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Normed modules and the categorification of integrations, series expansions, and differentiations

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Abstract We explore the assignment of norms to Λ-modules over a finite-dimensional algebra Λ, resulting in the establishment of normed Λ-modules. Our primary contribution lies in constructing two new categories $\mathscr{N}or^p$ and $\mathscr{A}^p$, where each object in $\mathscr{N}or^p$ is a normed Λ-module N limited by a special element $v_N \in N$ and a special Λ-homomorphism $\delta_N : N^{\oplus 2^{\dim \Lambda}} \to N$, the morphism in $\mathscr{N}or^p$ is a Λ-homomorphism $\theta : N \to M$ such that $\theta(v_N) = v_M$ and $\theta\delta_N = \delta_M\theta^{\oplus 2^{\dim \Lambda}}$, and $\mathscr{A}^p$ is a full subcategory of $\mathscr{N}or^p$ generated by all Banach modules. By examining the objects and morphisms in these categories, we establish a framework for understanding the categorification of integration, series expansions, and derivatives. Furthermore, we obtain the Stone-Weierstrass approximation theorem in the sense of $\mathscr{A}^p$.

Keywords categorification, finite-dimensional algebras, Lebesgue integration, normed modules, Banach spaces

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1 Introduction

Mathematical analysis, encompassing branches such as integrations, differentiations, and series expansions, is an integral component of mathematics and serves as an indispensable tool in various scientific domains including physics, engineering, and life sciences. Traditionally founded on the ϵ–δ definition of limits and the theory of completeness of the real numbers, mathematical analysis provides a rich and diverse array of research topics within its sub-disciplines. However, adaptation to different applications often obscures a unified understanding of its branches and their interconnections. For example, Lebesgue integration, introduced by Lebesgue in 1902 (see [28]), represents a critical advancement in mathematical analysis. Understanding Lebesgue's approach to integrability on the real line involves methodical and incremental steps beginning with the definition of measurable sets and null sets, followed by exploring

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measure convergence. The journey continues through the exploration of step functions and simple functions, progressing to sequences of their convergence and culminating in the sophisticated construction of spaces for integrable functions and consistent integration methods. This path, while comprehensive, paves a detailed route to fully appreciate the depth of Lebesgue integration, as discussed in foundational texts such as [9, 23]. However, the intricate methodologies developed do not directly translate to other branches of analysis, making it challenging to apply these achievements uniformly across the field.

Category theory has evolved far beyond its original scope, now permeating nearly all branches of mathematics. Initially formulated by Eilenberg and Mac Lane [13] in the mid-20th century within the realm of algebraic topology, a category fundamentally consists of objects and morphisms. This framework facilitates a systematic and structural approach to analyzing a wide range of mathematical entities, from algebraic structures to complex topological spaces. The true utility of category theory lies in its ability to abstractly model and examine mathematical concepts through functors and natural transformations. Functors are the "morphisms between categories", systematically relating the objects and morphisms of one category to those of another, thereby uncovering deep interconnections within mathematical frameworks. Natural transformations extend this by mapping between functors themselves, ensuring consistency across categorical representations. This level of abstraction proves invaluable in various mathematical applications, including the categorical descriptions of integration [10, 11, 22, 38] and differentiation [1, 7, 8, 20, 24, 30, 31], the categorical semantics of differential linear logic [6, 7], the Taylor series within Cartesian differential categories [32], preliminary categorifications of automorphic forms and the analytic continuation of L-functions [27], as well as providing cohesive frameworks for tackling complex problems such as quotient spaces, direct products, completions, and duality. Furthermore, recent research has begun to explore the synergy between category theory and mathematical analysis in the context of artificial intelligence. These advancements leverage categorical structures to enhance machine learning models and develop more abstract frameworks for AI algorithms [12]. Additionally, categorical semantics are being applied to better understand and design AI systems, providing a robust mathematical foundation for their development and analysis [39]. Through category theory, mathematicians gain a powerful tool for unifying and elucidating the intricacies of diverse mathematical concepts. Building on the foundational work of Leinster [29], we describe integration, series expansions, and differentiation using the unified category $\mathscr{A}^p$. Note that the Rota-Baxter algebra [5, 33, 34] provides another algebraic description of integration, but it is different from the categorification of integration given by the category $\mathscr{A}^p$.

As the landscape of integration theory expands, so too does the exploration into its algebraic facets, marking a significant evolution in the approach to integration. Algebraic approaches to integration can be traced back at least to Segal's work [38]. Building upon the foundational works of Escardó and Simpson [14], Freyd [16] and Leinster [29] constructed a special category $\mathscr{A}^p$, where p is a real number at least 1. In this category, objects are triples consisting of a Banach space V, an element v in V with $|v| \leqslant 1$, and a $\Bbbk$-linear map $\delta : V \oplus_p V \to V$ that satisfies $\delta(v, v) = v$. Here, the notation $V_1 \oplus_p V_2$ represents the direct sum of two normed spaces V_1 and V_2, where the norm is defined as

$$|(v_1, v_2)| = \left(\frac{1}{2}(|v_1|^p + |v_2|^p)\right)^{1/p}.$$

Furthermore, Leinster established three significant results as follows:

(1) $(L_p([0,1]), 1, \gamma)$ is the initial object in $\mathscr{A}^p$, where

$$\gamma : L_p([0,1]) \oplus_p L_p([0,1]) \to L_p([0,1])$$

is a special $\Bbbk$-linear map (indeed, γ is the map $\gamma_{\frac{1}{2}}$ given in Corollary 10.2);

(2) $(\mathbb{R}, 1, m)$ is an object in $\mathscr{A}^1$, where $m : \mathbb{R} \oplus_1 \mathbb{R} \to \mathbb{R}$ sends (x, y) to $\frac{1}{2}(x + y)$;

(3) there exists a unique morphism

$$H : (L_1([0,1]), 1, \gamma) \to (\mathbb{R}, 1, m) \quad \text{in } \mathscr{A}^1$$

(see [29, Theorem 2.1 and Proposition 2.2]). The homomorphism $H : L_1([0,1]) \to \Bbbk$ is a $\Bbbk$-linear map sending any function f in $L_1([0,1])$ to its Lebesgue integral, i.e.,

$$H(f) = (\mathrm{L}) \int_0^1 f d\mu_{\mathrm{L}},$$

where μ_{L} denotes the Lebesgue measure on $\mathbb{R}$. This profound relationship illustrates that Lebesgue integrability and integration are not merely abstract constructs; rather, they naturally emerge from the foundational principles of Banach spaces. Consequently, it can be logically inferred that the categorification of Lebesgue integration is inherently connected to, and can be derived from, the categorification of Banach spaces. However, we have discovered that Leinster's work can be extended to a more general setting of finite-dimensional algebras, and it encompasses not only definite and indefinite integrals, but also includes key areas of mathematical analysis such as weak derivatives, series expansions, and the Stone-Weierstrass approximation theorem.

Building upon Leinster's foundational work, we extend his categorical framework to encompass finite-dimensional algebras, thereby creating a more versatile and unified approach to integration theory. By incorporating normed modules over these algebras into our analysis, we bridge the gap between algebraic structures and analytical methods. This extension allows us to reinterpret classical concepts of integration, differentiation, and series expansions within a broader categorical context. Furthermore, our approach facilitates the seamless integration of algebraic techniques with analytical processes, offering a cohesive framework that enhances the applicability and depth of mathematical analysis. This novel categorical perspective not only unifies disparate areas of analysis but also opens new avenues for research and application in related scientific fields.

This study aims to explore and construct a comprehensive theoretical framework specifically tailored for normed modules in finite-dimensional algebras. We introduce and dissect a novel category, denoted by $\mathscr{N}\!or^p$, alongside its fully characterized subcategory, $\mathscr{A}^p$. This research endeavors to systematically categorize normed modules and their operations, aiming to enhance our understanding of fundamental mathematical procedures such as integration, series expansions, and differentiation. The specific research questions addressed are as follows.

Question 1.1. (1) How does the new categorical framework improve our comprehension of norm structures within various normed modules over an algebra?

(2) What contributions do morphisms in the subcategory $\mathscr{A}^p$ make towards advancing classical integration techniques?

(3) What implications does the categorification of normed modules hold for the broader mathematical analysis landscape and its practical applications?

The investigation of these questions not only broadens the scope of category theory in mathematical analysis and abstract algebra but also introduces novel theoretical tools and perspectives, potentially benefiting other disciplines such as physics and automation engineering. To comprehensively address the aforementioned questions, we delineate the following key topics in subsequent sections.

Firstly, we introduce functions defined on a finite-dimensional algebra Λ, along with the norm defined on Λ and any Λ-module M. It is pertinent to note that all Λ-modules considered in this paper are left Λ-modules. The specifics of these structures are elaborated in Subsections 3.1 and 4.1, respectively. A pivotal motivation for us to introduce normed modules is the pursuit of an integration definition that transcends the conventional reliance on L_p spaces. This approach is rooted in the understanding that an equivalent definition of L_p spaces can emerge through the integration itself. However, as highlighted by Leinster, the notion of Lebesgue integrals is intrinsically linked to Banach spaces. Consequently, our investigation also necessitates considering the completions of normed finite-dimensional algebras and normed modules (see Subsections 3.2 and 4.2).

Secondly, for a special subset $\mathbb{I}_\Lambda$ of Λ, denoted by $\mathbb{I}_\Lambda \subseteq \Lambda$, we construct the category $\mathscr{N}\!or^p$ in Subsection 5.1. Its object has the form (N, v, δ), where N is a normed Λ-module, v is an element in N satisfying $|v| \leqslant \mu(\mathbb{I}_\Lambda)$, μ is an arbitrary measure defined on $\mathbb{I}_\Lambda$ and $\delta : N^{\oplus_p 2^n} \to N$ is a Λ-homomorphism sending $(v, \dots, v)$ to v. The morphism $h : (N, v, \delta) \to (N', v', \delta')$ is induced by a special

Λ-homomorphism $N \to N'$ satisfying $h\delta = \delta'(h^{\oplus_p 2^n})$. Furthermore, we consider the full subcategory $\mathscr{A}^p$ of $\mathscr{N}or^p$, where each object (N, v, δ) consists of a Banach Λ-module N, an element $v \in N$ and a Λ-homomorphism $\delta : N^{\oplus_p 2^n} \to N$.

Thirdly, we investigate the set $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ of elementary simple functions (a special step function defined on Λ), where τ is a homomorphism between two $\Bbbk$-algebras. We demonstrate its structure as a Λ-module (Lemma 4.9). Consequently, we obtain an object $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ (Lemma 5.5) in $\mathscr{N}or^p$ and an object $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ in $\mathscr{A}^p$, where $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ is the completion of $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ and $\widehat{\gamma}_\xi$ is induced by γ_ξ.

Fourthly, we prove our main result in Section 6 to answer Question 1.1(1), which provides a unique homomorphism from the initial object in $\mathscr{A}^p$ to any normed module to describe the properties of normed representations of algebra.

Theorem 1.2 (Theorem 6.4 and Remark 6.5)**.** *The triple $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$, i.e., for any object (N, v, δ) in $\mathscr{A}^p$, there exists a unique morphism*

$$h \in \mathrm{Hom}_{\mathscr{N}or^p}((\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi), (N, v, \delta))$$

such that the diagram

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \arrow[r, "h"] \arrow[d, "\subseteq"'] & (N, v, \delta) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) \arrow[ur, "\widehat{h}"'] &
\end{tikzcd}$$

commutes, where $\widehat{h}$ is given by the completion of $h : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to N$.

Sections 7–9 realize integrations, series expansions and derivatives as three morphisms in $\mathscr{A}^1$.

In Section 7, we construct an object $(\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ in $\mathscr{A}^p$, where $m : \Bbbk^{\oplus_p 2^n} \to \Bbbk$ is a Λ-homomorphism whose definition is given in this section. Taking $(N, v, \delta) = (\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ as in Theorem 1.2, we obtain the following result to answer Question 1.1(2), which describes numerous integrations by using category $\mathscr{A}^p$ in a unified way since μ is an arbitrary measure.

Theorem 1.3 (Theorem 7.6)**.** *If $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ is an extension of $\mathbb{R}$, then there exists a unique Λ-homomorphism $T : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to \Bbbk$ such that*

$$T : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (\Bbbk, \mu(\mathbb{I}_\Lambda), m)$$

is a morphism in $\mathrm{Hom}_{\mathscr{N}or^p}((\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi), (\Bbbk, \mu(\mathbb{I}_\Lambda), m))$ and the diagram

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \arrow[r, "T"] \arrow[d, "\subseteq"'] & (\Bbbk, \mu(\mathbb{I}_\Lambda), m) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) \arrow[ur, "\widehat{T}"'] &
\end{tikzcd}$$

commutes, where $\widehat{T}$ is the unique morphism lying in $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (\Bbbk, \mu(\mathbb{I}_\Lambda), m))$. Furthermore, if $p = 1$, then we have the following three properties of $\widehat{T}$ by the direct limits

$$\varinjlim T_i : \widehat{T} = \varinjlim E_i \to \Bbbk$$

(*the definitions of E_i and T_i are given in Notation* 5.3 *and Section* 7*, respectively*):

(1) (Formula (7.1)) $\widehat{T}(\mathbf{1}) = \mu(\mathbb{I}_\Lambda)$.

(2) (Lemma 7.1) $\widehat{T} : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to \Bbbk$ *is a homomorphism of Λ-modules.*

(3) (Proposition 7.5) $|\widehat{T}(f)| \leqslant \widehat{T}(|f|)$.

The morphism $\widehat{T}$ provides a categorification of integration, and we define

$$\widehat{T}(f) =: (\mathscr{A}^1) \int_{\mathbb{I}_\Lambda} f d\mu. \tag{1.1}$$

The above (1)–(3) show that

$$(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda}\mathbf{1}d\mu=\mu(\mathbb{I}_\Lambda),$$

$$(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda}(\lambda_1f_1+\lambda_2f_2)d\mu=\lambda_1\cdot(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda}f_1d\mu+\lambda_2\cdot(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda}f_2d\mu,\quad\lambda_1,\lambda_2\in\Lambda,\tag{1.2}$$

and

$$\left|(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda}fd\mu\right|\leqslant(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda}|f|d\mu,$$

respectively.

Let $\Bbbk[X_1,\ldots,X_N]$ ($=\Bbbk[\boldsymbol{X}]$ for short) be the N variables polynomial ring over a field $\Bbbk$ with $N\geqslant\dim_{\Bbbk}\Lambda=n$. Then $\Bbbk[\boldsymbol{X}]$ can be seen as a normed left Λ-module, where the norm $\|\cdot\|_{\Bbbk[\boldsymbol{X}]}$ is either (8.1) or (8.3). In Section 8, we get two corollaries as follows to answer Question 1.1(3).

Corollary 1.4. *Let $\mathscr{A}^p$ satisfy $p=1$.*

(1) (Corollary 8.2/Weierstrass approximation theorem) *If $N=n$ and $\|\cdot\|_{\Bbbk[\boldsymbol{X}]}$ is defined by* (8.1)*, then the unique morphism in*

$$\widehat{E_{\mathrm{pow}}}\in\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\gamma_\xi),(\widehat{\Bbbk[\boldsymbol{X}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\Bbbk[\boldsymbol{X}]}}))$$

shows that for any function $f\in\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, there exists a sequence $\{P_i\}_{i\in\mathbb{N}}$ of polynomials such that

$$\widehat{E_{\mathrm{pow}}}(f)=\varinjlim P_i.$$

(2) (Corollary 8.5) *If $\Bbbk=\mathbb{C}$, $N=2n$ and $\|\cdot\|_{\Bbbk[\boldsymbol{X}]}$ is defined by* (8.3)*, then the unique morphism in*

$$\widehat{E_{\mathrm{Fou}}}\in\mathrm{Hom}_{\mathscr{A}^1}((\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi),(\widehat{\mathbb{C}[\mathrm{e}^{\pm2\pi\mathrm{i}\boldsymbol{X}}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm2\pi\mathrm{i}\boldsymbol{X}}]}}))$$

shows that for any function $f\in\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, there exists a sequence $\{P_i\}_{i\in\mathbb{N}}$ of triangulated polynomials such that

$$\widehat{E_{\mathrm{Fou}}}(f)=\varinjlim P_i.$$

Furthermore, we show the Stone-Weierstrass approximation theorem in Subsection 8.3 (see Corollary 8.8).

Corollary 1.5 (Corollary 8.8, Stone-Weierstrass approximation theorem)**.** *There exists a unique morphism*

$$E_{\mathrm{S-W}}:(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi)\to(\mathbf{W},\mathbf{1},\widehat{\gamma_{\xi^\dagger}})$$

in $\mathrm{Hom}_{\mathscr{N}\!or^1}((\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\widehat{\gamma}_\xi),(\mathbf{W},\mathbf{1},\widehat{\gamma_{\xi^\dagger}}))$ such that the diagram

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi) \arrow[r, "E_{\mathrm{S-W}}"] \arrow[d, "\subseteq"'] & (\mathbf{W},\mathbf{1},\widehat{\gamma_{\xi^\dagger}}) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi) \arrow[ur, "\widehat{E_{\mathrm{S-W}}}"'] &
\end{tikzcd}$$

commutes, where the definition of $\mathbf{W}$ is a direct limit defined in Subsection 8.3; *$\widehat{E_{\mathrm{S-W}}}$ is the unique extension of $E_{\mathrm{S-W}}$ lying in $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi),\ (\mathbf{W},\mathbf{1},\widehat{\gamma_{\xi^\dagger}}))$.*

In Section 9, we recall some works of Leinster and Meckes (see [29]). Based on their work, we show the following theorem.

Theorem 1.6. *Let $p=1$, $\Lambda=\mathbb{R}=\Bbbk$, $\tau=\mathrm{id}:\mathbb{R}\to\mathbb{R},x\mapsto x$, $\mathbb{I}_\Lambda=[0,1]$ and $\xi=\frac{1}{2}$; for simplification, we write $\widehat{\mathbf{S}}:=\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$.*

(1) (Theorem 9.3) (i) *A morphism in $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}},\mathbf{1},\widehat{\gamma}_{\frac{1}{2}}),(N,v,\delta))$ is zero if and only if $v=0$.*

(ii) *Furthermore, there is no morphism in* $\mathscr{A}^1$ *starting with* $(\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}})$ *such that this morphism sends any almost everywhere differentiable function* $f(x)$ *to its weak derivative* $\frac{df}{dx}$.

(2) (Theorem 9.5) *The differentiation* D *is a morphism in* $\mathscr{A}^p$ *ending with the initial object* $(\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}})$.

Recall that any function $f : \mathbb{I}_\Lambda = [0,1] \to \mathbb{R}$ in $\mathbf{S}$ is a *step function*, i.e., there is a dissection $[0,1] = \bigcup_{i=1}^t \mathbb{I}_i$ of $[0,1]$ with $\mathbb{I}_i \cap \mathbb{I}_j = \emptyset$ for any $1 \leqslant i \neq j \leqslant t$, such that each $f|_{\mathbb{I}_i}(x)$ is a constant in $\mathbb{R}$. Then $\frac{df}{dx}$ almost everywhere equals 0. It follows that the completion D of $\frac{d}{dx}$ sends every function in $\widehat{\mathbf{S}}$ to zero. Furthermore, if D is a morphism in $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}), (N, v, \delta))$, then we have $v = D(\mathbf{1}) = 0$, and it follows that $D = 0$ by Theorem 1.6(1)(i), which is a contradiction. Therefore, Theorem 1.6(1)(i) shows that differentiation, i.e., the homomorphism D, is not a morphism in $\mathscr{A}^1$ with the domain the initial object of $\mathscr{A}^1$. Then we obtain Theorem 1.6(1)(ii). Naturally, we would ask whether D can be characterized by $\mathscr{A}^p$? To do this, we prove Theorem 1.6(2) in Subsection 9.3, and show that D is a morphism in a category $\mathscr{A}^p$.

Finally, we provide some applications for our main results in Section 10. In Subsection 10.1, we assume $\Bbbk = \mathbb{R}$, $(\Lambda, \prec, \|\cdot\|_\Lambda) = (\mathbb{R}, \leqslant, |\cdot|)$, $B_\mathbb{R} = \{1\}$, $\mathfrak{n} : B_\mathbb{R} \to \{1\} \subseteq \mathbb{R}^{\geqslant 0}$, $\mathbb{I}_\mathbb{R} = [0,1]$, $\xi = \frac{1}{2}$, $\kappa_0(x) = \frac{x}{2}$, $\kappa_1(x) = \frac{x+1}{2}$ and $\tau = \mathrm{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$, and let μ_L be the Lebesgue measure. Then (1.1) is a Lebesgue integration

$$(\mathscr{A}^1) \int_{\mathbb{I}_\mathbb{R}=[0,1]} f d\mu_\mathrm{L} = (\mathrm{L}) \int_0^1 f d\mu_\mathrm{L},$$

and (1.2) shows that Lebesgue integration is $\mathbb{R}$-linear. This result provides a categorification of Lebesgue integration. In Subsection 10.2, we provide two examples for Corollary 1.4 to show that the Taylor series and Fourier series can be realized as two morphisms in $\mathscr{A}^1$ with the domain the initial object.

2 Preliminaries

In this section, we recall some concepts in the category theory and representation theory of algebras, including limits in the category theory (see [35, Chapter 5] and [26, Chapter III, pp. 62–74]), $\Bbbk$-algebras (see [2, Chapter I]), and some methods to establish topologies on algebras (see [3, Chapter 10]). These concepts are familiar to algebraists but may not be as familiar to those in the field of analysis.

2.1 Categories and limits

Recall that a *category* $\mathcal{C}$ consists of three ingredients: a class of *objects*, a set $\mathrm{Hom}_\mathcal{C}(X,Y)$ of *morphisms* for any objects X and Y in $\mathcal{C}$, and the *composition* $\mathrm{Hom}_\mathcal{C}(X,Y) \times \mathrm{Hom}_\mathcal{C}(Y,Z) \to \mathrm{Hom}_\mathcal{C}(X,Z)$, denoted by

$$(f : X \to Y, g : Y \to Z) \mapsto gf : X \to Z,$$

for any objects X, Y and Z in $\mathcal{C}$. These ingredients are subject to the following axioms:

(1) The Hom sets are pairwise disjoint.

(2) For any object X, the *identity morphism* $1_X : X \to X$ in $\mathrm{Hom}_\mathcal{C}(X,X)$ exists.

(3) The composition is associative: given morphisms

$$U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} X,$$

we have

$$h(gf) = (hg)f.$$

Next, we review the limits in the category theory.

Definition 2.1 (See [35, Chapter 5, Subsection 5.2]). Let $\mathfrak{I} = (\mathfrak{I}, \preceq)$ be a partially ordered set, and let $\mathcal{C}$ be a category. A *direct system* in $\mathcal{C}$ over $\mathfrak{I}$ is an ordered pair $((M_i)_{i\in\mathfrak{I}}, (\varphi_{ij})_{i\preceq j})$, where $(M_i)_{i\in\mathfrak{I}}$ is an indexed family of objects in $\mathcal{C}$ and $(\varphi_{ij} : M_i \to M_j)_{i \preceq j}$ is an indexed family of morphisms for which $\varphi_{ii} = 1_{M_i}$ for all i, such that the diagram

$$\begin{tikzcd}
M_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & M_k \\
& M_j \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

commutes whenever $i \preceq j \preceq k$. Furthermore, for the above direct system $((M_i)_{i\in\mathfrak{I}}, (\varphi_{ij})_{i\preceq j})$, the *direct limit* (also called *inductive limit* or *colimit*) is an object, i.e., $\varinjlim M_i$, and *insertion morphisms* $(\alpha_i : M_i \to \varinjlim M_i)_{i\in\mathfrak{I}}$ such that

(1) $\alpha_j \varphi_{ij} = \alpha_i$ whenever $i \preceq j$;

(2) for any object X in $\mathcal{C}$ such that there are given morphisms $f_i : M_i \to X$ satisfying $f_j \varphi_{ij} = f_i$ for all $i \preceq j$, there exists a unique morphism $\theta : \varinjlim M_i \to X$ making the diagram

$$\begin{tikzcd}
\varinjlim M_i \arrow[rr, dashed, "\theta", "(\exists!)"'] & & X \\
& M_i \arrow[ul, "\alpha_i"] \arrow[ur, "f_i"'] \arrow[d, "\varphi_{ij}", "(i\preceq j)"'] & \\
& M_j \arrow[uul, bend left, "\alpha_j"] \arrow[uur, bend right, "f_j"'] &
\end{tikzcd}$$

commutes.

Example 2.2. Let $\{x_n\}_{n\in\mathbb{N}^+}$ be a monotonically increasing sequence of real numbers, and let $\mathbb{R}$ be the partially ordered category $(\mathbb{R}, \leqslant)$, in which the elements are real numbers and the morphisms are of the form $\leqslant_{r_2 r_1} : r_1 \to r_2$ $(r_2 \leqslant r_1)$. If $\{x_n\}_{n\in\mathbb{N}^+}$ has limit x as in analysis, i.e., for any $\epsilon > 0$, there exists $N \in \mathbb{N}^+$ such that $|x_n - x| < \epsilon$ holds for all $n > N$, then $x = \varinjlim x_n$. Indeed, for any $x' \in \mathbb{R}$ such that the morphisms $(\alpha_i = \leqslant_{x_i x'} : x_i \to x')_{i\in\mathbb{N}^+}$ exist, there is a morphism $\theta = \leqslant_{xx'} : x \to x'$ such that the diagram

$$\begin{tikzcd}
x \arrow[rr, dashed, "\theta = \leqslant_{xx'}"] & & x' \\
& x_i \arrow[ul, "\leqslant_{x_i x}"] \arrow[ur, "\leqslant_{x_i x'}"'] \arrow[d, "\leqslant_{x_i x_j}"] & \\
& x_j \arrow[uul, bend left, "\leqslant_{x_j x}"] \arrow[uur, bend right, "\leqslant_{x_j x'}"'] &
\end{tikzcd}$$

commutes. It is clear that the morphism θ is unique in this example. Furthermore, $x \leqslant x'$ holds because if $x' < x$, then we can find some x_t such that $x_t > x'$, i.e., $\alpha_t \in \mathrm{Hom}_{(\mathbb{R},\leqslant)}(x', x_t) = \emptyset$, which is a contradiction.

Definition 2.3 (See [35, Chapter 5, Subsection 5.2])**.** Let $\mathfrak{I} = (\mathfrak{I}, \preceq)$ be a partially ordered set, and let $\mathcal{C}$ be a category. An *inverse system* in $\mathcal{C}$ over $\mathfrak{I}$ is an ordered pair $((M_i)_{i\in\mathfrak{I}}, (\psi_{ij})_{j\preceq i})$, where $(M_i)_{i\in\mathfrak{I}}$ is an indexed family of objects in $\mathcal{C}$ and $(\psi_{ij} : M_j \to M_i)_{j\preceq i}$ is an indexed family of morphisms for which $\psi_{ii} = 1_{M_i}$ for all i, such that the diagram

$$\begin{tikzcd}
M_i & & M_k \arrow[ll, "\psi_{ik}"'] \arrow[dl, "\psi_{jk}"] \\
& M_j \arrow[ul, "\psi_{ij}"] &
\end{tikzcd}$$

commutes whenever $i \preceq j \preceq k$. Furthermore, for the above direct system $((M_i)_{i\in\mathfrak{I}}, (\psi_{ij})_{j\preceq i})$, the *inverse limit* (also called *projective limit* or *limit*) is an object, say $\varprojlim M_i$, and *projects morphisms* $(\alpha_i : \varprojlim M_i \to M_i)_{i\in\mathfrak{I}}$ such that

(1) $\psi_{ji}\alpha_j = \alpha_i$ whenever $i \preceq j$;

(2) for any object X in $\mathcal{C}$ such that there are given morphisms $f_i : X \to M_i$ satisfying $\psi_{ji} f_j = f_i$ for all $i \preceq j$, there exists a unique morphism $\vartheta : X \to \varinjlim M_i$ making the diagram

$$\begin{tikzcd}
\varprojlim M_i \arrow[dr, "\alpha_i"] \arrow[ddr, bend right, "\alpha_j"'] & & X \arrow[ll, dashed, "\vartheta", "(\exists!)"'] \arrow[dl, "f_i"'] \arrow[ddl, bend left, "f_j"] \\
& M_i & \\
& M_j \arrow[u, "\psi_{ij}"', "(i\preceq j)"] &
\end{tikzcd}$$

commutes.

Example 2.4. Let $\{x_n\}_{n\in\mathbb{N}^+}$ be a monotonically decreasing sequence of real numbers, and let $\mathbb{R}$ be the partially ordered category $(\mathbb{R}, \leqslant)$. If $\{x_n\}_{n\in\mathbb{N}^+}$ has limit x as in analysis, then we have $x = \varprojlim x_n$ by a way similar to that in Definition 2.3.

2.2 $\Bbbk$-algebras and their completions

Let $\Bbbk$ be a field. In this subsection, we recall the definitions of $\Bbbk$-algebras and the completions of $\Bbbk$-algebras. All concepts in this subsection are parallel to those in [3, Chapter 10, Subsection 10.1] which extracts some important results about the completions of Abelian groups.

2.2.1 $\Bbbk$-*algebras*

Definition 2.5. A $\Bbbk$-*algebra* A defined over $\Bbbk$ is both a ring and a $\Bbbk$-vector space such that

$$k(aa') = (ka)a' = a(ka').$$

In particular,

(1) if A is a commutative ring, i.e., $a_1a_2 = a_2a_1$ holds for all $a_1, a_2 \in A$, then we call that A is *commutative*; otherwise, we call that it is *non-commutative*;

(2) if the $\Bbbk$-dimension $\dim_{\Bbbk} A$ of A, i.e., the dimension of A as a $\Bbbk$-vector space, is finite, then we call that A is a *finite-dimensional* $\Bbbk$-*algebra*; otherwise, we call that it is *an infinite-dimensional* $\Bbbk$-*algebra*.

In this paper, we do not require the commutativity of $\Bbbk$-algebras, but we always suppose that every $\Bbbk$-algebra in our paper is a finite-dimensional $\Bbbk$-algebra with identity 1.

Recall that an *idempotent* of a $\Bbbk$-algebra A is an element e in A such that $e^2 = e$. Obviously, 0 and 1 are idempotents. If an idempotent e has a decomposition

$$e = e' + e''$$

such that

(1) e' and e'' are non-zero idempotents;

(2) e' and e'' are *orthogonal*, i.e., $e'e'' = 0 = e''e'$,

then we call e *decomposable*. We call e a *primitive idempotent* if it is not decomposable. Furthermore, one can prove that 1 has a decomposition

$$1 = e_1 + e_2 + \cdots + e_t$$

such that all e_i are primitive idempotents and $e_ie_j = 0$ holds for all $i \neq j$, and we call $\{e_1, \ldots, e_t\}$ a *complete set of primitive orthogonal idempotents* (see [2, Chapter I, p. 18]).

Let $e_1, \ldots, e_t$ be the complete set of primitive orthogonal idempotents. Then A has a decomposition $A = \bigoplus_{i=1}^{t} Ae_i$, where each direct summand Ae_i is an indecomposable left A-module. We say that A is *basic* if $Ae_i \ncong Ae_j$ for all $1 \leqslant i \neq j \leqslant t$.

Example 2.6. The set $\mathbf{M}_n(\Bbbk)$ of all $n \times n$ matrices over $\Bbbk$, the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, and the field $\Bbbk$ itself are $\Bbbk$-algebras. In particular, $\mathbf{M}_n(\Bbbk)$ and $\Bbbk$ are finite-dimensional, and $\Bbbk[x_1, \ldots, x_n]$ is infinite-dimensional.

Recall that a quiver is a quadruple $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$, where $\mathcal{Q}_0$ is the set of vertices, $\mathcal{Q}_1$ is the set of arrows, and $\mathfrak{s}, \mathfrak{t} : \mathcal{Q}_1 \to \mathcal{Q}_0$ are functions respectively sending each arrow to its starting point and ending point. Then any vertex $v \in \mathcal{Q}_0$ can be seen as a path on $\mathcal{Q}$ whose length is zero, and any arrow $\alpha \in \mathcal{Q}_1$ can be seen as a path on $\mathcal{Q}$ whose length is one. A path $\wp$ of length l, denoted by $\ell(\wp)$, is the composition $\alpha_l \cdots \alpha_2\alpha_1$ of arrows $\alpha_1, \ldots, \alpha_l$, where $\mathfrak{t}(\alpha_i) = \mathfrak{s}(\alpha_{i+1})$ for all $1 \leqslant i < l$. Then, naturally, we define the composition of two paths $\wp_1 = \alpha_l \cdots \alpha_1$ and $\wp_2 = \beta_\ell \cdots \beta_1$ as

$$\wp_2\wp_1 = \beta_\ell \cdots \beta_1\alpha_l \cdots \alpha_1$$

provided that the ending point $\mathfrak{t}(\wp_1)$ of $\wp_1$ coincides with the starting point $\mathfrak{s}(\wp_2)$ of $\wp_2$; otherwise (i.e., $\mathfrak{t}(\wp_1) \neq \mathfrak{s}(\wp_2)$), the composition is defined to be zero. Consequently, let $\mathcal{Q}_l$ be the set of all paths of length l. Then $\Bbbk\mathcal{Q} := \mathrm{span}_{\Bbbk}(\bigcup_{l \geqslant 0} \mathcal{Q}_l)$, known as the *path algebra* of $\mathcal{Q}$, is a $\Bbbk$-algebra whose multiplication is defined as follows:

$$\Bbbk\mathcal{Q} \times \Bbbk\mathcal{Q} \to \Bbbk\mathcal{Q} \quad \text{via } (k_1\wp_1, k_2\wp_2) \mapsto \begin{cases} k_1k_2 \cdot \wp_2\wp_1, & \text{if } \mathfrak{t}(\wp_1) = \mathfrak{s}(\wp_2), \\ 0, & \text{otherwise.} \end{cases}$$

The following result shows that we can describe all finite-dimensional $\Bbbk$-algebras using quivers (see [36, p. 43] and [4, Theorem 1.9]). The idea of such a graphical representation seems to go back to Gabriel [17], Grothendieck [21], and Thrall [40], but it became widespread in the early seventies, mainly due to Gabriel [18, 19].

Theorem 2.7 (See [2, Chapter II, Theorem 3.7])**.** *For any finite-dimensional $\Bbbk$-algebra A, there is a finite quiver $\mathcal{Q}$, i.e., the vertex set and arrow set are finite sets, and an admissible ideal*[1] *$\mathcal{I}$ of $\Bbbk\mathcal{Q}$ such that the module category of A is equivalent to that of $\Bbbk\mathcal{Q}/\mathcal{I}$. Furthermore, if A is basic, we have $A \cong \Bbbk\mathcal{Q}/\mathcal{I}$.*

Remark 2.8. We provide a remark for the isomorphism $A \cong \Bbbk\mathcal{Q}/\mathcal{I}$ given in Theorem 2.7 here: the existence of the quiver $\mathcal{Q}$ is unique if A is basic and $\mathcal{I}$ is admissible; the definition of admissible can be found in [2, Chapter I, Subsection I.6].

2.2.2 *Topologies on $\Bbbk$-algebras*

Now we recall the topologies of $\Bbbk$-algebras A (not necessarily basic or finite-dimensional). Let $\mathfrak{i}(A)$ be the set of all ideals of A, which forms a partially ordered set $\mathfrak{i}(A) = (\mathfrak{i}(A), \preceq)$ with the partial order defined by the inclusion, i.e., for any $A_1, A_2 \in \mathfrak{i}(A)$, we have

$$A_1 \preceq A_2 \quad \text{if and only if} \quad A_1 \subseteq A_2.$$

Notice that A has two trivial ideal 0 and A, and then we have $\mathfrak{i}(A) \neq \emptyset$ and have a descending chain $A_0 = A \succeq A_1 = 0 \succeq A_2 = 0 \succeq \cdots$. Thus, there is at least one descending chain of ideals. Let $\mathcal{J}$ be a descending chain

$$A_0 = A \succeq A_1 \succeq A_2 \succeq \cdots$$

of ideals. We say that a subset U of A satisfies the *N-condition*, if it meets the following criteria:

(N1) U contains the zero of A;

[1] An admissible ideal $\mathcal{I}$ of $\Bbbk\mathcal{Q}$ is an ideal such that $R_{\mathcal{Q}}^m \subseteq \mathcal{I} \subseteq R_{\mathcal{Q}}^2$ holds for some $m \geqslant 2$ (see [2, Chapter II, Subsection II.1, p. 53]), where $R_{\mathcal{Q}}^t$ is the ideal of $\Bbbk\mathcal{Q}$ generated by all paths of length greater than or equal to t.

(N2) there exists some $j \in \mathbb{N}$ such that $U \supseteq A_j$.

Furthermore, we denote by $\mathfrak{U}_A(0)$ the set of all subsets satisfying the N-condition, which forms a partially ordered set with the partial order $\preceq$ given by $\subseteq$.

Lemma 2.9. *The set $\mathfrak{U}_A(0)$ is a topology defined on A; in other words, it satisfies the following four conditions*:

(1) *For any $U \in \mathfrak{U}_A(0)$, we have $0 \in U$.*

(2) *$\mathfrak{U}_A(0)$ is closed under finite intersection, i.e., for any $U_1, \dots, U_t \in \mathfrak{U}_A(0)$, we have*

$$\bigcap_{1 \leqslant j \leqslant t} U_j \in \mathfrak{U}_A(0).$$

(3) *If $U \in \mathfrak{U}_A(0)$ and $U \subseteq V \subseteq A$, then $V \in \mathfrak{U}_A(0)$.*

(4) *If $U \in \mathfrak{U}_A(0)$, then there is a set $V \in \mathfrak{U}_A(0)$ such that $V \subseteq U$ and $U - y := \{u - y \mid u \in U\} \in \mathfrak{U}_A(0)$ for all $y \in V$.*

Proof. First, (1) is trivial by the condition (N1).

Second, for arbitrary two subset U_1 and U_2, there are A_{j_1} and A_{j_2} such that $U_1 \supseteq A_{j_1}$ and $U_2 \supset A_{j_2}$. Then $U_1 \cap U_2 \supseteq A_{j_1} \cap A_{j_2}$. By the definition of A_j, we have $A_{j_1} \cap A_{j_2} = A_{\min\{j_1,j_2\}}$, i.e.,

$$U_1 \cap U_2 \supseteq A_{\min\{j_1,j_2\}}.$$

Since $0 \in U_1 \cap U_2$ trivially, we have $U_1 \cap U_2 \in \mathfrak{U}_A(0)$. By induction, we obtain (2).

Third, assume $U \in \mathfrak{U}_A(0)$ and $U \subseteq V \subseteq A$. By the definition of $\mathfrak{U}_A(0)$, we have $0 \in U$ and $U \supseteq A_j$ for some j. Thus, $0 \in V$ and $V \supseteq A_j$, so we obtain (3).

Finally, for each $U \in \mathfrak{U}_A(0)$, we can find V in the following way. There exists an index $\jmath$ such that $U \not\supseteq A_{\jmath-1}$ and $U \supseteq A_\jmath \supseteq A_{\jmath+1} \supseteq \cdots$. Take $V = \bigcap_{j \leqslant \jmath} A_j$ $(= A_\jmath \subseteq U)$. For any $y \in V$, we have (N1), i.e., $0 = y - y \in U - y = \{u - y \mid u \in U\}$ by $y \in V \subseteq U$, and have (N2) since $a = (a + y) - y$ holds for any $a \in V$ and $a + y \in V$. Then we obtain $U - y \in \mathfrak{U}_A(0)$, i.e., (4) holds. $\square$

Definition 2.10. The set $\mathfrak{U}_A(0)$ is called the *$\mathcal{J}$-topology* of A. Furthermore, we can define open sets on A.

(1) The subset in $\mathfrak{U}_A(0)$ is called a *neighborhood* of 0. For any $U \in \mathfrak{U}_A(0)$, the union $\bigcup_V V$ of all subsets V given in Lemma 2.9(4) is called the *interior* of U and denote $\bigcup_V V$ by U°.

(2) A neighborhood U is called *open* if $U = U^\circ$. An *open set* O defined on A is one of the following cases:

(a) O equals either A or $\emptyset$;

(b) O is the intersection of a finite number of open neighborhoods;

(c) O is the union of any number of open neighborhoods.

It induces the definitions of continuous homomorphisms of $\Bbbk$-algebras.

Definition 2.11. Let A_1 and A_2 be two $\Bbbk$-algebras, and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two descending chains of ideals in A_1 and A_2, respectively. Let $\mathfrak{U}_{A_1}(0)$ and $\mathfrak{U}_{A_2}(0)$ be the $\mathcal{J}_1$-topology and $\mathcal{J}_2$-topology given by $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. A homomorphism $h : A_1 \to A_2$ of $\Bbbk$-algebras is called *continuous* if the preimage of an arbitrary open set on A_2 is an open set on A_1.

Lemma 2.12. *Let A be a $\Bbbk$-algebra with a $\mathcal{J}$-topology. Then the addition $+ : A \times A \to A$ and each $\Bbbk$-linear transformation $h_\lambda : A \to A$ defined by $a \mapsto \lambda a$ $(\lambda \in A)$ are continuous.*

Proof. It is obvious that $\mathrm{id}_A = h_1 : A \to A$ via $a \mapsto a$ is continuous. The continuity of h_λ can be given by id_A.

Let $\mathcal{J}$ be

$$A = A_0 \succeq A_1 \succeq A_2 \succeq \cdots.$$

For any open neighborhood U of 0, its preimage is

$$+^{-1}(U) = \{(x_1, x_2) \mid x_1 + x_2 \in U\} =: \widetilde{U}.$$

We need to show that $\widetilde{U} \in \mathfrak{U}_{A\times A}((0,0))$ and $\widetilde{U}^\circ = \widetilde{U}$ in the case for $A \times A$ being a $\Bbbk$-algebra, where the descending chain, i.e., $\mathcal{J}_{A\times A}$, of $A \times A$ is induced by $\mathcal{J}$ as follows:

$$A \times A = A_0 \times A_0 \succeq A_1 \times A_1 \succeq A_2 \times A_2 \succeq \cdots .$$

First of all, the zero element of $A\times A$ is $(0,0)$ which satisfies $0 \in U$ and $0+0 = 0 \in U$, and then $(0,0) \in \widetilde{U}$.

Secondly, since U is a neighborhood of 0, there exists an ideal A_j of $\mathcal{J}$ such that $U \supseteq A_j$. Then for any $x_1, x_2 \in A_j$, we have $x_1 + x_2 \in A_j \subseteq U$, i.e., $(x_1, x_2) \in \widetilde{U}$. It follows that $A_j \times A_j \subseteq \widetilde{U}$. We obtain $\widetilde{U} \in \mathfrak{U}_{A\times A}((0,0))$.

Thirdly, for any $(y_1, y_2) \in \widetilde{U}$, we have $y_1 + y_2 \in U$ by the definition of $\widetilde{U}$, and then

$$(0,0) = (y_1 - y_1, y_2 - y_2) \in \widetilde{U} - (y_1, y_2) = \{(x_1 - y_1, x_2 - y_2) \mid x_1 + x_2 \in U\},$$

i.e., (N1) holds. On the other hand, for any $(z_1, z_2) \in A_j \times A_j$, we have

$$(z_1, z_2) = ((z_1 + y_1) - y_1, (z_2 + y_2) - y_2).$$

Note that $z_1 + y_1 + z_2 + y_2 = (y_1 + y_2) + (z_1 + z_2)$ is an element lying in $U + (z_1 + z_2)$. Since U is open, we have

$$U + (z_1 + z_2) = U^\circ - (-(z_1 + z_2)) = \{u + (z_1 + z_2) \mid u \in U\} \in \mathfrak{U}_A(0)$$

by Lemma 2.9(4) and Definition 2.10, i.e., $U + (z_1 + z_2)$ is a set satisfying Lemma 2.9(4). Then

$$U^\circ = \bigcup_{\substack{V\subseteq U,\, V \text{ satisfies} \\ \text{Lemma 2.9(4)}}} V \supseteq U + (z_1 + z_2),$$

and so we obtain $(y_1+y_2)+(z_1+z_2) \in U+(z_1+z_2) \subseteq U^\circ$, i.e., $(y_1+y_2)+(z_1+z_2) \in U$. Thus, $(z_1, z_2) \in \widetilde{U}$. It follows that $A_j \times A_j \subseteq \widetilde{U} - (y_1, y_2)$, and thus (N2) holds. Therefore, $\widetilde{U} - (y_1, y_2) \in \mathfrak{U}_{A\times A}((0,0))$. In summary, we have that $\widetilde{U}$ satisfies Lemma 2.9(4), and so by Definition 2.10, it is clear that $\widetilde{U}^\circ = \widetilde{U}$. □

Definition 2.13 (See [3, Chapter 10, p. 101])**.** A *topological* $\Bbbk$*-algebra* is a $\Bbbk$-algebra equipped with a topology such that the addition $+ : A \times A \to A$ and each $\Bbbk$-linear transformation $-h_1 : A \to A$ via $a \mapsto -a$ are continuous.

The following result is a consequence of Lemma 2.12.

Proposition 2.14. *Given an arbitrary* $\Bbbk$*-algebra* A *and its descending chain* $\mathcal{J}$ *of ideals, then* A *becomes a topological* $\Bbbk$*-algebra with the* $\mathcal{J}$*-topology* $\mathfrak{U}_A(0)$*.*

In this paper, we refer to A as a $\mathcal{J}$*-topological* $\Bbbk$*-algebra*.

2.2.3 *Completions induced by* $\mathcal{J}$*-topologies*

Assume that $|\cdot| : \Bbbk \to \mathbb{R}^{\geqslant 0}$ is a norm defined on the field $\Bbbk$ in this subsection, i.e., $|\cdot|$ is a map satisfying

(1) $|k| = 0$ if and only if $k = 0$;

(2) $|k_1k_2| = |k_1||k_2|$ holds for all $k_1, k_2 \in \Bbbk$;

(3) the triangle inequality $|k_1 + k_2| \leqslant |k_1| + |k_2|$ holds for all $k_1, k_2 \in \Bbbk$.

Then $\{\mathfrak{B}_r = \{a \in \Bbbk \mid |a| < r\} \mid r \in \mathbb{R}^+\}$ induces a standard topology $\mathfrak{U}_\Bbbk(0)$ on $\Bbbk$ whose elements are called the *neighborhoods* of $0 \in \Bbbk$.

Let A be a $\mathcal{J}$-topological $\Bbbk$-algebra whose dimension is finite and let $B_A = \{b_1, \ldots, b_n\}$ be a basis of A. Then, naturally, we can define the Cauchy sequence by the $\mathcal{J}$-topology. More precisely, a sequence $\{x_i\}_{i\in\mathbb{N}}$ in A is called a $\mathcal{J}$*-Cauchy sequence* if for any U, lying in $\mathfrak{U}_A(0)$, and containing some subset $\sum_{i=1}^n \mathfrak{u}_i b_i$ of A with $\mathfrak{u}_i \in \mathfrak{U}_\Bbbk(0)$ $(1 \leqslant i \leqslant n)$, there is $m \in \mathbb{N}$ such that $x_s - x_t \in U$ holds for all $s, t \geqslant m$. Two $\mathcal{J}$-Cauchy sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are called *equivalent*, denoted by $\{x_i\}_{i\in\mathbb{N}} \sim \{y_i\}_{i\in\mathbb{N}}$, if for any $U \in \mathfrak{U}_A(0)$, there is an integer $m \in \mathbb{N}$ such that $x_i - y_i \in U$ holds for all $i \geqslant m$. It is easy to see that "$\sim$" is an equivalence relation. We use $[\{x_i\}_{i\in\mathbb{N}}]$ to denote the equivalence class containing $\{x_i\}_{i\in\mathbb{N}}$, and use $\mathfrak{C}_{\mathcal{J}}(A)$ to denote the set of all equivalence classes of $\mathcal{J}$-Cauchy sequences. Then we have three families of A-homomorphisms:

(1) $(\varphi_{ji} : A/A_j \to A/A_i)_{j \geqslant i}$, where all φ_{ji} are naturally induced by $A_i \supseteq A_j$;

(2) $(p_i : \mathfrak{C}_{\mathcal{J}}(A) \to A/A_i)_{i \in \mathbb{N}}$, where $p_i(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) = x_i + A_i$ (p_i is called the i-th *projection*);

(3) $(u_i : A/A_i \to \mathfrak{C}_{\mathcal{J}}(A))_{i \in \mathbb{N}}$, where $u_i(a + A_i) = (0, \ldots, \overset{i-1}{0}, a, \overset{i+1}{0}, 0 \ldots)$ (u_i is called the i-th *injection*).

Let $\mathcal{X}$ be the category whose object set is $\{A/A_i \mid i \in \mathbb{N}\} \cup \{\mathfrak{C}_{\mathcal{J}}(A)\}$ and the morphism set is the collection of all A-homomorphisms as above. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{C}_{\mathcal{J}}(A) \arrow[dr, "p_i"] \arrow[ddr, bend right, "p_j"'] & & A/A_h. \arrow[ll, dashed, "u_h"', "(\exists!)"] \arrow[dl, "\varphi_{hi}"'] \arrow[ddl, bend left, "\varphi_{hj}"] \\
& A/A_i & \\
& A/A_j. \arrow[u, "(i \leqslant j)", "\varphi_{ji}"'] &
\end{tikzcd}$$

It follows from the above construction that the following proposition holds.

Proposition 2.15 (See [3, Chapter 10, p. 103])**.** *Using the notations as above, we have*

$$\varprojlim A/A_i \cong \mathfrak{C}_{\mathcal{J}}(A).$$

We write $\widehat{A} := \mathfrak{C}_{\mathcal{J}}(A)$ and call it the *completion* of A. We say that A is *complete* if $\widehat{A} = A$. In particular, if $A = \Bbbk$, then the descending chain $\mathcal{J} : A_0 = \Bbbk \succeq A_1 = 0$ induces a $\mathcal{J}$-topology

$$\mathfrak{U}_{\Bbbk}(0) = \{\text{the neighborhood of } 0\}$$

of $\Bbbk$. In this case, the $\mathcal{J}$-Cauchy sequence coincides with the usual Cauchy sequence.

Proposition 2.16. *Let A be a basic finite-dimensional $\Bbbk$-algebra and let $\mathcal{J}$ be the descending chain*

$$A_0 = A = \mathrm{rad}^0 A \succeq A_1 = \mathrm{rad} A \succeq A_2 = \mathrm{rad}^2 A \succeq \cdots.$$

Then A is complete (*in the sense of $\mathcal{J}$-topology*) *if and only if $\Bbbk$ is complete.*

Proof. Let A be a basic finite-dimensional $\Bbbk$-algebra. Then, by Theorem 2.7, there are a finite quiver $\mathcal{Q}$ and an ideal $\mathcal{I}$ of $\Bbbk\mathcal{Q}$ such that

$$A \cong \Bbbk\mathcal{Q}/\mathcal{I} = \bigoplus_{l \in \mathbb{N}} \Bbbk\mathcal{Q}_l.$$

Thus, up to isomorphism, each element $a \in A$ can be written as $\sum_{j=1}^{n} k_j \wp_j$, where n is the dimension of A, $k_u \in \Bbbk$ and $\wp_u$ is a path on $\mathcal{Q}$.

Assume that $\Bbbk$ is complete. Since A is finite-dimensional, we have $\mathrm{rad}^l A = \mathrm{span}_{\Bbbk}\{\mathcal{Q}_i \mid i \geqslant l\}$. Thus, $\mathrm{rad}^{L+1} A = 0$, where $L = \max_{\wp \in \mathcal{Q}_{\geqslant 0}} \ell(\wp)$, i.e.,

$$\mathcal{J} = A \succeq \mathrm{rad} A \succeq \mathrm{rad}^2 A \succeq \cdots \succeq \mathrm{rad}^L A \succeq 0 \succeq 0 \succeq \cdots.$$

Let $\{x_i = \sum_{j=1}^{n} k_{ij} \wp_j\}_{i \in \mathbb{N}}$ be a $\mathcal{J}$-Cauchy sequence in A. Take

$$U = \left\{ \sum_{\ell(\wp) = L} k_\wp \wp \;\middle|\; k_\wp \text{ lie in some neighborhood in } \mathfrak{U}_{\Bbbk}(0) \right\} \ (\supsetneq \mathrm{rad}^{L+1} A = 0).$$

Then, there is $N(U) \in \mathbb{N}$ such that

$$x_s - x_t = \sum_{j=1}^{n} (k_{sj} - k_{tj}) \wp_j \in \mathrm{rad}^L A \quad \text{holds for all } s, t \geqslant N(U).$$

Thus, $k_{sj} - k_{tj}$ lies in some neighborhood in $\mathfrak{U}_{\Bbbk}(0)$, and so for all i, $\{k_{ij}\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $\Bbbk$. Then it is clear that A is complete.

Conversely, if A is complete, we assume that $\Bbbk$ is not complete, and $\widehat{\Bbbk}$ is the completion of $\Bbbk$. Then we have a natural $\Bbbk$-linear embedding $\mathfrak{e} : \Bbbk \to \widehat{\Bbbk}$ sending $k \in \Bbbk$ to $\{k_i\}_{i\in\mathbb{N}}$, where $k_1 = k_2 = \cdots = k$. Then there is a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}} \in \widehat{\Bbbk}\backslash\mathfrak{e}(\Bbbk)$. Consider the sequence $\{x_i \cdot \wp\}_{i\in\mathbb{N}}$ in A, where $\wp \in \mathrm{rad}^L A$ is a path of length L. Then $\{x_i \cdot \wp\}_{i\in\mathbb{N}}$ is a $\mathcal{J}$-Cauchy sequence in A. However, we have $\{x_i \cdot \wp\}_{i\in\mathbb{N}} \in \widehat{A}\backslash A$ in this case, which contradicts that A is complete. □

2.3 The total order of $\Bbbk$-algebras

Recall that a field $\Bbbk$ equipped with a total order $\preceq$ is an *ordered field* if it satisfies the following four conditions:

(1) for any $a, b \in \Bbbk$, either $a \preceq b$, $b \preceq a$ or $a = b$ holds;

(2) if $a \preceq b$ and $b \preceq c$, then $a \preceq c$;

(3) if $a \preceq b$, then $a + c \preceq b + c$ for all $c \in \Bbbk$;

(4) if $a \preceq b$ and $0 \preceq c$, then $ac \preceq bc$.

In order to give the definition of integration defined on a finite-dimensional $\Bbbk$-algebra Λ, we need to assume that $\Bbbk$ is a field with the total order $\preceq$. However, it is well known that $\Bbbk$ might not always be an ordered field, as the case for $\Bbbk$ being the complex field $\mathbb{C}$. Interestingly, for our purposes, the existence of such a total order is not a prerequisite. We only require that the finite-dimensional $\Bbbk$-algebra involved in our study encompasses certain partially ordered subsets. Specifically, the subset $\mathbb{I}_\Lambda$ outlined in Subsection 3.3 is sufficient. For the sake of simplicity, we assume that $\Bbbk$ is fully ordered, although this assumption does not sacrifice generality. This simplification aids in our definition of integration within the context of category theory.

Remark 2.17. We provide a remark to show that if $\Bbbk$ is totally ordered, then any finite-dimensional $\Bbbk$-algebra Λ can be endowed with a total order. Let $B_\Lambda = \{b_i \mid 1 \leqslant i \leqslant n\}$ be a $\Bbbk$-basis of Λ. If B_Λ is totally ordered (assuming $b_i \preceq b_j$ if and only if $i \leqslant j$), then we can define a total order for Λ as follows.

Step 1. For any two elements $a, a' \in \Lambda$, we define $a \prec_p a'$ if and only if $\varphi(a) < \varphi(a')$, where φ is a map $\varphi : \Lambda \to \mathbb{R}^{\geqslant 0}$ (for example, φ is the norm $\|\cdot\|_p$ defined in Section 3).

Step 2. Assume $a = \sum_{i=1}^m k_i b_i$ and $a' = \sum_{i=1}^m k'_i b_i$ $(0 \leqslant m \leqslant n)$ such that $k_i = k'_i$ holds for all $i < m$. If $\varphi(a) = \varphi(a')$, then we define $a \preceq_p a'$ if and only if $k_m \preceq k'_m$.

3 Normed $\Bbbk$-algebras

In this section, let Λ be a finite-dimensional $\Bbbk$-algebra with a $\Bbbk$-basis $B_\Lambda = \{b_i \mid 1 \leqslant i \leqslant n\}$. Then any element $a \in \Lambda$ is of the form $a = \sum_{i=1}^n k_i b_i$. In this section, we define some algebraic structures on Λ.

3.1 Norms of $\Bbbk$-algebras

For a map $\mathfrak{n} : B_\Lambda \to \mathbb{R}^+$ and any $p \geqslant 1$, we have $\|\cdot\|_p : \Lambda \to \mathbb{R}^{\geqslant 0}$ as the function

$$\|a\|_p = \left\| \sum_{i=1}^n k_i b_i \right\|_p := ((|k_1|\mathfrak{n}(b_1))^p + \cdots + (|k_n|\mathfrak{n}(b_n))^p)^{\frac{1}{p}}. \tag{3.1}$$

Proposition 3.1. *Any triple* $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ $(= \Lambda$ *for short*) *is a normed* $\Bbbk$*-vector space.*

Proof. First of all, for any $a = \sum_{i=1}^n k_i b_i \in \Lambda$, we have $\|a\|_p \geqslant 0$ because $\mathfrak{n}(b_i) > 0$ and $|k_i| \geqslant 0$ $(1 \leqslant i \leqslant n)$. In particular, if $\|a\|_p = 0$, then

$$(|k_1|\mathfrak{n}(b_1))^p + \cdots + (|k_n|\mathfrak{n}(b_n))^p = 0.$$

Since $|k_i|\mathfrak{n}(b_i) \geqslant 0$ and $\mathfrak{n}(b_i) > 0$ hold for all $1 \leqslant i \leqslant n$, we obtain $|k_i|\mathfrak{n}(b_i) = 0$, and so $k_i = 0$. Thus, $a = \sum_{i=1}^n 0 b_i = 0$. Then it is easy to see that $\|a\|_p = 0$ if and only if $a = 0$.

Next, for any $k \in \Bbbk$ and $a = \sum_{i=1}^n k_i b_i \in \Lambda$, we have

$$\|ka\|_p = \|k(k_1 b_1 + \cdots + k_n b_n)\|_p$$

$$= \left(\sum_{i=1}^{n}(|kk_i|\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n}|k|^p(|k_i|\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}}$$
$$= |k|\left(\sum_{i=1}^{n}(|k_i|\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}} = |k|\cdot\|a\|_p.$$

Finally, we prove the triangle inequality $\|a+a'\|_p \leqslant \|a\|_p + \|a'\|_p$ for arbitrary two elements $a = \sum_{i=1}^n k_ib_i$ and $a' = \sum_{i=1}^n k_i'b_i$. It can be induced by the discrete Minkowski inequality

$$\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}} \geqslant \left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{p}}$$

as follows:

$$\|a\|_p + \|a'\|_p = \left(\sum_{i=1}^{n}(|k_i|\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}(|k_i'|\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}}$$
$$\geqslant \left(\sum_{i=1}^{n}(|k_i|\mathfrak{n}(b_i) + k_i'\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}}$$
$$= \left(\sum_{i=1}^{n}(|k_i + k_i'|\mathfrak{n}(b_i))^p\right)^{\frac{1}{p}} = \|a+a'\|_p.$$

Therefore, $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ is a normed space. □

Definition 3.2. A *normed* $\Bbbk$-*algebra* is a triple $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$, where $\mathfrak{n}: B_\Lambda \to \mathbb{R}^+$ and $\|\cdot\|_p : \Lambda \to \mathbb{R}^{\geqslant 0}$ are called the *normed basis function* and *norm* of Λ, respectively.

3.2 Completions of normed $\Bbbk$-algebras

We can define open neighborhoods $B(0,r)$ of 0 for any normed $\Bbbk$-algebra $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ by

$$B(0,r) := \{a \in \Lambda \mid \|a\|_p < r\}.$$

Let $\mathfrak{U}_\Lambda^B(0)$ be the class of all subsets U of Λ satisfying the following conditions:

(1) U is the intersection of a finite number of $B(0,r)$.

(2) U is the union of any number of $B(0,r)$.

Then $\mathfrak{U}_\Lambda^B(0)$ is a topology defined on Λ called the $\|\cdot\|_p$-topology, and we can define Cauchy sequences called $\|\cdot\|_p$-Cauchy sequences by the above topology.

Recall that Λ has a $\mathcal{J}$-topology $\mathfrak{U}_\Lambda(0)$ given by the descending chain

$$\Lambda = \mathrm{rad}^0\Lambda \succeq \mathrm{rad}^1\Lambda \succeq \mathrm{rad}^2\Lambda \succeq \cdots.$$

Thus, we obtain two completions $\widehat{\Lambda}^B$ and $\widehat{\Lambda}$ by the $\|\cdot\|_p$-topology and the $\mathcal{J}$-topology, respectively. The following lemma establishes the relation between $\widehat{\Lambda}^B$ and $\widehat{\Lambda}$ in the case of $\Bbbk$ being complete.

Proposition 3.3. *Assume that* $\Bbbk$ *is complete. Let* $\Lambda = (\Lambda, \mathfrak{n}, \|\cdot\|_p)$ *be an* n*-dimensional normed* $\Bbbk$*-algebra with the* $\mathcal{J}$*-topology* $\mathfrak{U}_\Lambda(0)$ *given by* $\Lambda = \mathrm{rad}^0\Lambda \succeq \mathrm{rad}^1\Lambda \succeq \mathrm{rad}^2\Lambda \succeq \cdots$ ($\|\cdot\|_p$ *is the norm defined on* Λ *given in Proposition* 3.1). *Then* $\widehat{\Lambda}^B = \widehat{\Lambda}$.

Proof. Similar to Proposition 2.16, we can show that $\widehat{\Lambda}^B = \Lambda$ (i.e., Λ is complete) if and only if $\widehat{\Bbbk} = \Bbbk$. By using Proposition 2.16 again, we have that $\widehat{\Lambda} = \Lambda$ if and only if $\widehat{\Bbbk} = \Bbbk$. Then $\widehat{\Bbbk} = \Bbbk$ if and only if $\widehat{\Lambda}^B = \Lambda = \widehat{\Lambda}$. Equivalently,

$$\widehat{\Lambda}^B = \widehat{\left(\sum_{i=1}^{n}\Bbbk b_i\right)}^B = \sum_{i=1}^{n}\widehat{\Bbbk}b_i = \widehat{\sum_{i=1}^{n}\Bbbk b_i} = \widehat{\Lambda}.$$

This completes the proof. □

Remark 3.4. (1) Note that the norms defined on Λ is not unique. In Section 4, we introduce normed Λ-modules N over any finite-dimensional normed $\Bbbk$-algebra Λ. In this case, we need a homomorphism $\tau : \Lambda \to \Lambda'$ between two finite-dimensional normed $\Bbbk$-algebras Λ and Λ', and the norms $\|\cdot\|$ and $\|\cdot\|'$ respectively defined on Λ and Λ' may not necessarily be of the form $\|\cdot\|_p$.

(2) If $\Lambda = \Bbbk$ and $\mathfrak{n}(1) = 1$, then the norm $\|\cdot\|_p$ given in Proposition 3.1 is the norm $|\cdot|$, i.e., $\|a\|_p = (|a|^p)^{\frac{1}{p}} = |a|$.

3.3 Elementary simple functions

Let $\mathbb{I}$ be a subset of $\Bbbk$. Denote $\mathbb{I}_\Lambda$ by the subset

$$\left\{ \sum_{i=1}^{n} k_i b_i \;\middle|\; k_i \in \mathbb{I} \right\} \xleftrightarrow{1-1} \prod_{i=1}^{n} (\mathbb{I} \times \{b_i\})$$

of Λ. A *function defined on* $\mathbb{I}_\Lambda$ is a map $f : \mathbb{I}_\Lambda \to \Bbbk$. Since $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ is a normed space, Λ is also a topological space induced by the norm $\|\cdot\|_p$, and so is $\mathbb{I}_\Lambda$. Thus, we can define an open set for every subset of Λ, including $\mathbb{I}_\Lambda$. The function f is said to be *continuous* if the preimage of any open subset of $\Bbbk$ is an open set of $\mathbb{I}_\Lambda$.

Let $\mathbb{I} := [a, b]_\Bbbk$ be a fully ordered subset of $\Bbbk$ whose minimal element and maximal element are a and b, respectively. In our paper, we assume that $\Bbbk$ and $[a, b]_\Bbbk$ are infinite sets and consider only the case for $\mathbb{I} = [a, b]_\Bbbk$ with $a \prec b$ such that there exists an element ξ with $a \prec \xi \prec b$ and the order-preserving bijections $\kappa_a : \mathbb{I} \to [a, \xi]_\Bbbk$ and $\kappa_b : \mathbb{I} \to [\xi, b]_\Bbbk$ exist (for example, the case of the cardinal number of $\mathbb{I}$ is either $\aleph_0$ or $\aleph_1$).

An *elementary simple function* on $\mathbb{I}_\Lambda$ is a finite sum $\sum_{i=1}^{t} k_i \mathbf{1}_{I_i}$, where

(1) for any $1 \leqslant i \leqslant t$, $k_i \in \Bbbk$;

(2) $I_i = I_{i1} \times \cdots \times I_{in}$, and for any $1 \leqslant j \leqslant n$, I_{ij} is a subset of $\mathbb{I}$ which is one of the following forms:

(a) $(c_{ij}, d_{ij})_\Bbbk := \{k \in \Bbbk \mid c_{ij} \prec k \prec d_{ij}\}$,

(b) $[c_{ij}, d_{ij})_\Bbbk := \{k \in \Bbbk \mid c_{ij} \preceq k \prec d_{ij}\}$,

(c) $(c_{ij}, d_{ij}]_\Bbbk := \{k \in \Bbbk \mid c_{ij} \prec k \preceq d_{ij}\}$,

(d) $[c_{ij}, d_{ij}]_\Bbbk := \{k \in \Bbbk \mid c_{ij} \preceq k \preceq d_{ij}\}$,

where $a \preceq c_{ij} \prec d_{ij} \preceq b$;

(3) $\mathbf{1}_{I_i}$ is the function $I_i \to \{1\}$ such that $I_i \cap I_j = \emptyset$ holds for all $1 \leqslant i \neq j \leqslant t$.

We denote by $\mathbf{S}(\mathbb{I}_\Lambda)$ the set of all elementary simple functions. Then $\mathbf{S}(\mathbb{I}_\Lambda)$ is a $\Bbbk$-vector space, and $\mathbf{S}(\mathbb{I}_\Lambda)$ induces the direct sum $\mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n}$ whose element can be seen as the sequence

$$\left\{ f_{(\delta_1,\ldots,\delta_n)}\left(\sum_{i=1}^{n} k_i b_i \right) \right\}_{(\delta_1,\ldots,\delta_n)\in\{a,b\}\times\cdots\times\{a,b\}} =: \boldsymbol{f}(k_1, \ldots, k_n),$$

$\sum_{i=1}^{n} k_i b_i$ is written as $(k_1, \ldots, k_n)$ since $\{b_1 \mid 1 \leqslant i \leqslant n\} = B_\Lambda$ is the $\Bbbk$-basis of Λ. Then we can characterize $\mathbf{S}(\mathbb{I}_\Lambda)$ together with two further pieces of data: the function $\mathbf{1}_{\mathbb{I}_\Lambda} : \mathbb{I}_\Lambda \to \{1\}$ and the map

$$\gamma_\xi : \mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n} \to \mathbf{S}(\mathbb{I}_\Lambda), \tag{3.2}$$

called the *juxtaposition map*, sending $\boldsymbol{f}$ to the function

$$\gamma_\xi(\boldsymbol{f})(k_1, \ldots, k_n) = \sum_{(\delta_1,\ldots,\delta_n)} \mathbf{1}_{\kappa_{\delta_1}(\mathbb{I})\times\cdots\times\kappa_{\delta_n}(\mathbb{I})} \cdot f_{(\delta_1,\ldots,\delta_n)}(\kappa_{\delta_1}^{-1}(k_1), \ldots, \kappa_{\delta_n}^{-1}(k_n)) \ (k_1 \neq \xi, \ldots, k_n \neq \xi),$$

where ξ is an element with $a \prec \xi \prec b$ such that the order-preserving bijections

$$\kappa_a : \mathbb{I} \to [a, \xi]_\Bbbk \quad \text{and} \quad \kappa_b : \mathbb{I} \to [\xi, b]_\Bbbk$$

exist.

Example 3.5. (1) Take Λ be the $\Bbbk$-algebra whose dimension is 2, and assume that $\{b_1, b_2\}$ is a basis of Λ. Then $\mathbb{I}_\Lambda \cong_\Bbbk [a,b]_\Bbbk b_1 \times [a,b]_\Bbbk b_2$. For any element

$$\boldsymbol{f} = (f_{(a,a)}, f_{(b,a)}, f_{(a,b)}, f_{(b,b)}) \in \mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 4},$$

where $f_{(\delta_1,\delta_2)} : \mathbb{I}_\Lambda \to \Bbbk$ is a function in $\mathbf{S}(\mathbb{I}_\Lambda)$ sending each $k_1b_1 + k_2b_2$ to the element $f_{(\delta_1,\delta_2)}(k_1,k_2)$ in $\Bbbk$, and $(\delta_1,\delta_2) \in \{a,b\} \times \{a,b\} = \{(a,a),(b,a),(a,b),(b,b)\}$, γ_ξ juxtaposes $f_{(a,a)}$, $f_{(b,a)}$, $f_{(a,b)}$ and $f_{(b,b)}$ into a new function

$$\gamma_\xi(f_{(a,a)}, f_{(b,a)}, f_{(a,b)}, f_{(b,b)})(k_1,k_2) = \tilde{f}_{(a,a)}(k_1,k_2) + \tilde{f}_{(b,a)}(k_1,k_2) + \tilde{f}_{(a,b)}(k_1,k_2) + \tilde{f}_{(b,b)}(k_1,k_2)$$

as shown in Figure 1, where

$$\begin{aligned}
\tilde{f}_{(a,a)}(k_1,k_2) &= \mathbf{1}_{[a,\xi)\times[a,\xi)} \cdot f_{(a,a)}(\kappa_a^{-1}(k_1), \kappa_a^{-1}(k_2)),\\
\tilde{f}_{(b,a)}(k_1,k_2) &= \mathbf{1}_{(\xi,b]\times[a,\xi)} \cdot f_{(b,a)}(\kappa_b^{-1}(k_1), \kappa_a^{-1}(k_2)),\\
\tilde{f}_{(a,b)}(k_1,k_2) &= \mathbf{1}_{[a,\xi)\times(\xi,b]} \cdot f_{(a,b)}(\kappa_a^{-1}(k_1), \kappa_b^{-1}(k_2)),\\
\tilde{f}_{(b,b)}(k_1,k_2) &= \mathbf{1}_{(\xi,b]\times(\xi,b]} \cdot f_{(b,b)}(\kappa_b^{-1}(k_1), \kappa_b^{-1}(k_2)).
\end{aligned}$$

(2) This example is used to establish the relation between Banach spaces and Lebesgue intersections in [29]. Take $\Bbbk = \mathbb{R}$, $\mathbb{I} = [0,1]$, $\xi = \frac{1}{2}$, $\Lambda = \mathbb{R}$ and the order-preserving bijections $\kappa_0 : \mathbb{I} = [0,1] \to \Bbbk = \mathbb{R}$ and $\kappa_1 : \mathbb{I} = [0,1] \to \Bbbk = \mathbb{R}$ are given by $x \mapsto \frac{x}{2}$ and $\frac{1+x}{2}$, respectively. Then $\mathbf{S}(\mathbb{I}_\mathbb{R}) = \mathbf{S}([0,1])$ is a normed space together with two further pieces of data: the function $\mathbf{1}_{[0,1]} : [0,1] \to \{1\}$ and the juxtaposition map

$$\gamma_{\frac{1}{2}} : \mathbf{S}([0,1]) \oplus \mathbf{S}([0,1]) \to \mathbf{S}([0,1])$$

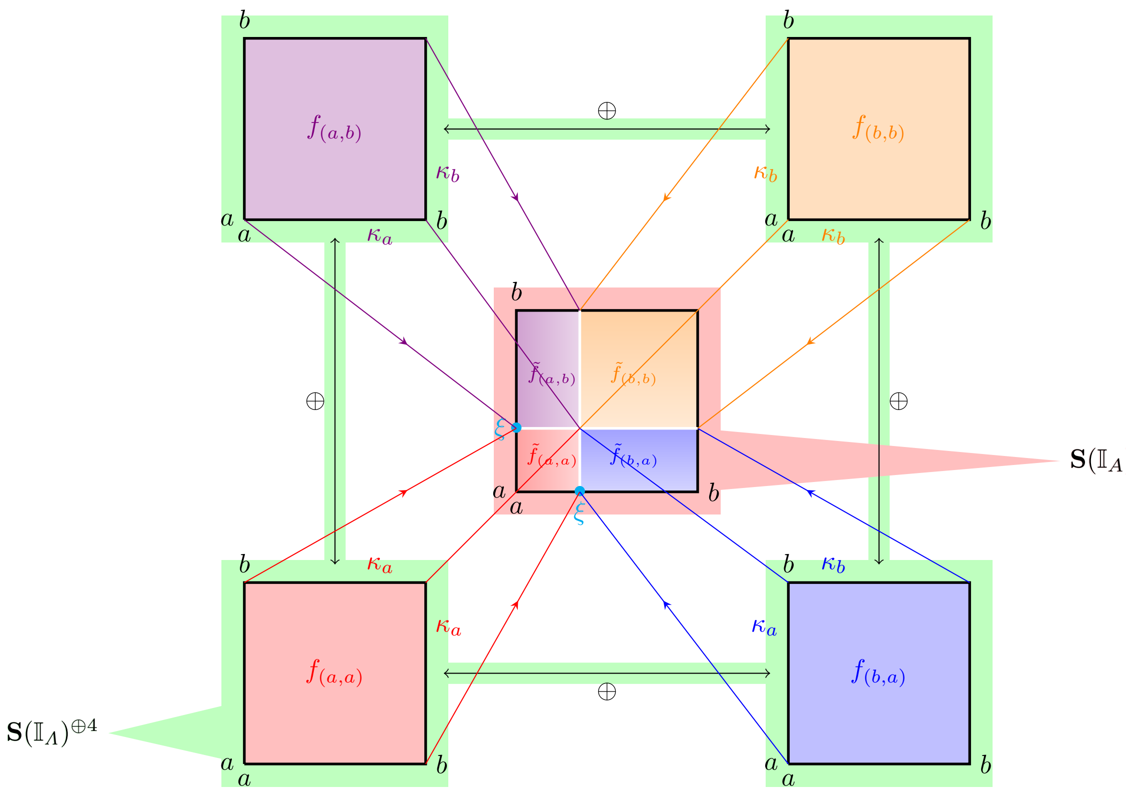

Figure 1 (Color online) The juxtaposition map

sending (f_1, f_2) to the following function:

$$\gamma_{\frac{1}{2}}(f_1, f_2)(x) = \mathbf{1}_{\kappa_0([0,1))} \cdot f_1(\kappa_0^{-1}(x)) + \mathbf{1}_{\kappa_1((0,1])} \cdot f_1(\kappa_1^{-1}(x))$$
$$= \begin{cases} f_1(2x), & x \in \kappa_0([0,1)) = \left[0, \dfrac{1}{2}\right), \\ f_2(2x-1), & x \in \kappa_1((0,1]) = \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

Lemma 3.6. *The map γ_ξ is a $\Bbbk$-linear map.*

Proof. Take $a, b \in \Bbbk$, $f, g \in \mathbf{S}(\mathbb{I}_\Lambda)$ and let $(k_i)_i$, $\mathbf{1}$ and $(\delta_i)_i$ be the element $(k_1, \ldots, k_n)$ in $\mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n}$, the identity function $\mathbf{1}_{\kappa_{\delta_1}(\mathbb{I}) \times \cdots \times \kappa_{\delta_n}(\mathbb{I})}$ and the n-multiple $(\delta_1 \times \cdots \times \delta_n)$, respectively. Then

$$\begin{aligned}\gamma_\xi(af+bg)((k_i)_i) &= \sum_{(\delta_i)_i} \mathbf{1} \cdot (af+bg)_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) \\ &= \sum_{(\delta_i)_i} \big(\mathbf{1} \cdot af_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) + \mathbf{1} \cdot bg_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i)\big) \\ &= a\sum_{(\delta_i)_i} \mathbf{1} \cdot f_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) + b\sum_{(\delta_i)_i} \mathbf{1} \cdot g_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) \\ &= a\gamma_\xi(f)((k_i)_i) + b\gamma_\xi(g)((k_i)_i).\end{aligned}$$

Thus, γ_ξ is a $\Bbbk$-linear map. □

4 Normed modules over $\Bbbk$-algebras

Let $\mathbb{I}$ be a subset of the field $\Bbbk = (\Bbbk, \preceq)$ with totally ordered $\preceq$. Then $\mathbb{I}$ is also a totally ordered set. For simplicity, we denote by $[x, y]_\Bbbk$ the set of all elements $k \in \Bbbk$ with $x \preceq k \preceq y$, i.e.,

$$[x, y]_\Bbbk := \{k \in \Bbbk \mid x \preceq k \preceq y\}.$$

In particular, if $x = y$ then $[x, y]_\Bbbk = \{x\} = \{y\}$ is a set containing only one element.

In this section, we introduce the category $\mathscr{N}or^p$, which is used to explore the categorification of integration.

4.1 Norms of Λ-modules

Recall that a *left A-module* (= A-module for short) over a $\Bbbk$-algebra A is a $\Bbbk$-vector space V with a $\Bbbk$-linear map $h : A \to \mathrm{End}_\Bbbk V$ sending a to h_a. Thus, h provides a right action $A \times V \to V$, $(a, v) \mapsto va := h_a(v)$ which satisfies the following properties:

(1) $a(v + v') = av + av'$ for any $v, v' \in V$ and $a \in A$;
(2) $(a + a')v = av + a'v$ for any $v \in V$ and $a, a' \in A$;
(3) $a'(av) = (a'a)v$ for any $v \in V$ and $a, a' \in A$;
(4) $1v = v$ for any $v \in V$;
(5) $(ka)v = k(av) = a(kv)$ for any $v \in V$, $a \in A$ and $k \in \Bbbk$.

Take $A = \Lambda$ to be the normed $\Bbbk$-algebra whose norm $\|\cdot\|_p : \Lambda \to \mathbb{R}^+$ given by (3.1), where the $\Bbbk$-basis of Λ is $B_\Lambda = \{b_i \mid 1 \leqslant i \leqslant n = \dim_\Bbbk \Lambda\}$.

Definition 4.1. Let $\tau : \Lambda \to \Bbbk$ be a homomorphism between two normed $\Bbbk$-algebras $(\Lambda, \|\cdot\|_p)$ and $(\Bbbk, |\cdot|)$. A *τ-normed Λ-module* is a Λ-module M with a norm $\|\cdot\| : M \to \mathbb{R}^{\geqslant 0}$ such that

$$\|am\| = |\tau(a)| \cdot \|m\| \quad \text{holds for all } a \in \Bbbk \text{ and } m \in M. \tag{4.1}$$

Thus, each normed Λ-module can be seen as a triple $(M, h, \|\cdot\|)$ of the $\Bbbk$-vector space M, the $\Bbbk$-linear map $h : M \to \mathrm{End}_\Bbbk M$ and a norm $\|\cdot\| : M \to \mathbb{R}^{\geqslant 0}$. For simplification, τ-normed modules are called *normed modules*.

The norms of Λ-modules yield the following fact.

Fact 4.2. (1) Note that $\|\cdot\|_p$ defined by (3.1) is the norm of Λ as a $\Bbbk$-vector space. It is easy to see that Λ is also a left Λ-module, called the *regular module*, where the scalar multiplication is given by the multiplication $\Lambda \times \Lambda \to \Lambda, (a, x) \mapsto ax$ of Λ as a finite-dimensional $\Bbbk$-algebra. Thus, it is natural to ask whether $\|\cdot\|_p$ is a norm of Λ as a Λ-module. Indeed, the norm of Λ as a finite-dimensional $\Bbbk$-algebra may not be equal to the norm $\|\cdot\|$ of Λ as a regular module. However, if Λ as the left Λ-module defined by

$$\Lambda \times \Lambda \to \Lambda, \quad (a, x) \mapsto a \star x := \tau(a)x, \tag{4.2}$$

where $\tau(a)x$ is defined by the scalar multiplication of Λ as the $\Bbbk$-vector space ${}_{\Bbbk}\Lambda$, then for any $x = \sum_{i=1}^{n} k_i b_i \in \Lambda$, we obtain

$$\begin{aligned}\|a \star x\|_p &= \left\|\tau(a)\sum_{i=1}^{n} k_i b_i\right\|_p = \left(\sum_{i=1}^{n} |\tau(a)k_i|^p \mathfrak{n}(b_i)^p\right)^{\frac{1}{p}} \\ &= |\tau(a)|\left(\sum_{i=1}^{n} |k_i|^p \mathfrak{n}(b_i)^p\right)^{\frac{1}{p}} = |\tau(a)|\|x\|_p.\end{aligned}$$

To be more precise, Λ is a (Λ, Λ)-bimodule with two norms, and Λ is a normed module satisfying Definition 4.1 when it is considered as a module defined in (4.2).

(2) For any Λ-homomorphism $f: M \to N$ of two Λ-modules M and N, if M and N are normed Λ-modules, i.e., $M = (M, h_M, \|\cdot\|_M)$ and $N = (N, h_N, \|\cdot\|_N)$, then we have

$$\|f(am)\|_N = \|af(m)\|_N = |\tau(a)| \cdot \|f(m)\|_N.$$

Example 4.3. Let

$$\Lambda = \begin{pmatrix} \Bbbk & 0 \\ \Bbbk & \Bbbk \end{pmatrix}.$$

Then a $\Bbbk$-basis of Λ is $B_\Lambda = \{\boldsymbol{E}_{11}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}\}$, where $\boldsymbol{E}_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\boldsymbol{E}_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\boldsymbol{E}_{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Take $\mathfrak{n}$ to be the map $B_\Lambda \to \mathbb{R}^+$ defined by $\mathfrak{n}(\boldsymbol{E}_{11}) = \mathfrak{n}(\boldsymbol{E}_{21}) = \mathfrak{n}(\boldsymbol{E}_{22}) = 1$, and then for any element $x = \left(\begin{smallmatrix} k_{11} & 0 \\ k_{21} & k_{22} \end{smallmatrix}\right)$ in Λ, we have $\|x\|_p = (|k_{11}|^p + |k_{21}|^p + |k_{22}|^p)^{\frac{1}{p}}$. There are three indecomposable Λ-modules up to Λ-isomorphisms:

$$P(1) = \begin{pmatrix} \Bbbk & 0 \\ \Bbbk & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & \Bbbk \\ 0 & \Bbbk \end{pmatrix}, \quad P(2) = \begin{pmatrix} 0 & 0 \\ 0 & \Bbbk \end{pmatrix},$$

and the cokernel $\operatorname{coker}(P(2) \to P(1) \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$. Then each Λ-module M is isomorphic to the direct sum $P(1)^{\oplus t_1} \oplus P(2)^{\oplus t_2} \oplus (P(1)/P(2))^{\oplus t_3}$ for some $t_1, t_2, t_3 \in \mathbb{N}$. Assume that $M = (M, h_M, \|\cdot\|_M)$ and $N = (N, h_N, \|\cdot\|_N)$ are two normed Λ-modules. Then, naturally, $M \oplus N$ is also a Λ-module, where the left Λ-action is the map

$$h_M \oplus h_N := \begin{pmatrix} h_M & 0 \\ 0 & h_N \end{pmatrix} : \Lambda \times M \oplus N \to M \oplus N$$

which sends $(a, \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right))$ to $\left(\begin{smallmatrix} h_M & 0 \\ 0 & h_N \end{smallmatrix}\right)\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = \left(\begin{smallmatrix} (h_M)_a(m) \\ (h_N)_a(n) \end{smallmatrix}\right) = \left(\begin{smallmatrix} am \\ an \end{smallmatrix}\right)$. Furthermore, we can use the τ-norms of M and N, i.e., $\|\cdot\|_M$ and $\|\cdot\|_N$, to define a τ-norm $\|\cdot\|_{M\oplus N}$ of $M \oplus N$ by

$$\|(m, n)\|_{M\oplus N} := (|k|(\|m\|_M^p + \|n\|_N^p))^{\frac{1}{p}} \quad \text{for given } k \in \Bbbk\backslash\{0\}.$$

Then we have

$$\begin{aligned}\|a(m, n)\|_{M\oplus N} &= (|k|(\|am\|_M^p + \|an\|_N^p))^{\frac{1}{p}} = (|k|(|\tau(a)|^p\|m\|_M^p + |\tau(a)|^p\|n\|_N^p))^{\frac{1}{p}} \\ &= |\tau(a)|\,(|k|(\|m\|_M^p + \|n\|_N^p))^{\frac{1}{p}} = |\tau(a)|\|(m, n)\|_{M\oplus N}\end{aligned}$$

for any $a \in \Lambda$.

Example 4.4. The quiver of the $\Bbbk$-algebra Λ given in Example 4.3 is

$$\mathcal{Q} = 1 \xrightarrow{\alpha} 2.$$

By representation theory, all Λ-modules M can be represented by

$$M_1 \xrightarrow{\varphi_a} M_2,$$

where M_1 and M_2 are two $\Bbbk$-vector spaces and φ_a is a $\Bbbk$-linear map. Indeed, the identity element of Λ is $\boldsymbol{E} = \boldsymbol{E}_{11} + \boldsymbol{E}_{22}$, where $\{\boldsymbol{E}_{11}, \boldsymbol{E}_{22}\}$ is the complete set of primitive orthogonal idempotents. Thus, M, as a $\Bbbk$-vector space, has a decomposition $M = \boldsymbol{E}_{11}M \oplus \boldsymbol{E}_{22}M$ (because $\boldsymbol{E}_{11}\boldsymbol{E}_{22} = 0$ yields $\boldsymbol{E}_{11}M \cap \boldsymbol{E}_{22}M = 0$). For any $a = k_{11}\boldsymbol{E}_{11} + k_{22}\boldsymbol{E}_{22} + k_{21}\boldsymbol{E}_{21}$ and $m \in M$, we have

$$\begin{aligned} am &= (k_{11}\boldsymbol{E}_{11} + k_{22}\boldsymbol{E}_{22} + k_{21}\boldsymbol{E}_{21})(\boldsymbol{E}_{11}m + \boldsymbol{E}_{22}m) \\ &= k_{11}\boldsymbol{E}_{11}(\boldsymbol{E}_{11}m) + k_{22}\boldsymbol{E}_{22}(\boldsymbol{E}_{22}m) + k_{21}\boldsymbol{E}_{21}(\boldsymbol{E}_{11}m) \\ &= k_{11}(h_M)_{\boldsymbol{E}_{11}}(\boldsymbol{E}_{11}m) + k_{22}(h_M)_{\boldsymbol{E}_{22}}(\boldsymbol{E}_{22}m) + k_{21}(h_M)_{\boldsymbol{E}_{21}}(\boldsymbol{E}_{11}m) \\ &= (h_M)_{\boldsymbol{E}_{11}}(k_{11}\boldsymbol{E}_{11}m) + (h_M)_{k_{22}\boldsymbol{E}_{22}}(\boldsymbol{E}_{22}m) + (h_M)_{\boldsymbol{E}_{21}}(k_{21}\boldsymbol{E}_{11}m), \end{aligned} \tag{4.3}$$

where

(a) $h_M : \Lambda \to \mathrm{End}_{\Bbbk}M$ is a homomorphism of $\Bbbk$-algebras sending a to $(h_M)_a$, which satisfies $\mathbf{1}_M = (h_M)_{\boldsymbol{E}} = (h_M)_{\boldsymbol{E}_{11}} + (h_M)_{\boldsymbol{E}_{22}}$;

(b) $(h_M)_{\boldsymbol{E}_{ii}} = \mathbf{1}_{\boldsymbol{E}_{ii}M}$ $(i = 1, 2)$;

(c) $(h_M)_{\boldsymbol{E}_{12}} : \boldsymbol{E}_{11}M \to \boldsymbol{E}_{22}M$ is a $\Bbbk$-linear map (this is equivalent to (4.3)).

Therefore, we obtain that the representation corresponding to $M = \boldsymbol{E}_{11}M \oplus \boldsymbol{E}_{22}M$ is

$$\boldsymbol{E}_{11}M \xrightarrow{\boldsymbol{E}_{21}} \boldsymbol{E}_{22}M.$$

Generally,

$$M_1 \xrightarrow{\varphi_a} M_2$$

corresponds to the module $M_1 \oplus M_2$, where the Λ-action $\Lambda \times M_1 \oplus M_2 \to M_1 \oplus M_2$ is defined by $\boldsymbol{E}_{11}(m_1, m_2) = (m_1, 0)$, $\boldsymbol{E}_{22}(m_1, m_2) = (0, m_2)$ and $\boldsymbol{E}_{12}(m_1, m_2) = (0, \varphi_\alpha(m_1))$. Without loss of generality, for any representation

$$M_1 \xrightarrow{\varphi_a} M_2$$

of $\mathcal{Q}$, we assume that $M_1 = \Bbbk^{\oplus t_1}$, $M_2 = \Bbbk^{\oplus t_2}$ and $\varphi_a \in \mathbf{Mat}_{t_2 \times t_1}(\Bbbk)$ (up to Λ-isomorphism), and for any $i = 1, 2$, M_i is a normed space equipping with the norm $\|\cdot\|_{M_i} : M_i = \Bbbk^{\oplus t_i} \to \mathbb{R}^+$ sending $m_i = (m_{ij})_{1 \leqslant j \leqslant t_i}$ to $(\sum_{j=1}^{t_i} |m_{ij}|^p)^{\frac{1}{p}}$. Then we can define a norm $\|\cdot\|_{M_1 \oplus M_2}$ by

$$\|(m_1, m_2)\|_{M_1 \oplus M_2} = (|k|(\|m_1\|_{M_1}^p + \|m_2\|_{M_2}^p))^{\frac{1}{p}},$$

where k is a given element in $\Bbbk \backslash \{0\}$. The direct sum $\oplus$ of $\Bbbk$-vector spaces is the p powers of the norm preserving in the case for $k = 1$, i.e., $\|(m_1, m_2)\|_{M_1 \oplus M_2}^p = \|m_1\|_{M_1}^p + \|m_2\|_{M_2}^p$. Furthermore, if $\|\cdot\|_{M_1}$ and $\|\cdot\|_{M_2}$ are τ-norms of M_1 and M_2, respectively, then for any $a \in \Lambda$, we have

$$\begin{aligned} \|a(m_1, m_2)\|_{M_1 \oplus M_2} &= \big(|k|(\|am_1\|_{M_1}^p + \|am_2\|_{M_2}^p)\big)^{\frac{1}{p}} \\ &= \big(|k|(|\tau(a)|^p\|m_1\|_{M_1}^p + |\tau(a)|^p\|m_2\|_{M_2}^p)\big)^{\frac{1}{p}} \\ &= |\tau(a)|\big(|k|(\|m_1\|_{M_1}^p + \|m_2\|_{M_2}^p)\big)^{\frac{1}{p}} \\ &= |\tau(a)|\|a(m_1, m_2)\|_{M_1 \oplus M_2}. \end{aligned}$$

4.2 Completions of normed $\mathit{\Lambda}$-modules

Let $N = (N, h, \|\cdot\|)$ be a normed $\mathit{\Lambda}$-module. In this subsection, we construct its completion. For us, we only need the completion in the finite-dimensional $\Bbbk$-algebra $\mathit{\Lambda}$ case. Otherwise, there is at least one $\mathit{\Lambda}$-module which is not complete, for example, $\mathit{\Lambda}$ is a non-complete $\mathit{\Lambda}$-module. Therefore, we assume that $\Bbbk$ is complete in this subsection by Propositions 2.16 and 3.3.

Similar to finite-dimensional $\Bbbk$-algebras, we can define open neighborhoods $B(0, r)$ of 0 for any normed $\mathit{\Lambda}$-module $N = (N, h, \|\cdot\|)$ by

$$B(0, r) := \{x \in N \mid \|x\| < r\}.$$

Let $\mathfrak{U}_N^B(0)$ be the class of all subsets U of N satisfying the following conditions:

(1) U is the intersection of a finite number of $B(0, r)$;

(2) U is the union of any number of $B(0, r)$.

Then $\mathfrak{U}_N^B(0)$ is a topology defined on N, and we can define the Cauchy sequence by the above topology.

Lemma 4.5. *Let $\mathfrak{C}^*(N)$ be the set of all Cauchy sequences in the normed $\mathit{\Lambda}$-module $N = (N, h, \|\cdot\|)$. Then $\mathfrak{C}^*(N)$ is a $\mathit{\Lambda}$-module.*

Proof. First of all, $\mathfrak{C}^*(N)$ is a $\Bbbk$-vector space whose addition and $\Bbbk$-action are given by $\{x_i\}_{i\in\mathbb{N}} + \{y_i\}_{i\in\mathbb{N}} = \{x_i + y_i\}_{i\in\mathbb{N}}$ ($\forall\, \{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}} \in \mathfrak{C}^*(N)$) and $k\{x_i\}_{i\in\mathbb{N}} = \{kx_i\}_{i\in\mathbb{N}}$ ($\forall\, k \in \Bbbk$), respectively. Furthermore, define

$$\mathit{\Lambda} \times \mathfrak{C}^*(N) \to \mathfrak{C}^*(N), \quad (a, \{x_i\}_{i\in\mathbb{N}}) \mapsto a \cdot \{x_i\}_{i\in\mathbb{N}} := \{a \cdot x_i\}_{i\in\mathbb{N}},$$

where $a \cdot x_i = h_a(x_i)$. Then $\mathfrak{C}^*(N)$ is a $\mathit{\Lambda}$-module. □

Two Cauchy sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ in N are called *equivalent*, denoted by $\{x_i\}_{i\in\mathbb{N}} \sim \{y_i\}_{i\in\mathbb{N}}$, if for any $U \in \mathfrak{U}_N^B(0)$, there is $r \in \mathbb{N}$ such that $x_s - x_t \in U$ holds for all $s, t \geqslant r$. It is easy to see that "$\sim$" is an equivalence relation. Let $[\{x_i\}_{i\in\mathbb{N}}]$ be the equivalent class of Cauchy sequences containing $\{x_i\}_{i\in\mathbb{N}}$ and let $\mathfrak{C}(N)$ be the set of all equivalent classes. We naturally obtain a map

$$h : \mathfrak{C}^*(N) \to \mathfrak{C}(N), \quad \{x_i\}_{i\in\mathbb{N}} \mapsto [\{x_i\}_{i\in\mathbb{N}}].$$

We can show that $\mathfrak{C}(N)$ is a $\mathit{\Lambda}$-module by using an argument similar to that in the proof of Lemma 4.5, and further obtain $\mathrm{Ker}(h : \mathfrak{C}^*(N) \to \mathfrak{C}(N)) = [\{0\}_{i\in\mathbb{N}}]$. Thus we have

$$\mathfrak{C}(N) \cong \mathfrak{C}^*(N)/[\{0\}_{i\in\mathbb{N}}].$$

Then $\mathfrak{C}(N)$ is complete, and we call it the *completion* of N. We use $\widehat{N}$ to denote the completion $\mathfrak{C}(N)$ of N. The $\mathit{\Lambda}$-module $\widehat{N}$ is a normed $\mathit{\Lambda}$-module, where the norm defined on $\widehat{N}$ is induced by the norm $\|\cdot\| : N \to \mathbb{R}^{\geqslant 0}$ defined on N.

Definition 4.6. Assume that $\mathit{\Lambda}$ is complete. A normed $\mathit{\Lambda}$-module N is called a *Banach $\mathit{\Lambda}$-module* if $\widehat{N} = N$ (i.e., N is complete).

4.3 $\boldsymbol{\sigma}$-algebras and the elementary simple function set $\mathbf{S}_{\boldsymbol{\tau}}(\mathbb{I}_{\mathit{\Lambda}})$

Lemma 4.7. *Take τ to be a homomorphism of $\Bbbk$-algebras $\tau : \mathit{\Lambda} \to \Bbbk$. Then the elementary simple function set $\mathbf{S}(\mathbb{I}_{\mathit{\Lambda}})$ with the above homomorphism τ, denoted by $\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}})$, is a $\mathit{\Lambda}$-module, where the $\mathit{\Lambda}$-action $\mathit{\Lambda} \times \mathbf{S}(\mathbb{I}_{\mathit{\Lambda}}) \to \mathbf{S}(\mathbb{I}_{\mathit{\Lambda}})$ is given by*

$$\left(a, f = \sum_{i=1}^{t} k_i \mathbf{1}_{I_i}\right) \mapsto af := \sum_{i=1}^{t} \tau(a) k_i \mathbf{1}_{I_i}.$$

Proof. For all $a \in \mathit{\Lambda}$, $a' \in \mathit{\Lambda}$, $k \in \Bbbk$, $f = \sum_i k_i \mathbf{1}_{I_i} \in \mathbf{S}(\mathbb{I}_{\mathit{\Lambda}})$ and $f' = \sum_j k'_j \mathbf{1}_{I'_j} \in \mathbf{S}(\mathbb{I}_{\mathit{\Lambda}})$, the following conditions are satisfied:

(1) $a(f + f') = af + af'$ (trivial).

(2) $(a+a')f = af + a'f$ (trivial).

(3) $(aa')f = a(a'f)$ because

$$(aa')f = (aa')\sum_i k_i\mathbf{1}_{I_i} = \sum_i \tau(aa')k_i\mathbf{1}_{I_i} = \sum_i \tau(a)\tau(a')k_i\mathbf{1}_{I_i}$$
$$= a\sum_i \tau(a')k_i\mathbf{1}_{I_i} = a\bigg(a'\sum_i k_i\mathbf{1}_{I_i}\bigg) = a(a'f).$$

(4) $1f = f$ (trivial).

(5) We have

- $(ka)f = (ka)\sum_i k_i\mathbf{1}_{I_i} = \sum_i \tau(ka)(k_i\mathbf{1}_{I_i})$;
- $k(af) = k(a\sum_i k_i\mathbf{1}_{I_i}) = k\sum_i \tau(a)k_i\mathbf{1}_{I_i} = \sum_i k(\tau(a)(k_i\mathbf{1}_{I_i}))$;
- $a(kf) = a\sum_i k(k_i\mathbf{1}_{I_i}) = \sum_i \tau(a)(k(k_i\mathbf{1}_{I_i}))$.

Since τ is a homomorphism of $\Bbbk$-algebras, we have

$$\tau(ka)(k_i\mathbf{1}_{I_i}) = k(\tau(a)(k_i\mathbf{1}_{I_i})) = \sum_i \tau(a)(k(k_i\mathbf{1}_{I_i})) = \sum_i kk_i\tau(a)\mathbf{1}_{I_i}$$

for all i. Then $(ka)f = k(af) = a(kf)$. □

Now, we introduce a norm for $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ such that it is a normed Λ-module. To do this, we first recall the definition of σ-algebras. The main use of σ-algebras is in the definition of measures. It is important in mathematical analysis and probability theory. In mathematical analysis, it is the foundation for Lebesgue integration, and in probability theory, it is interpreted as the collection of events that can be assigned probabilities (see, for example, [15, p. 12], [25, p. 10] and [37, p. 8]).

Definition 4.8. Let S be a set and let $P(S)$ be the set of all subsets of S, which is called the power set of S. A *σ-algebra* is a subset $\mathcal{A}$ of $P(S)$ satisfying the following conditions:

(1) $\emptyset$ and S lie in $\mathcal{A}$;

(2) for any $X \in \mathcal{A}$, the complement set $X^c := S\backslash X$ of X lies in $\mathcal{A}$;

(3) for any $X_1, \dots, X_n, \dots \in \mathcal{A}$, the union $\bigcup_{i=1}^\infty X_i$ is an element in $\mathcal{A}$.

For a class $\mathcal{C}$ of some sets lying in $P(S)$, we call $\mathcal{A}$ a *σ-algebra generated by* $\mathcal{C}$ if $\mathcal{A}$ is the minimal σ-algebra containing $\mathcal{C}$.

Let $\Sigma_\Bbbk$ be the σ-algebra generated by $\{(a,b)_\Bbbk, [a,b)_\Bbbk, (a,b]_\Bbbk, [a,b]_\Bbbk \mid a \preceq b\}$, and let $\mu : \Sigma_\Bbbk \to \mathbb{R}^{\geqslant 0}$ be a *measure* such that $\mu(\{k\}) = 0$ holds for any $k \in \Bbbk$, i.e., μ is a function satisfying the following conditions:

(1) $\mu(\emptyset) = 0$;

(2) $\mu(\bigcup_{i\in\mathbb{N}} X_i) = \sum_{i\in\mathbb{N}} \mu(X_i)$ holds for all sets $X_1, X_2, \dots$ satisfying $X_i \cap X_j = \emptyset$ $(i \neq j)$.

Any two functions f and g in $\mathbf{S}(\mathbb{I}_\Lambda)$ are called *equivalent* if

$$\mu(\{\boldsymbol{k} = (k_1, \dots, k_n) \in \Bbbk^{\oplus n} \mid f(\boldsymbol{k}) \neq g(\boldsymbol{k})\}) = 0.$$

The equivalent class containing f is written as $[f]$. Then we obtain an epimorphism

$$\mathbf{S}(\mathbb{I}_\Lambda) \to \overline{\mathbf{S}(\mathbb{I}_\Lambda)} := \{[f] \mid f \in \mathbf{S}(\mathbb{I}_\Lambda)\}$$

sending each function to its equivalent classes. It is easy to see that the kernel of the above epimorphism is $[0]$. Then we have

$$\overline{\mathbf{S}(\mathbb{I}_\Lambda)} \cong \mathbf{S}(\mathbb{I}_\Lambda)/[0].$$

For simplification, we do not differentiate between two equivalent functions under the above isomorphism. Therefore, we treat $\mathbf{S}(\mathbb{I}_\Lambda)$ and the quotient $\overline{\mathbf{S}(\mathbb{I}_\Lambda)}$ equivalently.

Lemma 4.9. *Let $\tau : \Lambda \to \Bbbk$ be a homomorphism between two $\Bbbk$-algebras. Then the Λ-module $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ with the map*

$$\|\cdot\|_p : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to \mathbb{R}^{\geqslant 0}, \quad f = \sum_{i=1}^t k_i\mathbf{1}_{I_i} \mapsto \bigg(\sum_{i=1}^t (|k_i|\mu(I_i))^p\bigg)^{\frac{1}{p}}$$

is normed.

Proof. Let f be an arbitrary function lying in $\mathbf{S}(\mathbb{I}_\Lambda)$. It is trivial that $\|f\|_p$ is non-negative. Let a be an arbitrary element in Λ and assume $f=\sum_{i=1}^t k_i\mathbf{1}_{I_i}$. We have

$$\begin{aligned}\|af\|_p&=\Big\|\sum_{i=1}^t\tau(a)k_i\mathbf{1}_{I_i}\Big\|_p=\bigg(\sum_{i=1}^t|\tau(a)k_i|^p\mu(\mathbf{1}_{I_i})^p\bigg)^{\frac{1}{p}}\\&=|\tau(a)|\cdot\bigg(\sum_{i=1}^t|k_i|^p\mu(\mathbf{1}_{I_i})^p\bigg)^{\frac{1}{p}}=|\tau(a)|\cdot\|f\|_p,\end{aligned}$$

which satisfies the formula (4.1). In particular, if $\|f\|_p=0$, then so is $(|k_i|\mu(I_i))^p=0$ for all i, and we have $|k_i|=0$ in the case for $\mu(I_i)\neq0$. If $\mu(I_j)=0$ holds for some $j\in J$ ($\subseteq\{1,2,\dots,t\}$), then we have $f=\sum_{j\in J}k_j\mathbf{1}_{I_j}$. Clearly,

$$\mu(\{x\in\mathbb{I}_\Lambda\mid f(x)\neq0\})=\sum_{j\in J}\mu(I_j)=0,$$

i.e., $f=0$ in treating $\mathbf{S}(\mathbb{I}_\Lambda)$ and the quotient $\overline{\mathbf{S}(\mathbb{I}_\Lambda)}$ equivalently. Thus, $\|f\|_p=0$ if and only if $f=0$.

Next, we prove the triangle inequality. For two arbitrary functions $f=\sum_i k_i\mathbf{1}_{I_i}$ and $g=\sum_j l_j\mathbf{1}_{I'_j}$, we have

$$f+g=\sum_i k_i\mathbf{1}_{I_i\backslash\bigcup_j I'_j}+\sum_j l_j\mathbf{1}_{I'_j\backslash\bigcup_i I_i}+\sum_{I_i\cap I'_j\neq\emptyset}(k_i\mathbf{1}_{I_i\cap I'_j}+l_j\mathbf{1}_{I_i\cap I'_j}) \tag{4.4}$$

by $I_i\cap I_\imath=\emptyset$ ($\forall\, i\neq\imath$) and $I'_j\cap I'_\jmath=\emptyset$ ($\forall\, j\neq\jmath$). Then we can compute the norm of $f+g$ by (4.4) as the following formula:

$$\|f+g\|_p=(R+G+B)^{\frac{1}{p}},$$

where

$$\begin{aligned}R&=\sum_i|k_i|^p\mu\bigg(I_i\backslash\bigcup_j I'_j\bigg)^p,\\G&=\sum_j|l_j|^p\mu\bigg(I'_j\backslash\bigcup_i I_i\bigg)^p,\\B&=\sum_{I_i\cap I'_j\neq\emptyset}(|k_i|^p+|l_j|^p)\mu(I_i\cap I'_j)^p.\end{aligned}$$

On the other hand, we have the following inequality by the discrete Minkowski inequality:

$$\begin{aligned}\|f\|_p+\|g\|_p&=\bigg(\sum_i|k_i|^p\mu(I_i)^p\bigg)^{\frac{1}{p}}+\bigg(\sum_j|l_i|^p\mu(I'_i)^p\bigg)^{\frac{1}{p}}\\&\geqslant\bigg(\sum_i|k_i|^p\mu(I_i)^p+\sum_j|l_i|^p\mu(I'_i)^p\bigg)^{\frac{1}{p}}=:\mathfrak{S}.\end{aligned}\tag{4.5}$$

Since by the definition of measure, $\mu(X\cup Y)=\mu(X)+\mu(Y)$ holds for any X and Y with $X\cap Y=\emptyset$, we obtain

$$\mu(X\cup Y)^p\geqslant\mu(X)^p+\mu(Y)^p,\tag{4.6}$$

and then

$$\mu(I_i)^p\geqslant\mu\bigg(I_i\backslash\bigcup_j I'_j\bigg)^p+\mu\bigg(I_i\cap\bigcup_j I'_j\bigg)^p.$$

Thus,

$$\sum_i|k_i|^p\mu(I_i)^p\geqslant\sum_i|k_i|^p\mu\bigg(I_i\backslash\bigcup_j I'_j\bigg)^p+\sum_i|k_i|^p\mu\bigg(I_i\cap\bigcup_j I'_j\bigg)^p$$

$$= R + \sum_i |k_i|^p \Bigg(\sum_{\substack{j \\ I_i \cap I'_j \neq \emptyset}} \mu(I_i \cap I'_j) \Bigg)^p$$

$$\overset{(4.6)}{\geqslant} R + \sum_{I_i \cap I'_j \neq \emptyset} |k_i|^p \mu(I_i \cap I'_j)^p. \tag{4.7}$$

Similarly,

$$\sum_j |l_j|^p \mu(I'_j)^p \geqslant G + \sum_{I'_j \cap I_i \neq \emptyset} |l_j|^p \mu(I'_j \cap I_i)^p. \tag{4.8}$$

Notice that

$$\sum_{I_i \cap I'_j \neq \emptyset} |k_i|^p \mu(I_i \cap I'_j)^p + \sum_{I'_j \cap I_i \neq \emptyset} |l_j|^p \mu(I'_j \cap I_i)^p = \sum_{I_i \cap I'_j \neq \emptyset} (|k_i|^p + |l_j|^p) \mu(I_i \cap I'_j)^p = B,$$

and then (4.7)+(4.8) induces $\mathfrak{S}^p \geqslant R + G + B$. Thus, the triangle inequality $\|f\|_p + \|g\|_p \geqslant \|f+g\|_p$ holds. $\square$

5 The categories $\mathscr{N}\!or^p$ and $\mathscr{A}^p$

Recall that a measure defined on $\Sigma_{\Bbbk}$ is a countable additive function $\mu : \Sigma_{\Bbbk} \to \mathbb{R}^{\geqslant 0}$ with $\mu(\emptyset) = 0$. Naturally, it induces a measure, still written as μ, defined on some σ-algebra of $\varLambda$ such that for any $\sum_{i=1}^n I_i b_i$ ($I_i \in \Sigma_{\Bbbk}$ is measurable), the equation $\mu(\sum_{i=1}^n I_i b_i) = \prod_{i=1}^n \mu(I_i)$ holds.

Let $\dim_{\Bbbk} \varLambda = n$, and let N be a normed $\varLambda$-module equipped with two additional pieces of data: an element $v \in N$ such that $\|v\| \leqslant \mu(\mathbb{I}_{\varLambda})$, and a continuous $\varLambda$-homomorphism $\delta : N^{\oplus_p 2^n} \to N$. Here, $\oplus_p$ denotes the direct sum of 2^n normed $\varLambda$-modules $X_1, \ldots, X_{2^n}$ with the norm defined as follows:

$$\|\cdot\|_p : \bigoplus_{i=1}^{2^n}{}_p\, X_i \to \mathbb{R}^{\geqslant 0}, \quad (x_1, x_2, \ldots, x_{2^n}) \mapsto \Bigg(\bigg(\frac{\mu(\mathbb{I})}{\mu(\mathbb{I}_{\varLambda})} \bigg)^n \sum_{i=1}^{2^n} \|x_i\|^p \Bigg)^{\frac{1}{p}}.$$

5.1 The categories $\mathscr{N}\!or^p$ and $\mathscr{A}^p$

Let $\mathscr{N}\!or^p$ be a class of triples which are of the form (N, v, δ), where N is a normed $\varLambda$-module, $v \in N$ is an element with $\|v\|_p \leqslant \mu(\mathbb{I}_{\varLambda})$ and $\delta : N^{\oplus_p 2^n} \to N$ is a $\varLambda$-homomorphism satisfying $\delta(v, v, \ldots, v) = v$ such that for any Cauchy sequence $\{x_i\}_{i \in \mathbb{N}} \in \widehat{N^{\oplus_p 2^n}} \cong \widehat{N}^{\oplus_p 2^n}$, the commutativity

$$\varprojlim \delta(x_i) = \delta(\varprojlim x_i) \tag{5.1}$$

of the inverse limit and the $\varLambda$-homomorphism holds. For any two triples (N, v, δ) and (N', v', δ') in $\mathscr{N}\!or^p$, we define the morphism $(N, v, \delta) \to (N', v', \delta')$ to be the $\varLambda$-homomorphism $\theta : N \to N'$ with $\theta(v) = v'$ such that the diagram

$$\begin{tikzcd} N^{\oplus_p 2^n} \arrow[r, "\delta"] \arrow[d, "\theta^{\oplus 2^n} = \begin{pmatrix} \theta & & \\ & \ddots & \\ & & \theta \end{pmatrix}_{2^n \times 2^n}"'] & N \arrow[d, "\theta"] \\ N'^{\oplus_p 2^n} \arrow[r, "\delta'"'] & N' \end{tikzcd}$$

commutes, i.e., for any $(v_1, \ldots, v_{2^n}) \in N^{\oplus_p 2^n}$, $\theta(\delta(v_1, \ldots, v_{2^n})) = \delta'(\theta(v_1), \ldots, \theta(v_{2^n}))$. Then it is easy to check that $\mathscr{N}\!or^p$ is a category.

Lemma 5.1. *Let*

(1) ξ *be an element in* $\mathbb{I} = [a, b]_{\Bbbk}$ *with* $a \prec \xi \prec b$ *such that the order-preserving bijections* $\kappa_a : \mathbb{I} \to [a, \xi]_{\Bbbk}$ *and* $\kappa_b : \mathbb{I} \to [\xi, b]_{\Bbbk}$ *exist*;

(2) $\mathbf{1}$ *be the identity function* $\mathbf{1}_{\mathbb{I}_{\mathit{\Lambda}}} : \mathbb{I}_{\mathit{\Lambda}} \to \{1\}$;

(3) γ_ξ *be the map given in* (3.2);

(4) $\tau : \mathit{\Lambda} \to \Bbbk$ *be the homomorphism of* $\Bbbk$*-algebras given in Lemma* 4.9.

Then the following statements hold:

(a) $\gamma_\xi(\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$;

(b) γ_ξ *is a* $\mathit{\Lambda}$*-homomorphism.*

First, we provide a remark for the above lemma.

Remark 5.2. Indeed, $(\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}}), \mathbf{1}, \gamma_\xi)$ is an object in the category $\mathscr{N}\!or^p$. However, Lemma 5.1 points out that $\gamma_\xi(\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$ and γ_ξ is a $\mathit{\Lambda}$-homomorphism. Thus, we need to show that the commutativity of the inverse limit and γ_ξ holds. We prove this result in the following content, as shown in Lemma 5.5.

Next, we prove Lemma 5.1.

Proof of Lemma 5.1. (a) We have that $\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}})$ is a normed $\mathit{\Lambda}$-module by Lemma 4.9, and γ_ξ is a $\Bbbk$-linear map by Lemma 3.6. The formula $\gamma_\xi(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$ can be directly induced by the definition of γ_ξ.

(b) Take $\lambda \in \mathit{\Lambda}$, $f \in \mathbf{S}(\mathbb{I}_{\mathit{\Lambda}})$ and let $(k_i)_i$, $\mathbf{1}$ and $(\delta_i)_i$ be an arbitrary element $(k_1, \ldots, k_n)$ in $\mathbf{S}(\mathbb{I}_{\mathit{\Lambda}})^{\oplus 2^n}$, the identity function $\mathbf{1}_{\kappa_{\delta_1}(\mathbb{I}) \times \cdots \times \kappa_{\delta_n}(\mathbb{I})}$ and the n-multiple $(\delta_1 \times \cdots \times \delta_n)$, respectively. Then we have

$$\begin{aligned}\gamma_\xi(\lambda \cdot f)((k_i)_i) &= \sum_{(\delta_i)_i} \mathbf{1} \cdot (\tau(\lambda) f)_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) \\ &= \tau(\lambda)\gamma_\xi(f)((k_i)_i) \text{ (similar to Lemma 3.6)} \\ &= \lambda \cdot \gamma_\xi(f)((k_i)_i).\end{aligned}$$

Thus γ_ξ is a $\mathit{\Lambda}$-homomorphism. □

Let $\mathscr{A}^p$ denote a class of triples which are of the form $(\widehat{N}, v, \widehat{\delta})$, where $\widehat{N}$ is a Banach $\mathit{\Lambda}$-module (see Definition 4.6), $v \in \widehat{N}$ is an element with $\|v\| \leqslant \mu(\mathbb{I}_{\mathit{\Lambda}})$ and $\widehat{\delta} : \widehat{N}^{\oplus_p 2^n} \to \widehat{N}$ is a $\mathit{\Lambda}$-homomorphism satisfying $\widehat{\delta}(v, v, \ldots, v) = v$. Obviously, $\mathscr{A}^p$ is a full subcategory of $\mathscr{N}\!or^p$.

5.2 The triple $(\mathbf{S}_{\boldsymbol{\tau}}(\mathbb{I}_{\mathit{\Lambda}}), \mathbf{1}, \boldsymbol{\gamma_\xi})$

Let (N, v, δ) be an object in $\mathscr{N}\!or^p$ and $\widehat{N}$ be the completion of the $\mathit{\Lambda}$-module N. Then $\widehat{N}$, as a $\Bbbk$-vector space, is a Banach space which is a Banach $\mathit{\Lambda}$-module. In addition, naturally, we obtain the $\mathit{\Lambda}$-homomorphism

$$\widehat{\delta} : \widehat{N}^{\oplus_p 2^n} \to \widehat{N}$$

induced by the $\mathit{\Lambda}$-homomorphism δ. Furthermore, we have that $(\widehat{N}, v, \widehat{\delta})$ is also an object in $\mathscr{N}\!or^p$, and there is a naturally embedding morphism

$$\mathrm{emb} : (N, v, \delta) \hookrightarrow (\widehat{N}, v, \widehat{\delta})$$

which is induced by $N \subseteq \widehat{N}$.

Notation 5.3. Keep the notations $\xi =: \xi_{11}$, κ_a, κ_b, $\mathbf{1}$, γ_ξ and τ as in Lemma 5.1. Then ξ_{11} divides $\mathbb{I} =: \mathbb{I}^{(01)}$ into two subsets $[a, \xi_{11}]_{\Bbbk} =: \mathbb{I}^{(11)}$ and $[\xi_{11}, b]_{\Bbbk} =: \mathbb{I}^{(12)}$. Next, let $\xi_{22} = \xi_{11}$ $(= \xi)$, and denote by ξ_{21} and ξ_{23} the two elements in $\mathbb{I}_{\mathit{\Lambda}}$ such that

- $a \prec \xi_{21} = \kappa_a\kappa_a(b) = \kappa_a\kappa_b(a) = \kappa_b\kappa_a(a) = \kappa_a(\xi_{11}) \prec \xi_{22}$;
- $\xi_{22} \prec \xi_{23} = \kappa_b\kappa_b(a) = \kappa_b\kappa_a(b) = \kappa_b\kappa_a(b) = \kappa_b(\xi_{11}) \prec b$.

Then $\mathbb{I}$ is divided into four subsets, which are of the form $\mathbb{I}^{(2\ t+1)} = [\xi_{2t}, \xi_{2\ t+1}]_{\Bbbk}$ $(0 \leqslant t \leqslant 3)$ by $a = \xi_{20} \prec \xi_{21} \prec \xi_{22} \prec \xi_{23} \prec \xi_{24} = b$. Repeating the above step t times, we obtain a sequence of $2^t - 1$ elements lying in $\mathbb{I}_{\mathit{\Lambda}}$:

$$a = \xi_{t0} \prec \xi_{t1} \prec \xi_{t2} \prec \cdots \prec \xi_{t2^t} = b,$$

all 2^t subsets which are of the form $\mathbb{I}^{(t\ s+1)} = [\xi_{ts}, \xi_{t\ s+1}]_{\Bbbk}$, and 2^t order-preserving bijections $\kappa_{\xi_{ts}}$: $\mathbb{I}^{(t\ s+1)} \to \mathbb{I}^{(01)}$.

For any family of subsets $(\mathbb{I}^{(u_i v_i)})_{1\leqslant i\leqslant n}$ $(1 \leqslant v_i \leqslant 2^{u_i})$, we denote by $\mathbf{1}_{(u_i v_i)_i}$ the function

$$\mathbf{1}_{(u_i v_i)_i} := \mathbf{1}_{\mathbb{I}_\Lambda}\big|_{\prod_{i=1}^n \mathbb{I}^{(u_i v_i)}} : \mathbb{I}_\Lambda \to \{0,1\}, \quad x \mapsto \begin{cases} 1, & x \in \prod\limits_{i=1}^n \mathbb{I}^{(u_i v_i)}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbb{I}^{(u_i v_i)} \cong \mathbb{I}^{(u_i v_i)} \times \{b_i\} \subseteq \mathbb{I}_\Lambda$ holds for all i and $B_\Lambda = \{b_i \mid 1 \leqslant i \leqslant n\}$ is the $\Bbbk$-basis of Λ.

Let E_u be the set of all step functions constant on each of $\prod_{i=1}^n \mathbb{I}^{(u_i v_i)}$ $(1 \leqslant v_i \leqslant 2^{u_i}$ for all $i)$, i.e., every step function in E_u is of the form

$$\sum_{(u_i v_i)_i} k_{(u_i v_i)_i} \mathbf{1}_{(u_i v_i)_i},$$

where each $k_{(u_i v_i)_i}$ lies in $\Bbbk$, the number of summands is $(2^u)^n = 2^{un}$, and each $(u_i v_i)_i$ corresponds to the Cartesian product $\prod_{i=1}^n \mathbb{I}^{(u_i v_i)}$. Then it is easy to check that each E_u is a normed submodule of $\mathbf{S}(\mathbb{I}_\Lambda)$, and $E_u \subseteq E_{u+1}$ because each step function constant on each of $\mathbb{I}^{(uv)}$ is equivalent to a step function constant on each of $\mathbb{I}^{(u+1\ v)}$. Thus,

$$\Bbbk \cong E_0 \subseteq E_1 \subseteq \cdots \subseteq E_t \subseteq \cdots \subseteq \mathbf{S}(\mathbb{I}_\Lambda) \subseteq \widehat{\mathbf{S}(\mathbb{I}_\Lambda)}.$$

Moreover, for any $\mathbb{I}^{(uv)} = [\xi_{u\ v-1}, \xi_{uv}]_{\Bbbk}$, we have two cases (i) $\xi_{uv} \preceq \xi$ and (ii) $\xi \preceq \xi_{u\ v-1}$ by the definition of E_u. Therefore, we obtain a map

$$\mathfrak{p} : \{\mathbb{I}^{(uv)} \mid u \in \mathbb{N}\} \to \{a, b\}, \quad \mathbb{I}^{(uv)} \mapsto \begin{cases} a, & \mathbb{I}^{(uv)} \text{ lies in the case (i),} \\ b, & \mathbb{I}^{(uv)} \text{ lies in the case (ii).} \end{cases}$$

Now we use the above map to prove the following lemma.

Lemma 5.4. *The map $\gamma_\xi : \mathbf{S}(\mathbb{I}_\Lambda)^{\oplus_p 2^n} \to \mathbf{S}(\mathbb{I}_\Lambda)$ induces the following $\Bbbk$-linear map*:

$$\gamma_\xi : E_u^{\oplus_p 2^n} \xrightarrow{\cong} E_{u+1}$$

which is an isomorphism of Λ-modules.

Proof. The $\Bbbk$-vector space E_u is a Λ-module, where $\Lambda \times E_u \to E_u$ is defined by

$$\left(a, f = \sum_i 1 \cdot \mathbf{1}_{I_i}\right) \mapsto a \cdot f = \sum_i \tau(a) \cdot \mathbf{1}_{I_i}.$$

Then it is easy to see that γ_ξ is a Λ-homomorphism. Since $\mathrm{Ker}(\gamma_\xi) = 0$, we have that γ_ξ is injective. Next, we prove that it is also surjective.

Any step function $f : \Bbbk^{\oplus n} \to \Bbbk$ lying in E_{u+1} can be written as

$$f(k_1, \ldots, k_n) = \sum_{(u_i v_i)_i} f_i = \sum_{(\omega_1,\ldots,\omega_n)\in\{a,b\}\times\cdots\times\{a,b\}} f_{(\omega_1,\ldots,\omega_n)},$$

where

- $f_i = k_{(u_i v_i)_i} \mathbf{1}_{(u_i v_i)_i}$;
- $$f_{(\omega_1,\ldots,\omega_n)}(k_1, \ldots, k_n) = \sum_{\prod_{i=1}^n \mathfrak{p}(\mathbb{I}^{(u_i v_i)})=(\omega_1,\ldots,\omega_n)} f_i,$$

and thus the number of all summands of it is $(2^u)^n = 2^{un}$;

- the number of all summands of $\sum_{(\omega_1,\ldots,\omega_n)\in\{a,b\}\times\cdots\times\{a,b\}} f_{(\omega_1,\ldots,\omega_n)}$ is 2^n (thus the number of all summands of $\sum_{(u_i v_i)_i} f_i$ is $2^{un} \cdot 2^n = 2^{(u+1)n}$).

Then

$$\tilde{f}_{(\omega_1,\ldots,\omega_n)}(k_1,\ldots,k_n) = f_{(\omega_1,\ldots,\omega_n)}(\kappa_{\omega_1}^{-1}(k_1),\ldots,\kappa_{\omega_n}^{-1}(k_n)) \in E_u,$$

and γ_ξ sends $\{f_{(\omega_1,\ldots,\omega_n)}\}_{(\omega_1,\ldots,\omega_n)\in\{a,b\}\times\cdots\times\{a,b\}}$ to f by the definition of γ_ξ (see (3.2)). We obtain that γ_ξ is surjective. Therefore, γ_ξ is a Λ-isomorphism. □

By Lemma 5.4, the following result holds.

Lemma 5.5. *The commutativity of the inverse limit and the map* $\widehat{\gamma}_\xi : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ *induced by the completion of* $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ *holds, i.e., for any sequence* $\{\boldsymbol{f}_i\}_{i\in\mathbb{N}^+}$ *in* $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n}$*, if its inverse limit exists, then we have*

$$\widehat{\gamma}_\xi(\varprojlim \boldsymbol{f}_i) = \varprojlim \widehat{\gamma}_\xi(\boldsymbol{f}_i).$$

Furthermore, $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ *is an object in* $\mathscr{N}\!or^p$.

Proof. Since γ_ξ is a Λ-isomorphism, it is clear that $\widehat{\gamma}_\xi$ is also a Λ-isomorphism. Then, the commutativity of the inverse limit and the map $\widehat{\gamma}_\xi$ holds. Thus, for any sequence $\{\boldsymbol{f}_i\}_{i\in\mathbb{N}^+}$ in $\mathbf{S}_\tau(\mathbb{I}_\Lambda)^{\oplus_p 2^n}$, if its inverse limit exists, then this inverse limit is also an element in $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n}$, and so

$$\gamma_\xi(\varprojlim \boldsymbol{f}_i) = \widehat{\gamma}_\xi(\varprojlim \boldsymbol{f}_i) \overset{\spadesuit}{=} \varprojlim \widehat{\gamma}_\xi(\boldsymbol{f}_i) = \varprojlim \gamma_\xi(\boldsymbol{f}_i),$$

where ♠ holds since $\widehat{\gamma}_\xi$ is a Λ-isomorphism (see Lemma 5.4). Therefore, by Lemma 5.1, $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ is an object in $\mathscr{N}\!or^p$. □

5.3 $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ is a direct limit

Let $\mathsf{nor}\Lambda$ be the category of normed Λ-modules and Λ-homomorphisms between them. Then it is easy to check that all E_u are objects in $\mathsf{nor}\Lambda$. Furthermore, for any $u \leqslant v$, we have a Λ-homomorphism $\varphi_{uv} : E_u \to E_v$ which is induced by $E_u \subseteq E_v$. Thus we obtain a direct system $((E_i)_{i\in\mathbb{N}}, (\varphi_{uv})_{u\leqslant v})$ in $\mathsf{nor}\Lambda$ over $\mathbb{N}$. Let $\mathsf{Ban}(\Lambda)$ be the category of Banach Λ-modules and continuous Λ-homomorphisms between them. Then $\mathsf{Ban}(\Lambda)$ is a full subcategory of $\mathsf{nor}(\Lambda)$, and so, naturally, we obtain a direct system $((E_i)_{i\in\mathbb{N}}, (\varphi_{uv})_{u\leqslant v})$ in $\mathsf{Ban}(\Lambda)$ if Λ is a complete $\Bbbk$-algebra.

The following lemma establishes the relation between E_n and $\mathbf{S}(\mathbb{I}_\Lambda)$.

Lemma 5.6. *Let* Λ *be a complete* $\Bbbk$*-algebra. Consider the category* $\mathsf{Ban}(\Lambda)$ *and take* $(\alpha_i : E_i \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)})_{i\in\mathbb{N}}$*, where every* α_i *is the embedding given by* $E_i \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$*. Then*

$$\varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}.$$

Proof. Let X be an arbitrary object in $\mathsf{nor}\Lambda$ such that there is $(f_i : E_i \to X)_{i\in\mathbb{N}}$ satisfying $f_i\varphi_{ij} = f_j$ for all $i \leqslant j$. Then we can find the Λ-homomorphism $\theta : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \to X$ in the following way.

For any $x \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, there exists a sequence $\{x_t\}_{t\in\mathbb{N}}$ in $\bigcup_i E_i$ such that $\{\|x_t - x\|_p\}_t$ is a monotonically decreasing sequence of positive real numbers. Then we have

$$\varprojlim \{\|x_t - x\|_p\}_t = 0$$

by Example 2.4 which induces $\varprojlim x_t = x$. Since α_i, α_j and φ_{ij} ($\forall\, i \leqslant j$) are Λ-homomorphisms induced by $\subseteq$ (thus they are $\Bbbk$-linear maps induced by $\subseteq$) and every x_t has a preimage in some $E_{u(t)}$, Λ-homomorphisms $(f_i)_{i\in\mathbb{N}}$ send $\{x_t\}_{t\in\mathbb{N}}$ to a sequence $\{f_{u(t)}(x_t)\}_{t\in\mathbb{N}}$ in X. By the completeness of X, $\varprojlim f_{u(t)}(x_t) \in X$ holds. Define

$$\theta(x) = \varprojlim f_{u(t)}(x_t) = \varprojlim f|_{E_{u(t)}}(x_t) = \varprojlim f(x_t),$$

where f is the map $\varprojlim E_u \to X$ induced by the direct limit of $((E_i)_{i\in\mathbb{N}}, (\varphi_{uv})_{u\leqslant v})$. Then one can check that θ is well-defined and is a $\varLambda$-homomorphism making the following diagram commute:

$$\begin{tikzcd}
\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)} \arrow[rr, dashed, "\theta"] & & X. \\
& E_i \arrow[ul, "\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "(i\preceq j)"', "\varphi_{ij}"] & \\
& E_j \arrow[uul, bend left, "\alpha_j"] \arrow[uur, bend right, "f_j"'] &
\end{tikzcd}$$

Next, we show that the existence of θ is unique. Assume that θ' is also a $\varLambda$-homomorphism with $\theta'\alpha_i = f_i$ for all i. Note that all morphisms in $\mathsf{Ban}(\varLambda)$ are continuous, which ensure the commutativity $\varprojlim \vartheta(x_i) = \vartheta(\varprojlim x_i)$ between the inverse limit and any morphism ϑ starting from $\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}$. Then for any $x \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}$, taking the sequence $\{x_t\}_{t\in\mathbb{N}}$ in $\bigcup_i E_i$ satisfying $\varprojlim x_t = x$, we have

$$\theta'(x) = \theta'\big(\varprojlim \alpha_i(x_t)\big) = \varprojlim \theta'(\alpha_i(x_t)) = \varprojlim f_i(x_t) = \varprojlim \theta(\alpha_i(x_t)) = \theta\big(\varprojlim \alpha_i(x_t)\big) = \theta(x),$$

i.e., $\theta = \theta'$. Therefore, by the definition of direct limits, we have $\varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}$. □

6 The $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$

Let $\mathcal{C}$ be a category. Recall that an object O in $\mathcal{C}$ is an *initial object* if for any object Y, we have that $\mathrm{Hom}_{\mathcal{C}}(O, Y)$ contains only one morphism, i.e., there is a unique morphism $O \to Y$ in $\mathcal{C}$. Obviously, if $\mathcal{C}$ has initial objects, then the initial object is unique up to isomorphism (see [35, Chapter 5, Lemma 5.3]). Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. An object $C \in \mathcal{C}$ is called a $\mathcal{D}$*-initial object* if for any $D \in \mathcal{D}$, there is a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "\subseteq"'] & D \\
D' \arrow[ur, "h'"'] &
\end{tikzcd}$$

commutes, where D' is an initial object in $\mathcal{D}$ and h' is a morphism in $\mathcal{D}$ (see [35, p. 216]). It is trivial that an initial object in $\mathcal{C}$ is a $\mathcal{C}$-initial object.

Let $\varLambda$ be a complete $\Bbbk$-algebra. In this section, we show that $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$. The proof is divided into two parts: (1) there is at least one morphism from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ to any object in $\mathscr{A}^p$; (2) the above morphism is unique.

6.1 The existence of morphism from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$

In this subsection, we show that $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta))$ is not empty for every object (V, v, δ) in $\mathscr{A}^p$.

Lemma 6.1. *For any object* $(V, v, \delta) \in \mathscr{A}^p$*, we have*

$$\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta)) \neq \emptyset.$$

Proof. For each $u \in \mathbb{N}$, consider the map $\theta_u : E_u \to V$ as follows:

(i) $\theta_0 : E_0 \to V$ is a map induced by the $\Bbbk$-linear map $\Bbbk \to V$ sending 1 to v (note that $E_0 \cong \Bbbk$). Then one can check that θ is a $\varLambda$-homomorphism.

(ii) θ_{u+1} is induced by θ_u through the composition

$$\theta_{u+1} := (E_{u+1} \xrightarrow{\gamma_\xi^{-1}} E_u^{\oplus_p 2^n} \xrightarrow{\theta_u^{\oplus 2^n}} V^{\oplus_p 2^n} \xrightarrow{\delta} V),$$

where the inverse γ_ξ^{-1} of the map γ_ξ is given in Lemma 5.4.
Notice that $\gamma_\xi^{-1}(f) \in E_{u-1}$ for any $f \in E_u \subseteq E_{u+1}$, and then for the case $u = 0$, we have that $f = k\mathbf{1}_{E_0}$ is a constant defined on E_0, and

$$\theta_1(f) = \delta(\theta_0^{\oplus 2^n}(\gamma_\xi^{-1}(f))) = \delta(\theta_0(k\mathbf{1}_{E_0}), \theta_0(k\mathbf{1}_{E_0}), \dots, \theta_0(k\mathbf{1}_{E_0})) = kv,$$

i.e., θ_1 is an extension of θ_0. It yields $\theta_1(\mathbf{1}_{E_1}) = v$ by $\theta_0(\mathbf{1}_{E_0}) = v$ (see (i)). Furthermore, we can check that θ_{u+1} is an extension of θ_u and

$$\theta_u(\mathbf{1}_{E_u}) = v \ (\forall\, u \in \mathbb{N}) \tag{6.1}$$

by induction, i.e., the diagram

$$\begin{tikzcd}
\varinjlim E_i & & V \\
& E_u \arrow[ul, hook', "\alpha_u"'] \arrow[ur, "\theta_u"] \arrow[d, hook, "\alpha_{u\,u+1}"] & \\
& E_{u+1} \arrow[uul, hook', bend left, "\alpha_{u+1}"] \arrow[uur, bend right, "\theta_{u+1}"'] &
\end{tikzcd}$$

commutes, where $\alpha_i : E_i \to \varinjlim E_i$ and $\alpha_{ij} : E_i \to E_j$ $(i \leqslant j)$ are the embeddings induced by $E_i \subseteq \varinjlim E_i$ and $E_i \subseteq E_j$, respectively. Then, for any $i \leqslant j$, there is a unique Λ-homomorphism θ such that the diagram

$$\begin{tikzcd}
\varinjlim E_i \arrow[rr, dashed, "\theta"] & & V \\
& E_i \arrow[ul, hook', "\alpha_i"'] \arrow[ur, "\theta_i"] \arrow[d, hook, "\alpha_{ij}"] & \\
& E_j \arrow[uul, hook', bend left, "\alpha_j"] \arrow[uur, bend right, "\theta_j"'] &
\end{tikzcd}$$

commutes. By Lemma 5.6, we have that $\theta : \varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \to V$ is a Λ-homomorphism in $\mathrm{Hom}_\Lambda(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, V)$.

Next, we prove that θ is a morphism in $\mathscr{N}\!or^p$. First of all, we have

$$\theta(\mathbf{1}) = \varprojlim \theta|_{E_i}(\mathbf{1}_{E_i}) = \varprojlim \theta(\alpha_i(\mathbf{1}_{E_i})) = \varprojlim \theta_i(\mathbf{1}_{E_i}) \overset{(6.1)}{=\!=} \varprojlim v = v.$$

In the following, we show that the diagram

$$\begin{tikzcd}
\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \arrow[r, "\gamma_\xi"] \arrow[d, "\theta^{\oplus 2^n}"'] & \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \arrow[d, "\theta"] \\
V^{\oplus_p 2^n} \arrow[r, "\delta"'] & V
\end{tikzcd} \tag{6.2}$$

commutes. Notice that each $\boldsymbol{f} = (f_1, \dots, f_{2^n}) \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n}$ can be seen as the inverse limit $\varprojlim \boldsymbol{f}_i$ of some sequence $\{\boldsymbol{f}_i = (f_{1i}, \dots, f_{2^n i})\}_{i \in \mathbb{N}}$ in $\bigcup_{u \in \mathbb{N}} E_u^{\oplus_p 2^n}$, where $f_{ji} \in E_{u_i}$ $(1 \leqslant j \leqslant 2^n)$, $u_i \in \mathbb{N}$ such that for any $i \leqslant j$, we have $u_i \leqslant u_j$. Thus, naturally, we need to consider the diagram

$$\begin{tikzcd}
E_{u_i}^{\oplus_p 2^n} \arrow[r, "\gamma_\xi|_{E_{u_i}^{\oplus_p 2^n}}", "\cong"'] \arrow[d, hook, "e_{u_i}^{\oplus 2^n}"'] \arrow[dd, bend right=60, "\theta_{u_i}^{\oplus 2^n}"'] & E_{u_i+1} \arrow[d, hook, "e_{u_i+1}"] \arrow[dd, bend left=60, "\theta_{u_i}"] \\
\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \arrow[r, "\gamma_\xi"] \arrow[d, "\theta^{\oplus 2^n}"'] & \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \arrow[d, "\theta"] \\
V^{\oplus_p 2^n} \arrow[r, "\delta"'] & V,
\end{tikzcd}$$

where $(e_{u_i} : E_{u_i} \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)})$ is the embedding induced by $E_{u_i} \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. Since

$$\begin{aligned}
\theta(\gamma_\xi(\boldsymbol{f})) &= \varprojlim \theta(\gamma_\xi(e_{u_i}^{\oplus 2^n}(\boldsymbol{f}_i))) \\
&= \varprojlim \theta(e_{u_i+1}(\gamma_\xi|_{E^{\oplus_p 2^n}}(\boldsymbol{f}_i))) \ \ (\gamma_\xi e_{u_i}^{\oplus 2^n} = e_{u_i+1}\gamma_\xi|_{E^{\oplus_p 2^n}}) \\
&= \varprojlim \theta_{u_i}(\gamma_\xi|_{E^{\oplus_p 2^n}}(\boldsymbol{f}_i)) \ \ (\theta e = \theta_{u_i}) \\
&= \varprojlim \delta(\theta_{u_i}^{\oplus 2^n}(\boldsymbol{f}_i)) \ \ (\theta_{u_i}\gamma_\xi\big|_{E^{\oplus_p 2^n}} = \delta\theta_{u_i}^{\oplus 2^n}) \\
&= \varprojlim \delta(\theta^{\oplus 2^n}(e_{u_i}^{\oplus 2^n}(\boldsymbol{f}_i))) \ \ (\theta_u^{\oplus 2^n} = \theta^{\oplus 2^n} e_{u_i}^{\oplus 2^n}) \\
&= \delta(\theta^{\oplus 2^n}(\varprojlim e_{u_i+1}^{\oplus 2^n}(\boldsymbol{f}_i))) = \delta(\theta^{\oplus 2^n}(\boldsymbol{f})) \ \ (\text{by } (5.1)),
\end{aligned}$$

the assertion follows. $\square$

6.2 The uniqueness of morphism from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$

Now, we show that for any object (V, v, δ) in $\mathscr{A}^p$, if the morphism in the category $\mathscr{A}^p$ from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ exists, then it is unique.

Lemma 6.2. *Let $(V, v, \delta) \in \mathscr{A}^p$ be an object in $\mathscr{A}^p$. If*

$$\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta)) \neq \emptyset,$$

then $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta))$ *contains a unique morphism.*

Proof. Let θ and θ' be two Λ-homomorphisms from $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ to (V, v, δ) in $\mathscr{A}^p$. Then $\theta(\mathbf{1}) = v = \theta'(\mathbf{1})$. Since θ and θ' are maps in $\mathscr{A}^p$, the square

$$\begin{tikzcd}
E_u^{\oplus_p 2^n} \arrow[r, "\gamma_\xi|_{E_u^{\oplus 2^n}}", "\cong"'] \arrow[d, "(\theta|_{E_u} - \theta'|_{E_u})^{\oplus 2^n}"'] & E_{u+1} \arrow[d, "\theta|_{E_{u+1}} - \theta'|_{E_{u+1}}"] \\
V^{\oplus_p 2^n} \arrow[r, "\delta"'] & V
\end{tikzcd}$$

commutes. Then for any $f \in E_{u+1}$, we have

$$(\theta|_{E_{u+1}} - \theta'|_{E_{u+1}})(f) = (\delta \circ (\theta|_{E_u} - \theta'|_{E_u})^{\oplus 2^n} \circ (\gamma_\xi|_{E_u^{\oplus 2^n}})^{-1})(f),$$

i.e., $\theta|_{E_{u+1}} - \theta'|_{E_{u+1}}$ is determined by $\theta|_{E_u} - \theta'|_{E_u}$. Considering the case for $u = 0$, since $\theta|_{E_0}$ and $\theta'|_{E_0} : E_0 \to V$ are defined by $\theta_0(\mathbf{1}_{E_0}) = v$, we have

$$(\theta|_{E_0} - \theta'|_{E_0})(k\mathbf{1}_{E_0}) = k(\theta|_{E_0}(\mathbf{1}_{E_0}) - \theta'|_{E_0}(\mathbf{1}_{E_0})) = k(v - v) = 0.$$

Therefore, $\theta|_{E_u} - \theta'|_{E_u} = 0$ for all $u \in \mathbb{N}$ by induction.

On the other hand, considering the embeddings $e_u : E_u \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ and $e_{uv} : E_u \to E_v$ $(u \leqslant v)$ induced by $\subseteq$ and the direct system

$$((E_u^{\oplus_p 2^n})_{u\in\mathbb{N}}, (e_u^{\oplus 2^n} : E_u^{\oplus_p 2^n} \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n})_{u\in\mathbb{N}}),$$

we have the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} & & V. \\
& E_i^{\oplus_p 2^n} \arrow[ul, hook', "e_i^{\oplus 2^n}"'] \arrow[ur, "\theta|_{E_i}-\theta'|_{E_i}", "(=0)"'] \arrow[d, hook, "e_{ij}^{\oplus 2^n}"] & \\
& E_{ij}^{\oplus_p 2^n} \arrow[uul, hook', bend left, "e_j^{\oplus 2^n}"] \arrow[uur, bend right, "\theta|_{E_j}-\theta'|_{E_j}=0"'] &
\end{tikzcd}$$

Since

$$\varinjlim E_i^{\oplus_p 2^n} \cong (\varinjlim E_i)^{\oplus_p 2^n} \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n},$$

there is a unique Λ-homomorphism $\phi : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \to V$ such that the diagram

$$\begin{tikzcd}
\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \arrow[rr, dashed, "\phi"] & & V \\
& E_i^{\oplus_p 2^n} \arrow[ul, hook', "e_i^{\oplus 2^n}"'] \arrow[ur, "\theta|_{E_i}-\theta'|_{E_i}", "(=0)"'] \arrow[d, hook, "e_{ij}^{\oplus 2^n}"] & \\
& E_{ij}^{\oplus_p 2^n} \arrow[uul, hook', bend left, "e_j^{\oplus 2^n}"] \arrow[uur, bend right, "\theta|_{E_j}-\theta'|_{E_j}=0"'] &
\end{tikzcd}$$

commutes. Since $(\theta - \theta')e_u^{\oplus 2^n} = \theta|_{E_i} - \theta'|_{E_j}$, we know that the case for $\phi = \theta - \theta'$ makes the above diagram commute. On the other hand, the case for $\phi = 0$ makes the above diagram commute. Thus $\theta - \theta' = 0$ and $\theta = \theta'$. $\square$

6.3 The $\mathscr{A}^{\boldsymbol{p}}$-initial object in $\mathscr{N}\!or^{\boldsymbol{p}}$

Lemma 6.3. *Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a subcategory of $\mathcal{C}$, and let D' be an initial object in $\mathcal{D}$. If an object C is a subobject of D' in $\mathcal{C}$, then C is a $\mathcal{D}$-initial object.*

Proof. For any object D in $\mathcal{D}$, there is a unique morphism $h' \in \mathrm{Hom}_{\mathcal{D}}(D', D)$ since D' is an initial object in $\mathcal{D}$. Let e be the embedding $C \to D'$ obtained by C being a subobject of D'. Then we obtain a morphism $h'e \in \mathrm{Hom}_{\mathcal{C}}(C, D)$. Next, we assume that h_0 is any morphism in $\mathrm{Hom}_{\mathcal{C}}(C, D)$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "h_0"] \arrow[d, "e"'] & D \\
D' \arrow[ur, "h_0'"'] &
\end{tikzcd}$$

commutes, where h_0' is a morphism in $\mathcal{D}$. Since D' is an initial object in $\mathcal{D}$, we have $h_0' = h'$, and thus $h_0 = h_0'e = h'e$. $\square$

Now, we can prove the following main result of this paper.

Theorem 6.4. *The triple $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ is an $\mathscr{A}^{\boldsymbol{p}}$-initial object in $\mathscr{N}\!or^{\boldsymbol{p}}$.*

Proof. For any object (V,v,δ) in $\mathscr{A}^p$, the existence of morphisms in $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi),(V,v,\delta))$ is proved in Lemma 6.1, and its uniqueness is proved in Lemma 6.2. Thus, the triple $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi)$, as an object in $\mathscr{A}^p$, is an initial object in $\mathscr{A}^p$. It follows that $(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\widehat{\gamma}_\xi)$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}\!or^p$ by Lemma 6.3. □

We give a remark for Theorem 6.4.

Remark 6.5. For any object (V,v,δ) in $\mathscr{A}^p$, there is a unique morphism

$$h:(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi)\to(V,v,\delta)$$

in $\mathscr{N}\!or^p$, which can be extended to $\widehat{h}:(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi)\to(V,v,\delta)$. In other words, if there exists a morphism h making the diagram

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi) \arrow[r, "h"] \arrow[d, "\subseteq"'] & (V,v,\delta) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi) \arrow[ur, "\widehat{h}"']
\end{tikzcd}$$

commute, then the existence of h is guaranteed to be unique.

7 The categorification of integration

Take $\Bbbk=(\Bbbk,|\cdot|,\preceq)$ to be an extension of $\mathbb{R}$ and $p=1$. Recall the symbols given in Notation 5.3, any step function f in E_u can be written as

$$f=\sum_{(u_iv_i)_i}k_{(u_iv_i)_i}\mathbf{1}_{(u_iv_i)_i}.$$

We define the map $T_u:E_u\to\Bbbk$ by

$$T_u(f)=\sum_{(u_iv_i)_i}k_{(u_iv_i)}\mu\bigg(\prod_i\mathbb{I}^{(u_iv_i)}\bigg) \tag{7.1}$$

(note that if all coefficients $k_{(u_iv_i)}$ are equal to 1, then $T_u(f)=\mu(E_u)$).

Then the Λ-isomorphism γ_ξ shown in Lemma 5.4 points out the following fact: there is a map m_u : $\Bbbk^{\oplus_p2^n}\to\Bbbk$ such that the diagram

$$\begin{tikzcd}
E_u^{\oplus_p2^n} \arrow[r, "\gamma_\xi"] \arrow[d, "T_u^{\oplus2^n}"'] & E_{u+1} \arrow[d, "T_{u+1}"] \\
\Bbbk^{\oplus_p2^n} \arrow[r, "m_u"'] & \Bbbk
\end{tikzcd} \tag{7.2}$$

commutes. Indeed, for the function $f_k=\frac{k}{\mu(\mathbb{I}_\Lambda)}\mathbf{1}_{\mathbb{I}_\Lambda}$ with $k\in\Bbbk$, we have

$$T_u(f_k)=T_u\bigg(\frac{k}{\mu(\mathbb{I}_\Lambda)}\mathbf{1}_{\mathbb{I}_\Lambda}\bigg)=\frac{k}{\mu(\mathbb{I}_\Lambda)}T_u(\mathbf{1}_{\mathbb{I}_\Lambda})=k$$

by (7.1). Then for any $\boldsymbol{k}=(k_1,\dots,k_{2^n})\in\Bbbk^{\oplus_p2^n}$, $\boldsymbol{f_k}=(f_{k_1},\dots,f_{k_{2^n}})\in E_u^{\oplus_p2^n}$ is a preimage of $\boldsymbol{k}$ under the $\Bbbk$-linear map $T_u^{\oplus2^n}$. We define μ_u as follows:

$$m_u(\boldsymbol{k})=T_{u+1}(\gamma_\xi(\boldsymbol{f_k})).$$

It is easy to see that m_u is a $\Bbbk$-linear map. In particular, for the constant function $\mathbf{1}_{\mathbb{I}_\Lambda}$ given by the measure $\mu(\mathbb{I}_\Lambda)$ of $\mathbb{I}_\Lambda$, we obtain that $f_{\mu(E_u)}$ is a preimage of $\mu(\mathbb{I}_\Lambda)\in\Bbbk$, and then

$$m_u(\mu(\mathbb{I}_\Lambda),\dots,\mu(\mathbb{I}_\Lambda))=T_{u+1}\gamma_\xi(\mathbf{1}_{\mathbb{I}_\Lambda},\dots,\mathbf{1}_{\mathbb{I}_\Lambda})=\sum_{(u_iv_i)_i}1\cdot\mu\bigg(\prod_i\mathbb{I}^{(u_iv_i)}\bigg)=\mu(\mathbb{I}_\Lambda).$$

Lemma 7.1. *Let $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ be an extension of $\mathbb{R}$. Then $T_u : E_u \to \Bbbk$ is a Λ-homomorphism.*

Proof. Note that $\Bbbk$ is a Λ-module given by

$$\Lambda \times \Bbbk \to \Bbbk, \quad (\lambda, k) \mapsto \lambda \cdot k := \tau(\lambda)k.$$

For arbitrary two elements $\lambda_1, \lambda_2 \in \Lambda$ and arbitrary two functions $f = \sum_i k_i \mathbf{1}_{I_i}$ and $g = \sum_j k'_j \mathbf{1}_{I'_j} \in E_u$, we have

$$\begin{aligned} T_u(\lambda_1 \cdot f + \lambda_2 \cdot g) &= T_u\left(\sum_i \tau(\lambda_1) k_i \mathbf{1}_{I_i} + \sum_j \tau(\lambda_2) k'_j \mathbf{1}_{I'_j}\right) \\ &= \tau(\lambda_1) T_u\left(\sum_i k_i \mathbf{1}_{I_i}\right) + \tau(\lambda_2) T_u\left(\sum_j k'_j \mathbf{1}_{I'_j}\right) \\ &= \tau(\lambda_1) T_u(f) + \tau(\lambda_2) T_u(g) \\ &= \lambda_1 \cdot T_u(f) + \lambda_2 \cdot T_u(g). \end{aligned}$$

This completes the proof. □

Lemma 7.2. *Let $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ be an extension of $\mathbb{R}$ and let m_u be the $\Bbbk$-linear map given in the diagram* (7.2)*. Then m_u is a Λ-homomorphism.*

Proof. We can prove that m_u is a Λ-homomorphism by using an argument similar to proving that T_u is a $\Bbbk$-linear mapping as in Lemma 7.1. □

Remark 7.3. Since $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_u \subseteq \cdots \subseteq \mathbf{S}_\tau(\mathbb{I}_\Lambda) \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} = \varinjlim E_i$, we have that μ is independent of u. Thus, we can use m to present all maps m_u ($u \in \mathbb{N}$) because $m_0 = m_1 = m_2 = \cdots$.

Proposition 7.4. *Let $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ be an extension of $\mathbb{R}$. Then the triple $(\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ is an object in $\mathscr{N}\!or^p$. Furthermore, since Λ is complete, so is $\Bbbk$. Then $\Bbbk^{\oplus_p 2n}$ is a Banach Λ-module, and so $(\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ is an object in $\mathscr{A}^p$.*

Proof. It follows from Lemmas 7.1 and 7.2 and Remark 7.3. □

The following proposition shows that T_u satisfies the triangle inequality.

Proposition 7.5. *If $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ is an extension of $\mathbb{R}$, then for any $f \in E_u$, the following inequality holds for all $u \in \mathbb{N}$:*

$$|T_u(f)| \leqslant T_u(|f|). \tag{7.3}$$

Proof. Assume that $f = \sum_i k_i \mathbf{1}_{I_i} \in E_u$, where $I_i \cap I_j = \emptyset$ for all $i \neq j$. Then $|f| = |\sum_i k_i \mathbf{1}_{I_i}|$ is also a step function in E_u, and we have

$$\begin{aligned} T_u(|f|) = T_u\left(\left|\sum_i k_i \mathbf{1}_{I_i}\right|\right) &\overset{(\star)}{=\!=} T_u\left(\sum_i |k_i| \mathbf{1}_{I_i}\right) \\ &= \sum_i |k_i| \mu\left(\prod_i \mathbb{I}^{(u_i v_i)_i}\right) \\ &\geqslant \left|\sum_i k_i \mu\left(\prod_i \mathbb{I}^{(u_i v_i)_i}\right)\right| = |T_u(f)|, \end{aligned}$$

where $(\star)$ is given by $I_i \cap I_j = \emptyset$. □

Theorem 7.6. *If $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ is an extension of $\mathbb{R}$, then there exists a unique morphism*

$$T : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (\Bbbk, \mu(\mathbb{I}_\Lambda), m)$$

in $\mathrm{Hom}_{\mathscr{N}or^p}((\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi),(\Bbbk,\mu(\mathbb{I}_\Lambda),m))$ *such that the diagram*

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi) \arrow[r, "T"] \arrow[d, "\subseteq"'] & (\Bbbk,\mu(\mathbb{I}_\Lambda),m) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi) \arrow[ur, "\widehat{T}"'] &
\end{tikzcd}$$

commutes, where $\widehat{T}$ *is the is the unique extension of* T *lying in* $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)},\mathbf{1},\widehat{\gamma}_\xi),\ (\Bbbk,\mu(\mathbb{I}_\Lambda),m))$. *Furthermore,* $\widehat{T}$ *is given by the direct limit* $\varinjlim T_i : \varinjlim E_i \to \Bbbk$.

Proof. Denote by $\alpha_{ij} : E_i \to E_j$ $(i \leqslant j)$ and $\alpha_i : E_i \to \varinjlim E_i$ the monomorphism induced by $E_i \subseteq E_j \subseteq \varinjlim E_i$. Then there is a unique morphism $\varinjlim T_i : \varinjlim E_i \to \Bbbk$ such that the diagram

$$\begin{tikzcd}
\varinjlim E_i \arrow[rr, dashed, "\varinjlim T_i"] & & \Bbbk \\
& E_i \arrow[ul, hook', "\alpha_i"] \arrow[ur, "T_i"'] \arrow[d, hook, "\alpha_{ij}"] & \\
& E_j \arrow[uul, bend left, hook', "\alpha_j"] \arrow[uur, bend right, "T_j"'] &
\end{tikzcd}$$

commutes. By Lemma 5.6, we have $\varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, and then $\varinjlim T_i$ induces a morphism in $\mathscr{A}^p$ from $(\mathbf{S}_\tau(\mathbb{I}_\Lambda),\mathbf{1},\gamma_\xi)$ to $(\Bbbk,\mu(\mathbb{I}_\Lambda),m)$. Theorem 6.4 and Remark 6.5 show that $\varinjlim T_i = \widehat{T}$ and $T = \widehat{T}|_{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. $\square$

Definition 7.7. Let $\Bbbk$ be a field with the norm $|\cdot| : \Bbbk \to \mathbb{R}^{\geqslant 0}$ and the total ordered $\preceq$, and let $f : \mathbb{I}_\Lambda \to \Bbbk$ be a function in $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. We call that f is an *integrable function* on $\mathbb{I}_\Lambda$, and its integral, denoted by $(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f d\mu$, is defined as follows:

$$(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f d\mu := \widehat{T}(f).$$

By using the limit $\varinjlim T_i : \varinjlim E_i \to \Bbbk$ given in Theorem 7.6, we see that the formula (7.1), Lemma 7.1 and Proposition 7.5 show that

$$(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} 1 d\mu = \mu(\mathbb{I}_\Lambda),$$
$$(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} (\lambda_1 \cdot f_1 + \lambda_2 \cdot f_2)\mu = \lambda_1 \cdot (\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f_1\mu + \lambda_2 \cdot (\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f_2\mu \ (\lambda_1,\lambda_2 \in \Lambda)$$

and

$$\left|(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f d\mu\right| \leqslant (\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} |f| d\mu,$$

respectively.

In Subsection 10.1, we point out that Theorem 7.6 and Definition 7.7 provide a categorification of Lebesgue integration.

8 Series expansions of functions

Set $n := \dim_\Bbbk \Lambda$ and define the n *variables polynomial ring* $\Bbbk[X_1,\ldots,X_N]$ $(= \Bbbk[\boldsymbol{X}]$ for short) over $\Bbbk$ to be the set of all N variables polynomial rings $(N \geqslant n)$. Then $\Bbbk[\boldsymbol{X}]$ is a left Λ-module whose left Λ-action is defined as

$$\Lambda \times \Bbbk[\boldsymbol{X}] \to \Bbbk[\boldsymbol{X}], \quad (a, P(x)) \mapsto \tau(a)P(x).$$

8.1 Realizing power series expansions of functions as morphisms in $\mathscr{A}^p$

Take $N=n$. In this subsection, we define the map

$$\|\cdot\|:\Bbbk[\boldsymbol{X}]\to\mathbb{R}^{\geqslant 0},\quad P\mapsto\left((\mathscr{A}^p)\int_{\mathbb{I}_\varLambda}|P|^p d\mu\right)^{\frac{1}{p}},\tag{8.1}$$

where $|P|$ is defined by the norm $|\cdot|:\Bbbk\to\mathbb{R}^{\geqslant 0}$ defined on $\Bbbk$ and $|P(\boldsymbol{X})|$ for any $\boldsymbol{X}\in\mathbb{I}_\varLambda\subseteq\varLambda$.

Lemma 8.1. *The polynomial ring* $\Bbbk[\boldsymbol{X}]$ *with the map* (8.1) *is a normed left* $\varLambda$*-module.*

Proof. Each polynomial can be seen as a function lying in $\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}$. Then, by using the norm $|\cdot|:\Bbbk\to\mathbb{R}^{\geqslant 0}$, the map (8.1) induces a norm as required since $\|a\cdot P\|=\|\tau(a)P\|=|\tau(a)|\cdot\|P\|$. □

By using Lemma 8.1, we see that the Banach left $\varLambda$-module $\widehat{\Bbbk[\boldsymbol{X}]}$, as the completion of $\Bbbk[\boldsymbol{X}]$, provides a triple $(\widehat{\Bbbk[\boldsymbol{X}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\Bbbk[\boldsymbol{X}]}})$, which is an object of $\mathscr{A}^1$. Thus, by Theorem 7.6, the following result holds.

Corollary 8.2 (Weierstrass approximation theorem)**.** *There exists a unique morphism*

$$E_{\mathrm{pow}}:(\mathbf{S}_\tau(\mathbb{I}_\varLambda),\mathbf{1},\gamma_\xi)\to(\widehat{\Bbbk[\boldsymbol{X}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\Bbbk[\boldsymbol{X}]}})$$

in $\mathrm{Hom}_{\mathscr{N}\!or^1}((\mathbf{S}_\tau(\mathbb{I}_\varLambda),\mathbf{1},\widehat{\gamma}_\xi),(\widehat{\Bbbk[\boldsymbol{X}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\Bbbk[\boldsymbol{X}]}}))$ *such that the diagram*

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\varLambda),\mathbf{1},\gamma_\xi) \arrow[r, "E_{\mathrm{pow}}"] \arrow[d, "\subseteq"'] & (\widehat{\Bbbk[\boldsymbol{X}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\Bbbk[\boldsymbol{X}]}}) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)},\mathbf{1},\widehat{\gamma}_\xi) \arrow[ur, "\widehat{E_{\mathrm{pow}}}"'] &
\end{tikzcd}$$

commutes, where $\widehat{E_{\mathrm{pow}}}$ *is the unique extension of* E_{pow} *lying in*

$$\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)},\mathbf{1},\widehat{\gamma}_\xi),(\widehat{\Bbbk[\boldsymbol{X}]},\mathbf{1},\widehat{\gamma}_\xi|_{\widehat{\Bbbk[\boldsymbol{X}]}})).$$

The above corollary shows that for any function $f\in\widehat{\mathbf{S}_\tau(\mathbb{I}_\varLambda)}$, there exists a sequence $\{P_i\}_{i\in\mathbb{N}}$ of polynomials such that

$$\widehat{E_{\mathrm{pow}}}(f)=\varprojlim P_i\in\widehat{\Bbbk[\boldsymbol{X}]}\subseteq\Bbbk[[\boldsymbol{X}]].$$

This formula is called a *power series expansion* of f.

Remark 8.3. In the case for $N=2n$, if $\Bbbk[\boldsymbol{X}]=\Bbbk[Y_j,Y_j^{-1}\mid 1\leqslant j\leqslant n]$, where $X_u=Y_u$ holds for any $1\leqslant u\leqslant n$, and $X_{n+v}=Y_v^{-1}$ holds for any $1\leqslant v\leqslant n$, then we can obtain the Laurent series of analytic functions.

8.2 Realizing Fourier series expansions of functions as morphisms in $\mathscr{A}^p$

Consider the case for $N=2n$ and $\Bbbk=\mathbb{C}$ in this subsection. Let $\triangle$ be the $\mathbb{C}$-linear map

$$\triangle:\mathbb{C}[\boldsymbol{X}]\to\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]:=\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}X_j}\mid 1\leqslant j\leqslant n]$$

induced by

$$X_j\mapsto\begin{cases}\mathrm{e}^{2\pi\mathrm{i}X_j}, & \text{if } 1\leqslant j\leqslant n,\\ \mathrm{e}^{-2\pi\mathrm{i}X_j}, & \text{if } n+1\leqslant j\leqslant 2n,\end{cases}\tag{8.2}$$

and define the map

$$\|\cdot\|:\mathbb{C}[\boldsymbol{X}]\to\mathbb{R}^{\geqslant 0},\quad P\mapsto\left((\mathscr{A}^p)\int_{\mathbb{I}_\varLambda}|\triangle(P)|^p d\mu_{\mathrm{L}}\right)^{\frac{1}{p}}.\tag{8.3}$$

Lemma 8.4. *The $\mathbb{C}$-linear map $\triangle$ is a Λ-isomorphism, and $\mathbb{C}[\boldsymbol{X}] \cong \mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]$ with the map (8.3) is a normed left Λ-module.*

Proof. It is trivial that $\triangle$ is a $\mathbb{C}$-linear isomorphism by (8.2). Thus, the assertion that $\triangle$ is a Λ-isomorphism follows from the fact that the formula

$$\triangle(a \cdot P) = \triangle(\tau(a)P) = \tau(a) \triangle (P) = a \cdot \triangle(P)$$

holds for any $a \in \Lambda$. Furthermore, we can prove that the polynomial ring $\Bbbk[\boldsymbol{X}]$ with the map (8.3) is a normed left Λ-module by the way similar to that in Lemma 8.1. $\square$

Next, by Lemma 8.4, we obtain that

$$(\widehat{\mathbb{C}[\boldsymbol{X}]}, \mathbf{1}, \widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}}) \cong (\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}, \mathbf{1}, \widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}})$$

is an object in $\mathscr{A}^p$. Then the following corollary follows from Theorem 7.6.

Corollary 8.5. *There exists a unique morphism*

$$E_{\mathrm{Fou}} : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}, \mathbf{1}, \widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}})$$

in $\mathrm{Hom}_{\mathscr{N}or^1}((\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \widehat{\gamma}_\xi), (\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}, \mathbf{1}, \widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}}))$ *such that the diagram*

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \arrow[r, "E_{\mathrm{Fou}}"] \arrow[d, "\subseteq"'] & (\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}, \mathbf{1}, \widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}}) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) \arrow[ur, "\widehat{E_{\mathrm{Fou}}}"'] &
\end{tikzcd}$$

commutes, where $\widehat{E_{\mathrm{Fou}}}$ is the unique extension of E_{Fou} lying in

$$\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}, \mathbf{1}, \widehat{\gamma}_\xi|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}})).$$

The above corollary shows that for any function $f \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, there exists a sequence $\{P_i\}_{i\in\mathbb{N}}$ of triangulated polynomials such that

$$\widehat{E_{\mathrm{Fou}}}(f) = \varprojlim P_i \in \widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi\mathrm{i}\boldsymbol{X}}]}.$$

This formula is called a *Fourier series expansion* of f.

8.3 Stone-Weierstrass theorem in $\mathscr{A}^p$

Let $\mathbf{W}_0$ be a normed left Λ-module generated by some functions lying in $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ such that $\widehat{\mathbf{W}_0}$ and $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, as left Λ-modules, are isomorphic preserving $\mathbf{1}$. For any $u \in \mathbb{N}$, define

$$\mathbf{W}_u = \{\widehat{\gamma}_\xi|_{\mathbf{W}_{u-1}}(\boldsymbol{f}) \mid \boldsymbol{f} = (f_1, \ldots, f_{2^n}) \in \mathbf{W}_{u-1}^{\oplus_p 2^n}\}.$$

Then we obtain a family of canonical embeddings

$$\mathbf{W}_0 \xrightarrow{\subseteq} \mathbf{W}_1 \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} \mathbf{W}_u \xrightarrow{\subseteq} \cdots \ (\subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}),$$

which induced a direct limit

$$\varinjlim \mathbf{W}_u =: \mathbf{W}.$$

Lemma 8.6. *For any complete extension $\mathbf{W}_\dagger$ of $\mathbf{W}$, i.e., the Banach Λ-module satisfying $\mathbf{W} \subseteq \mathbf{W}_\dagger$, there exists a Λ-monomorphism*

$$\widehat{E_{\mathrm{S-W}}} : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \to \mathbf{W}_\dagger$$

between two left Λ-modules $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ and $\mathbf{W}$ such that $E_{\mathrm{S-W}}(\mathbf{1}) = \mathbf{1}$ holds in the case for $\mathbf{1} \in \mathbf{W}$.

Proof. Since $\mathbf{W}_i \subseteq \mathbf{W}_j \subseteq \mathbf{W}$ for any $i, j \in \mathbb{N}$ with $i \leqslant j$, we have $\widehat{\mathbf{W}}_i \subseteq \widehat{\mathbf{W}}_j \subseteq \widehat{\mathbf{W}}$. On the other hand, $\mathbf{W} \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ yields $\widehat{\mathbf{W}} \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. It follows that

$$\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \cong \widehat{\mathbf{W}}_0 \subseteq \widehat{\mathbf{W}}_u \subseteq \widehat{\mathbf{W}} \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}.$$

Therefore, we get a Λ-isomorphism $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \cong \mathbf{W}\ (= \widehat{\mathbf{W}})$ since the isomorphism $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \cong \widehat{\mathbf{W}}_0$ preserves $\mathbf{1}$. The composition

$$\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \xrightarrow{\cong} \mathbf{W} \xrightarrow{\subseteq} \mathbf{W}_\dagger$$

is the desired Λ-monomorphism. □

Lemma 8.7. *There exists a Λ-homomorphism $\widehat{\gamma}_{\xi\dagger} : \mathbf{W}_\dagger^{\oplus_p 2^n} \to \mathbf{W}_\dagger$ such that the diagram*

$$\begin{tikzcd}
\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \arrow[r, "\widehat{\gamma}_\xi"] \arrow[d, "E_{\mathrm{S-W}}^{\oplus 2^n}"'] & \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \arrow[d, "E_{\mathrm{S-W}}"] \\
\mathbf{W}_\dagger^{\oplus_p 2^n} \arrow[r, "\widehat{\gamma}_{\xi\dagger}"'] & \mathbf{W}_\dagger
\end{tikzcd}$$

commutes and $\widehat{\gamma}_{\xi\dagger}(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$ holds.

Proof. The composition $\widehat{\gamma}_{\xi\dagger} := E_{\mathrm{S-W}} \circ \widehat{\gamma}_\xi \circ (E_{\mathrm{S-W}}^{\oplus 2^n})^{-1}$ is the desired Λ-homomorphism. □

Corollary 8.8 (Stone-Weierstrass approximation theorem)**.** *There exists a unique morphism*

$$E_{\mathrm{S-W}} : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (\mathbf{W}, \mathbf{1}, \widehat{\gamma_{\xi\dagger}})$$

in $\mathrm{Hom}_{\mathscr{N}or^1}((\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \widehat{\gamma}_\xi), (\mathbf{W}, \mathbf{1}, \widehat{\gamma_{\xi\dagger}}))$ *such that the diagram*

$$\begin{tikzcd}
(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \arrow[r, "E_{\mathrm{S-W}}"] \arrow[d, "\subseteq"'] & (\mathbf{W}, \mathbf{1}, \widehat{\gamma_{\xi\dagger}}) \\
(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) \arrow[ur, "\widehat{E_{\mathrm{S-W}}}"'] &
\end{tikzcd}$$

commutes, where $\widehat{E_{\mathrm{S-W}}}$ is the unique extension of $E_{\mathrm{S-W}}$ lying in $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (\mathbf{W}, \mathbf{1}, \widehat{\gamma_{\xi\dagger}}))$.

9 Differentiations

In this section, let $\mathscr{A}^p$ satisfy $\Lambda = \Bbbk$ which is an extension of $\mathbb{R}$, and take $\tau = \mathrm{id}$, $\xi = \frac{1}{2}$, $\mu = \mu_{\mathrm{L}}$, $\mathbb{I}_\Lambda = [0, 1]$, and $\xi = \frac{1}{2}$. In this case, the initial object of $\mathscr{A}^p$ is $(\widehat{\mathbf{S}}, \mathbf{1}, \gamma_{\frac{1}{2}})$, where $\widehat{\mathbf{S}} = \widehat{\mathbf{S}_{\mathrm{id}}([0, 1])}$.

9.1 Realizing variable upper limit integration as a morphism in $\mathscr{A}^1$

We recall some works of Leinster [29, Section 2]. Let $C_*([0, 1])$ be the set of all continuous functions $F : [0, 1] \to \Bbbk$ such that $F(0) = 0$, with the sup norm

$$\|\cdot\| : C_*([0, 1]) \to \mathbb{R}^{\geqslant 0}, \quad f \mapsto \sup_{x \in [0,1]} |f(x)|.$$

Then the triple $(C_*([0, 1]), \mathrm{id}, \kappa)$ of the $\Bbbk$-module $C_*([0, 1])$, the identity function $\mathrm{id}(x) = x$, and the $\Bbbk$-homomorphism $\kappa : C_*([0, 1])^{\oplus 2} \to C_*([0, 1])$ defined by

$$\kappa(F_1, F_2) = \begin{cases} \dfrac{1}{2} F_1(2x), & 0 \leqslant x \leqslant \dfrac{1}{2}, \\ \dfrac{1}{2}(F_1(1) + F_2(2x - 1)), & \dfrac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

is an object in $\mathscr{A}^1$. Then the following proposition, first proved by Meckes (see [29]), holds.

Proposition 9.1 (See [29, Proposition 2.4])**.** *There exists a unique morphism*

$$\widehat{T}_{[0,x]} : (\widehat{\mathbf{S}}, \mathbf{1}, \gamma_{\frac{1}{2}}) \to (\widehat{C_*([0,1])}, \mathrm{id}, \widehat{\kappa})$$

in $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}}, \mathbf{1}, \gamma_{\frac{1}{2}}), (\widehat{C_*([0,1])}, \mathrm{id}, \widehat{\kappa}))$ *sending each function* $f \in \widehat{\mathbf{S}}$ *to the variable upper limit integration* $F(x) = (\mathrm{L}) \int_0^x f d\mu_{\mathrm{L}}$.

9.2 Realizing differentiation as a preimage of a morphism in $\mathscr{A}^1$

It follows from Proposition 9.1 that for any function $F \in \mathrm{Im}(\widehat{T}_{[0,x]})$, there exists an element $f \in \widehat{\mathbf{S}}$ such that

(1) if F is a differentiable function (in the classical sense), then

$$\frac{dF}{dx} = f \quad \text{holds for all } x \in [0,1];$$

here, f is seen as a function in some equivalence class lying in $\widehat{\mathbf{S}}$, and, strictly speaking, $\frac{dF}{dx}$ is an element lying in the equivalence class containing f;

(2) otherwise, there exists a function f such that

$$\int_0^1 F(x)\phi(x) d\mu_{\mathrm{L}} = -\int_0^1 f(x)\varPhi(x) d\mu_{\mathrm{L}}$$

holds for any differentiable function $\varPhi : [0,1] \to \Bbbk$ (in the classical sense) satisfying $\varPhi(0) = \varPhi(1) = 0$. Thus, we can define the weak derivatives for functions lying in $\mathrm{Im}(\widehat{T}_{[0,x]})$ by using the preimage of the $\Bbbk$-homomorphism $\widehat{T}_{[0,x]}$ as follows.

Definition 9.2. All functions lying in the preimage $\widehat{T}^{-1}_{[0,x]}(F)$ of $F \in \mathrm{Im}(\widehat{T}_{[0,x]})$ are called *weak derivatives* of F, and written $\widehat{T}^{-1}_{[0,x]}(F(x))$ as $\frac{dF}{dx}$.

The following theorem shows that we cannot define the weak derivatives of a function by using the morphism in $\mathscr{A}^1$ starting with $(\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}})$.

Theorem 9.3. (1) *A morphism in* $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}), (N, v, \delta))$ *is zero if and only if* $v = 0$.

(2) *Furthermore, there is no morphism* D *in* $\mathscr{A}^1$ *starting with* $(\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}})$ *such that* D *sends any almost everywhere differentiable function* $f(x)$ *to its weak derivative* $\frac{df}{dx}$.

Proof. (1) For any $h \in \mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}), (N, v, \delta))$, the diagram

$$\begin{tikzcd} \widehat{\mathbf{S}}^{\oplus 2} \arrow[r, "\widehat{\gamma}"] \arrow[d, "h^{\oplus 2}"'] & \widehat{\mathbf{S}} \arrow[d, "h"] \\ N^{\oplus 2} \arrow[r, "\delta"'] & N \end{tikzcd}$$

commutes.

If $v = 0$, then $h(\mathbf{1}) = v = 0$, and the map $0 : \widehat{\mathbf{S}} \to N, f \mapsto 0$ is a $\Bbbk$-homomorphism such that the above diagram commutes. By using $\mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}), (N, v, \delta))$ to be a set containing only one morphism (see Theorem 6.4), we obtain $h = 0$.

Conversely, if $h = 0$, then by the definition of morphism in $\mathscr{A}^1$, we have $v = h(\mathbf{1}) = 0$.

(2) If there is an object (N, v, δ) such that $D : (\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}) \to (N, v, \delta)$ is a morphism in $\mathscr{A}^1$ sending each almost everywhere differentiable function $f(x)$ to its weak derivative $\frac{df}{dx}$, then by the definition of morphism in $\mathscr{A}^1$, we have $v = D(\mathbf{1}) = \frac{d\mathbf{1}}{dx} = 0$. It follows from (1) that $D = 0$, which is a contradiction. $\square$

9.3 Realizing differentiation as a morphism in $\mathscr{A}^1$

In this subsection, we provide a description of differentiation by another morphism in $\mathscr{A}^1$.

Consider the triple $(\widehat{\mathbf{S}}, \mathrm{id}, \widehat{\kappa})$, where $\mathrm{id} : [0,1] \to \Bbbk, x \mapsto x$ is the function given in Subsection 9.1, and $\widehat{\kappa} : \widehat{\mathbf{S}}^{\oplus 2} \to \widehat{\mathbf{S}}$ is a $\Bbbk$-homomorphism defined as

$$\widehat{\kappa}(F_1, F_2) = \begin{cases} \frac{1}{2}F_1(2x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}(F_1(1) + F_2(2x-1)), & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

which is a natural extension of the $\Bbbk$-homomorphism κ (the definition of κ is given in Subsection 9.1).

Lemma 9.4. *The triple* $(\widehat{\mathbf{S}}, \mathrm{id}, \widehat{\kappa})$ *is an object in* $\mathscr{A}^1$.

Proof. It is clear that $\widehat{\kappa}$ sends $(\mathrm{id}, \mathrm{id})$ to id by using the definition of $\widehat{\kappa}$. Now, let $\{(F_{1,n}, F_{2,n})\}_{n\in\mathbb{N}}$ be any Cauchy sequence in $\widehat{\mathbf{S}}^{\oplus 2}$ whose limits is (F_1, F_2). We need to prove

$$\varprojlim \kappa(F_{1,n}, F_{2,n}) = \kappa(\varprojlim (F_{1,n}, F_{2,n})).$$

Indeed, we have

$$\begin{aligned} \varprojlim \kappa(F_{1,n}, F_{2,n}) &= \begin{cases} \frac{1}{2}\varprojlim F_{1,n}(2x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}(F_{2,n}(1) + \varprojlim F_{2,n}(2x-1)), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \\ &= \begin{cases} \frac{1}{2}F_1(2x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}(F_2(1) + F_2(2x-1)), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \\ &= \kappa(F_1, F_2) \\ &= \kappa(\varprojlim (F_{1,n}, F_{2,n})), \end{aligned}$$

as required. $\square$

Theorem 9.5. *There exists a morphism*

$$D \in \mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{S}}, \mathrm{id}, \widehat{\kappa}), (\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}))$$

in $\mathscr{A}^1$ *sending each element* $f \in \widehat{\mathbf{S}}$ *to its weak derivative.*

Proof. First of all, the diagram

$$\begin{tikzcd} \widehat{\mathbf{S}}^{\oplus 2} \arrow[r, "\widehat{\kappa}"] \arrow[d, "D^{\oplus 2}"'] & \widehat{\mathbf{S}} \arrow[d, "D"] \\ \widehat{\mathbf{S}}^{\oplus 2} \arrow[r, "\widehat{\gamma}"'] & \widehat{\mathbf{S}} \end{tikzcd}$$

commutes since for any $F_1(x), F_2(x) \in \widehat{\mathbf{S}}$, we have

$$\begin{aligned} D \circ \widehat{\kappa}(F_1, F_2) &= \begin{cases} \frac{1}{2}\frac{d}{dx}F_1(2x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}\frac{d}{dx}(F_1(1) + F_2(2x-1)), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \\ &= \begin{cases} f_1(2x), & 0 \leqslant x \leqslant \frac{1}{2}, \\ f_2(2x-1), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \end{aligned}$$

$$= \widehat{\gamma}(f_1, f_2) = \widehat{\gamma} \circ D^{\oplus 2}(F_1, F_2),$$

where $\frac{d}{dx}F_i(x) = f_i(x)$ and $i \in \{1, 2\}$. Moreover, it is obvious that $D(\mathrm{id}) = \frac{d}{dx}\mathrm{id} = 1$, and thus D is a morphism in $\mathscr{A}^1$. $\square$

10 Applications and examples

10.1 Lebesgue integration

We assume the following assumption holds in this subsection.

Assumption 10.1. Take $\Bbbk = \mathbb{R}$, $(\Lambda, \prec, \|\cdot\|_\Lambda) = (\mathbb{R}, \leqslant, \|\cdot\|_\mathbb{R})$, $B_\mathbb{R} = \{1\}$ and $\mathfrak{n} : B_\mathbb{R} \to \{1\} \subseteq \mathbb{R}^{\geqslant 0}$. Then $\dim \mathbb{R} = 1$, $\mathbb{R}$ is a normed $\mathbb{R}$-algebra with the norm $\|\cdot\|_\mathbb{R} = |\cdot| : \mathbb{R} \to \mathbb{R}^{\geqslant 0}$ sending each real number r to its absolute value $|r|$, and any normed $\mathbb{R}$-module is a normed $\Bbbk$-vector space. Take $\mathbb{I}_\mathbb{R} = [0, 1]$, $\xi = \frac{1}{2}$, $\kappa_0(x) = \frac{x}{2}$, $\kappa_1(x) = \frac{x+1}{2}$ and $\tau = \mathrm{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$. Then any object (N, v, δ) in $\mathscr{N}or^p$ is a triple of a normed $\Bbbk$-module $N = (N, h_N, \|\cdot\|)$, an element $v \in N$ with $\|v\|_1$ and the $\Bbbk$-linear map $\delta : N \oplus_1 N \to N$, where the norm $\|\cdot\|$ satisfies $\|rx\| = |\tau(r)| \cdot \|x\| = |r| \cdot \|x\|$ for any $r \in \Lambda = \mathbb{R}$ and $x \in N$.

Under the above assumption, we have the following properties for $\mathscr{N}or^p$.

(L1) The normed $\Bbbk$-module $\mathbf{S}_\tau(\mathbb{I}_\Lambda) = \mathbf{S}_{\mathbf{1}_\mathbb{R}}([0, 1])$ ($= \mathbf{S}$ for short) is a $\Bbbk$-vector space of all elementary simple functions which are of the form

$$f = \sum_{x=i}^{t} k_i \mathbf{1}_{[x_i, y_i]},$$

where $[x_i, y_i] \cap [x_j, y_j] = \emptyset$ for any $i \neq j$, and for any $f(r), g(r) \in \mathbf{S}$, it holds that

$$\gamma_{\frac{1}{2}}(f, g) = \begin{cases} f(2r), & 1 \leqslant r < \dfrac{1}{2}, \\ g(2r-1), & \dfrac{1}{2} < r \leqslant 1, \end{cases}$$

by the definition of γ_ξ (see (3.2)).

(L2) $\mathscr{A}^p$ is a full subcategory, $(\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}})$ is an object in $\mathscr{N}or^p$, but is not an object in $\mathscr{A}^p$ because $\mathbf{S}$ is not complete.

(L3) Let $\widehat{\mathbf{S}}$ be the completion of $\mathbf{S}$, and let $\widehat{\gamma}_{\frac{1}{2}}$ be the map $\widehat{\mathbf{S}} \oplus_1 \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$ induced by $\gamma_{\frac{1}{2}}$. Then $(\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}})$ is an object in $\mathscr{A}^p$.

By Theorem 6.4, we obtain the following result directly.

Corollary 10.2. *The triple $(\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}})$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$.*

Remark 10.3. It follows from Theorem 6.4 that $(\widehat{\mathbf{S}}, \mathbf{1}[0, 1], \widehat{\gamma}_{\frac{1}{2}})$ is an initial object in $\mathscr{A}^p$, and then Corollary 10.2 holds. In [29], Leinster showed that the initial object in $\mathscr{A}^p$ is $(L^p([0, 1]), \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}})$. Then we obtain $L^p([0, 1]) \cong \widehat{\mathbf{S}}$ by the uniqueness (up to isomorphism) of initial objects in arbitrary categories.

Consider the triple $(\mathbb{R}, 1, m)$ of the normed $\mathbb{R}$-module $\mathbb{R}$, the constant function and the map

$$m : \mathbb{R} \oplus_p \mathbb{R} \to \mathbb{R}$$

sending (x, y) to $\frac{1}{2}(x + y)$. Then $(\mathbb{R}, 1, m)$ is an object in $\mathscr{A}^p$, and there is a family of $\mathbb{R}$-linear maps $(L_i : E_i \to \Bbbk)_{i \in \mathbb{N}}$ such that the diagram

$$\begin{tikzcd} E_i \oplus_p E_i \arrow[r, "\gamma_{\frac{1}{2}}"] \arrow[d, "\left(\begin{smallmatrix} L_i & 0 \\ 0 & L_i \end{smallmatrix}\right)"'] & E_{i+1} \arrow[d, "L_{i+1}"] \\ \Bbbk \oplus_p \Bbbk \arrow[r, "m_i"'] & \Bbbk \end{tikzcd}$$

commutes, where E_i is the set of all step function constants on each $(\frac{t-1}{2^i}, \frac{t}{2^i})$, L_i sends $f = \sum_i k_i \mathbf{1}_{[a_i, b_i]}$ to $\sum_i k_i |b_i - a_i|$, and $m = \varinjlim m_i$. Furthermore, we have the following result.

Corollary 10.4. *There exists a unique morphism*

$$L : (\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}}) \to (\mathbb{R}, 1, m)$$

in $\mathrm{Hom}_{\mathscr{N}or^1}((\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}}), (\mathbb{R}, 1, m))$ *such that the diagram*

$$\begin{tikzcd}
(\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}}) \arrow[r, "L"] \arrow[d, "\subseteq"'] & (\mathbb{R}, 1, m) \\
(\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}}) \arrow[ur, "\widehat{L}"'] &
\end{tikzcd}$$

commutes, where $\widehat{L}$ *is the unique extension of* L *lying in* $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}}), (\mathbb{R}, 1, m))$*. Furthermore,* $\widehat{L}$ *is given by the direct limit* $\varinjlim L_i : \varinjlim E_i \to \Bbbk$*.*

Proof. It is an immediate consequence of Theorem 7.6. □

The morphism $\widehat{L}$ induces a $\Bbbk$-linear map sending f to $\widehat{L}(f)$. Furthermore, if $\mu = \mu_{\mathrm{L}}$ is a Lebesgue measure, then $\widehat{L}(f)$ is Lebesgue integration $(\mathrm{L})\int$ of f, i.e.,

$$\widehat{L}(f) = (\mathrm{L})\int_0^1 f d\mu_{\mathrm{L}},$$

where μ_{L} is the Lebesgue measure in this case (see [29, Proposition 2.2]).

Next, as an application, we establish the Cauchy-Schwarz inequality for the morphism $\widehat{T}$ in $\mathscr{N}or^1$. We need the following lemma for arbitrary complete finite-dimensional $\mathbb{R}$-algebras.

Lemma 10.5. *If* $f \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ *is non-negative, then so is* $\widehat{T}(f)$*, i.e.,* $f \geqslant 0$ *yields*

$$(\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f d\mu \geqslant 0.$$

Proof. By $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} = \varinjlim E_u$, there is a monotonically increasing sequence $\{f_t\}_{t\in\mathbb{N}^+}$ of non-negative functions with $f_t = \sum_{i=1}^{2^{u_t}} k_{ti}\mathbf{1}_{I_{ti}} \in E_{u_t}$, such that $I_{ti} \cap I_{tj} = \emptyset$ for any $i \neq j$, $t_1 < t_2$ yields $u_{t_1} < u_{t_2}$ and $f_{t_1} \leqslant f_{t_2}$, and $f = \varinjlim f_t$. Thus, for any $1 \leqslant i \leqslant 2^{u_t}$ and $t \in \mathbb{N}^+$, we have $k_{ti\geqslant 0}$, and then the inequality

$$\widehat{T}(f_t) = T_{u_t}(f_t) = \sum_{i=1}^{2^{u_t}} k_{ti}\mu(I_{ti}) \geqslant 0$$

holds. Furthermore, we obtain

$$\widehat{T}(f) = \varinjlim T_{u_t}(f_t) = \varinjlim T|_{E_{u_t}}(f_t) = \varinjlim T(f_t) \geqslant 0$$

as required, where

$$\varinjlim T(f_t) = \lim_{t\to+\infty} T(f_t)$$

is the usual limit in $\mathbb{R}$ in analysis. □

Proposition 10.6 (Cauchy-Schwarz inequality)**.** *Let* f *and* g *be two functions lying in* $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$*. Then*

$$\bigg((\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} fg d\mu\bigg)^2 \leqslant \bigg((\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} f^2 d\mu\bigg)\bigg((\mathscr{A}^1)\int_{\mathbb{I}_\Lambda} g^2 d\mu\bigg). \tag{10.1}$$

Proof. Indeed, consider the quadratic function

$$\varphi(t) = \widehat{T}(f^2)\cdot t^2 + 2\widehat{T}(fg)\cdot t + \widehat{T}(g^2) \ (t \in \mathbb{R}).$$

Notice that $\widehat{T}$ is a Λ-homomorphism, and thus it is also an $\mathbb{R}$-linear map. Then

$$\varphi(t) = \widehat{T}(f^2 \cdot (t\mathbf{1}_{\mathbb{R}})^2 + 2fg \cdot (t\mathbf{1}_{\mathbb{R}}) + g^2) = \widehat{T}((f \cdot (t\mathbf{1}_{\mathbb{R}}) + g)^2).$$

Notice that $(f \cdot (t\mathbf{1}_{\mathbb{R}}) + g)^2$, written as h, is also a function defined on $\mathbb{I}_\Lambda$ lying in $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$, and thus for any $x \in \mathbb{I}_\Lambda$, we have $h(x) = (tf(x) + g(x))^2 \geqslant 0$. Then $\varphi(t) \geqslant 0$ by Lemma 10.5. It follows that the discriminant $(2\widehat{T}(fg))^2 - 4\widehat{T}(f^2)\widehat{T}(g^2)$ of $\varphi(x)$ is at most zero, i.e., (10.1) holds. $\square$

The above inequality yields that if $\mathscr{N}\!or^p$ satisfies the conditions (L1)–(L3) given in the subsection, then the Cauchy-Schwarz inequality

$$\left((\mathrm{L})\int_0^1 fg d\mu_{\mathrm{L}}\right)^2 \leqslant \left((\mathrm{L})\int_0^1 f^2 d\mu_{\mathrm{L}}\right)\left((\mathrm{L})\int_0^1 g^2 d\mu_{\mathrm{L}}\right)$$

holds.

10.2 Series expansions of functions

We provide two examples for Corollaries 8.2 and 8.5 in this subsection.

Example 10.7 (Taylor series)**.** Assume that $\mathscr{A}^1$ satisfies Assumption 10.1. Then the Λ-homomorphism $\widehat{E_{\mathrm{pow}}}$ in Corollary 8.2 is

$$\widehat{E_{\mathrm{pow}}} : (\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}) \to (\widehat{\Bbbk[x]}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}|_{\widehat{\Bbbk[x]}}).$$

Now we show that for any analytic function $f(x) \in \widehat{\mathbf{S}}$, we have

$$\widehat{E_{\mathrm{pow}}} : f(x) \mapsto \sum_{k=0}^{+\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} x^k.$$

To do this, let $\mathbf{A}_0$ be the set of all analytic functions defined on $[0,1]$, and define

$$\mathbf{A}_u = \{\widehat{\gamma}_{\frac{1}{2}}(f,g) \mid (f,g) \in \mathbf{A}_{u-1}^{\oplus 2}\}$$

for any $u \in \mathbb{N}$. Then we have

$$\Bbbk[x] \subseteq \mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \cdots \subseteq \widehat{\mathbf{S}} \cong L^1([0,1]).$$

Let $\mathfrak{E}_0 : \mathbf{A}_0 \to \widehat{\mathbf{S}}$ be the map sending each analytic function $f(x)$ to its Taylor series

$$\sum_{k=0}^{+\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} x^k \in \widehat{\Bbbk[x]}.$$

Then $\mathfrak{E}_0$ is a $\Bbbk$-linear map since for any $a, b \in \Lambda = \mathbb{R}$ and $f, g \in \widehat{\mathbf{S}}$, the $\mathbb{R}$-linear formula $\mathfrak{E}_0(a \cdot f + b \cdot g) = a\mathfrak{E}_0(f) + b\mathfrak{E}_0(g)$ holds. Furthermore, one can check that $\mathfrak{E}_0$ is a Λ-homomorphism. For any $u \in \mathbb{N}$, any function f in $\mathbf{A}_u$ can be seen as two functions f_1 and f_2 lying in $\mathbf{A}_{u-1}$ such that

$$f = \widehat{\gamma}_{\frac{1}{2}}(f_1, f_2) = \begin{cases} f_1(2x), & 0 \leqslant x < \dfrac{1}{2}, \\ f_2(2x-1), & \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

Thus, we can inductively define

$$\mathfrak{E}_u : \mathbf{A}_u \to \widehat{\Bbbk[x]}, \quad f \mapsto \widehat{\gamma}_{\frac{1}{2}}(\mathfrak{E}_{u-1}(f_1), \mathfrak{E}_{u-1}(f_2)).$$

Let $\mathbf{A}$ be the direct limit $\varinjlim \mathbf{A}_u$ given by $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \cdots$. The following statements (a) and (b) show that $\mathfrak{E} := \varinjlim \mathfrak{E}_u : \mathbf{A} \to \widehat{\Bbbk[x]}$, induced by $\varinjlim \mathbf{A}_u = \mathbf{A}$, is a homomorphism in $\mathscr{N}\!or^1$.

(a) First of all, it is obvious that $\mathfrak{E}(\mathbf{1}) = \varinjlim \mathfrak{E}_u(\mathbf{1}) = \varinjlim \mathbf{1} = \mathbf{1}$.

(b) Next, for any two functions $f_1(x)$ and $f_2(x)$ in $\mathbf{A}$, the diagram

$$\begin{tikzcd} \mathbf{A}^{\oplus 2} \arrow[r, "\widehat{\gamma}_{\frac{1}{2}}|_{\mathbf{A}}"] \arrow[d, "\left(\begin{smallmatrix} \mathfrak{E} & 0 \\ 0 & \mathfrak{E} \end{smallmatrix}\right)"'] & \mathbf{A} \arrow[d, "\mathfrak{E}"] \\ \widehat{\Bbbk[x]}^{\oplus 2} \arrow[r, "\widehat{\gamma}_{\frac{1}{2}}"'] & \widehat{\Bbbk[x]} \end{tikzcd}$$

commutes since

$$\mathfrak{E}(\widehat{\gamma}_{\frac{1}{2}}|_{\mathbf{A}}(f(x), g(x))) = \begin{cases} \mathfrak{E}(f(2x)), & 0 \leqslant x < \dfrac{1}{2}, \\ \mathfrak{E}(g(2x-1)), & \dfrac{1}{2} < x \leqslant 1 \end{cases}$$
$$= \widehat{\gamma}_{\frac{1}{2}}(\mathfrak{E}(f(x)), \mathfrak{E}(g(x))).$$

Thus, the completion $\widehat{\mathbf{A}}$ of $\mathbf{A}$ induces a Λ-homomorphism $\widehat{\mathfrak{E}} : \widehat{\mathbf{A}} \to \widehat{\Bbbk[x]}$ which provides a morphism

$$\widehat{\mathfrak{E}} \in \mathrm{Hom}_{\mathscr{A}^1}((\widehat{\mathbf{A}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}|_{\mathbf{A}}), (\widehat{\Bbbk[x]}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}|_{\Bbbk[x]}))$$

in the category $\mathscr{A}^1$.

On the other hand, for any polynomial $P(x) \in \Bbbk[x]$, there exists a monotonically increasing sequence $\{s_i(x)\}_{i=0}^{+\infty}$ of elementary simple functions such that $\varinjlim s_i(x) = P(x)$. Then we obtain that $\Bbbk[x]$ is dense in $\widehat{\mathbf{S}}$. It follows that $\widehat{\mathbf{A}}$ is dense in $\widehat{\mathbf{S}}$ by $\Bbbk[x] \subseteq \widehat{\mathbf{A}}$. Thus, we have an isomorphism

$$\eta : (\widehat{\mathbf{A}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}|_{\mathbf{A}}) \xrightarrow{\cong} (\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}})$$

and an isomorphism

$$\widehat{\mathfrak{E}}\eta^{-1} : (\widehat{\mathbf{A}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}|_{\mathbf{A}}) \xrightarrow{\cong} (\widehat{\Bbbk[x]}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}|_{\Bbbk[x]})$$

in the category $\mathscr{A}^1$ such that

$$\widehat{E_{\mathrm{pow}}}(f) = (\widehat{\mathfrak{E}}\eta^{-1})|_{\mathbf{A}_0}(f) = \mathfrak{E}_0(f) = \sum_{k=0}^{+\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} x^k$$

holds for any analytic function f by using $(\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}})$ to be an initial object of $\mathscr{A}^1$ (see Theorem 6.4).

Example 10.8 (Fourier series)**.** Assume that $\mathscr{A}^1$ satisfies Assumption 10.1. Then the Λ-homomorphism $\widehat{E_{\mathrm{Fou}}}$ in Corollary 8.5 is

$$\widehat{E_{\mathrm{Fou}}} : (\widehat{\mathbf{S}}, \mathbf{1}, \widehat{\gamma}_{\frac{1}{2}}) \to (\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi \mathrm{i} x}]}, \mathbf{1}, \widehat{\gamma_{\frac{1}{2}}}|_{\widehat{\mathbb{C}[\mathrm{e}^{\pm 2\pi \mathrm{i} x}]}}),$$

which sends each function f satisfying the Dirichlet condition to its Fourier series. The proof of the above statement is similar to that of Example 10.7 by using $\mathbb{C}[\mathrm{e}^{\pm 2\pi \mathrm{i} x}]$ to be a dense subspace of $\widehat{\mathbf{S}}$. In particular, $\widehat{E_{\mathrm{Fou}}}$ induces an isomorphism in $\mathscr{A}^1$.

11 Conclusions

In this paper, we have significantly expanded the theoretical landscape of mathematical analysis by extending the domain of classical Lebesgue integration beyond the real numbers and establishing a robust framework for the major branches of analysis—differentiation, integration, and series—over finite-dimensional $\Bbbk$-algebras. By developing the categories $\mathscr{N}\!or^p$ and $\mathscr{A}^p$, we have introduced a structured methodology for examining norms and integration within an algebraic context. This approach not only enhances our understanding of these processes but also provides a unified perspective across various analytical branches.

Our study has not only reinforced existing mathematical theories within a generalized algebraic setting, but has also paved the way for exploring how these concepts interact within the realms of category theory. The categorification of key analytical operations such as differentiation and integration through normed modules and their morphisms in $\mathscr{A}^p$ illustrates a significant theoretical advance, bridging various analytical disciplines through a common categorical framework.

The implications of this work extend beyond the theoretical, suggesting applications in fields that benefit from a deep understanding of the algebraic underpinnings of analysis, such as computational mathematics and theoretical physics. Looking forward, the exploration of higher-dimensional normed modules within this categorical framework promises to open new research avenues in areas such as quantum field theory and numerical methods for differential equations.

In summary, our research not only deepens the mathematical understanding of the interplay between algebra and analysis, but also lays a solid foundation for further explorations. Future work can extend these methods to more complex algebraic structures and explore their practical applications in science and engineering, thereby continuing to bridge the gap between abstract theory and real-world problem-solving.

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