{T}} c_{j+1} \cdots c_r$, denoted by c, corresponds to the string

$$\gamma \xrightarrow{\alpha(c_{j+1})} \gamma'' \xrightarrow{\alpha(c_{j+2})} \cdots.$$

Thus, $M(c) \cong \alpha(c_j)A \leqslant_{\oplus} \mathrm{rad}M(c^{\gamma'}_{\mathrm{proj}})$ is Gorenstein projective by Theorem 6.1.

On the other hand, the side $\mathfrak{v}(\overline{x+1})$ of Δ_i satisfies that

- γ and $\mathfrak{v}(\overline{x+1})$ are two adjacent sides of Δ_i (the common endpoint of them is denoted by q), and
- γ and $\mathfrak{v}(\overline{x+1})$ surround q in the counterclockwise order.

Then $c \not\simeq c^{\gamma}_{\mathrm{proj}}$ and thus $M(c)$ is not projective by Lemma 4.5. □

Proposition 6.3 (Gorenstein projective modules)**.** *Let $A = kQ/I$ be a gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. For a permissible curve c, the indecomposable module $M(c)$ is Gorenstein projective if and only if one of the following holds*:

- (GP1) *c is a projective permissible curve.*
- (GP2) *$c \simeq c_1 \cdots c_r$ consecutively crosses arcs $\gamma_1, \ldots, \gamma_{r-1}$, which have a common endpoint*

$$p = p(\gamma_1, \gamma_2, c) = \cdots = p(\gamma_{r-2}, \gamma_{r-1}, c)$$

such that

- *γ_{i+1} appears after γ_i $(1 \leqslant i \leqslant r - 2)$ in the counterclockwise order surrounding p;*
- *γ_1 is a side of some ∞-elementary polygon Δ_j with $1 \leqslant j \leqslant d$;*
- *the end segment of c between $c(0^+)$ and γ_1 lies in the inner of Δ_j.*

Proof. We first prove the sufficiency. By Lemma 4.5, if c is a projective permissible curve, then it is trivial that $M(c)$ is a Gorenstein projective module.

If c satisfies (GP2), then there is a unique arc γ' such that γ' and $\gamma := \gamma_1$ satisfy the conditions given in Lemma 6.2(1). Suppose that $c^{\gamma'}_{\mathrm{proj}} = c_1 \cdots c_s$ consecutively crosses $\tilde{\gamma}_1 \cdots \tilde{\gamma}_{s-1}$ $(s \geqslant r+1)$. Then there is a unique h with $1 \leqslant h \leqslant s-1$ such that $\tilde{\gamma}_{h-1} = \gamma'$, $\tilde{\gamma}_h = \gamma_1$, $\tilde{\gamma}_{h+1} = \gamma_2, \ldots, \tilde{\gamma}_{s-1} = \gamma_{r-1}$. To be more precise, $c^{\gamma'}_{\mathrm{proj}}$ corresponds to the following string:

$$\begin{gathered}\tilde{\gamma}_1 \longleftarrow \cdots \longleftarrow \tilde{\gamma}_{h-2} \longleftarrow \tilde{\gamma}_{h-1} \xrightarrow{\alpha} \tilde{\gamma}_h \longrightarrow \tilde{\gamma}_{h+1} \longrightarrow \cdots \longrightarrow \tilde{\gamma}_{s-1}\\ (\tilde{\gamma}_{h-1} = \gamma' \text{ and } \tilde{\gamma}_h = \gamma \text{ are adjacent sides of } \Delta_j).\end{gathered}$$

We have that c consecutively crosses $\tilde{\gamma}_h, \tilde{\gamma}_{h+1}, \ldots, \tilde{\gamma}_{s-1}$, where $\tilde{\gamma}_h \in \mathfrak{S}(\Delta_j)$ and $\tilde{\gamma}_m \notin \mathfrak{S}(\Delta_j)$ for any $h+1 \leqslant m \leqslant s-1$. Thus, $M(c) \cong \alpha A$ is a Gorenstein projective module satisfying $M(c) \leqslant_{\oplus} \mathrm{rad} M(c^{\tilde{\gamma}_{h-1}}_{\mathrm{proj}})$. In this case, $M(c)$ is non-projective.

Next, we prove the necessity. By Theorem 6.1, if $M(c)$ is Gorenstein projective, then $M(c)$ is isomorphic to either a projective module or an αA, where α is an arrow on some oriented cycle with full relations.

If $M(c)$ is projective, then c is a projective permissible curve. Now suppose that $M(c)$ is isomorphic to αA. Let $v := s(\alpha)$. It follows from Lemma 4.5 that

$$\alpha A \cong \mathrm{rad} P(v) \cong \mathrm{rad} M(c^{\tilde{\gamma}(v)}_{\mathrm{proj}})$$

for some arc $\tilde{\gamma}(v) \in \Gamma_A$, where $c^{\tilde{\gamma}(v)}_{\mathrm{proj}} = c_1 \cdots c_s$ is a projective permissible curve consecutively crossing arcs $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{s-1}$. Then v corresponds to some $\tilde{\gamma}_h$ with $2 \leqslant h \leqslant s-2$ through $\mathfrak{v} : Q_0 \to \Gamma_A$, i.e., $\tilde{\gamma}_h = \tilde{\gamma}(v)$ (note that $h \neq 1$ and $h \neq s-1$ because v is a vertex on the oriented cycle with full relations). Then $c^{\tilde{\gamma}(v)}_{\mathrm{proj}} = c^{\tilde{\gamma}_h}_{\mathrm{proj}}$ corresponds to the string

$$\tilde{\gamma}_1 \longleftarrow \cdots \longleftarrow \tilde{\gamma}_{h-1} \xleftarrow{\alpha_1} \tilde{\gamma}_h \xrightarrow{\alpha_2} \tilde{\gamma}_{h+1} \longrightarrow \cdots \longrightarrow \tilde{\gamma}_{s-1},$$

and the arrow α is either α_1 or α_2. Without loss of generality, suppose $\alpha = \alpha_1$. Since $M(c) \cong \alpha_1 A$ is Gorenstein projective, the arcs $\tilde{\gamma}_{h-1}$ and $\tilde{\gamma}_h$ are adjacent sides of some ∞-elementary polygon Δ_j of $\mathcal{S}_A$. Let $\tilde{\gamma}_{h-1} = \gamma_1$, $\tilde{\gamma}_{h-2} = \gamma_2, \ldots, \tilde{\gamma}_1 = \gamma_{h-1}$. Then the permissible curve c consecutively crossing $\gamma_1, \ldots, \gamma_{h-1}$ satisfies the condition (GP2). $\square$

Example 6.4. Let A be a gentle algebra whose marked ribbon surface $\mathcal{S}_A$ is given in Figure 19. We can find 12 indecomposable Gorenstein projective modules including

- 9 indecomposable projective modules (see green permissible curves)

$$\begin{smallmatrix} & 1 & \\ 4 & & 2 \\ 5 & & 6 \\ & & 7 \end{smallmatrix},\ \begin{smallmatrix} & 2 & \\ 6 & & 3 \\ 7 & & 8 \\ & & 9 \end{smallmatrix},\ \begin{smallmatrix} & 3 & \\ 8 & & 1 \\ 9 & & 4 \\ & & 5 \end{smallmatrix},\ \begin{smallmatrix} 4 \\ 5 \end{smallmatrix},\ \begin{smallmatrix} 6 \\ 7 \end{smallmatrix},\ \begin{smallmatrix} 8 \\ 9 \end{smallmatrix},\ 5,\ 7,\ 9;$$

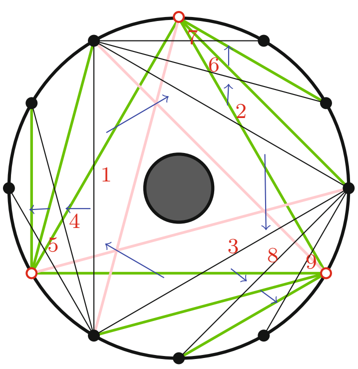


Figure 19 (Color online) An example for calculating all the indecomposable Gorenstein projective modules

- 3 indecomposable non-projective Gorenstein projective modules (see pink permissible curves)

$$\begin{smallmatrix}2\\6\\7\end{smallmatrix}\Big(\leqslant_{\oplus}\mathrm{rad}\Big(\begin{smallmatrix} & 1 & \\ 4 & & 2\\ 5 & & 6\\ & & 7\end{smallmatrix}\Big)\Big),\ \ \begin{smallmatrix}3\\8\\9\end{smallmatrix}\Big(\leqslant_{\oplus}\mathrm{rad}\Big(\begin{smallmatrix} & 2 & \\ 6 & & 3\\ 7 & & 8\\ & & 9\end{smallmatrix}\Big)\Big),\ \ \begin{smallmatrix}1\\4\\5\end{smallmatrix}\Big(\leqslant_{\oplus}\mathrm{rad}\Big(\begin{smallmatrix} & 3 & \\ 8 & & 1\\ 9 & & 4\\ & & 5\end{smallmatrix}\Big)\Big).$$

6.2 The descriptions of the self-injective dimension

Geiß and Reiten [14, Theorem 3.4] showed that gentle algebras are Gorenstein. In this subsection, we provide a geometric interpretation of this result by marked ribbon surfaces. We use $\mathsf{f}\text{-}\Delta$ to denote the set of all the non-∞-elementary polygons, and use $\mathsf{f}\text{-}\mathcal{F}_A$ to denote the set of all the forbidden threads of finite length. We first prove the following proposition.

Proposition 6.5. *Let $A = kQ/I$ be a non-simple gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. Then*

$$\mathrm{inj.dim}_A A = \mathrm{inj.dim} A_A = \begin{cases} 0, & \text{if } Q \text{ is an oriented cycle with full relations},\\ \max\limits_{\Delta_i\in \mathsf{f}\text{-}\Delta}\{1, \mathfrak{C}(\Delta_i)-1\}, & \text{otherwise.}\end{cases}$$

Proof. If Q is an oriented cycle with full relations, then A is a self-injective Nakayama algebra.

Now suppose that Q is not an oriented cycle with full relations. Since A is a gentle algebra, it follows from [14, Theorem 3.4] that A is Gorenstein. Then by [18, Theorem] and [19, Theorem 1.4], we have $\mathrm{inj.dim}_A A = \mathrm{inj.dim} A_A = \sup\{\text{G-proj.dim} S \mid S \text{ is simple}\}$.

The proof is divided into the following two cases: (i) (Q, I) has at least one oriented cycle with full relations. (ii) (Q, I) has no oriented cycle with full relations.

(i) Note that if $v \in Q_0$ is a vertex on the oriented cycle with full relations, then $\mathrm{proj.dim} S(v) = \infty$. Consider the arrow α on this oriented cycle with full relations such that its sink is v. We have that αA is an indecomposable non-projective Gorenstein projective module, and moreover, it is the Gorenstein projective cover of $S(v)$. In the marked ribbon surface $\mathcal{S}_A$, v corresponds to a side, denoted by γ_1, of an ∞-elementary polygon Δ_j and $M^{-1}(\alpha A)$ is such a permissible curve c consecutively crossing $\gamma_1, \ldots, \gamma_r$ ($r \geqslant 1$), where $\gamma_1, \ldots, \gamma_r$ satisfy the condition (GP2) given in Proposition 6.3 and $\gamma_1 = \mathfrak{v}(v)$.

In the case where $r = 1$, it is obvious that $S(v)$ is a Gorenstein projective module and $\text{G-proj.dim} S(v) = 0$. Suppose $r > 1$. Then there exists an elementary polygon $\Delta_{j'}$ such that γ_1 and γ_2 are two sides of $\Delta_{j'}$. If $\Delta_{j'}$ is not ∞-elementary, then we get $\text{G-proj.dim} S(v) = \mathfrak{C}(\Delta_{j'})-1$ by using an argument similar to that of Proposition 5.6 (the arcs γ_1 and γ_2 can be viewed as γ^0 and γ^1 shown in Proposition 5.6, respectively). If $\Delta_{j'}$ is ∞-elementary, then consider the permissible curve c' consecutively crossing $\gamma_2, \ldots, \gamma_r$ (in this case $\gamma_2 \in \mathfrak{S}(\Delta_{j'})$) where $\gamma_2, \ldots, \gamma_r$ satisfy (GP2). We have that $M(c')$ is the kernel of $M(c) \twoheadrightarrow S(v) \cong M(c^{\gamma_1}_{\mathrm{simp}})$ by Theorem 3.12(1) because the positional relationship of c', c and $c^{\gamma_1}_{\mathrm{simp}}$ satisfies the conditions given in Proposition 3.6. Then $\text{G-proj.dim} S(v) = 1$. Consequently, we have

$$\mathrm{inj.dim}_A A = \mathrm{inj.dim} A_A = \max\Big\{1, \max_{\Delta_i\in\mathsf{f}\text{-}\Delta}\mathfrak{C}(\Delta_i)-1\Big\}.$$

(ii) If (Q, I) has no oriented cycle with the relation, then A has finite global dimension by Theorem 5.10. In this case, we have

$$\mathrm{inj.dim}_A A = \mathrm{inj.dim} A_A = \mathrm{gl.dim} A = \max_{1\leqslant i\leqslant d}\mathfrak{C}(\Delta_i)-1,$$

where $\{\Delta_i \mid 1 \leqslant i \leqslant d\} = \mathsf{f}\text{-}\Delta$ is the set of all the elementary polygons of $\mathcal{S}_A$. □

The proof of Proposition 6.5 depends on Theorem 6.1. In the following, we will give another proof which does not need to use the properties of Gorenstein projective modules. We assume that the quiver of A is neither an oriented cycle with full relations nor a point.

Lemma 6.6. *Let $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be a marked ribbon surface of a gentle algebra A, and γ be an arc on $\mathcal{S}_A$ with endpoints p and q. As shown in Figure 20, suppose that $c^{\gamma}_{\mathrm{inj}}$ consecutively crosses arcs $\gamma^0_1, \ldots, \gamma^0_{i-1}, \gamma$, $\tilde{\gamma}^0_{j-1}, \ldots, \tilde{\gamma}^0_1$, and $c^{\gamma}_{\mathrm{proj}}$ consecutively crosses arcs $\tilde{\gamma}^0_n, \ldots, \tilde{\gamma}^0_{j+1}, \gamma, \gamma^0_{i+1}, \ldots, \gamma^0_m$, where p is*

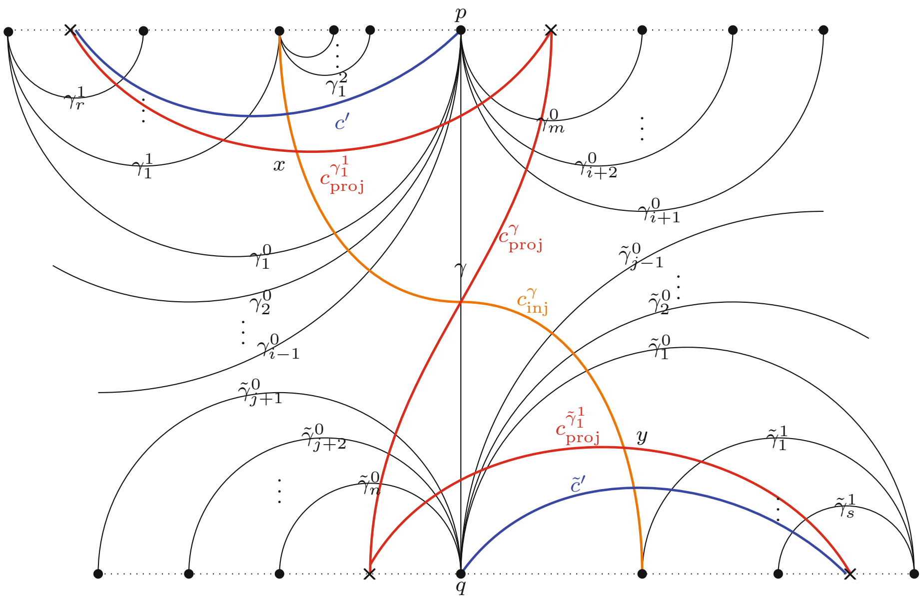


Figure 20 (Color online) The formal direct sum $c_{\mathrm{proj}}^{\gamma_1^1} \oplus c_{\mathrm{proj}}^{\tilde{\gamma}_1^1}$ of permissible curves provides the projective cover of the injective module $M(c_{\mathrm{inj}}^{\gamma})$

the common endpoint of $\gamma_1^0, \ldots, \gamma_m^0$ *and* q *is the common endpoint of* $\tilde{\gamma}_1^0, \ldots, \tilde{\gamma}_n^0$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$ *and* $\gamma = \gamma_i^0 = \tilde{\gamma}_j^0)$. *Then*

(1) *the projective cover of* $M(c_{\mathrm{inj}}^{\gamma})$ *is* $p(M(c_{\mathrm{inj}}^{\gamma})) : M(c_{\mathrm{proj}}^{\gamma_1^0} \oplus c_{\mathrm{proj}}^{\tilde{\gamma}_1^0}) \twoheadrightarrow M(c_{\mathrm{inj}}^{\gamma})$;

(2) $\mathrm{Ker}p(M(c_{\mathrm{inj}}^{\gamma})) \cong M(c' \oplus c_{\mathrm{proj}}^{\gamma} \oplus \tilde{c}')$, *where* c' *and* c'' *are two permissible curves shown in Figure* 20.

Proof. (1) This follows from Proposition 4.3 and Theorem 4.6.

(2) Indeed, $\mathrm{top}M(c_{\mathrm{inj}}^{\gamma}) = M(c_{\mathrm{simp}}^{\gamma_1^0}) \oplus M(c_{\mathrm{simp}}^{\tilde{\gamma}_1^0})$, and there exists no arc γ^0 (resp. $\tilde{\gamma}^0$) with the endpoint p (resp. q) such that $\gamma^0, \gamma_1^0, \ldots, \gamma_{i-1}^0, \gamma$ (resp. $\gamma, \tilde{\gamma}_{j-1}^0, \ldots, \tilde{\gamma}_1^0, \tilde{\gamma}$) surround p (resp. q) in the counterclockwise order. Thus, $M(c_{\mathrm{inj}}^{\gamma})$ corresponds to the string

$$\gamma_1^0 \longrightarrow \gamma_2^0 \longrightarrow \cdots \longrightarrow \gamma_{i-1}^0 \longrightarrow \gamma \longleftarrow \tilde{\gamma}_{j-1}^0 \longleftarrow \cdots \longleftarrow \tilde{\gamma}_2^0 \longleftarrow \tilde{\gamma}_1^0,$$

and $M(c_{\mathrm{simp}}^{\gamma_1^0})$ and $M(c_{\mathrm{simp}}^{\tilde{\gamma}_1^0})$ correspond to the strings

$$\gamma_r^1 \longleftarrow \cdots \longleftarrow \gamma_1^1 \longleftarrow \gamma_1^0 \longrightarrow \gamma_2^0 \longrightarrow \cdots \longrightarrow \gamma_{i-1}^0 \longrightarrow \gamma \longrightarrow \gamma_{i+1}^0 \longrightarrow \cdots \longrightarrow \gamma_m^0$$

and

$$\tilde{\gamma}_n^0 \longleftarrow \cdots \longleftarrow \tilde{\gamma}_{j+1}^0 \longleftarrow \gamma \longleftarrow \tilde{\gamma}_{j-1}^0 \longleftarrow \cdots \longleftarrow \tilde{\gamma}_2^0 \longleftarrow \tilde{\gamma}_1^0 \longrightarrow \tilde{\gamma}_1^1 \longrightarrow \cdots \longrightarrow \tilde{\gamma}_s^1,$$

respectively. Then the kernel of $p(M(c_{\mathrm{inj}}^{\gamma}))$ is isomorphic to $M_1 \oplus M_2 \oplus M_3$, where M_1, M_2 and M_3 correspond to the strings

$$\gamma_r^1 \longleftarrow \cdots \longleftarrow \gamma_1^1, \quad \tilde{\gamma}_n^0 \longleftarrow \cdots \longleftarrow \tilde{\gamma}_{j+1}^0 \longleftarrow \gamma \longrightarrow \gamma_{i+1}^0 \longrightarrow \cdots \longrightarrow \gamma_m^0$$

and $\tilde{\gamma}_1^1 \longrightarrow \cdots \longrightarrow \tilde{\gamma}_s^1$, respectively. Since $c_{\mathrm{proj}}^{\gamma_1^0}$ (resp. $c_{\mathrm{proj}}^{\tilde{\gamma}_1^0}$) is projective, both conditions (LMC) and (RMC) are satisfied. The condition (RMC) (resp. (LMC)) shows that there is no arc γ_{m+1}^0 (resp. $\tilde{\gamma}_{n+1}^0$) such that $\gamma_1^0, \ldots, \gamma_m^0, \gamma_{m+1}^0$ (resp. $\tilde{\gamma}_1^0, \ldots, \tilde{\gamma}_n^0, \tilde{\gamma}_{n+1}^0$) have the same endpoint p (resp. q) and surround p (resp. q) in the counterclockwise order. Then the string of M_2 corresponds to the permissible curve $c_{\mathrm{proj}}^{\gamma}$ which is projective, and the strings of M_1 and M_3 correspond to c' and $\tilde{c}'$, respectively. □

Remark 6.7. If γ, γ_1^0 and $\tilde{\gamma}_1^0$ satisfy the P-condition (we have $i < m$ and $j < n$ in this case), then in $\mathcal{S}_A \backslash \partial\mathcal{S}_A$, $c_{\mathrm{inj}}^{\gamma}$ and $c_{\mathrm{proj}}^{\gamma_1^0}$ have an intersection x, and $c_{\mathrm{inj}}^{\gamma}$ and $c_{\mathrm{proj}}^{\tilde{\gamma}_1^0}$ have an intersection y. The permissible curves c', $\tilde{c}'$ and $c_{\mathrm{proj}}^{\gamma}$ can be understood as obtained by "negative rotating respect to $c_{\mathrm{proj}}^{\gamma_1^1} \oplus c_{\mathrm{proj}}^{\tilde{\gamma}_1^1}$" of

c_{inj}^{γ}. To be precise, c_{inj}^{γ} is divided into three parts c_{I}, c_{II} and c_{III}, x is the common endpoint of c_{I} and c_{II}, and y is the common endpoint of c_{II} and c_{III}. Then c' (resp. $\tilde{c}'$) can be obtained as follows:

Step 1. Following the positive direction of boundary, we move the endpoint of c_{I} (resp. c_{III}) lying in $\mathcal{M}_A$ to the previous vertex of the elementary polygon $\Delta_{\gamma_1^0}$ (resp. $\Delta_{\tilde{\gamma}_1^0}$), where γ_1^0 and γ_1^1 (resp. $\tilde{\gamma}_1^0$ and $\tilde{\gamma}_1^1$) are sides of $\Delta_{\gamma_1^0}$ (resp. $\Delta_{\tilde{\gamma}_1^0}$).

Step 2. Move x (resp. y) to the endpoint of $c_{\text{proj}}^{\gamma_1^1}$ (resp. $c_{\text{proj}}^{\tilde{\gamma}_1^1}$) such that c_{I} (resp. c_{III}) makes a negative rotation.

Proposition 6.8. *Keeping the notations in Lemma* 6.6*, we have*

$$\text{proj.dim} M(c_{\text{inj}}^{\gamma}) = \max\{1, \mathfrak{C}(\Delta_{\gamma_1^0}) - 1, \mathfrak{C}(\Delta_{\tilde{\gamma}_1^0}) - 1\},$$

where $\Delta_{\gamma_1^0}$ (resp. $\Delta_{\tilde{\gamma}_1^0}$) is the elementary polygon with sides γ_1^0 and γ_1^1 (resp. $\tilde{\gamma}_1^0$ and $\tilde{\gamma}_1^1$). To be more precise,

(1) *if γ_1^0 and γ_1^1 (resp. $\tilde{\gamma}_1^0$ and $\tilde{\gamma}_1^1$) exist, then* $\text{proj.dim} M(c_{\text{inj}}^{\gamma}) = \max\{\mathfrak{C}(\Delta_{\gamma_1^0}), \mathfrak{C}(\Delta_{\tilde{\gamma}_1^0})\} - 1$;

(2) *if at least one of γ_1^0 and γ_1^1 does not exist and at least one of $\tilde{\gamma}_1^0$ and $\tilde{\gamma}_1^1$ does not exist, then* $\text{proj.dim} M(c_{\text{inj}}^{\gamma}) = 1$.

Proof. Suppose that γ_1^0 and γ_1^1 (resp. $\tilde{\gamma}_1^0$ and $\tilde{\gamma}_1^1$) exist. Similar to Proposition 5.6, regard γ_1^0 and γ_1^1 as the functions $\gamma_1^0, \gamma_1^1 : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$, where $\gamma_1^0(0^+) = p$ and $\gamma_1^0(1^-) = \gamma_1^1(0^+)$. If there are arcs $\gamma_1^2, \ldots, \gamma_1^{\theta} : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$ such that $\gamma_1^0, \ldots, \gamma_1^{\theta} \in \mathfrak{C}(\Delta_{\gamma_1^0})$ satisfy (PD1)–(PD3), then the minimal projective resolution of $M(c')$ is of the form

$$\cdots \longrightarrow M(c_{\text{proj}}^{\gamma_1^{\theta}}) \longrightarrow \cdots \longrightarrow M(c_{\text{proj}}^{\gamma_1^2}) \longrightarrow M(c_{\text{proj}}^{\gamma_1^1}) \longrightarrow M(c') \longrightarrow 0.$$

We claim that $\Delta_{\gamma_1^0}$ is not ∞-elementary. Otherwise, there exists an arc γ_1^{ϑ} with $\vartheta \geqslant \theta$ such that $\gamma_1^{\vartheta}(1^-) = \gamma_1^0(0^+)$, and this leads to a contradiction because c_{inj}^{γ} is injective. Since $\sharp \mathcal{M}_A < \infty$, there is a sequence of arcs $\gamma_1^{\theta}, \gamma_1^{\theta+1}, \ldots, \gamma_1^d$ $(d \geqslant \theta)$ such that $\gamma_1^0, \gamma_1^1, \ldots, \gamma_1^d$ satisfy (PD) by Proposition 5.6. Then the minimal projective resolution of $M(c')$ is as follows:

$$0 \longrightarrow M(c_{\text{proj}}^{\gamma_1^d}) \longrightarrow \cdots \longrightarrow M(c_{\text{proj}}^{\gamma_1^2}) \longrightarrow M(c_{\text{proj}}^{\gamma_1^1}) \longrightarrow M(c') \longrightarrow 0. \tag{6.1}$$

Without loss of generality, suppose that

- there are arcs $\gamma_1^0, \gamma_1^1, \ldots, \gamma_1^d$ with $d = \mathfrak{C}(\Delta_{\gamma_1^0})$ satisfying (PD);
- dually, there are arcs $\tilde{\gamma}_1^0, \tilde{\gamma}_1^1, \ldots, \tilde{\gamma}_1^{\tilde{d}}$ with $\tilde{d} = \mathfrak{C}(\Delta_{\tilde{\gamma}_1^0})$ satisfying (PD$'$).

Similarly, we get the minimal projective resolution of $M(\tilde{c}')$ as follows:

$$0 \longrightarrow M(c_{\text{proj}}^{\tilde{\gamma}_1^{\tilde{d}}}) \longrightarrow \cdots \longrightarrow M(c_{\text{proj}}^{\tilde{\gamma}_1^2}) \longrightarrow M(c_{\text{proj}}^{\tilde{\gamma}_1^1}) \longrightarrow M(\tilde{c}') \longrightarrow 0. \tag{6.2}$$

Thus,

$$\text{proj.dim} M(c_{\text{inj}}^{\gamma}) = \max\{\text{proj.dim} M(c'), \text{proj.dim} M(\tilde{c}')\} - 1 = \max\{d, \tilde{d}\} - 1.$$

Now suppose that at least one of γ_1^0 and γ_1^1 does not exist and at least one of $\tilde{\gamma}_1^0$ and $\tilde{\gamma}_1^1$ does not exist. Then $M(c_{\text{proj}}^{\gamma_1^1}) = 0 = M(c_{\text{proj}}^{\tilde{\gamma}_1^1})$ in (6.1) and (6.2), and hence $M(c') = 0 = M(\tilde{c}')$. By Lemma 6.6(2), we have that $\operatorname{Ker} p(M(c_{\text{inj}}^{\gamma})) \cong M(c_{\text{proj}}^{\gamma})$ is projective and $\text{proj.dim} M(c_{\text{inj}}^{\gamma}) = 1$. □

For any basic finite-dimensional k-algebra A, we know $A \cong \bigoplus_{i \in I} e_i A$ and $D(A) \cong \bigoplus_{i \in I} D(Ae_i)$, where $D = \operatorname{Hom}_k(-, k)$, $\{e_i \mid i \in I\}$ is a complete set of primitive orthogonal idempotents of A, each $e_i A$ is an indecomposable projective right A-module, and each $D(Ae_i)$ is an indecomposable injective right A-module. Thus, Proposition 6.8 provides a method for calculating the projective dimension of $D(A)$. Similarly, we can establish the dual of Proposition 6.8 and calculate the injective dimension of A.

Theorem 6.9. *Let $A = kQ/I$ be a non-simple gentle algebra and $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ be its marked ribbon surface. If the quiver (Q, I) of A is not an oriented cycle with full relations, then*

$$\text{inj.dim}_A A = \text{inj.dim} A_A = \max_{\Delta_j \in \mathsf{f}\text{-}\Delta} \{1, \mathfrak{C}(\Delta_j) - 1\} = \max_{\mathit{\Pi} \in \mathsf{f}\text{-}\mathcal{F}_A} l(\mathit{\Pi}). \tag{6.3}$$

Proof. Let $A = kQ/I$ be a gentle algebra. Then $\text{inj.dim}_A A = \text{inj.dim} A_A = \text{proj.dim} D(A)$. Similar to the proofs of Proposition 5.6 and Theorem 5.10, we get

$$\text{proj.dim}D(A) = \max\left\{1, \max_{\Delta_i \in \text{f-}\Delta} \mathfrak{C}(\Delta_i) - 1\right\} = \max_{\Delta_j \in \text{f-}\Delta}\{1, \mathfrak{C}(\Delta_j) - 1\} = \max_{\Pi \in \text{f-}\mathcal{F}_A} l(\Pi)$$

by Proposition 6.8. □

Note that the number of all the elements in $\mathfrak{C}(\Delta_i)$ is equal to that of all the sides of Δ_i minus 1. Thus, Theorem 6.9 shows that the self-injective dimension of A is equal to the number of sides of element polygon(s) with the largest number of sides minus 2. Moreover, by [14, Theorem 3.4], any gentle algebra is Gorenstein. Thus, the finitistic dimension of a gentle algebra equals its left and right self-injective dimensions (see [16]). Thus, Theorem 6.9 provides a description of the finitistic dimension of a gentle algebra.

6.3 The number of indecomposable Gorenstein projective modules

Recall that an algebra is said to be Cohen-Macaulay (CM)-finite if the number of indecomposable Gorenstein projective modules (up to isomorphism) is finite. It is known that gentle algebras are CM-finite [10]. We use $\text{ind}(\text{nproj-}\mathcal{GP}(A))$ to denote the subcategory of mod$A$ consisting of all the indecomposable non-projective Gorenstein projective modules. In this subsection, we obtain a formula for calculating $\sharp\text{ind}(\text{nproj-}\mathcal{GP}(A))$ by AG-invariants.

Lemma 6.10. *Let $A = kQ/I$ be a gentle algebra, and $c_1 = \alpha_1\alpha_2\cdots\alpha_m$ and $c_2 = \beta_1\beta_2\cdots\beta_n$ be two oriented cycles with full relations of A. If c_1 and c_2 have a common vertex v_0, i.e., there are arrows α_h, α_{h+1}, $\beta_\hbar$ and $\beta_{\hbar+1}$ such that $v_0 = t(\alpha_h) = t(\beta_\hbar) = s(\alpha_{h+1}) = s(\beta_{\hbar+1})$, then $\alpha_h A$ and $\beta_\hbar A$ are indecomposable Gorenstein projective modules satisfying* $\text{top}(\alpha_h A) = \text{top}(\beta_\hbar A) \cong M(c_{\text{simp}}^{\gamma_0}) = S(v_0)$.

Proof. We know that $\alpha_h A$ and $\beta_\hbar A$ are indecomposable non-projective Gorenstein projective by Theorem 6.1. Now we show that $\text{top}(\alpha_h A) = \text{top}(\beta_\hbar A) \cong M(c_{\text{simp}}^{\gamma_0}) = S(v_0)$ by using the marked ribbon surface $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ of A. Let γ_0 be the arc in Γ_A corresponding to v_0 by the bijection $\mathfrak{v} : Q_0 \to \Gamma_A$. Then γ_0 is a common side of two ∞-elementary polygons Δ_1 and Δ_2, i.e., there are arcs $\gamma_{-1}, \gamma_1, \gamma_2, \ldots, \gamma_a$ and $\tilde{\gamma}_{-1}, \tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_b$ such that

- $\gamma_{-1}, \gamma_0\gamma_1, \ldots, \gamma_a$ (resp. $\tilde{\gamma}_{-1}, \tilde{\gamma}_0 = \gamma_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_b$) have a common endpoint p (resp. q) and γ_{i+1} (resp. $\tilde{\gamma}_{j+1}$) appears after γ_i (resp. $\tilde{\gamma}_j$) for any $-1 \leqslant i \leqslant a-1$ (resp. $-1 \leqslant j \leqslant b-1$);
- $\gamma_{-1}, \gamma_0 \in \mathfrak{S}(\Delta_2)$, $\gamma_0, \gamma_1 \in \mathfrak{S}(\Delta_1)$ and $\gamma_i \notin \mathfrak{S}(\Delta_1)$ for any $3 \leqslant i \leqslant a$;
- $\tilde{\gamma}_{-1}, \tilde{\gamma}_0 \in \mathfrak{S}(\Delta_1)$, $\tilde{\gamma}_0, \tilde{\gamma}_1 \in \mathfrak{S}(\Delta_2)$ and $\gamma_j \notin \mathfrak{S}(\Delta_2)$ for any $3 \leqslant j \leqslant b$;
- $\mathfrak{v}^{-1}(\gamma_1) = t(\alpha_{h+1})$ and $\mathfrak{v}^{-1}(\tilde{\gamma}_1) = t(\beta_{\hbar+1})$.

For the permissible curve c consecutively crossing $\gamma_0, \gamma_1, \ldots, \gamma_a$ and $\tilde{c}$ consecutively crossing $\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_b$ such that $p = p(\gamma_0, \gamma_1, c) = p(\gamma_1, \gamma_2, c) = \cdots = p(\gamma_{a-1}, \gamma_a, c)$ and $q = p(\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{c}) = p(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{c}) = \cdots = p(\tilde{\gamma}_{b-1}, \tilde{\gamma}_b, \tilde{c})$, we have $M(c) \cong \alpha_h A$, $M(\tilde{c}) \cong \beta_\hbar A$, and c and $\tilde{c}$ correspond to the strings

$$\mathfrak{v}^{-1}(\gamma_a) \longleftarrow \cdots \xleftarrow{\alpha_{h+2}} \mathfrak{v}^{-1}(\gamma_1) \xleftarrow{\alpha_{h+1}} \mathfrak{v}^{-1}(\gamma_0) \quad \text{and} \quad \mathfrak{v}^{-1}(\tilde{\gamma}_0) \xrightarrow{\beta_{\hbar+1}} \mathfrak{v}^{-1}(\tilde{\gamma}_1) \xrightarrow{\beta_{\hbar+2}} \cdots \longrightarrow \mathfrak{v}^{-1}(\tilde{\gamma}_b),$$

respectively, where $\mathfrak{v}^{-1}(\gamma_0) = \mathfrak{v}^{-1}(\tilde{\gamma}_0) = v_0$. So $\text{top}(\alpha_h A) \cong \text{top}M(c) \cong M(c_{\text{simp}}^{\gamma_0})$ and $\text{top}(\beta_\hbar A) \cong \text{top}M(\tilde{c}) \cong M(c_{\text{simp}}^{\tilde{\gamma}_0})$. Notice that $\gamma_0 = \tilde{\gamma}_0$, and thus $M(c_{\text{simp}}^{\gamma_0}) = M(c_{\text{simp}}^{\tilde{\gamma}_0})$ is isomorphic to the simple module $S(\mathfrak{v}^{-1}(\gamma_0)) = S(v_0)$. □

Lemma 6.11. *Let $A = kQ/I$ be a gentle algebra, and Q_0^{ocf} be the set of all the vertices on the oriented cycle with full relations. Then the map $f : \text{ind}(\text{nproj-}\mathcal{GP}(A)) \to Q_0^{\text{ocf}}$ via $\alpha A \mapsto t(\alpha)$ is surjective*; *furthermore, if $v \in Q_0^{\text{ocf}}$ is a common vertex of two oriented cycles with full relations, then $\sharp f^{-1}(v) = \sharp\{\alpha A \in \text{ind}(\text{nproj-}\mathcal{GP}(A)) \mid t(\alpha) = v\} = 2$; if otherwise, then $\sharp f^{-1}(v) = 1$.*

Proof. For any $v \in Q_0^{\text{ocf}}$, there exists at least one oriented cycle with full relations $\alpha_0\cdots\alpha_{l-1}$ such that $v = t(\alpha_{\bar{i}}) = s(\alpha_{\overline{i+1}})$ for some $0 \leqslant i \leqslant l$, where $\bar{i}$ is equal to i modulo l. Then by Theorem 6.1, we have that $\alpha_{\bar{i}} A$ is an indecomposable non-projective Gorenstein projective module and f is surjective.

If v is a common vertex of two oriented cycles with full relations $\alpha_0\alpha_1\cdots\alpha_{l-1}$ and $\beta_0\beta_1\cdots\beta_{\ell-1}$, then there are arrows $\alpha_{\bar{i}}$, $\alpha_{\overline{i+1}}$, $\beta_{\underline{j}}$ and $\beta_{\underline{j+1}}$ such that

$$v = t(\alpha_{\bar{i}}) = s(\alpha_{\overline{i+1}}) = t(\beta_{\underline{j}}) = s(\beta_{\underline{j+1}}),$$

where $\bar{i}$ is equal to i modulo l and $\underline{j}$ is equal to j modulo ℓ. Thus we have $\{\alpha_{\bar{i}}A, \beta_{\underline{j}}A\} \subseteq f^{-1}(v)$ by Lemma 6.10; furthermore, $\sharp f^{-1}(v) \geqslant 2$. Notice that A is gentle, and thus there is no other arrow whose source or sink is v. Therefore, $\sharp f^{-1}(v) \leqslant 2$ and $\sharp f^{-1}(v) = 2$.

It is obvious that if there exists a unique oriented cycle $\mathsf{s} = \alpha_0\alpha_1\cdots\alpha_l$ with full relations of A such that $v \in Q_0^{\mathrm{ocf}}$ is a vertex of s, then $\sharp f^{-1}(v) = 1$. Indeed, suppose $t(\alpha_i) = v$ with $0 \leqslant i \leqslant l$. Then $f^{-1}(v) = \{\alpha_i A\} \subseteq \mathrm{ind}(\text{nproj-}\mathcal{GP}(A))$. □

On the marked ribbon surface $\mathcal{S}_A = (\mathcal{S}_A, \mathcal{M}_A, \Gamma_A)$ of a gentle algebra $A = kQ/I$, for each ∞-elementary polygon, its each side provides an indecomposable non-projective Gorenstein projective module by the permissible curve satisfying Proposition 6.3(GP2), and the arc which is a common side of two ∞-elementary polygons provides two indecomposable non-projective Gorenstein projective modules. From this observation, we can also prove Lemma 6.11. Moreover, we have the following proposition.

Proposition 6.12. *Let $A = kQ/I$ be a gentle algebra with the AG-invariant ϕ_A, and $L \subseteq \mathbb{N}^+$ be a finite set such that $\phi_A(0,\ell) > 0$ if $\ell \in L$, and $\phi_A(0,\ell) = 0$ otherwise. Then we have*

(1) $\sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A)) = \sum_{\ell\in L} \ell\cdot\phi_A(0,\ell)$, *i.e.,* $\sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A))$ *is the number of arrows on oriented cycles with full relations.*

(2) *Furthermore, if two gentle algebras A and A' are AG-equivalent, then* $\sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A)) = \sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A'))$.

Proof. Recall from the proof of Proposition 5.12 that for each $\ell \in L$, the number of oriented cycles with full relations of length ℓ is $\phi_A(0,\ell)$. Thus we can suppose that A has $t = \sum_{\ell\in L}\phi_A(0,\ell)$ oriented cycles with full relations $\mathsf{s}^i = \alpha_1^i\alpha_2^i\cdots\alpha_{\ell_i}^i$ with $1 \leqslant i \leqslant t$. Then there exist the following maps:

$$Q_1^{\mathrm{ocf}} \xrightarrow[\sim]{g} \mathrm{ind}(\text{nproj-}\mathcal{GP}(A)) \xrightarrow{f} Q_0^{\mathrm{ocf}},$$

where $Q_1^{\mathrm{ocf}} = \bigcup_{1\leqslant i\leqslant t}\{\alpha_j^i \mid 1 \leqslant j \leqslant \ell_i\}$ and f is surjective (see Lemma 6.11), and g defined by $\alpha \mapsto \alpha A$ is bijective (see Theorem 6.1). Thus, $\sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A'))$ is equal to the number $\sum_{\ell\in L}\ell\cdot\phi_A(0,\ell)$ of arrows on oriented cycles with full relations.

Let A' be a gentle algebra such that $\phi_{A'} = \phi_A$. Then

$$\sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A')) = \sum_{\ell\in L}\ell\cdot\phi_{A'}(0,\ell) = \sum_{\ell\in L}\ell\cdot\phi_A(0,\ell) = \sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A)).$$

The second equality is obtained by $\phi_{A'} = \phi_A$. □

Corollary 6.13 (See [10,20])**.** *Gentle algebras are CM-finite.*

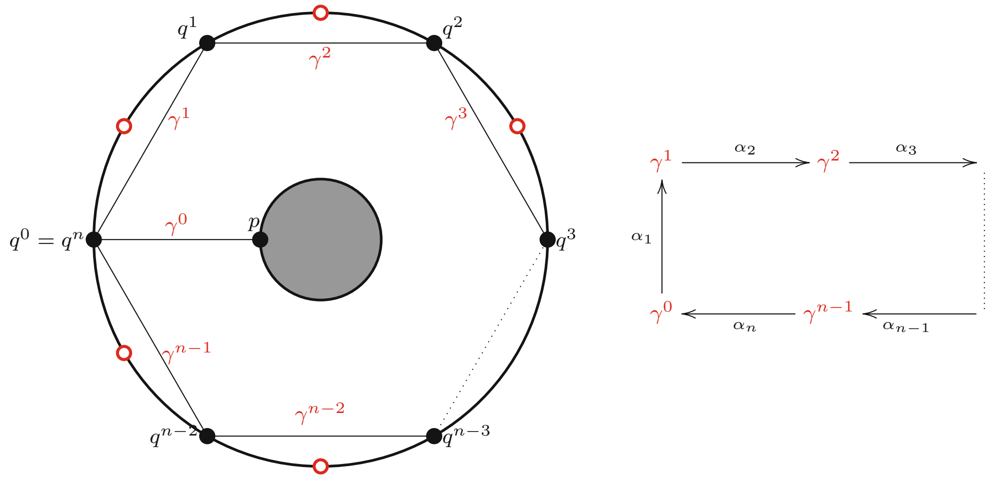
Proof. Let $A = kQ/I$ be a gentle algebra. Since (Q, I) is a finite quiver, the number of oriented cycles with full relations is finite, so there exists a finite set $L \subseteq \mathbb{N}^+$ such that $0 < \phi_A(0,\ell) < \infty$ for any $\ell \in L$, and $\phi_A(0,\ell) = 0$ for any $\ell \notin L$. By Proposition 6.12, we have $\sharp\mathrm{ind}(\text{nproj-}\mathcal{GP}(A)) = \sum_{\ell\in L}\ell\cdot\phi_A(0,\ell) < \infty$. □

7 Examples

7.1 Some examples for calculating the global dimension of gentle algebras

In this subsection, we provide some examples to calculate the global dimension of gentle algebras and the projective dimension of simple modules.

Example 7.1. For any $n \geqslant 1$, there exists a gentle one-cycle algebra A, i.e., its quiver has exactly one cycle, such that $\mathrm{gl.dim}A = n$. Indeed, if A is such a gentle one-cycle algebra whose marked ribbon surface $\mathcal{S}_A$ is shown in Figure 21(a), then $A \cong kQ/I$, where $Q_0 = \{\gamma^0, \gamma^1, \ldots, \gamma^{n-1}\}$, Q is shown in Figure 21(b), and $I = \langle\alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_{n-1}\alpha_n\rangle$.



(a) (b)

Figure 21 (Color online) (b) is the gentle algebra whose marked ribbon surface is shown in (a)

Thus gl.dim$A = n$ by Theorem 5.10. Let γ^0 be the arc $\gamma^0 : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$ such that $\gamma^0(0^+) = p$ and $\gamma^0(1^-) = q^0$, and γ^i be the arc $\gamma^i : (0,1) \to \mathcal{S}_A \backslash \partial \mathcal{S}_A$ such that $\gamma^i(0^+) = q^{i-1}$ and $\gamma^i(1^-) = q^i$ for any $1 \leqslant i \leqslant n-1$, and γ^n be the arc $\gamma^0(1-x)$ (note: $\gamma^n = \gamma^0(1-x) \simeq \gamma^0(x) = \gamma^0$). Then the arcs $\gamma^0, \gamma^1, \ldots, \gamma^{n-1}, \gamma^n$ satisfy the condition (PD), and the minimal projective resolution of the simple module $M(c^{\gamma^0}_{\mathrm{simp}})$ can be determined by Proposition 5.6 as shown in Figure 22, where $c_i \simeq c^{\gamma^i}_{\mathrm{proj}}$; c'_{i+1} is the permissible curve corresponding to the kernel of δ_i $(1 \leqslant i \leqslant n-1)$ and $c_n \simeq c'_n \simeq c^{\gamma^n}_{\mathrm{proj}}$.

Furthermore, by (5.1), we obtain

$$0 \longrightarrow M(c^{\gamma^n}_{\mathrm{proj}}) \xrightarrow{\delta_n} M(c^{\gamma^{n-1}}_{\mathrm{proj}}) \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} M(c^{\gamma^1}_{\mathrm{proj}}) \xrightarrow{\delta_1} M(c^{\gamma^0}_{\mathrm{proj}}) \xrightarrow{\delta_0} M(c^{\gamma^0}_{\mathrm{simp}}) \longrightarrow 0,$$

where $M(c^{\gamma^{n-1}}_{\mathrm{proj}}) \cong \begin{smallmatrix} \gamma^{n-1} \\ \gamma^0 \\ \gamma^1 \end{smallmatrix}$ and Ker$\delta \cong \begin{smallmatrix} \gamma^0 \\ \gamma^1 \end{smallmatrix} \cong M(c^{\gamma^n}_{\mathrm{proj}})$. Thus, proj.dim$M(c^{\gamma^0}_{\mathrm{simp}}) = n$. Similarly, proj.dim$M(c^{\gamma^i}_{\mathrm{simp}}) = n - i$ for any $0 \leqslant i \leqslant n-1$.

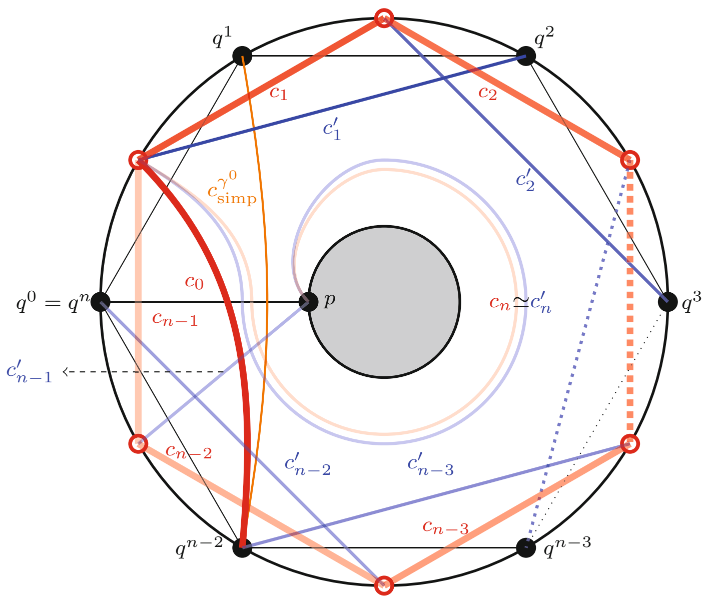


Figure 22 (Color online) The minimal projective resolution of $M(c^{\gamma^0}_{\mathrm{simp}})$

Example 7.2. See Figure 23(a), which is the marked ribbon surface of the gentle algebra $A = kQ/I$, where Q is shown in (b) and

$$I = \langle a_1a_3, a_5a_4, a_8a_7, a_7a_6, a_{10}a_2, a_9a'_{10}, a'_{10}a_{14}, a_{14}a_{13}\rangle.$$

The elementary polygon of the marked ribbon surface gives the projective dimension of each simple module. For example, for the arc $S(8)$, by Proposition 5.6, each term in the minimal projective resolution of $S(8)$ corresponds to an arc of the elementary polygon Δ_j whose arc set $\mathfrak{S}(\Delta_j) = \{6, 7, 8, 9\}$. Then the minimal projective resolution of $S(8)$ is

$$0 \longrightarrow M(c^6_{\text{proj}}) \longrightarrow M(c^7_{\text{proj}}) \longrightarrow M(c^8_{\text{proj}}) \longrightarrow S(8) = M(c^8_{\text{simp}}) \to 0,$$

and thus proj.dim$S(8) = 2$. Similarly, we get

$$\begin{array}{lllll} \text{proj.dim}S(1) = 2, & \text{proj.dim}S(2) = 0, & \text{proj.dim}S(4) = 0, & \text{proj.dim}S(5) = 1, & \text{proj.dim}S(6) = 2, \\ \text{proj.dim}S(7) = 1, & \text{proj.dim}S(11) = 1, & \text{proj.dim}S(12) = 1, & \text{proj.dim}S(13) = 1, & \text{proj.dim}S(14) = 2. \end{array}$$

The simple module $S(9) = M(c^9_{\text{simp}})$ corresponds to the arc 9 which satisfies the P-condition. Its minimal projective resolution is

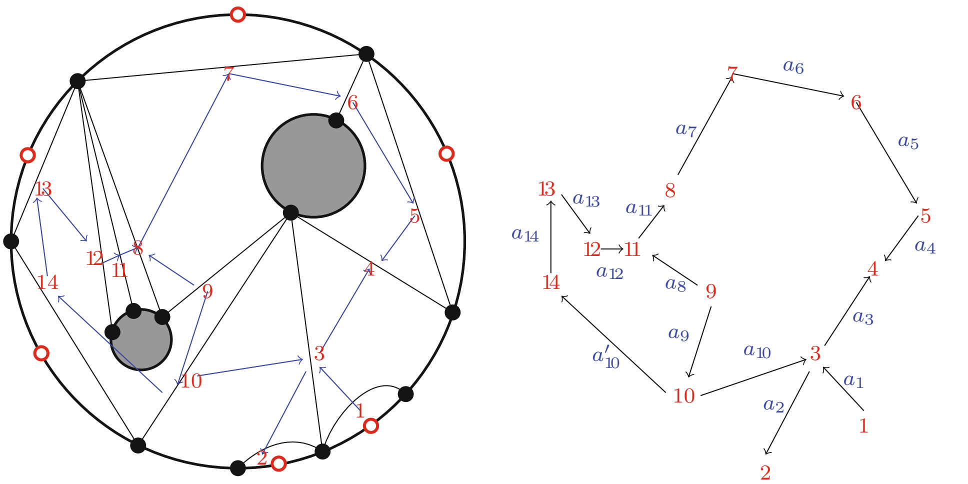
$$\begin{aligned} 0 &\longrightarrow M(c^{12}_{\text{proj}}) \longrightarrow M(c^6_{\text{proj}}) \oplus M(c^{14}_{\text{proj}}) \longrightarrow M(c^7_{\text{proj}}) \oplus M(c^{14}_{\text{proj}}) \\ &\longrightarrow M(c^8_{\text{proj}}) \oplus M(c^{10}_{\text{proj}}) \longrightarrow M(c^9_{\text{proj}}) \longrightarrow S(9) = M(c^9_{\text{simp}}) \longrightarrow 0, \end{aligned}$$

and thus proj.dim$S(9) = 4$. Similarly, we get proj.dim$S(3) = 1$ and proj.dim$S(10) = 3$. Consequently, we conclude that gl.dim$A = 4$. It should be pointed out that we can get gl.dim$A = 4$ by Theorem 5.10 directly.

Example 7.3. Let A be a gentle algebra whose marked ribbon surface $\mathcal{S}_A$ is given in Example 6.4. Then gl.dim$A = \infty$ by Theorem 6.9 because $\mathcal{S}_A$ has an ∞-elementary polygon with the arc set $\{1, 2, 3\}$. Indeed, the minimal projective resolution of $S(1)$ is

$$\cdots \to P(3) \to P(2) \to P(1) \to P(3) \to P(2) \to P(1) \to S(1) \to 0$$

by Proposition 5.6, so proj.dim$S(1) = \infty$.



(a) (b)

Figure 23 (Color online) (b) is the gentle algebra whose marked ribbon surface is shown in (a)

7.2 Some examples for calculating the self-injective dimension of gentle algebras

In this subsection, we provide some examples to calculate the projective (resp. injective) dimension of indecomposable injective (resp. projective) modules, and the self-injective dimension of gentle algebras.

Example 7.4. Consider the gentle algebra A given in Example 7.2. Then $\mathrm{gl.dim}A = 4$. Since

$$I(8) = {}^{\substack{13\\12\\11}}{}_{8}{}^{9} = M(c^{8}_{\mathrm{inj}})$$

is an indecomposable injective module with the top $M(c^{13}_{\mathrm{simp}}) \oplus M(c^{9}_{\mathrm{simp}}) \cong S(13) \oplus S(9)$, its projective cover is $M(c^{13}_{\mathrm{proj}} \oplus c^{9}_{\mathrm{proj}})$. Consider the elementary polygons Δ_9 with sides $\{9, 10, 14, 13, 12\}$ and Δ_{12} with the side $\{12\}$. We have $\mathfrak{C}(\Delta_9) = 5$ and $\mathfrak{C}(\Delta_{12}) = 1$, and thus $\mathrm{proj.dim}I(8) = \max\{5, 1\} - 1 = 4$. Furthermore, by the proof of Proposition 6.8, the minimal projective resolution of $I(8)$ is as follows:

$$\begin{aligned} 0 &\longrightarrow M(c^{12}_{\mathrm{proj}}) \longrightarrow M(c^{13}_{\mathrm{proj}}) \longrightarrow M(c^{14}_{\mathrm{proj}}) \longrightarrow M(c^{8}_{\mathrm{proj}}) \oplus M(c^{10}_{\mathrm{proj}}) \\ &\longrightarrow M(c^{13}_{\mathrm{proj}}) \oplus M(c^{9}_{\mathrm{proj}}) \longrightarrow I(8) \to 0. \end{aligned}$$

Thus, we have $\mathrm{inj.dim}_A A = \mathrm{inj.dim}A_A = \mathrm{proj.dim}D(A) = 4 = \mathrm{gl.dim}A$.

In Example 7.4, for the gentle algebra A, its global and self-injective dimensions are identical. Now we give two examples of gentle algebras with infinite global dimension.

Example 7.5. (1) Let A be the gentle algebra given in Example 6.4. Then $\mathrm{gl.dim}A = \infty$ by Example 7.3. Since $\max_{\Delta_j \in \mathsf{f}\text{-}\Delta} \mathfrak{C}(\Delta_j) = 2$, we have $\mathrm{inj.dim}_A A = \mathrm{inj.dim}A_A = \max_{\Delta_i \in \mathsf{f}\text{-}\Delta}\{1, \mathfrak{C}(\Delta_i) - 1\} = 1$ by Theorem 6.9.

(2) Let A be the gentle algebra whose marked ribbon surface is shown in the Figure 24(I). Then $\mathrm{gl.dim}A = \infty$ by Theorem 5.10 (because the elementary polygon with sides $\{3, 4, 5, 6\}$ and the elementary polygon with sides $\{5, 7, 8\}$ are ∞-elementary), and $\mathrm{inj.dim}_A A = \mathrm{inj.dim}A_A = 2$ by Theorem 6.9. Furthermore, since its AG-invariant $\phi_A = [(9, 4), (0, 4), (0, 3)]$, we have

$$\sharp\mathrm{ind}(\mathrm{nproj}\text{-}\mathcal{GP}(A)) = 4\phi_A(0, 4) + 3\phi_A(0, 3) = 4 \cdot 1 + 3 \cdot 1 = 7$$

by Proposition 6.12. Indeed, we have $\mathrm{ind}(\mathrm{nproj}\text{-}\mathcal{GP}(A)) = \{3,\ 4,\ {\substack{5\\7\\9}},\ {\substack{7\\9}},\ 8,\ {\substack{5\\6}},\ 6\}$ by Proposition 6.3.

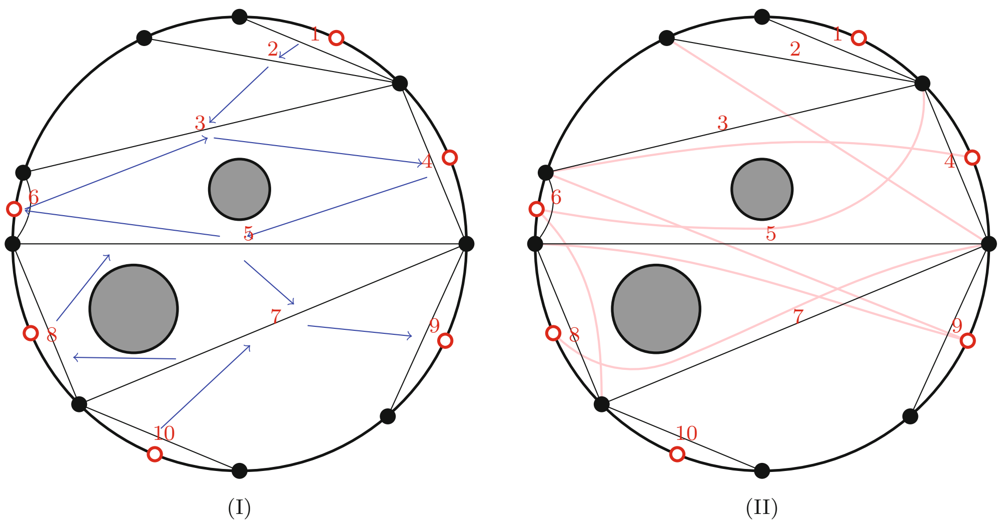


Figure 24 (Color online) The marked ribbon surface of gentle algebra given in Example 7.5 (I) and all the permissible curves corresponding to indecomposable non-projective Gorenstein projective modules (II)

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