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Special precovered categories of Gorenstein categories

Tiwei Zhao & Zhaoyong Huang*

Department of Mathematics, Nanjing University, Nanjing 210093, *China*
Email: tiweizhao@hotmail.com, huangzy@nju.edu.cn

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Abstract Let $\mathscr{A}$ be an abelian category and $\mathscr{P}(\mathscr{A})$ be the subcategory of $\mathscr{A}$ consisting of projective objects. Let $\mathscr{C}$ be a full, additive and self-orthogonal subcategory of $\mathscr{A}$ with $\mathscr{P}(\mathscr{A})$ a generator, and let $\mathcal{G}(\mathscr{C})$ be the Gorenstein subcategory of $\mathscr{A}$. Then the right 1-orthogonal category $\mathcal{G}(\mathscr{C})^{\perp_1}$ of $\mathcal{G}(\mathscr{C})$ is both projectively resolving and injectively coresolving in $\mathscr{A}$. We also get that the subcategory $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ of $\mathscr{A}$ consisting of objects admitting special $\mathcal{G}(\mathscr{C})$-precovers is closed under extensions and $\mathscr{C}$-stable direct summands (∗). Furthermore, if $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$, then we have that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is the minimal subcategory of $\mathscr{A}$ containing $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C})$ with respect to the property (∗), and that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving in $\mathscr{A}$ with a $\mathscr{C}$-proper generator $\mathscr{C}$.

Keywords Gorenstein categories, right 1-orthogonal categories, special precovers, special precovered categories, projectively resolving, injectively coresolving

MSC(2010) 18G25, 18E10

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1 Introduction

As a generalization of finitely generated projective modules, Auslander and Bridger [3] introduced the notion of finitely generated modules of Gorenstein dimension zero over commutative Noetherian rings. Then Enochs and Jenda [7] generalized it to arbitrary modules over a general ring and introduced the notion of Gorenstein projective modules and its dual (i.e., the notion of Gorenstein injective modules). Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ an additive and full subcategory of $\mathscr{A}$. Recently, Sather-Wagstaff et al. [14] introduced the notion of the Gorenstein subcategory $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$, which is a common generalization of the notions of modules of Gorenstein dimension zero (see [3]), Gorenstein projective modules, Gorenstein injective modules (see [7]), V-Gorenstein projective modules and V-Gorenstein injective modules (see [9]), and so on.

Let R be an associative ring with identity, and let $\mathrm{Mod}\, R$ be the category of left R-modules and $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ the subcategory of $\mathrm{Mod}\, R$ consisting of Gorenstein projective modules. Let $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)))$ and $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)))$ be the subcategories of $\mathrm{Mod}\, R$ consisting of modules admitting a $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$-precover and admitting a special $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$-precover, respectively. The following question in relative homological algebra still remains open: Does $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))) = \mathrm{Mod}\, R$ always

* Corresponding author

hold true? Several authors have given some partially positive answers to this question (see [2, 4, 5, 16]). Note that in these references, $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R))) = \mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R)))$ (see Example 4.8 below for details). In particular, any module in $\mathrm{Mod}\,R$ with finite Gorenstein projective dimension admits a $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R))$-precover which is also special (see [10]). In fact, it is unknown whether $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R))) = \mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R)))$ always holds true. Based on the above, it is necessary to study the properties of these two subcategories.

Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ be an additive and full subcategory of $\mathscr{A}$. We use $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ to denote the subcategory of $\mathscr{A}$ consisting of objects admitting special $\mathcal{G}(\mathscr{C})$-precovers. The aim of this paper is to investigate the structure of $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ in terms of the properties of the right 1-orthogonal category $\mathcal{G}(\mathscr{C})^{\perp_1}$ of $\mathcal{G}(\mathscr{C})$. This paper is organized as follows.

In Section 2, we give some terminologies and some preliminary results.

Assume that $\mathscr{C}$ is self-orthogonal and the subcategory of $\mathscr{A}$ consisting of projective objects is a generator for $\mathscr{C}$. In Section 3, we prove that $\mathcal{G}(\mathscr{C})^{\perp_1}$ is both projectively resolving and injectively coresolving in $\mathscr{A}$. We also characterize when all objects in $\mathscr{A}$ are in $\mathcal{G}(\mathscr{C})^{\perp_1}$.

In Section 4, we prove that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under extensions and $\mathscr{C}$-stable direct summands $(*)$. Furthermore, if $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$, then we get the following two results: (1) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is the minimal subcategory of $\mathscr{A}$ containing $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C})$ with respect to the property $(*)$; and (2) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving in $\mathscr{A}$ with a $\mathscr{C}$-proper generator $\mathscr{C}$.

2 Preliminaries

Throughout this paper, $\mathscr{A}$ is an abelian category and all subcategories of $\mathscr{A}$ are full, additive and closed under isomorphisms. We use $\mathscr{P}(\mathscr{A})$ (resp. $\mathscr{I}(\mathscr{A})$) to denote the subcategory of $\mathscr{A}$ consisting of projective (resp. injective) objects. For a subcategory $\mathscr{C}$ of $\mathscr{A}$ and an object A in $\mathscr{A}$, the *$\mathscr{C}$-dimension* $\mathscr{C}\text{-}\dim A$ of A is defined as $\inf\{n \geqslant 0 \mid$ there exists an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to A \to 0$ in $\mathscr{A}$ with all C_i in $\mathscr{C}\}$. Set $\mathscr{C}\text{-}\dim A = \infty$ if no such integer exists (see [12]). For a non-negative integer n, we use $\mathscr{C}^{\leqslant n}$ (resp. $\mathscr{C}^{<\infty}$) to denote the subcategory of $\mathscr{A}$ consisting of objects with $\mathscr{C}$-dimension at most n (resp. finite $\mathscr{C}$-dimension).

Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$. Recall that a sequence in $\mathscr{A}$ is called $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-*exact* if it is exact after applying the functor $\mathrm{Hom}_{\mathscr{A}}(X, -)$ for any $X \in \mathscr{X}$. Dually, the notion of a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$-*exact sequence* is defined. Set

$$\mathscr{X}^{\perp} := \{M \mid \mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(X, M) = 0 \text{ for any } X \in \mathscr{X}\}, \quad {}^{\perp}\mathscr{X} := \{M \mid \mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(M, X) = 0 \text{ for any } X \in \mathscr{X}\},$$

and

$$\mathscr{X}^{\perp_1} := \{M \mid \mathrm{Ext}^{1}_{\mathscr{A}}(X, M) = 0 \text{ for any } X \in \mathscr{X}\}, \quad {}^{\perp_1}\mathscr{X} := \{M \mid \mathrm{Ext}^{1}_{\mathscr{A}}(M, X) = 0 \text{ for any } X \in \mathscr{X}\}.$$

We call $\mathscr{X}^{\perp_1}$ (resp. ${}^{\perp_1}\mathscr{X}$) the *right* (*resp. left*) 1-*orthogonal category of* $\mathscr{X}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\mathscr{A}$. We write $\mathscr{X} \perp \mathscr{Y}$ if $\mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Definition 2.1 (See [6])**.** Let $\mathscr{X} \subseteq \mathscr{Y}$ be subcategories of $\mathscr{A}$. The morphism $f : X \to Y$ in $\mathscr{A}$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$ is called an *$\mathscr{X}$-precover* of Y if $\mathrm{Hom}_{\mathscr{A}}(X', f)$ is epic for any $X' \in \mathscr{X}$. An $\mathscr{X}$-precover $f : X \to Y$ is called *special* if f is epic and $\mathrm{Ker}\, f \in \mathscr{X}^{\perp_1}$. $\mathscr{X}$ is called *special precovering* in $\mathscr{Y}$ if any object in $\mathscr{Y}$ admits a special $\mathscr{X}$-precover. Dually, the notions of a (*special*) *$\mathscr{X}$-*(*pre*)*envelope* and a *special preenveloping subcategory* are defined.

Definition 2.2 (See [10])**.** A subcategory of $\mathscr{A}$ is called *projectively resolving* if it contains $\mathscr{P}(\mathscr{A})$ and is closed under extensions and under kernels of epimorphisms. Dually, the notion of *injectively coresolving subcategories* is defined.

From now on, assume that $\mathscr{C}$ is a given subcategory of $\mathscr{A}$.

Definition 2.3 (See [14]). The *Gorenstein subcategory* $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$ is defined as $\mathcal{G}(\mathscr{C}) = \{M$ is an object in $\mathscr{A} \mid$ there exists an exact sequence:

$$\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots \tag{2.1}$$

in $\mathscr{C}$, which is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact, such that $M \cong \mathrm{Im}(C_0 \to C^0)\}$; in this case, (2.1) is called a *complete $\mathscr{C}$-resolution* of M.

In what follows, R is an associative ring with identity, $\mathrm{Mod}\, R$ is the category of left R-modules and $\mathrm{mod}\, R$ is the category of finitely generated left R-modules.

Remark 2.4. (1) Let R be a left and right Noetherian ring. Then $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))$ coincides with the subcategory of $\mathrm{mod}\, R$ consisting of modules with Gorenstein dimension zero (see [3]).

(2) $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ (resp. $\mathcal{G}(\mathscr{I}(\mathrm{Mod}\, R))$) coincides with the subcategory of $\mathrm{Mod}\, R$ consisting of Gorenstein projective (resp. injective) modules (see [7]).

(3) Let R be a left Noetherian ring, S be a right Noetherian ring and ${}_RV_S$ be a dualizing bimodule. Put $\mathscr{W} = \{V \bigotimes_S P \mid P \in \mathscr{P}(\mathrm{Mod}\, S)\}$ and $\mathscr{U} = \{\mathrm{Hom}_S(V, E) \mid E \in \mathscr{I}(\mathrm{Mod}\, S^{\mathrm{op}})\}$. Then $\mathcal{G}(\mathscr{W})$ (resp. $\mathcal{G}(\mathscr{U})$) coincides with the subcategory of $\mathrm{Mod}\, R$ consisting of V-Gorenstein projective (resp. injective) modules (see [9]).

Definition 2.5 (See [14]). Let $\mathscr{X} \subseteq \mathscr{T}$ be subcategories of $\mathscr{A}$. Then $\mathscr{X}$ is called a *generator* (resp. *cogenerator*) for $\mathscr{T}$ if for any $T \in \mathscr{T}$, there exists an exact sequence $0 \to T' \to X \to T \to 0$ (resp. $0 \to T \to X \to T' \to 0$) in $\mathscr{T}$ with $X \in \mathscr{X}$; and $\mathscr{X}$ is called a *projective generator* (resp. *an injective cogenerator*) for $\mathscr{T}$ if $\mathscr{X}$ is a generator (resp. cogenerator) for $\mathscr{T}$ and $\mathscr{X} \perp \mathscr{T}$ (resp. $\mathscr{T} \perp \mathscr{X}$).

We have the following easy observation.

Lemma 2.6. *Assume that $\mathscr{C} \perp \mathscr{C}$ and $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$. Then for any $G \in \mathcal{G}(\mathscr{C})$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact exact sequence $0 \to G' \to P \to G \to 0$ in $\mathscr{A}$ with $P \in \mathscr{P}(\mathscr{A})$ and $G' \in \mathcal{G}(\mathscr{C})$.*

Proof. Let $G \in \mathcal{G}(\mathscr{C})$. Then there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact exact sequence $0 \to G_1 \to C_0 \to G \to 0$ in $\mathscr{A}$ with $C_0 \in \mathscr{C}$ and $G_1 \in \mathcal{G}(\mathscr{C})$. Because $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$ by assumption, there exists an exact sequence $0 \to C' \to P \to C_0 \to 0$ in $\mathscr{A}$ with $P \in \mathscr{P}(\mathscr{A})$ and $C' \in \mathscr{C}$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C' & = & C' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & G' & \dashrightarrow & P & \dashrightarrow & G & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G_1 & \longrightarrow & C_0 & \longrightarrow & G & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By [11, Lemma 2.5], the middle row is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact, and hence $G' \in \mathcal{G}(\mathscr{C})$ by [11, Proposition 4.7], i.e., the middle row is the desired sequence. $\square$

The following result is useful in the sequel.

Proposition 2.7. *Assume that $\mathscr{C} \perp \mathscr{C}$ and $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$. Then*

(1) $\mathcal{G}(\mathscr{C})^{\perp_1} = \mathcal{G}(\mathscr{C})^{\perp}$.

(2) $\mathcal{G}(\mathscr{C}) \subseteq {}^{\perp}\mathscr{C} \cap \mathscr{C}^{\perp}$.

Proof. (1) It suffices to prove that $\mathcal{G}(\mathscr{C})^{\perp_1} \subseteq \mathcal{G}(\mathscr{C})^{\perp}$. Let $M \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $G \in \mathcal{G}(\mathscr{C})$. By Lemma 2.6, we have an exact sequence $0 \to G' \to P \to G \to 0$ in $\mathscr{A}$ with $P \in \mathscr{P}(\mathscr{A})$ and $G' \in \mathcal{G}(\mathscr{C})$. It induces

$\mathrm{Ext}^2_{\mathscr{A}}(G, M) \cong \mathrm{Ext}^1_{\mathscr{A}}(G', M) = 0$, and hence $\mathrm{Ext}^2_{\mathscr{A}}(G', M) = 0$ and $\mathrm{Ext}^3_{\mathscr{A}}(G, M) \cong \mathrm{Ext}^2_{\mathscr{A}}(G', M) = 0$. Repeating this process, we get $\mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(G, M) = 0$.

(2) See [11, Lemma 5.7]. $\square$

We remark that if $\mathscr{A}$ has enough projective objects, and if $\mathscr{P}(\mathscr{A}) \subseteq \mathscr{C}$ and $\mathscr{C}$ is closed under kernels of epimorphisms, then $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$.

3 The right 1-orthogonal category of $\mathcal{G}(\mathscr{C})$

In the rest of this paper, assume that the subcategory $\mathscr{C}$ is self-orthogonal (i.e., $\mathscr{C} \perp \mathscr{C}$) and $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$. In this section, we mainly investigate the homological properties of $\mathcal{G}(\mathscr{C})^{\perp_1}$. We begin with some examples of $\mathcal{G}(\mathscr{C})^{\perp_1}$.

Example 3.1. (1) By Proposition 2.7 and [11, Theorem 5.8], we have $\mathscr{P}(\mathscr{A}) \subseteq \mathscr{C} \subseteq \mathscr{C}^{<\infty} \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$.

(2) $\mathscr{P}(\mathscr{A})^{<\infty} \cup \mathscr{I}(\mathscr{A})^{<\infty} \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$.

(3) If the global dimension of R is finite, then $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathrm{Mod}\, R$.

(4) By [8, Theorem 11.5.1] and [1, Theorem 31.9], we have that R is quasi-Frobenius if and only if $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathscr{I}(\mathrm{Mod}\, R)$, and if and only if $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathscr{P}(\mathrm{Mod}\, R) = \mathscr{I}(\mathrm{Mod}\, R)$.

For a non-negative integer n, recall that a left and right Noetherian ring R is called *n-Gorenstein* if the left and right self-injective dimensions of R are at most n. The following result is a generalization of Example 3.1(4).

Example 3.2. If R is n-Gorenstein, then $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathscr{P}(\mathrm{Mod}\, R)^{\leqslant n} = \mathscr{P}(\mathrm{Mod}\, R)^{<\infty} = \mathscr{I}(\mathrm{Mod}\, R)^{\leqslant n} = \mathscr{I}(\mathrm{Mod}\, R)^{<\infty}$.

Proof. By [13, Theorem 2] and Example 3.1(2), we have $\mathscr{P}(\mathrm{Mod}\, R)^{\leqslant n} = \mathscr{P}(\mathrm{Mod}\, R)^{<\infty} = \mathscr{I}(\mathrm{Mod}\, R)^{\leqslant n} = \mathscr{I}(\mathrm{Mod}\, R)^{<\infty} \subseteq \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1}$.

Now let $M \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1}$ and $N \in \mathrm{Mod}\, R$. Since R is n-Gorenstein, there exists an exact sequence $0 \to G_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ in $\mathrm{Mod}\, R$ with all P_i in $\mathscr{P}(\mathrm{Mod}\, R)$ and $G_n \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ by [8, Theorem 11.5.1]. Then we have $\mathrm{Ext}^{n+1}_R(N, M) \cong \mathrm{Ext}^1_R(G_n, M) = 0$ and $M \in \mathscr{I}(\mathrm{Mod}\, R)^{\leqslant n}$, and thus $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} \subseteq \mathscr{I}(\mathrm{Mod}\, R)^{\leqslant n}$. $\square$

The following result shows that $\mathcal{G}(\mathscr{C})^{\perp_1}$ behaves well.

Theorem 3.3. (1) *$\mathcal{G}(\mathscr{C})^{\perp_1}$ is closed under direct products, direct summands and extensions.*

(2) *$\mathcal{G}(\mathscr{C})^{\perp_1}$ is projectively resolving in $\mathscr{A}$.*

(3) *$\mathcal{G}(\mathscr{C})^{\perp_1}$ is injectively coresolving in $\mathscr{A}$.*

Proof. (1) It is trivial.

(2) By Example 3.1(1), $\mathscr{P}(\mathscr{A}) \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$. Let $G \in \mathcal{G}(\mathscr{C})$ and $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathscr{A}$ with $M, N \in \mathcal{G}(\mathscr{C})^{\perp_1}$. By Proposition 2.7(1), we have $\mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(G, M) = 0 = \mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(G, N)$. Then $\mathrm{Ext}^{\geqslant 2}_{\mathscr{A}}(G, L) = 0$. Because $G \in \mathcal{G}(\mathscr{C})$, we have an exact sequence $0 \to G \to C^0 \to G^1 \to 0$ in $\mathscr{A}$ with $C^0 \in \mathscr{C}$ and $G^1 \in \mathcal{G}(\mathscr{C})$. For C^0, there exists an exact sequence $0 \to C^{-1} \to P^0 \to C^0 \to 0$ in $\mathscr{A}$ with $P^0 \in \mathscr{P}(\mathscr{A})$ and $C^{-1} \in \mathscr{C}$. Consider the following pullback diagram:

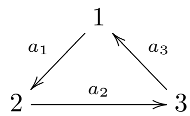
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C^{-1} & = & C^{-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & G^0 & \dashrightarrow & P^0 & \dashrightarrow & G^1 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G & \longrightarrow & C^0 & \longrightarrow & G^1 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By the above argument, we have $\mathrm{Ext}^1_{\mathscr{A}}(G^0, L) \cong \mathrm{Ext}^2_{\mathscr{A}}(G^1, L) = 0$. Because the leftmost column splits by Proposition 2.7(2), G is isomorphic to a direct summand of G^0 and $\mathrm{Ext}^1_{\mathscr{A}}(G, L) = 0$, which shows that $L \in \mathcal{G}(\mathscr{C})^{\perp_1}$.

(3) It is trivial that $\mathscr{I}(\mathscr{A}) \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$. By Proposition 2.7, we have that $\mathcal{G}(\mathscr{C})^{\perp_1}$ is closed under cokernels of monomorphisms. Thus $\mathcal{G}(\mathscr{C})^{\perp_1}$ is injectively coresolving. $\square$

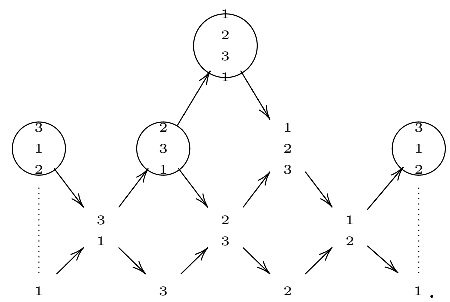
Before giving some applications of Theorem 3.3(2), consider the following example.

Example 3.4. Let Q be a quiver



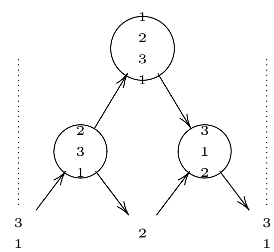
and $I = \langle a_1a_3a_2, a_2a_1a_3\rangle$. Let $R = kQ/I$ with k a field. Then the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, R)$ of $\mathrm{mod}\, R$ is as follows:

$\Gamma(\mathrm{mod}\, R)$:

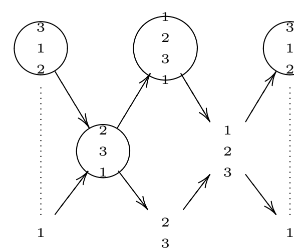


By a direct computation, we have

$\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))$:



, $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))^{\perp_1}$:



,

where the terms marked by circles are indecomposable projective modules in $\mathrm{mod}\, R$. Then we have $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R)) \cap \mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))^{\perp_1} = \mathscr{P}(\mathrm{mod}\, R)$.

In general, we have the following corollary.

Corollary 3.5. *If $\mathscr{C}$ is closed under direct summands, then for any $n \geqslant 0$, we have $\mathcal{G}(\mathscr{C})^{\leqslant n} \cap \mathcal{G}(\mathscr{C})^{\perp_1} = \mathscr{C}^{\leqslant n}$.*

Proof. By Example 3.1(1), we have $\mathscr{C}^{\leqslant n} \subseteq \mathcal{G}(\mathscr{C})^{\leqslant n} \cap \mathcal{G}(\mathscr{C})^{\perp_1}$.

Now let $M \in \mathcal{G}(\mathscr{C})^{\leqslant n} \cap \mathcal{G}(\mathscr{C})^{\perp_1}$. By [11, Theorem 5.8], there exists an exact sequence

$$0 \to K_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$$

in $\mathscr{A}$ with all C_i in $\mathscr{C}$ and $K_n \in \mathcal{G}(\mathscr{C})$. By Theorem 3.3(2), we have $K_n \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Because $\mathscr{C}$ is closed under direct summands by assumption, it follows easily from the definition of $\mathcal{G}(\mathscr{C})$ that $K_n \in \mathscr{C}$ and $M \in \mathscr{C}^{\leqslant n}$. $\square$

Proposition 3.6. *For any $M \in \mathscr{A}$, the following statements are equivalent:*

(1) *$M \in \mathcal{G}(\mathscr{C})^{\perp_1}$.*

(2) *The functor $\mathrm{Hom}_{\mathscr{A}}(-, M)$ is exact with respect to any short exact sequence in $\mathscr{A}$ ending with an object in $\mathcal{G}(\mathscr{C})$.*

(3) *Every short exact sequence starting with M is $\mathrm{Hom}_{\mathscr{A}}(\mathcal{G}(\mathscr{C}), -)$-exact.*

If, moreover, R is a commutative ring, $\mathscr{A} = \operatorname{Mod} R$ and $\mathscr{C} = \mathscr{P}(\operatorname{Mod} R)$, then the above conditions are equivalent to the following:

(4) $\operatorname{Hom}_R(Q, M) \in \mathcal{G}(\mathscr{P}(\operatorname{Mod} R))^{\perp_1}$ *for any* $Q \in \mathscr{P}(\operatorname{Mod} R)$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). It is easy.

Now let R be a commutative ring.

(1) $\Rightarrow$ (4). For any $G \in \mathcal{G}(\mathscr{P}(\operatorname{Mod} R))$, we have an exact sequence

$$0 \to K \xrightarrow{f} P \to G \to 0 \tag{3.1}$$

in $\operatorname{Mod} R$ with $P \in \mathscr{P}(\operatorname{Mod} R)$. Let $Q \in \mathscr{P}(\operatorname{Mod} R)$. Then $0 \to Q \otimes_R K \xrightarrow{1_Q \otimes f} Q \otimes_R P \to Q \otimes_R G \to 0$ is exact. It is easy to check that $Q \otimes_R G \in \mathcal{G}(\mathscr{P}(\operatorname{Mod} R))$. Then $\operatorname{Ext}^1_R(Q \otimes_R G, M) = 0$ by (1), and so $\operatorname{Hom}_R(1_Q \otimes f, M)$ is epic. By the adjoint isomorphism, we have that $\operatorname{Hom}_R(f, \operatorname{Hom}_R(Q, M))$ is also epic. So applying the functor $\operatorname{Hom}_R(-, \operatorname{Hom}_R(Q, M))$ to (3.1) we get $\operatorname{Ext}^1_R(G, \operatorname{Hom}_R(Q, M)) = 0$, and hence $\operatorname{Hom}_R(Q, M) \in \mathcal{G}(\mathscr{P}(\operatorname{Mod} R))^{\perp_1}$.

(4) $\Rightarrow$ (1). It is trivial by setting $Q = R$. □

In the following result, we characterize categories over which all objects are in $\mathcal{G}(\mathscr{C})^{\perp_1}$.

Proposition 3.7. *Assume that $\mathscr{C}$ is closed under direct summands. Consider the following conditions*:

(1) $\mathcal{G}(\mathscr{C})^{\perp_1} = \mathscr{A}$.

(2) $\mathcal{G}(\mathscr{C}) \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$.

(3) $\mathcal{G}(\mathscr{C}) = \mathscr{C}$.

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If $\mathscr{C}$ is a projective generator for $\mathscr{A}$, then all of them are equivalent.*

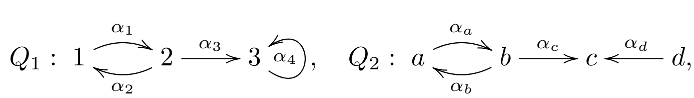
Proof. The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Let $G \in \mathcal{G}(\mathscr{C})$. Then there exists an exact sequence $0 \to G_1 \to C_0 \to G \to 0$ in $\mathscr{A}$ with $C_0 \in \mathscr{C}$ and $G_1 \in \mathcal{G}(\mathscr{C})$. By (2), we have that $G_1 \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and the above exact sequence splits. Thus as a direct summand of C_0, $G \in \mathscr{C}$ by assumption.

If $\mathscr{C}$ is a projective generator for $\mathscr{A}$, then the implication (3) $\Rightarrow$ (1) follows directly. □

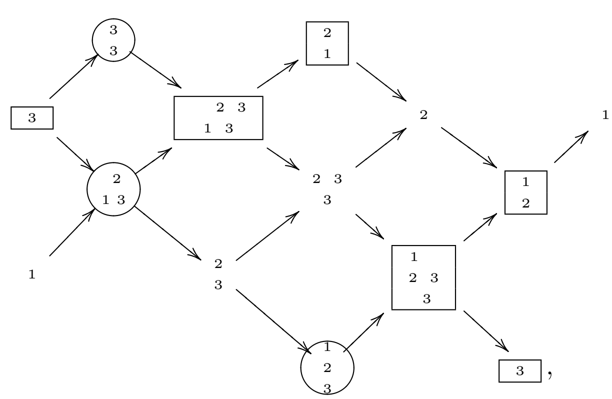
Let $\mathscr{X}$ be a subcategory of $\operatorname{mod} R$ containing $\mathscr{P}(\operatorname{mod} R)$. We use $\underline{\mathscr{X}}$ to denote the stable category of $\mathscr{X}$ modulo $\mathscr{P}(\operatorname{mod} R)$. We end this section by giving two examples about $\mathcal{G}(\mathscr{P}(\operatorname{mod} R))^{\perp_1}$.

Example 3.8. Let Q_1 and Q_2 be the following two quivers:

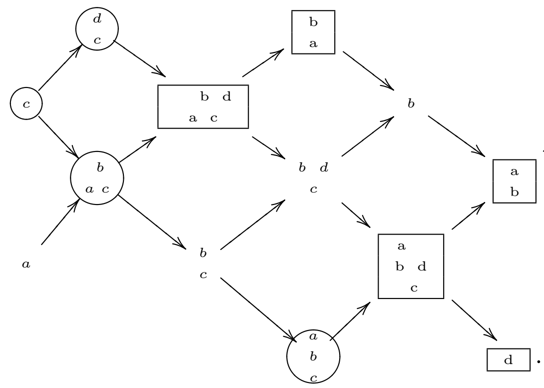


and let $I_1 = \langle \alpha_2\alpha_1, \alpha_1\alpha_2, \alpha_4\alpha_3, \alpha_4^2 \rangle$ and $I_2 = \langle \alpha_b\alpha_a, \alpha_a\alpha_b \rangle$. Let $R_1 = KQ_1/I_1$ and $R_2 = KQ_2/I_2$. Note that R_2 is Gorenstein and R_1 is not Gorenstein. The Auslander-Reiten quivers of $\operatorname{mod} R_1$ and $\operatorname{mod} R_2$ are as follows:

$\Gamma(\operatorname{mod} R_1)$:



$\Gamma(\mathrm{mod}\,R_2)$:



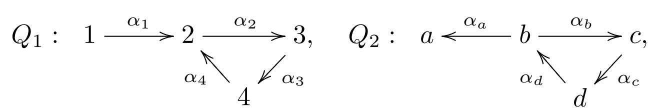
Then we have the following:

(1) The objects marked in a cycle or a box are indecomposable objects in $\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_i))^{\perp_1}$ $(i=1,2)$; in particular, the objects marked in a cycle are indecomposable objects in $\mathscr{P}(\mathrm{mod}\,R_i)$ $(i=1,2)$.

(2) $\underline{\mathrm{mod}}R_1 \simeq \underline{\mathrm{mod}}R_2$ and $\frac{\mathrm{mod}\,R_1}{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_1))^{\perp_1}} \simeq \frac{\mathrm{mod}\,R_2}{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_2))^{\perp_1}}$.

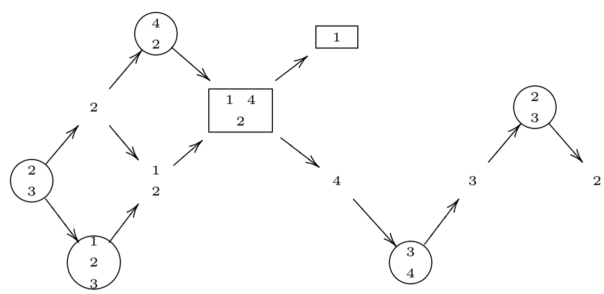
(3) $\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_1))^{\perp_1} \not\simeq \mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_2))^{\perp_1}$ and $\underline{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_1))^{\perp_1}} \simeq \underline{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_2))^{\perp_1}}$.

Example 3.9. Let Q_1 and Q_2 be the following two quivers:

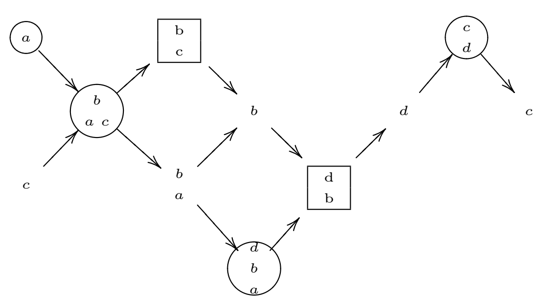


and let $I_1 = \langle \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_2\alpha_4 \rangle$ and $I_2 = \langle \alpha_c\alpha_b, \alpha_d\alpha_c, \alpha_b\alpha_d \rangle$. Let $R_1 = KQ_1/I_1$ and $R_2 = KQ_2/I_2$. Then the Auslander-Reiten quivers of $\mathrm{mod}\,R_1$ and $\mathrm{mod}\,R_2$ are as follows:

$\Gamma(\mathrm{mod}\,R_1)$:



$\Gamma(\mathrm{mod}\,R_2)$:



Then we have the following:

(1) The objects marked in a cycle or a box are indecomposable objects in $\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_i))^{\perp_1}$ $(i=1,2)$; in particular, the objects marked in a cycle are indecomposable objects in $\mathscr{P}(\mathrm{mod}\,R_i)$ $(i=1,2)$.

(2) $\underline{\mathrm{mod}}R_1 \not\simeq \underline{\mathrm{mod}}R_2$ and $\frac{\mathrm{mod}\,R_1}{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_1))^{\perp_1}} \simeq \frac{\mathrm{mod}\,R_2}{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_2))^{\perp_1}}$.

(3) $\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_1))^{\perp_1} \not\simeq \mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_2))^{\perp_1}$ and $\underline{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_1))^{\perp_1}} \simeq \underline{\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R_2))^{\perp_1}}$.

4 The special precovered category of $\mathcal{G}(\mathscr{C})$

In this section, we introduce and investigate the special precovered category of $\mathcal{G}(\mathscr{C})$ in terms of the properties of $\mathcal{G}(\mathscr{C})^{\perp_1}$.

Proposition 4.1. (1) *Let $M \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $f : C \twoheadrightarrow M$ be an epimorphism in $\mathscr{A}$ with $C \in \mathscr{C}$. Then* $\operatorname{Ker} f \in \mathcal{G}(\mathscr{C})^{\perp_1}$ *and f is a special $\mathcal{G}(\mathscr{C})$-precover of M.*

(2) *Consider an exact sequence*

$$0 \to M' \to C \to M \to 0. \tag{4.1}$$

If M' admits a special $\mathcal{G}(\mathscr{C})$-precover, then so is M. The converse is true if $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$ and (4.1) is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact.

Proof. (1) The assertion follows from Example 3.1(1) and Theorem 3.3(2).

(2) Assume that M' admits a special $\mathcal{G}(\mathscr{C})$-precover and $0 \to N \to G \to M' \to 0$ is an exact sequence in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$ and $N \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Combining it with the following $\operatorname{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact exact sequence:

$$0 \to G \xrightarrow{i} C^0 \xrightarrow{p} G^1 \to 0$$

in $\mathscr{A}$ with $C^0 \in \mathscr{C}$ and $G^1 \in \mathcal{G}(\mathscr{C})$, we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & N & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & G & \xrightarrow{i} & C^0 & \xrightarrow{p} & G^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & M' & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0. \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Adding the exact sequence

$$0 \longrightarrow 0 \longrightarrow C \xrightarrow{1_C} C \longrightarrow 0$$

to the middle row, we obtain the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & N & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & G & \xrightarrow{\binom{i}{0}} & C^0 \oplus C & \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1_C \end{smallmatrix}\right)} & G^1 \oplus C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle (g, 1_C)} & & \downarrow{\scriptstyle h'} & & \\ 0 & \longrightarrow & M' & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

which can be completed to a commutative diagram with exact columns and rows as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & C' & \to & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G & \xrightarrow{\binom{i}{0}} & C^0 \oplus C & \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & 1_C \end{pmatrix}} & G^1 \oplus C & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (g,1_C)} & & \downarrow{\scriptstyle h'} & & \\
0 & \to & M' & \to & C & \to & M & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that $G^1 \oplus C \in \mathcal{G}(\mathscr{C})$. Moreover, since $N \in \mathcal{G}(\mathscr{C})^{\perp_1}$, we have $M'' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Theorem 3.3(3). Thus the rightmost column in the above diagram is a special $\mathcal{G}(\mathscr{C})$-precover of M.

Now let $\mathscr{C}$ be a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$ and (4.1) be $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. Assume that M admits a special $\mathcal{G}(\mathscr{C})$-precover and $0 \to L \to G \to M \to 0$, $0 \to L' \to C' \to L \to 0$ are exact sequences in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$, $L \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $C' \in \mathscr{C}$. By [11, Lemma 3.1(1)], we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L' & \to & G' & \to & M' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C' & \to & C' \oplus C & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & G & \to & M & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Proposition 2.7(2) and Theorem 3.3(2), we have $L' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and the leftmost column is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. So the middle column is also $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. On the other hand, the middle column is $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{C})$-exact by Proposition 2.7(2). So $G' \in \mathcal{G}(\mathscr{C})$ by [11, Proposition 4.7(5)], and hence the upper row is a special $\mathcal{G}(\mathscr{C})$-precover of M'. □

We introduce the following definition.

Definition 4.2. We call $\mathrm{SPC}(\mathcal{G}(\mathscr{C})) := \{A \in \mathscr{A} \mid A \text{ admits a special } \mathcal{G}(\mathscr{C})\text{-precover}\}$ the *special precovered category* of $\mathcal{G}(\mathscr{C})$.

It is trivial that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is the largest subcategory of $\mathscr{A}$ such that $\mathcal{G}(\mathscr{C})$ is special precovering in it. In particular, $\mathrm{SPC}(\mathcal{G}(\mathscr{C})) = \mathscr{A}$ if and only if $\mathcal{G}(\mathscr{C})$ is special precovering in $\mathscr{A}$. For the sake of convenience, we say that a subcategory $\mathscr{X}$ of $\mathscr{A}$ is *closed under $\mathscr{C}$-stable direct summands* provided that the condition $X \oplus C \in \mathscr{X}$ with $C \in \mathscr{C}$ implies $X \in \mathscr{X}$.

Theorem 4.3. (1) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is closed under extensions.*

(2) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is closed under $\mathscr{C}$-stable direct summands.*

Proof. (1) Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathscr{A}$. Assume that L and N admit special $\mathcal{G}(\mathscr{C})$-precovers and $0 \to L' \to G_L \xrightarrow{f} L \to 0$, $0 \to N' \to G_N \xrightarrow{g} N \to 0$ are exact sequences in $\mathscr{A}$ with G_L, $G_N \in \mathcal{G}(\mathscr{C})$ and $L', N' \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
0 & \to & L & \to & Q & \to & G_N & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} & & \\
0 & \to & L & \to & M & \to & N & \to & 0.
\end{array}$$

Since $\mathrm{Ext}^2_R(G_N, L') = 0$ by Proposition 2.7(1), we get an epimorphism $\mathrm{Ext}^1_R(G_N, f) : \mathrm{Ext}^1_R(G_N, G_L) \to \mathrm{Ext}^1_R(G_N, L)$. It induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & G_L & \dashrightarrow & G_M & \dashrightarrow & G_N & \dashrightarrow & 0\\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & Q & \longrightarrow & G_N & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

Set $M' := \mathrm{Ker}\,\alpha\beta$. Then we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & L' & \dashrightarrow & M' & \dashrightarrow & N' & \dashrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_L & \longrightarrow & G_M & \longrightarrow & G_N & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0.\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that $G_M \in \mathcal{G}(\mathscr{C})$ (see [14, Corollary 4.5]) and $M' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ (see Theorem 3.3(1)). Thus the middle column in the above diagram is a special $\mathcal{G}(\mathscr{C})$-precover of M. This proves that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under extensions.

(2) Let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and $0 \to K \to G \to M \to 0$ be an exact sequence in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$ and $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Assume that $M \cong L \oplus C$ with $C \in \mathscr{C}$. We have an exact and split sequence $0 \to C \to M \to L \to 0$ in $\mathscr{A}$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K & \dashrightarrow & L' & \dashrightarrow & C & \dashrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0.\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $K, C \in \mathcal{G}(\mathscr{C})^{\perp_1}$, we have $L' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Theorem 3.3(1). Thus the middle column in the above diagram is a special $\mathcal{G}(\mathscr{C})$-precover of L. $\square$

The following question seems to be interesting.

Question 4.4. *Is* $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *closed under direct summands?*

The following result shows that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ possesses certain minimality, which generalizes [15, Theorem 6.8(1)].

Theorem 4.5. *Assume that* $\mathscr{C}$ *is a generator for* $\mathcal{G}(\mathscr{C})^{\perp_1}$. *Then we have the following*:

(1) $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C}) \subseteq \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *and* $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is closed under extensions and* $\mathscr{C}$*-stable direct summands.*

(2) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is the minimal subcategory with respect to the property* (1) *as above.*

To prove this theorem, we need the following lemma.

Lemma 4.6. *Let $0 \to K \to G \to M \to 0$ be an exact sequence in $\mathscr{A}$ with $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $G \in \mathcal{G}(\mathscr{C})$. Then there exists an exact sequence $0 \to G \to M \oplus C \to K' \to 0$ in $\mathscr{A}$ with $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $C \in \mathscr{C}$.*

Proof. Let $0 \to K \to G \to M \to 0$ be an exact sequence in $\mathscr{A}$ with $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $G \in \mathcal{G}(\mathscr{C})$. Since $G \in \mathcal{G}(\mathscr{C})$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence $0 \to G \to C \to G' \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $G' \in \mathcal{G}(\mathscr{C})$. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & K' & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $K, C \in \mathcal{G}(\mathscr{C})^{\perp_1}$, we have $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Theorem 3.3(3).

Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & K' & \longrightarrow & G' & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since the middle column in the first diagram is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact, so is the rightmost column in this diagram. Then the middle row in the second diagram is also $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact by [11, Lemma 2.4(1)], and in particular, it splits. Thus $Q \cong M \oplus C$ and the middle column in the second diagram is the desired exact sequence. □

Proof of Theorem 4.5. (1) It follows from Proposition 4.1(1) and Theorem 4.3.

(2) Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$ such that $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C}) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and $\mathscr{C}$-stable direct summands. Let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then by Lemma 4.6, we have an exact sequence $0 \to G \to M \oplus C \to K' \to 0$ in $\mathscr{A}$ with $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$, $G \in \mathcal{G}(\mathscr{C})$ and $C \in \mathscr{C}$. Because $G, K' \in \mathscr{X}$, we have that $M \oplus C \in \mathscr{X}$ and $M \in \mathscr{X}$. It follows that $\mathrm{SPC}(\mathcal{G}(\mathscr{C})) \subseteq \mathscr{X}$. □

As an immediate consequence of Theorem 4.5, we get the following corollary.

Corollary 4.7. *Assume that $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ is special precovering in $\mathrm{Mod}\, R$ and $\mathscr{X}$ is a subcategory of $\mathrm{Mod}\, R$. If $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} \cup \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and $\mathscr{P}(\mathrm{Mod}\, R)$-stable direct summands, then $\mathscr{X} = \mathrm{Mod}\, R$.*

Proof. By assumption, we have $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))) = \mathrm{Mod}\, R$. Now the assertion follows from Theorem 4.5. □

We collect some known classes of rings R satisfying that $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ is special precovering in $\mathrm{Mod}\, R$ as follows.

Example 4.8. For any one of the following rings R, $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R))$ is special precovering in $\mathrm{Mod}\,R$.

(1) Commutative Noetherian rings of finite Krull dimension (see [5, Remark 5.8]).

(2) Rings in which all projective left R-modules have finite injective dimension (see [16, Corollary 4.3]); especially, Gorenstein rings (i.e., n-Gorenstein rings for some $n \geqslant 0$).

(3) Right coherent rings in which all flat R-modules have finite projective dimension (see [2, Theorem 3.5] and [4, Proposition 8.10]); especially, right coherent and left perfect rings, and right Artinian rings.

We recall the following definition from [12].

Definition 4.9. Let $\mathscr{C}$, $\mathscr{T}$ and $\mathscr{E}$ be subcategories of $\mathscr{A}$ with $\mathscr{C} \subseteq \mathscr{T}$.

(1) $\mathscr{C}$ is called an *$\mathscr{E}$-proper generator* (resp. *$\mathscr{E}$-coproper cogenerator*) for $\mathscr{T}$ if for any object T in $\mathscr{T}$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},-)$ (resp. $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$)-exact exact sequence $0 \to T' \to C \to T \to 0$ (resp. $0 \to T \to C \to T' \to 0$) in $\mathscr{A}$ such that C is an object in $\mathscr{C}$ and T' is an object in $\mathscr{T}$.

(2) $\mathscr{T}$ is called *$\mathscr{E}$-preresolving* in $\mathscr{A}$ if the following conditions are satisfied:

(i) $\mathscr{T}$ admits an $\mathscr{E}$-proper generator.

(ii) $\mathscr{T}$ is *closed under $\mathscr{E}$-proper extensions*, i.e., for any $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},-)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, if both A_1 and A_3 are objects in $\mathscr{T}$, then A_2 is also an object in $\mathscr{T}$.

An $\mathscr{E}$-preresolving subcategory $\mathscr{T}$ of $\mathscr{A}$ is called *$\mathscr{E}$-resolving* if the following condition is satisfied:

(iii) $\mathscr{T}$ is *closed under kernels of $\mathscr{E}$-proper epimorphisms*, i.e., for any $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},-)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, if both A_2 and A_3 are objects in $\mathscr{T}$, then A_1 is also an object in $\mathscr{T}$.

In the following, we investigate when $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving. We need the following two lemmas.

Lemma 4.10. *For any $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to K \to C \to M \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$.*

Proof. Let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to K' \to G \to M \to 0$ in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$ and $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$. For G, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to G' \to C \to G \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $G' \in \mathcal{G}(\mathscr{C})$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & K & \dashrightarrow & C & \dashrightarrow & M & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K' & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By [11, Lemma 2.5], the middle row is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact, as desired. □

Lemma 4.11. *Assume that $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$. Given a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{A}$, we have the following*:

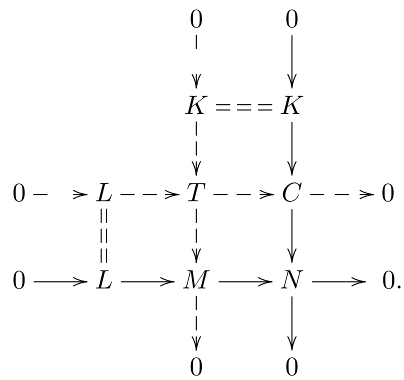
(1) *If $M, N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$, then $L \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$.*

(2) *If $L, M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence $0 \to K \to C \to N \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$, then $N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$.*

Proof. Let $0 \to L \to M \to N \to 0$ be a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence in $\mathscr{A}$.

(1) Assume that $M, N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. By Lemma 4.10, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact exact sequence

$0 \to K \to C \to N \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$. Consider the following pullback diagram:



By Proposition 4.1(2), $K \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then it follows from Theorem 4.3(1) and the exactness of the middle column that $T \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Notice that the middle row is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact by [11, Lemma 2.4(1)], so it splits and $T \cong L \oplus C$. Thus $L \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by Theorem 4.3(2).

(2) Assume $L, M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence $0 \to K \to C \to N \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$. As in the above diagram, since $L, C \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$, we have $T \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by Theorem 4.3(1). Moreover, the middle column is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact by [11, Lemma 2.4(1)]. So $K \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by (1), and hence $N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by Proposition 4.1(2). □

Now we are ready to prove the following theorem.

Theorem 4.12. *If $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$, then $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving in $\mathscr{A}$ with a $\mathscr{C}$-proper generator $\mathscr{C}$.*

Proof. Following Theorem 4.3(1) and Lemma 4.11(1), we know that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under $\mathscr{C}$-proper extensions and kernels of $\mathscr{C}$-proper epimorphisms. Now let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then by Lemma 4.10, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence $0 \to K \to C \to M \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$. By Proposition 4.1(2), we have $K \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. It follows that $\mathscr{C}$ is a $\mathscr{C}$-proper generator for $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is a $\mathscr{C}$-resolving. □

As a consequence, we get the following corollary.

Corollary 4.13. *If $\mathscr{C}$ is a projective generator for $\mathscr{A}$, then $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is projectively resolving and injectively coresolving in $\mathscr{A}$.*

Proof. Let $\mathscr{C}$ be a projective generator for $\mathscr{A}$. Because $\mathcal{G}(\mathscr{C})^{\perp_1}$ is projectively resolving by Theorem 3.3(2), $\mathscr{C}$ is also a projective generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$. It follows from Theorem 4.12 that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is projectively resolving. Now let I be an injective object in $\mathscr{A}$ and $0 \to K \to P \xrightarrow{f} I \to 0$ an exact sequence in $\mathscr{A}$ with $P \in \mathscr{C}$. Then it is easy to see that $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Example 3.1(1) and Theorem 3.3(2). So f is a special $\mathcal{G}(\mathscr{C})$-precover of I and $I \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. On the other hand, by Lemma 4.11(2), we have that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under cokernels of monomorphisms. Thus we conclude that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is injectively coresolving. □

The following corollary is an immediate consequence of Corollary 4.13, in which the second assertion generalizes [15, Theorem 6.8(2)].

Corollary 4.14. (1) $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)))$ *is projectively resolving and injectively coresolving in* $\mathrm{Mod}\, R$.

(2) *If R is a left Noetherian ring, then* $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R)))$ *is projectively resolving and injectively coresolving in* $\mathrm{mod}\, R$.

Let $\mathrm{SPE}(\mathcal{G}(\mathscr{C}))$ be the subcategory of $\mathscr{A}$ consisting of objects admitting special $\mathcal{G}(\mathscr{C})$-preenvelopes. We point out that the dual versions on ${}^{\perp_1}\mathcal{G}(\mathscr{C})$ and $\mathrm{SPE}(\mathcal{G}(\mathscr{C}))$ of all of the above results also hold true by using completely dual arguments.

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