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Silting Modules over Triangular Matrix Rings

Hanpeng Gao and Zhaoyong Huang*

Abstract. Let Λ, Γ be rings and $R = \left(\begin{smallmatrix} \Lambda & 0 \\ M & \Gamma \end{smallmatrix}\right)$ the triangular matrix ring with M a (Γ, Λ)-bimodule. Let X be a right Λ-module and Y a right Γ-module. We prove that $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a silting right R-module if and only if both X_Λ and Y_Γ are silting modules and $Y \otimes_\Gamma M$ is generated by X. Furthermore, we prove that if Λ and Γ are finite dimensional algebras over an algebraically closed field and X_Λ and Y_Γ are finitely generated, then $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a support τ-tilting R-module if and only if both X_Λ and Y_Γ are support τ-tilting modules, $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$ and $\operatorname{Hom}_\Lambda(e\Lambda, Y \otimes_\Gamma M) = 0$ with e the maximal idempotent such that $\operatorname{Hom}_\Lambda(e\Lambda, X) = 0$.

1. Introduction

Tilting modules are fundamental in the representation theory of algebras. It is important to construct a new tilting module from a given one and mutation of tilting modules is a very effective way to do it. Happel and Unger [15] gave some necessary and sufficient conditions under which mutation of tilting modules is possible; however, mutation of tilting modules may not be realized.

As a generalization of tilting modules, support τ-tilting modules over finite dimensional algebras were introduced by Adachi, Iyama and Reiten [1], and they showed that mutation of all support τ-tilting modules is possible. A new (support τ-)tilting module can be constructed by an algebra extension. For example, Assem, Happel and Trepode [5] studied how to extend and restrict tilting modules by given tilting modules for the one-point extension of an algebra by a projective module. Suarez [19] generalized this result to the case for support τ-tilting modules.

To generalize tilting modules over an arbitrary ring and support τ-tilting modules over a finite dimensional algebra, (partial) silting modules over an arbitrary ring were introduced by Angeleri Hügel, Marks and Vitória [3]. It was proved in [3, Proposition 3.15] that a finitely generated module is partial silting (resp. silting) if and only if it is τ-rigid

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*Corresponding author.

(resp. support τ-tilting) over a finite dimensional algebra. Silting modules share many properties with tilting modules and support τ-tilting modules, see [2, 4, 10, 11] and the references therein.

Let Λ, Γ be rings and M a (Γ, Λ)-bimodule. Then we can construct the triangular matrix ring $\left(\begin{smallmatrix} \Lambda & 0 \\ M & \Gamma \end{smallmatrix}\right)$ by the ordinary operation on matrices, see [8, p. 76]. If M and N are two Λ-modules with $\mathrm{Hom}_\Lambda(N, M) = 0$, then the endomorphism ring of $M \oplus N$ is exactly the triangular matrix ring $\left(\begin{smallmatrix} \mathrm{End}_\Lambda(M) & 0 \\ \mathrm{Hom}_\Lambda(M,N) & \mathrm{End}_\Lambda(N) \end{smallmatrix}\right)$. Moreover, a one-point extension of an algebra is a special triangular matrix algebra. In [12], Chen, Gong and Rump gave a criterion for lifting tilting modules from an arbitrary ring to its trivial extension ring, and they constructed tilting modules over triangular matrix rings under some conditions. The aim of this paper is to construct (partial) silting modules over triangular matrix rings. This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results. Let Λ, Γ be rings and $R = \left(\begin{smallmatrix} \Lambda & 0 \\ M & \Gamma \end{smallmatrix}\right)$ the triangular matrix ring with M a (Γ, Λ)-bimodule. In Section 3, for any X_Λ and Y_Γ, we investigate the relationship between the projective presentations of X_Λ and Y_Γ and the projective presentation of the right R-module $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$. Then we give a necessary and sufficient condition for constructing (partial) silting right R-modules from (partial) silting right Λ-modules and right Γ-modules (Theorem 3.4). As a consequence, we get that if ${}_\Gamma M$ is flat, then $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a tilting right R-module if and only if both X_Λ and Y_Γ are tilting and $Y \otimes_\Gamma M$ is generated by X (Theorem 3.8). In Section 4, Λ and Γ are finite dimensional k-algebras over an algebraically closed field k and all modules considered are finitely generated and basic. As an application of Theorem 3.4, we give a necessary and sufficient condition for constructing support τ-tilting right R-modules from support τ-tilting right Λ-modules and right Γ-modules (Theorem 4.3). Furthermore, we generalize this result to tensor algebras (Theorem 4.8). In Section 5, we give an example to illustrate our results; in particular, we may construct many support τ-tilting modules over triangular matrix algebras.

2. Preliminaries

Throughout this paper, all rings are associative with identities and all modules are unitary. For a ring Λ, $\mathrm{Mod}\,\Lambda$ is the category of right Λ-modules, $\mathrm{mod}\,\Lambda$ is the category of finitely generated right Λ-modules, and all subcategories of $\mathrm{Mod}\,\Lambda$ or $\mathrm{mod}\,\Lambda$ are full and closed under isomorphisms. We use $\mathrm{Proj}\,\Lambda$ (resp. $\mathrm{proj}\,\Lambda$) to denote the subcategory of $\mathrm{Mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda$) consisting of (resp. finitely generated) projective modules. For a module $M \in \mathrm{Mod}\,\Lambda$, $\mathrm{Add}\,M$ is the subcategory of $\mathrm{Mod}\,\Lambda$ consisting of direct summands of direct sums of copies of M and $\mathrm{Gen}\,M$ is the subcategory of $\mathrm{Mod}\,\Lambda$ consisting of quotients of direct sums of copies of M.

2.1. Triangular matrix rings

Let Λ, Γ be rings and ${}_\Gamma M_\Lambda$ a (Γ, Λ)-bimodule. Then the triangular matrix ring

$$R := \begin{pmatrix} \Lambda & 0 \\ M & \Gamma \end{pmatrix}$$

can be defined by the ordinary operation on matrices. Let $\mathcal{C}_R$ be the category whose objects are the triples $(X, Y)_f$ with $X \in \operatorname{Mod}\Lambda$, $Y \in \operatorname{Mod}\Gamma$ and $f \in \operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, X)$ (sometimes, f is omitted). The morphisms from $(X, Y)_f$ to $(X', Y')_{f'}$ are pairs of (α, β) such that the following diagram

$$\begin{tikzcd} Y \otimes_\Gamma M \arrow[r, "f"] \arrow[d, "\beta\otimes M"'] & X \arrow[d, "\alpha"] \\ Y' \otimes_\Gamma M \arrow[r, "f'"] & X' \end{tikzcd}$$

commutes, where $\alpha \in \operatorname{Hom}_\Lambda(X, X')$ and $\beta \in \operatorname{Hom}_\Gamma(Y, Y')$.

It is well known that there exists an equivalence of categories between $\operatorname{Mod} R$ and $\mathcal{C}_R$ [14]. Hence we can view an R-module as a triples $(X, Y)_f$ with $X \in \operatorname{Mod}\Lambda$ and $Y \in \operatorname{Mod}\Gamma$. Moreover, a sequence

$$0 \to (X_1, Y_1) \xrightarrow{(\alpha_1,\beta_1)} (X_2, Y_2) \xrightarrow{(\alpha_2,\beta_2)} (X_3, Y_3) \to 0$$

in $\operatorname{Mod} R$ is exact if and only if

$$0 \to X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \to 0$$

is exact in $\operatorname{Mod}\Lambda$ and

$$0 \to Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} Y_3 \to 0$$

is exact in $\operatorname{Mod}\Gamma$. All indecomposable projective modules in $\operatorname{Mod} R$ are exactly of the forms $(P_\Lambda, 0)$ and $(Q_\Gamma \otimes_\Gamma M, Q_\Gamma)_{\mathrm{id}}$, where P_Λ is an indecomposable projective Λ-module and Q_Γ is an indecomposable projective Γ-module.

2.2. Silting modules

Let Λ be a ring and

$$\sigma\colon P_1 \to P_0$$

a homomorphism in $\operatorname{Mod}\Lambda$ with $P_1, P_0 \in \operatorname{Proj}\Lambda$. We write

$$D_\sigma := \{A \in \operatorname{Mod}\Lambda \mid \operatorname{Hom}_\Lambda(\sigma, A) \text{ is epic}\}.$$

Recall that a subcategory $\mathcal{T}$ of $\operatorname{Mod}\Lambda$ is called a *torsion class* if it is closed under images, direct sums and extensions (cf. [7, Chapter VI]).

Definition 2.1. [3, Definition 3.7] Let $T \in \operatorname{Mod}\Lambda$.

(1) T is called *partial silting* if there exists a projective presentation σ of T such that D_σ is a torsion class and $T \in D_\sigma$.

(2) T is called *silting* if there exists a projective presentation σ of T such that $\operatorname{Gen} T = D_\sigma$.

Sometimes, we also say that T is a (*partial*) *silting* module with respect to σ.

By [3, Lemma 3.6(1)], D_σ is always closed under images and extensions. Hence, D_σ is a torsion class if and only if it is closed under direct sums. This is always true when σ is a map in $\operatorname{proj}\Lambda$. Moreover, it is trivial that $T \in D_\sigma$ implies $\operatorname{Gen} T \subseteq D_\sigma$.

Given a subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$, recall that a *left $\mathcal{X}$-approximation* of a module $M \in \operatorname{Mod}\Lambda$ is a homomorphism $\phi\colon M \to X$ with $X \in \mathcal{X}$ such that $\operatorname{Hom}_\Lambda(\phi, X')$ is epic for any $X' \in \mathcal{X}$. The following result establishes the relation between partial silting modules and silting modules.

Proposition 2.2. [3, Proposition 3.11] *Let $T \in \operatorname{Mod}\Lambda$ with a projective presentation σ. Then T is a silting module with respect to σ if and only if T is a partial silting module with respect to σ and there exists an exact sequence*

$$\Lambda \xrightarrow{\phi} T^0 \to T^1 \to 0$$

in $\operatorname{Mod}\Lambda$ with $T^0, T^1 \in \operatorname{Add} T$ and ϕ a left D_σ-approximation.

2.3. Support τ-tilting modules

In this subsection, Λ is a finite dimensional k-algebra over an algebraically closed field k. The Auslander-Reiten translation is denoted by τ. For a module $M \in \operatorname{mod}\Lambda$, $|M|$ is the number of pairwise non-isomorphic direct summands of M. All modules considered are finitely generated and basic.

Definition 2.3. [1, Definition 0.1] Let $M \in \operatorname{mod}\Lambda$.

(1) M is called *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$.

(2) M is called *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$.

(3) M is called *support τ-tilting* if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ.

Lemma 2.4. *If M is a τ-rigid Λ-module and $\operatorname{Hom}_\Lambda(e\Lambda, M) = 0$ for some idempotent e of Λ, then $|M| + |e\Lambda| \leq |\Lambda|$.*

Proof. Let M be a τ-rigid Λ-module and $\mathrm{Hom}_\Lambda(e\Lambda, M) = 0$ for some idempotent e of Λ. Then M is a τ-rigid $\Lambda/\Lambda e\Lambda$-module by [1, Lemma 2.1]. So $|M| + |e\Lambda| \leq |\Lambda|$. $\square$

Sometimes, it is convenient to view support τ-tilting modules and τ-rigid modules as certain pairs of modules in $\mathrm{mod}\,\Lambda$.

Definition 2.5. [1, Definition 0.3] Let (M, P) be a pair in $\mathrm{mod}\,\Lambda$ with $P \in \mathrm{proj}\,\Lambda$.

(1) (M, P) is called a *τ-rigid pair* if M is τ-rigid and $\mathrm{Hom}_\Lambda(P, M) = 0$.

(2) (M, P) is called a *support τ-tilting pair* if M is τ-rigid and $|M| + |P| = |\Lambda|$.

It was shown in [1, Proposition 2.3] that (M, P) is a support τ-tilting pair in $\mathrm{mod}\,\Lambda$ if and only if M is a τ-tilting $\Lambda/\Lambda e\Lambda$-module with $e\Lambda \cong P$. Recall that $M \in \mathrm{mod}\,\Lambda$ is called *sincere* if there does not exist a non-zero idempotent e of Λ that annihilates M. Notice that all τ-tilting modules are sincere, so M is a support τ-tilting Λ-module if and only if M is a τ-rigid Λ-module and $|M| + |e\Lambda| = |\Lambda|$, where e is the maximal idempotent such that $\mathrm{Hom}_\Lambda(e\Lambda, M) = 0$.

Lemma 2.6. [1, Proposition 2.4] *Let $X \in \mathrm{mod}\,\Lambda$ and*

$$P_1 \xrightarrow{f_0} P_0 \to X \to 0$$

a minimal projective presentation of X in $\mathrm{mod}\,\Lambda$. For any $Y \in \mathrm{mod}\,\Lambda$, $\mathrm{Hom}_\Lambda(f_0, Y)$ is epic if and only if $\mathrm{Hom}_\Lambda(Y, \tau X) = 0$.

Silting modules are intended to generalize support τ-tilting modules. In particular, when restricting to finitely generated modules over a finite dimensional k-algebra, they are equivalent.

Proposition 2.7. [3, Proposition 3.15] *Let $T \in \mathrm{mod}\,\Lambda$. Then we have*

(1) *T is a partial silting Λ-module if and only if T is a τ-rigid Λ-module.*

(2) *T is a silting Λ-module if and only if T is a support τ-tilting Λ-module.*

Let $T \in \mathrm{mod}\,\Lambda$ with a minimal projective presentation σ. Then T is support τ-tilting if and only if $\mathrm{Gen}\,T$ consists of Λ-modules M such that $\mathrm{Hom}_\Lambda(\sigma \oplus \sigma', M)$ is epic, where σ' is the complex $(e\Lambda \to 0)$ and e is a suitable idempotent of Λ [3, Theorem 2.5]. In fact, it follows from [3, Theorem 4.9] and [1, Theorem 3.2] that $(T, e\Lambda)$ is a support τ-tilting pair if and only if T is a silting module with respect to $\sigma \oplus \sigma'$.

3. Silting modules over triangular matrix rings

From now on, Λ, Γ are rings and ${}_\Gamma M_\Lambda$ a (Γ,Λ)-bimodule and

$$R := \begin{pmatrix} \Lambda & 0 \\ M & \Gamma \end{pmatrix}$$

is the corresponding triangular matrix ring.

Let $X \in \operatorname{Mod}\Lambda$ and $Y \in \operatorname{Mod}\Gamma$, and let

$$P_1 \xrightarrow{\sigma_X} P_0 \to X \to 0$$

and

$$Q_1 \xrightarrow{\sigma_Y} Q_0 \to Y \to 0 \tag{3.1}$$

be projective presentations of X and Y respectively, with $P_1, P_0 \in \operatorname{Proj}\Lambda$ and $Q_1, Q_0 \in \operatorname{Proj}\Gamma$. Applying the functor $-\otimes_\Gamma M$ to (3.1), we get the following exact sequence

$$Q_1 \otimes_\Gamma M \xrightarrow{\sigma_Y \otimes M} Q_0 \otimes_\Gamma M \to Y \otimes_\Gamma M \to 0.$$

Hence, we get a projective presentation of $(X,0) \oplus (Y \otimes_\Gamma M, Y)$ denoted by $\sigma = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$:

$$(P_1,0) \oplus (Q_1 \otimes_\Gamma M, Q_1) \xrightarrow{\sigma} (P_0,0) \oplus (Q_0 \otimes_\Gamma M, Q_0) \to (X,0) \oplus (Y \otimes_\Gamma M, Y) \to 0,$$

where $a = (\sigma_X, 0)$ and $b = (\sigma_Y \otimes M, \sigma_Y)$.

Lemma 3.1. *Let $X_1 \in \operatorname{Mod}\Lambda$ and $Y_1 \in \operatorname{Mod}\Gamma$.*

(1) $(X_1, Y_1)_h \in D_\sigma$ *if and only if* $X_1 \in D_{\sigma_X}$ *and* $Y_1 \in D_{\sigma_Y}$.

(2) *If* $X_1 \in D_{\sigma_X}$, *then* $(X_1, 0) \in D_\sigma$.

(3) *If* $Y_1 \in D_{\sigma_Y}$, *then* $(0, Y_1) \in D_\sigma$.

(4) *If* $Y_1 \in D_{\sigma_Y}$ *and* $Y_1 \otimes_\Gamma M \in D_{\sigma_X}$, *then* $(Y_1 \otimes_\Gamma M, Y_1)_{\mathrm{id}} \in D_\sigma$.

Proof. (1) Let $f \in \operatorname{Hom}_\Lambda(P_1, X_1)$. Then $((f,0),0) \in \operatorname{Hom}_R((P_1,0) \oplus (Q_1 \otimes_\Gamma M, Q_1), (X_1, Y_1))$. Since $(X_1, Y_1)_h \in D_\sigma$, there exists $((f',0),y)\colon (P_0,0) \oplus (Q_0 \otimes_\Gamma M, Q_0) \to (X_1, Y_1)$ with $f' \in \operatorname{Hom}_\Lambda(P_0, X_1)$ such that the following diagram

$$\begin{tikzcd}
(P_1,0) \oplus (Q_1 \otimes_\Gamma M, Q_1) \arrow[r, "\sigma"] \arrow[d, "((f,0),0)"'] & (P_0,0) \oplus (Q_0 \otimes_\Gamma M, Q_0) \arrow[dl, dotted, "((f',0),y)"] \\
(X_1, Y_1) &
\end{tikzcd}$$

commutes. So $((f,0),0) = ((f',0),y)\circ\sigma$, and hence $(f,0) = (f',0)\circ a = (f',0)\circ(\sigma_X,0)$ and $f = f'\circ\sigma_X$. This implies $X_1 \in D_{\sigma_X}$.

Let $g \in \mathrm{Hom}_\Gamma(Q_1, Y_1)$. Then $(0,(h\circ(g\otimes M),g)) \in \mathrm{Hom}_R((P_1,0)\oplus(Q_1\otimes_\Gamma M, Q_1), (X_1,Y_1))$ and there exists $(x,(f_1,g_1)) \in \mathrm{Hom}_R((P_0,0)\oplus(Q_0\otimes_\Gamma M, Q_0),(X_1,Y_1))$ with $f_1 \in \mathrm{Hom}_\Lambda(Q_0\otimes_\Gamma M, X_1)$ and $g_1 \in \mathrm{Hom}_\Gamma(Q_0, Y_1)$ such that the following diagram

$$\begin{tikzcd}
(P_1,0)\oplus(Q_1\otimes_\Gamma M, Q_1) \arrow[r, "\sigma"] \arrow[d, "(0,(h\circ(g\otimes M),g))"'] & (P_0,0)\oplus(Q_0\otimes_\Gamma M, Q_0) \arrow[dl, dotted, "(x,(f_1,g_1))"] \\
(X_1,Y_1)_f &
\end{tikzcd}$$

commutes. So $(0,(h\circ(g\otimes M),g)) = (x,(f_1,g_1))\circ\sigma$, and hence $(h\circ(g\otimes M),g) = (f_1,g_1)\circ b = (f_1,g_1)\circ(\sigma_Y\otimes_\Gamma M,\sigma_Y)$ and $g = g_1\circ\sigma_Y$. This implies $Y_1 \in D_{\sigma_Y}$.

Conversely, let $(x,y) \in \mathrm{Hom}_R((P_1,0)\oplus(Q_1\otimes_\Gamma M, Q_1),(X_1,Y_1)_h)$. Write $x = (f_2,0)$ and $y = (f_3,g_3)$ with $f_2 \in \mathrm{Hom}_\Lambda(P_1,X_1)$, $f_3 \in \mathrm{Hom}_\Lambda(Q_1\otimes_\Gamma M, X_1)$ and $g_3 \in \mathrm{Hom}_\Gamma(Q_1,Y_1)$. Then we have the following commutative diagram

$$\begin{tikzcd}
Q_1\otimes_\Gamma M \arrow[r, "\mathrm{id}"] \arrow[d, "g_3\otimes M"'] & Q_1\otimes_\Gamma M \arrow[d, "f_3"] \\
Y_1\otimes_\Gamma M \arrow[r, "h"] & X_1,
\end{tikzcd}$$

and so $f_3 = h\circ(g_3\otimes M)$. Since $X_1 \in D_{\sigma_X}$ and $Y_1 \in D_{\sigma_Y}$, there exist $f_2' \in \mathrm{Hom}_\Lambda(P_0, X_1)$ and $g_3' \in \mathrm{Hom}_\Gamma(Q_0,Y_1)$ such that $f_2 = f_2'\circ\sigma_X$ and $g_3 = g_3'\circ\sigma_Y$. Since $(h\circ(g_3'\otimes M), g_3') \in \mathrm{Hom}_R((Q_0\otimes_\Gamma M, Q_0),(X_1,Y_1))$ and the following equalities hold

$$\begin{aligned}
&\big((f_2',0),(h\circ(g_3'\otimes M),g_3')\big)\circ\sigma \\
&= \big((f_2',0),(h\circ(g_3'\otimes M),g_3')\big)\circ\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right) = \big((f_2',0)\circ a,(h\circ(g_3'\otimes M),g_3')\circ b\big) \\
&= \big((f_2',0)\circ(\sigma_X,0),(h\circ(g_3'\otimes M),g_3')\circ(\sigma_Y\otimes M,\sigma_Y)\big) \\
&= \big((f_2'\circ\sigma_X,0),\big(h\circ(g_3'\circ\sigma_Y)\otimes M\big),g_3'\circ\sigma_Y\big) \\
&= \big((f_2,0),(h\circ(g_3\otimes M),g_3)\big) = \big((f_2,0),(f_3,g_3)\big) = (x,y),
\end{aligned}$$

we have $(X_1,Y_1)_h \in D_\sigma$.

The assertions (2), (3) and (4) follow directly from (1). □

Let I be a set and $\{(X_i,Y_i)_{f_i}\}_{i\in I} \in \mathrm{Mod}\,R$ with all $X_i \in \mathrm{Mod}\,\Lambda$ and $Y_i \in \mathrm{Mod}\,\Gamma$. Since the tensor functor commutes with direct sums, we have

$$\bigoplus_{i\in I}(X_i,Y_i)_{f_i} \cong \left(\bigoplus_{i\in I} X_i, \bigoplus_{i\in I} Y_i\right)_{\bigoplus_{i\in I} f_i}.$$

Lemma 3.2. *D_σ is a torsion class if and only if both D_{σ_X} and D_{σ_Y} are torsion classes.*

Proof. Suppose that D_σ is a torsion class. Let $\{X_i\}_{i\in I}$ be a family of modules in D_{σ_X}. Then $(X_i,0)\in D_\sigma$ for any $i\in I$ by Lemma 3.1(2). So $\big(\bigoplus_{i\in I}X_i,0\big)\cong\bigoplus_{i\in I}(X_i,0)\in D_\sigma$, and hence $\bigoplus_{i\in I}X_i\in D_{\sigma_X}$ by Lemma 3.1(1). Thus D_{σ_X} is a torsion class by [3, Lemma 3.6(1)]. Similarly, D_{σ_Y} is also a torsion class.

Conversely, suppose that both D_{σ_X} and D_{σ_Y} are torsion classes. Let $\{(X_i,Y_i)_{f_i}\}_{i\in I}$ be a family of modules in D_σ with $X_i\in\operatorname{Mod}\Lambda$ and $Y_i\in\operatorname{Mod}\Gamma$. Then $X_i\in D_{\sigma_X}$ and $Y_i\in D_{\sigma_Y}$ for any $i\in I$ by Lemma 3.1(1). Hence $\bigoplus_{i\in I}X_i\in D_{\sigma_X}$ and $\bigoplus_{i\in I}Y_i\in D_{\sigma_Y}$. Let $(x,y)\in\operatorname{Hom}_R\big((P_1,0)\oplus(Q_1\otimes_\Gamma M,Q_1),\big(\bigoplus_{i\in I}X_i,\bigoplus_{i\in I}Y_i\big)_{\bigoplus_{i\in I}f_i}\big)$. Write $x=(f_1,0)$ and $y=(f_2,g_2)$ with $f_1\in\operatorname{Hom}_\Lambda\big(P_1,\bigoplus_{i\in I}X_i\big)$, $f_2\in\operatorname{Hom}_\Lambda\big(Q_1\otimes_\Gamma M,\bigoplus_{i\in I}X_i\big)$ and $g_2\in\operatorname{Hom}_\Gamma\big(Q_1,\bigoplus_{i\in I}Y_i\big)$. Then we have the following commutative diagram

$$\begin{tikzcd}
Q_1\otimes_\Gamma M \arrow[r, "\mathrm{id}"] \arrow[d, "g_2\otimes M"'] & Q_1\otimes_\Gamma M \arrow[d, "f_2"] \\
\big(\bigoplus_{i\in I}Y_i\big)\otimes_\Gamma M \arrow[r, "\bigoplus_{i\in I}f_i"] & \bigoplus_{i\in I}X_i,
\end{tikzcd}$$

and so $f_2=\big(\bigoplus_{i\in I}f_i\big)\circ(g_2\otimes M)$. Since $\bigoplus_{i\in I}X_i\in D_{\sigma_X}$ and $\bigoplus_{i\in I}Y_i\in D_{\sigma_Y}$, there exist $f_1'\in\operatorname{Hom}_\Lambda\big(P_0,\bigoplus_{i\in I}X_i\big)$ and $g_2'\in\operatorname{Hom}_\Gamma\big(Q_0,\bigoplus_{i\in I}Y_i\big)$ such that $f_1=f_1'\circ\sigma_X$ and $g_2=g_2'\circ\sigma_Y$.

Set $g_1':=\big(\bigoplus_{i\in I}f_i\big)\circ(g_2'\otimes M)$. Then $(g_1',g_2')\in\operatorname{Hom}_R\big((Q_0\otimes_\Gamma M,Q_0),\big(\bigoplus_{i\in I}X_i,\bigoplus_{i\in I}Y_i\big)\big)$. Since the following equalities hold

$$\begin{aligned}
\big((f_1',0),(g_1',g_2')\big)\circ\sigma &= \big((f_1',0),(g_1',g_2')\big)\circ\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)=\big((f_1',0)\circ a,(g_1',g_2')\circ b\big)\\
&= \big((f_1',0)\circ(\sigma_X,0),(g_1',g_2')\circ(\sigma_Y\otimes M,\sigma_Y)\big)\\
&= \big((f_1'\circ\sigma_X,0),(g_1'\circ(\sigma_Y\otimes M),g_2'\circ\sigma_Y)\big)\\
&= \left((f_1,0),\left(\left(\bigoplus_{i\in I}f_i\right)\circ(g_2'\otimes M)\circ(\sigma_Y\otimes M),g_2\right)\right)\\
&= \left((f_1,0),\left(\left(\bigoplus_{i\in I}f_i\right)\circ(g_2'\circ\sigma_Y)\otimes M,g_2\right)\right)\\
&= \left((f_1,0),\left(\left(\bigoplus_{i\in I}f_i\right)\circ(g_2\otimes M),g_2\right)\right)\\
&= \big((f_1,0),(f_2,g_2)\big)=(x,y),
\end{aligned}$$

we have $\bigoplus_{i\in I}(X_i,Y_i)_{f_i}\cong\big(\bigoplus_{i\in I}X_i,\bigoplus_{i\in I}Y_i\big)_{\bigoplus_{i\in I}f_i}\in D_\sigma$. Thus D_σ is a torsion class by [3, Lemma 3.6(1)] again. □

As an immediate consequence of Lemmas 3.1 and 3.2, we get the following result.

Proposition 3.3. *Let $X\in\operatorname{Mod}\Lambda$ with a projective presentation σ_X and $Y\in\operatorname{Mod}\Gamma$ with a projective presentation σ_Y. Then $(X,0)\oplus(Y\otimes_\Gamma M,Y)$ is a partial silting R-module with respect to σ if and only if the following conditions are satisfied.*

(i) *X is a partial silting Λ-module with respect to σ_X.*

(ii) *Y is a partial silting Γ-module with respect to σ_Y.*

(iii) *$Y \otimes_\Gamma M \in D_{\sigma_X}$.*

The following is the main result in this section.

Theorem 3.4. *Let $X \in \operatorname{Mod}\Lambda$ and $Y \in \operatorname{Mod}\Gamma$. Then $(X,0) \oplus (Y \otimes_\Gamma M, Y)$ is a silting R-module if and only if the following conditions are satisfied.*

(i) *X is a silting Λ-module.*

(ii) *Y is a silting Γ-module.*

(iii) *$Y \otimes_\Gamma M \in \operatorname{Gen} X$.*

Proof. Let $(X,0) \oplus (Y \otimes_\Gamma M, Y)$ be a silting R-module with respect to σ. Then

$$D_\sigma = \operatorname{Gen}\big((X,0) \oplus (Y \otimes_\Gamma M, Y)\big) = \operatorname{Gen}(X,0) \oplus \operatorname{Gen}(Y \otimes_\Gamma M, Y).$$

Since X is partial silting by Proposition 3.3, we have $\operatorname{Gen} X \subseteq D_{\sigma_X}$. Let $X_1 \in D_{\sigma_X}$. Then $(X_1, 0) \in D_\sigma$. If $(X_1, 0)$ has a direct summand $(X_1', 0)$ in $\operatorname{Gen}(Y \otimes_\Gamma M, Y)$, then we have the following commutative diagram with exact columns

$$\begin{tikzcd}
(Y \otimes_\Gamma M)^{(I)} \arrow[r, "\mathrm{id}"] \arrow[d, "0"'] & (Y \otimes_\Gamma M)^{(I)} \arrow[d, "\alpha"] \\
0 \arrow[r] \arrow[d] & X_1' \arrow[d] \\
0 & 0,
\end{tikzcd}$$

and so $X_1' = 0$. Thus $(X_1, 0) \in \operatorname{Gen}(X, 0)$ and $X_1 \in \operatorname{Gen} X$. It follows that $\operatorname{Gen} X = D_{\sigma_X}$ and X is a silting Λ-module. By Proposition 3.3 again, we have $Y \otimes_\Gamma M \in D_{\sigma_X} = \operatorname{Gen} X$. Similarly, we have that Y is a silting Γ-module.

Conversely, let X be a silting Λ-module with respect to σ_X and Y a silting Γ-module with respect to σ_Y such that $Y \otimes_\Gamma M \in \operatorname{Gen} X$. Then $(X,0) \oplus (Y \otimes_\Gamma M, Y)$ is a partial silting module with respect to σ by Proposition 3.3. According to Proposition 2.2, we have the following exact sequences

$$\Lambda \xrightarrow{\phi} T^1 \to T^2 \to 0 \quad \text{and} \quad \Gamma \xrightarrow{\psi} E^1 \to E^2 \to 0$$

with $T^1, T^2 \in \operatorname{Add} X$ and $E^1, E^2 \in \operatorname{Add} Y$ such that ϕ is a left D_{σ_X}-approximation and ψ is a left D_{σ_Y}-approximation. Set

$$a' := (\phi, 0), \quad b' := (\psi \otimes_\Gamma M, \psi) \quad \text{and} \quad \alpha := \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}.$$

Then we get the following exact sequence

$$(\Lambda, 0) \oplus (M, \Gamma) \xrightarrow{\alpha} (T^1, 0) \oplus (E^1 \otimes_\Gamma M, E^1) \to (T^2, 0) \oplus (E^2 \otimes_\Gamma M, E^2) \to 0.$$

Clearly, both $(T^1, 0) \oplus (E^1 \otimes_\Gamma M, E^1)$ and $(T^2, 0) \oplus (E^2 \otimes_\Gamma M, E^2)$ belong to $\operatorname{Add}((X, 0) \oplus (Y \otimes_\Gamma M, Y))$.

Let $(X_1, Y_1)_h \in D_\sigma$ and $(x, y) \in \operatorname{Hom}_R((\Lambda, 0) \oplus (M, \Gamma), (X_1, Y_1))$. By Lemma 3.1(1), we have $X_1 \in D_{\sigma_X}$ and $Y_1 \in D_{\sigma_Y}$. Write $x = (f_1, 0)$ and $y = (f_2, g_2)$, we have $f_2 = h \circ (g_2 \otimes M)$. Since ϕ is a left D_{σ_X}-approximation and ψ is a left D_{σ_Y}-approximation, there exist $f_1' \in \operatorname{Hom}_\Lambda(T^1, X_1)$ and $g_2' \in \operatorname{Hom}_\Gamma(E^1, Y_1)$ such that $f_1 = f_1' \circ \phi$ and $g_2 = g_2' \circ \psi$. Take $f_3 := h \circ (g_2' \otimes M)$ ($\in \operatorname{Hom}_\Lambda(E^1 \otimes_\Gamma M, X_1)$). Since the following equalities hold

$$\begin{aligned}
&\big((f_1', 0), (f_3, g_2')\big) \circ \alpha \\
&= \big((f_1', 0), (f_3, g_2')\big) \circ \left(\begin{smallmatrix} a' & 0 \\ 0 & b' \end{smallmatrix}\right) = \big((f_1', 0) \circ a', (f_3, g_2') \circ b'\big) \\
&= \big((f_1', 0) \circ (\phi, 0), (f_3, g_2') \circ (\psi \otimes M, \psi)\big) = \big((f_1' \circ \phi, 0), (f_3 \circ (\psi \otimes M), g_2' \circ \psi)\big) \\
&= \big((f_1, 0), (h \circ (g_2' \otimes M) \circ (\psi \otimes M), g_2)\big) = \big((f_1, 0), (h \circ (g_2' \circ \psi) \otimes M, g_2)\big) \\
&= \big((f_1, 0), (h \circ (g_2 \otimes M), g_2)\big) = \big((f_1, 0), (f_2, g_2)\big) = (x, y),
\end{aligned}$$

that is, the following diagram

$$\begin{tikzcd}
(\Lambda, 0) \oplus (M, \Gamma) \arrow[r, "\alpha"] \arrow[d, "{(x,y)}"'] & (T^1, 0) \oplus (E^1 \otimes_\Gamma M, E^1) \arrow[dl, dotted, "{((f_1',0),(f_3,g_2'))}"] \\
(X_1, Y_1) &
\end{tikzcd}$$

commutes, we have that α is a left D_σ-approximation. It follows from Proposition 2.2 that $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a silting R-module. $\square$

Remark 3.5. The sufficiency of the above theorem can be obtained by considering a special ring extension. Take $S = \Lambda \times \Gamma$. Then we have a split surjective morphism $R \to S$ whose kernel is ${}_S M_S$. Of course, we have $(X \oplus Y) \otimes R \cong (X, 0) \oplus (Y \otimes_\Gamma M, Y)$ and $R_S \cong S \oplus M$. If X is a silting Λ-module and Y is a silting Γ-module, then $X \oplus Y$ is a silting S-module. It follows from [3, Theorem 4.9] that σ_X and σ_Y are 2-term silting complexes. Hence $(X \oplus Y) \otimes R$ is a silting R-module if and only if σ is a 2-term silting complex, and if and only if $\operatorname{Hom}_R(R_S, (X \oplus Y) \otimes R) \in \operatorname{Gen}(X \oplus Y)$ by [9, Theorem 2.2]. Since

$$\operatorname{Hom}_R(R_S, (X \oplus Y) \otimes R) \cong (X \oplus Y) \otimes R_S \cong (X \oplus Y) \oplus (X \oplus Y) \otimes_S M,$$

we have that $\mathrm{Hom}_R(R_S, (X \oplus Y) \otimes R) \in \mathrm{Gen}(X \oplus Y)$ if and only if $(X \oplus Y) \otimes_S M \in \mathrm{Gen}(X \oplus Y)$, that is, $Y \otimes_\Gamma M \in \mathrm{Gen}\, X$ since $X \otimes_S M = 0$.

Corollary 3.6. *Let $X \in \mathrm{Mod}\,\Lambda$ and $Y \in \mathrm{Mod}\,\Gamma$. Then*

(1) $(X, 0)$ *is a silting R-module if and only if X is a silting Λ-module.*

(2) $(\Lambda, 0) \oplus (Y \otimes_\Gamma M, Y)$ *is a silting R-module if and only if Y is a silting Γ-module.*

Proof. (1) It is clear.

(2) Since Λ is a silting Λ-module and $\mathrm{Gen}\,\Lambda = \mathrm{Mod}\,\Lambda$, the assertion follows immediately from Theorem 3.4. □

Recall from [13] that a module $X \in \mathrm{Mod}\,\Lambda$ is called *tilting* if $\mathrm{Gen}\, X = \{N \in \mathrm{Mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(X, N) = 0\}$; or equivalently, if X satisfies the following conditions

(i) The projective dimension of X is at most one.

(ii) $\mathrm{Ext}^1_\Lambda(X, X^{(I)}) = 0$ for any set I.

(iii) There exists an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ in $\mathrm{Mod}\,\Lambda$ with $T^0, T^1 \in \mathrm{Add}\, X$.

It follows from [3, Proposition 3.13(1)] that a module $X \in \mathrm{Mod}\,\Lambda$ is tilting if and only if it is a silting with respect to a monomorphic projective presentation.

Corollary 3.7. *Let $X \in \mathrm{Mod}\,\Lambda$ and $Y \in \mathrm{Mod}\,\Gamma$ be silting with monomorphic projective presentations σ_X and σ_Y respectively. If $\sigma_Y \otimes_\Gamma M$ is monic and $Y \otimes_\Gamma M \in \mathrm{Gen}\, X$, then $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a tilting R-module.*

Proof. It follows that σ is monic when $\sigma_Y \otimes_\Gamma M$ is monic. □

Let X_Λ and Y_Γ be tilting modules, and let ε^X and ε^Y denote the counits of the corresponding adjunctions, see [12, p. 535]. It was proved in [12, p. 538, Corollary] that if ${}_\Gamma M$ is flat such that the functor $F = - \otimes_\Gamma M \colon \mathrm{Mod}\,\Gamma \to \mathrm{Mod}\,\Lambda$ satisfies $F\varepsilon^Y = \varepsilon^X F$, then $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a tilting R-module. The following theorem extends this result. The sufficiency of this theorem was obtained independently in [17, Theorem 5.2] by considering an epimorphism of rings which is split.

Theorem 3.8. *Let $X \in \mathrm{Mod}\,\Lambda$ and $Y \in \mathrm{Mod}\,\Gamma$. If ${}_\Gamma M$ is flat, then the following statements are equivalent.*

(1) $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ *is a tilting R-module.*

(2) *X is a tilting Λ-module, Y is a tilting Γ-module and $Y \otimes_\Gamma M \in \mathrm{Gen}\, X$.*

Proof. Since σ is monic if and only if both σ_X and σ_Y are monic when ${}_\Gamma M$ is flat, the assertion follows from Theorem 3.4. $\square$

Applying Theorems 3.4 and 3.8 to the special triangular matrix ring $\begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$, we get the following result.

Corollary 3.9. *Let $X, Y \in \operatorname{Mod}\Lambda$ and $R = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$. Then the following statements are equivalent.*

(1) *$(X,0) \oplus (Y,Y)$ is a silting (resp. tilting) R-module.*

(2) *Both X and Y are silting (resp. tilting) Λ-modules and $Y \in \operatorname{Gen} X$.*

In particular, $(X,0) \oplus (X,X)$ is a silting (resp. tilting) R-module if and only if X is a silting (resp. tilting) Λ-module.

4. Support τ-tilting modules over triangular matrix algebras

In this section, all modules considered are finitely generated modules over finite dimensional k-algebras over an algebraically closed field k.

Proposition 4.1. *Let $X \in \operatorname{mod}\Lambda$ and $Y \in \operatorname{mod}\Gamma$. Then $(X,0) \oplus (Y \otimes_\Gamma M, Y)$ is a τ-rigid R-module if and only if the following conditions are satisfied.*

(1) *X is a τ-rigid Λ-module.*

(2) *Y is a τ-rigid Γ-module.*

(3) $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$.

Proof. Considering the minimal projective presentation σ_X of X, we have that $Y \otimes_\Gamma M \in D_{\sigma_X}$ if and only if $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$ by Lemma 2.6. Thus the assertion follows from Propositions 2.7 and 3.3. $\square$

For any $X \in \operatorname{mod}\Lambda$, it is clear that $(X,0)$ is indecomposable if and only if so is X. Let $Y \in \operatorname{mod}\Gamma$. Then $(Y \otimes_\Gamma M, Y)_{\mathrm{id}}$ is indecomposable implies so is Y; conversely, assume that Y is indecomposable and write $(Y \otimes_\Gamma M, Y)_{\mathrm{id}} = (X_1, Y_1) \oplus (X_2, Y_2)$ with $X_1, X_2 \in \operatorname{mod}\Lambda$ and $Y_1, Y_2 \in \operatorname{mod}\Gamma$. Then either $Y_1 = 0$ or $Y_2 = 0$. If $Y_1 = 0$, then there exists a split epimorphism $(\alpha, 0)\colon (Y \otimes_\Gamma M, Y)_{\mathrm{id}} \to (X_1, 0)$ in $\operatorname{mod} R$, which implies $\alpha = 0$. So $X_1 = 0$ and $(X_1, Y_1) = 0$. Similarly, if $Y_2 = 0$, then $(X_2, Y_2) = 0$. Thus we conclude that $(Y \otimes_\Gamma M, Y)_{\mathrm{id}}$ is also indecomposable. This proves the following lemma.

Lemma 4.2. *For any $X \in \operatorname{mod}\Lambda$ and $Y \in \operatorname{mod}\Gamma$, we have*

$$|(X,0)| + |(Y \otimes_\Gamma M, Y)| = |X| + |Y|.$$

The following is the main result in this section.

Theorem 4.3. *Let $X \in \operatorname{mod}\Lambda$ and $Y \in \operatorname{mod}\Gamma$. Then $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a support τ-tilting R-module if and only if the following conditions are satisfied.*

(1) *X is a support τ-tilting Λ-module.*

(2) *Y is a support τ-tilting Γ-module.*

(3) $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$.

(4) $\operatorname{Hom}_\Lambda(e\Lambda, Y \otimes_\Gamma M) = 0$, *where e is the maximal idempotent such that* $\operatorname{Hom}_\Lambda(e\Lambda, X) = 0$.

Proof. Assume that $((X, 0) \oplus (Y \otimes_\Gamma M, Y), (e\Lambda, 0) \oplus (e'\Gamma \otimes_\Gamma M, e'\Gamma))$ is a support τ-tilting pair in $\operatorname{mod} R$, where e and e' are idempotents of Λ and Γ respectively. Then $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$ and both X and Y are τ-rigid by Proposition 4.1. Moreover, we have

$$\operatorname{Hom}_\Lambda(e\Lambda, X) = 0, \quad \operatorname{Hom}_\Gamma(e'\Gamma, Y) = 0, \quad \operatorname{Hom}_\Lambda(e\Lambda, Y \otimes_\Gamma M) = 0.$$

By Lemma 2.4, we have $|X| + |e\Lambda| \leq |\Lambda|$ and $|Y| + |e'\Gamma| \leq |\Gamma|$. Note that

$$\begin{aligned} |X| + |Y| + |e\Lambda| + |e'\Gamma| &= |(X, 0)| + |(Y \otimes_\Gamma M, Y)| + |(e\Lambda, 0)| + |(e'\Gamma \otimes_\Gamma M, e'\Gamma)| \\ &= |R| = |\Lambda| + |\Gamma| \end{aligned}$$

by Lemma 4.2. So $|X| + |e\Lambda| = |\Lambda|$ and $|Y| + |e'\Gamma| = |\Gamma|$, and hence both X and Y are support τ-tilting. Moreover, the support τ-tilting pair $(X, e\Lambda)$ implies that e is the maximal idempotent such that $\operatorname{Hom}_\Lambda(e\Lambda, X) = 0$.

Conversely, assume that the conditions (1)–(4) are satisfied. Then $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a τ-rigid R-module by Proposition 4.1. Moreover, $(X, e\Lambda)$ is a support τ-tilting pair in $\operatorname{mod}\Lambda$. Let $(Y, e'\Gamma)$ be a support τ-tilting pair in $\operatorname{mod}\Gamma$. Then $\operatorname{Hom}_R((e\Lambda, 0) \oplus (e'\Gamma \otimes_\Gamma M, e'\Gamma), (X, 0) \oplus (Y \otimes_\Gamma M, Y)) = 0$. Moreover, we have

$$\begin{aligned} &|(X, 0)| + |(Y \otimes_\Gamma M, Y)| + |(e\Lambda, 0)| + |(e'\Gamma \otimes_\Gamma M, e'\Gamma)| \\ &= |X| + |Y| + |e\Lambda| + |e'\Gamma| = |\Lambda| + |\Gamma| = |R| \end{aligned}$$

by Lemma 4.2. Thus $((X, 0) \oplus (Y \otimes_\Gamma M, Y), (e\Lambda, 0) \oplus (e'\Gamma \otimes_\Gamma M, e'\Gamma))$ is a support τ-tilting pair in $\operatorname{mod} R$. $\square$

We give another proof of Theorem 4.3 as follows.

Second proof of Theorem 4.3. By Theorem 3.4, we have that $(X, 0) \oplus (Y \otimes_\Gamma M, Y)$ is a silting R-module if and only if X is a silting Λ-module with respect to a projective presentation σ_X of X, Y is a silting Γ-module and $Y \otimes_\Gamma M \in \operatorname{Gen} X$ $(= D_{\sigma_X})$. By [3, Theorem 4.9], σ_X is a 2-term silting complex. Let e be the maximal idempotent such that $\operatorname{Hom}_\Lambda(e\Lambda, X) = 0$. Then by [1, Theorem 3.2], we have $\sigma_X = \sigma \oplus \sigma'$ with σ a minimal projective presentation of X and σ' the complex $e\Lambda \to 0$. So the condition $Y \otimes_\Gamma M \in D_{\sigma_X}$ is equivalent to that $Y \otimes_\Gamma M \in D_\sigma$ and $Y \otimes_\Gamma M \in D_{\sigma'}$, and hence is equivalent to that $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$ and $\operatorname{Hom}_\Lambda(e\Lambda, Y \otimes_\Gamma M) = 0$ by Lemma 2.6. Now Theorem 4.3 follows from Proposition 2.7. $\square$

Remark 4.4. (1) In fact, the maximal idempotent e in Theorem 4.3(4) is exactly the idempotent such that $(X, e\Lambda)$ is a support τ-tilting pair in $\operatorname{mod}\Lambda$ [1, Proposition 2.3(a)].

(2) Since τ-tilting modules are exactly sincere support τ-tilting modules by [1, Proposition 2.2(a)], it follows from Theorem 4.3 that $(X,0)\oplus(Y\otimes_\Gamma M, Y)$ is a τ-tilting R-module if and only if X is a τ-tilting Λ-module, Y is a τ-tilting Γ-module and $\operatorname{Hom}_\Lambda(Y\otimes_\Gamma M, \tau X) = 0$.

(3) Note that tilting modules are exactly τ-tilting modules whose projective dimension is at most one. Then by (2), we have that if ${}_\Gamma M$ is projective, then $(X,0) \oplus (Y \otimes_\Gamma M, Y)$ is a tilting R-module if and only if X is a tilting Λ-module, Y is a tilting Γ-module and $\operatorname{Hom}_\Lambda(Y \otimes_\Gamma M, \tau X) = 0$. This result can be induced directly from [6, Theorem A]. Take $S = \Lambda \times \Gamma$ as in Remark 3.5. Then $(X \oplus Y) \otimes R$ is a tilting R-module if and only if $X \oplus Y$ is a tilting S-module, $\operatorname{Hom}_R((X\oplus Y)\otimes_S M, \tau(X\oplus Y)) = 0$ (that is, $\operatorname{Hom}_\Lambda(Y\otimes_\Gamma M, \tau X) = 0$) and $\operatorname{Hom}_k({}_S M, k) \in \operatorname{Gen}(X \oplus Y)$. Since $\operatorname{Hom}_k({}_S M, k)$ is an injective S-module when ${}_\Gamma M$ is projective, the assertion now is obvious.

Corollary 4.5. *For any $Y \in \operatorname{mod}\Gamma$, $(Y \otimes_\Gamma M, Y)$ is a support τ-tilting R-module if and only if Y is a support τ-tilting Γ-module and $Y \otimes_\Gamma M = 0$.*

Putting $\Gamma = M = \Lambda$ in Theorem 4.3, we get the following result.

Corollary 4.6. *Let $X, Y \in \operatorname{mod}\Lambda$ and $R = \left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$. Then the following statements are equivalent.*

(1) *$(X,0) \oplus (Y,Y)$ is a support τ-tilting R-module.*

(2) *Both X and Y are support τ-tilting Λ-modules and $\operatorname{Hom}_\Lambda(Y, \tau X) = 0 = \operatorname{Hom}_\Lambda(e\Lambda, Y)$, where e is the maximal idempotent such that $\operatorname{Hom}_\Lambda(e\Lambda, X) = 0$.*

In particular, $(X,0) \oplus (Y,Y)$ is a tilting R-module if and only if both X and Y are tilting Λ-modules and $\operatorname{Hom}_\Lambda(Y, \tau X) = 0$.

Recall from [18, Chapter XV, Definition 1.1(a)] that the *one-point extension* of Λ by the module M_Λ is the special triangular algebra $\left(\begin{smallmatrix} \Lambda & 0 \\ {}_k M_\Lambda & k \end{smallmatrix}\right)$. There are only two support τ-tilting k-modules: 0 and k. Let e_a be the idempotent corresponding to the extension point a. Then we have $k \otimes_k M \cong M_\Lambda$ and $(k \otimes_k M, k) \cong (M, k) = e_a R$. As a consequence of Theorem 4.3, we get the following

Corollary 4.7. *Let $X \in \operatorname{mod} \Lambda$ and $R = \left(\begin{smallmatrix} \Lambda & 0 \\ {}_k M_\Lambda & k \end{smallmatrix}\right)$. Then we have*

(1) *X_R is support τ-tilting if and only if X_Λ is support τ-tilting.*

(2) *$X_R \oplus e_a R$ is support τ-tilting R-module if and only if $(X, e\Lambda)$ is a support τ-tilting pair in $\operatorname{mod} \Lambda$, $\operatorname{Hom}_\Lambda(M, \tau X) = 0 = \operatorname{Hom}_\Lambda(e\Lambda, M)$ for some idempotent e of Λ.*

(3) *$X_R \oplus e_a R$ is a tilting R-module if and only if X is a tilting Λ-module and $\operatorname{Hom}_\Lambda(M, \tau X) = 0$.*

Let Λ be an algebra and ${}_\Lambda M_\Lambda$ a (Λ, Λ)-bimodule. Recall that

$$T(\Lambda, M) := \Lambda \oplus {}_\Lambda M \oplus {}_\Lambda M^2 \oplus \cdots \oplus {}_\Lambda M^n \oplus \cdots$$

as an abelian group is called the *tensor algebra* of M over Λ, where M^n is the n-fold Λ-tensor product $M \oplus {}_\Lambda M \oplus \cdots \oplus {}_\Lambda M$. We will assume that $T(\Lambda, M)$ is finite dimensional (equivalently, M is nilpotent). Note that the triangular matrix algebra $\left(\begin{smallmatrix} \Lambda & 0 \\ M & \Gamma \end{smallmatrix}\right)$ can be viewed as the tensor algebra $T(\Lambda \times \Gamma, {}_{\Lambda\times\Gamma} M_{\Lambda\times\Gamma})$. The following result is a generalization of Theorem 4.3.

Theorem 4.8. *Let Λ be an algebra, ${}_\Lambda M_\Lambda$ a (Λ, Λ)-bimodule and $X \in \operatorname{mod} \Lambda$, and let $T(\Lambda, M)$ be the tensor algebra of M over Λ and e an idempotent of Λ. Then the following statements are equivalent.*

(1) *$(X \otimes_\Lambda T(\Lambda, M), eT(\Lambda, M))$ is a support τ-tilting pair in $\operatorname{mod} T(\Lambda, M)$.*

(2) *$(X, e\Lambda)$ is a support τ-tilting pair in $\operatorname{mod} \Lambda$ and*

$$\operatorname{Hom}_\Lambda(X \otimes {}_\Lambda N, \tau X) = 0 = \operatorname{Hom}_\Lambda(e\Lambda, X \otimes {}_\Lambda N),$$

where $N = M \oplus {}_\Lambda M^2 \oplus \cdots \oplus {}_\Lambda M^n \oplus \cdots$.

Proof. Assume that σ_X is a minimal projective presentation of X. Then $\sigma_X \otimes T(\Lambda, M)$ is a minimal projective presentation of $X \otimes_\Lambda T(\Lambda, M)$. Write $\sigma := \sigma_X \otimes T(\Lambda, M) \oplus (eT(\Lambda, M) \to 0)$. Note that there exists a natural projection from $T(\Lambda, M)$ to Λ. It follows from [1, Theorems 3.2] and [9, Theorem 2.2] that $(X \otimes_\Lambda T(\Lambda, M), eT(\Lambda, M))$ is a support τ-tilting pair in $\operatorname{mod} T(\Lambda, M)$ if and only if $(\sigma_X \oplus (e\Lambda \to 0)) \otimes T(\Lambda, M)$ is a 2-term silting complex, and if and only if $\sigma_X \oplus (e\Lambda \to 0)$ is a 2-term silting complex and $\operatorname{Hom}_{T(\Lambda,M)}(T(\Lambda, M)_\Lambda, X \otimes_\Lambda T(\Lambda, M)) \in \operatorname{Gen} X = D_{\sigma_X \oplus (e\Lambda \to 0)}$. Since

$$\operatorname{Hom}_{T(\Lambda,M)}(T(\Lambda, M)_\Lambda, X \otimes T(\Lambda, M)) \cong X \otimes T(\Lambda, M)_\Lambda \cong X \oplus X \otimes N,$$

we have that $\mathrm{Hom}_{T(\Lambda,M)}(T(\Lambda,M)_\Lambda, X\otimes_\Lambda T(\Lambda,M)) \in \mathrm{Gen}\,X$ if and only if $X\otimes_\Lambda N \in \mathrm{Gen}\,X$, that is, $X\otimes_\Lambda N \in D_{\sigma_X}$ and $X\otimes_\Lambda N \in D_{(e\Lambda\to 0)}$. Now the assertion follows from Lemma 2.6. $\square$

We recall some notions from [16]. A *pseudovalued graph* $(\mathcal{G},\mathcal{D})$ consists of

(i) a finite set $\mathcal{G}=\{1,2,\dots,n\}$ whose elements are called *vertices*; and

(ii) a correspondence taking any ordered pair $(i,j)\in\mathcal{G}\times\mathcal{G}$ to a non-negative integer d_{ij} such that if $d_{ij}\neq 0$ then $d_{ji}\neq 0$. If $d_{ij}\neq 0$, then such a pair (i,j) is called an *edge* between the vertices i and j.

The family $\mathcal{D}=\{(d_{ij},d_{ji}) \mid (i,j)\in\mathcal{G}\times\mathcal{G}\}$ is called a *valuation* of the graph $\mathcal{G}$. A *pseudovalued quiver* is a pseudovalued graph $(\mathcal{G},\mathcal{D})$ with an *orientation* which is given by prescribing for each edge an ordering, indicated by an oriented edge. A *path* from j to i of the pseudovalued quiver $(\mathcal{G},\mathcal{D})$ is a sequence $j=k_1,k_2,\dots,k_t=i$ of vertices such that there is a valued oriented edge from k_s to k_{s+1} for any $s=1,2,\dots,t-1$.

For an algebra Λ and a finitely generated left (resp. right) Λ-module M, the *rank* $\mathrm{rank}_\Lambda M$ (resp. $\mathrm{rank}\, M_\Lambda$) of M is defined as the minimal cardinal number of the sets generators of M as a left (resp. right) Λ-module. A *k-pseudomodulation* $\mathcal{M}=(\Lambda_i, {}_iM_j)$ of a pseudovalued graph $(\mathcal{G},\mathcal{D})$ is defined as a set of k-algebras $\{\Lambda_i\}_{i\in\mathcal{G}}$, together with a set $\{{}_iM_j\}_{(i,j)\in\mathcal{G}\times\mathcal{G}}$ of finitely generated (Λ_i,Λ_j)-bimodules ${}_iM_j$ such that $\mathrm{rank}({}_iM_j)_{\Lambda_j}=d_{ij}$ and $\mathrm{rank}_{\Lambda_i}({}_iM_j)=d_{ji}$.

As a consequence of Theorem 4.8, we have the following result.

Corollary 4.9. *Let $\mathcal{M}=(\Lambda_i, {}_iM_j)$ be a k-pseudomodulation of a pseudovalued quiver $(\mathcal{G},\mathcal{D})$ and $V_i\in\mathrm{mod}\,\Lambda_i$ for any $i\in\mathcal{G}$, and let*

$$\Lambda=\bigoplus_{i\in\mathcal{G}}\Lambda_i,\quad M=\bigoplus_{(i,j)\in\mathcal{G}\times\mathcal{G}}{}_iM_j,\quad X_i=\bigoplus_{j\in Q_i}V_j\otimes{}_jM_{k_2}\otimes\cdots\otimes{}_{k_{s-1}}M_i,$$

where $Q_i=\{j\in\mathcal{G}\mid$ there is a path from j to $i\}$ and $j=k_1,k_2,\dots,k_s=i$ is a path from j to i. Then the following statements are equivalent.

(1) *$\bigoplus_{i\in\mathcal{G}}(V_i\oplus X_i)$ is a support τ-tilting $T(\Lambda,M)$-module.*

(2) *$\bigoplus_{i\in\mathcal{G}}V_i$ is a support τ-tilting Λ-module and*

$$\mathrm{Hom}_\Lambda\Big(\bigoplus_{i\in\mathcal{G}}X_i,\tau\Big(\bigoplus_{i\in\mathcal{G}}V_i\Big)\Big)=0=\mathrm{Hom}_\Lambda\Big(e\Lambda,\bigoplus_{i\in\mathcal{G}}X_i\Big),$$

where e is an idempotent of Λ such that $\big(\bigoplus_{i\in\mathcal{G}}V_i, e\Lambda\big)$ is a support τ-tilting pair in $\mathrm{mod}\,\Lambda$.

(3) *For any $i \in \mathcal{G}$, V_i is a support τ-tilting Λ_i-module and*

$$\mathrm{Hom}_{\Lambda_i}(X_i, \tau V_i) = 0 = \mathrm{Hom}_{\Lambda_i}(e_i\Lambda_i, X_i),$$

where e_i is an idempotent of Λ_i such that $(V_i, e_i\Lambda_i)$ is a support τ-tilting pair in $\mathrm{mod}\,\Lambda_i$.

Proof. The assertion (2) $\Leftrightarrow$ (3) is clear.

Set $X := \bigoplus_{i\in\mathcal{G}} V_i$ and $N := M \oplus {}_\Lambda M^2 \oplus \cdots \oplus {}_\Lambda M^n \oplus \cdots$. Then we have $X \otimes_\Lambda N \cong \bigoplus_{i\in\mathcal{G}} X_i$, and hence

$$X \otimes_\Lambda T(\Lambda, M) \cong X \oplus (X \otimes_\Lambda N) \cong X \oplus \left(\bigoplus_{i\in\mathcal{G}} X_i\right) \cong \bigoplus_{i\in\mathcal{G}} (V_i \oplus X_i).$$

Now the assertion (1) $\Leftrightarrow$ (2) follows from Theorem 4.8. $\square$

Remark 4.10. Let $\mathcal{M}$ be a k-pseudomodulation of a pseudovalued quiver $(\mathcal{G}, \mathcal{D})$. Recall from [16] that a *representation* of $\mathcal{M}$ is an object $(V_i, {}_j\varphi_i)$, where to each vertex $i \in \mathcal{G}$ corresponds a Λ_i-module V_i and to each oriented edge $i \to j$ corresponds a Λ_j-homomorphism ${}_j\varphi_i \colon V_i \otimes_{\Lambda_i} {}_iM_j \to V_j$. If each V_i is finitely generated as Λ_i-module, then the representation $(V_i, {}_j\varphi_i)$ is called *finitely generated*. We use $\mathrm{rep}(\mathcal{M})$ to denote the category consisting of all finitely generated representations of $\mathcal{M}$. It was shown in [16, Theorem 3.2] that $\mathrm{rep}(\mathcal{M})$ is equivalent to $\mathrm{mod}\,T(\Lambda, M)$, where Λ and M are as in Corollary 4.9.

5. An example

In this section, for a finite dimensional k-algebra over an algebraically closed field k with the quiver Q, we use P_i (resp. S_i) to denote the indecomposable projective (resp. simple) module corresponding to the vertex i in Q, and use e_i to denote the idempotent corresponding to the vertex i. For brevity, the symbol $\oplus$ between modules will be omitted; for example, for modules M and N, we will replace $M \oplus N$ with MN. We illustrate some of our work with the following example.

Example 5.1. Let R be a finite dimensional k-algebra over k given by the following quiver

$$\begin{tikzcd}
 & 2 & & \\
1 \arrow[ur, "\delta"] & & 4 \arrow[ul, "\gamma"'] \arrow[r, "\beta"] & 5 \\
 & 3 \arrow[ul, "\varepsilon"] \arrow[ur, "\alpha"'] & &
\end{tikzcd}$$

with the relation $\alpha\gamma = \varepsilon\delta$ and $\alpha\beta = 0$. Let $e = e_1 + e_2$. Then

$$R = \begin{pmatrix} eRe & 0 \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Take

$$\begin{gathered} \Lambda := eRe \ (\cong k(1 \xrightarrow{\delta} 2)), \\ \Gamma := (1-e)R(1-e) \ (\cong k(3 \xrightarrow{\alpha} 4 \xrightarrow{\beta} 5) \text{ with } \alpha\beta = 0), \\ {}_{\Gamma}M_{\Lambda} = (1-e)Re. \end{gathered}$$

Then $M_{\Lambda} \cong \Lambda$ and ${}_{\Gamma}M \cong P_3S_4$.

For an indecomposable Λ-module X and an indecomposable Γ-module Y, we use (X, Y) to denote the corresponding indecomposable R-module. The Auslander-Reiten quiver of R is as follows:

$$\begin{tikzcd}[column sep=small]
(0,P_5) \arrow[dr] & & (P_2,S_4) \arrow[dr] & & (S_1,0) \arrow[dr] & & (0,P_3) \arrow[dr] & \\
& (P_2,P_4) \arrow[ur] \arrow[dr] & & (P_1,S_4) \arrow[ur] \arrow[r] \arrow[dr] & (P_1,P_3) \arrow[r] & (S_1,P_3) \arrow[ur] \arrow[dr] & & (0,S_3) \\
(P_2,0) \arrow[ur] \arrow[dr] & & (P_1,P_4) \arrow[ur] \arrow[dr] & & (0,S_4) \arrow[ur] & & (S_1,S_3) \arrow[ur] & \\
& (P_1,0) \arrow[ur] & & (0,P_4) \arrow[ur] & & & &
\end{tikzcd}$$

The Hasse quivers of Λ and Γ are as follows:

$Q(s\tau\text{-}\operatorname{tilt}\Lambda)$:

$$\begin{tikzcd}
& (P_2,P_1) \arrow[r] & (0,P_1P_2) \\
(P_1P_2,0) \arrow[ur] \arrow[r] & (P_2S_1,0) \arrow[r] & (S_1,P_2), \arrow[u]
\end{tikzcd}$$

$Q(s\tau\text{-}\operatorname{tilt}\Gamma)$:

$$\begin{tikzcd}
& P_3P_4S_4 \arrow[r] \arrow[drr] & P_3S_4 \arrow[r] \arrow[drr] & P_3S_3 \arrow[r] & S_3 \arrow[dr] & \\
P_3P_4P_5 \arrow[ur] \arrow[r] \arrow[dr] & P_3S_3P_5 \arrow[r] \arrow[urr] & S_3P_5 \arrow[urr] \arrow[drr] & P_4S_4 \arrow[r] & S_4 \arrow[r] & 0 \\
& P_4P_5 \arrow[urr] \arrow[rrr] & & & P_5. \arrow[ur] &
\end{tikzcd}$$

Now we list $T \otimes_{\Gamma} M$ for all support τ-tilting Γ-modules T_{Γ} in the following table.

T_i	$P_3P_4P_5$	$P_3P_4S_4$	$P_3S_3P_5$	P_4P_5	P_3S_4	S_3P_5	P_3S_3	P_4S_4	S_3	S_4	P_5	0
$T_i \otimes M$	P_1P_2	$P_1P_2P_2$	P_1S_1	P_2	P_1P_2	S_1	P_1S_1	P_2P_2	S_1	P_2	0	0

By Remark 4.4(1), we have that the maximal idempotent e in Theorem 4.3 is exactly the idempotent such that $(X, e\Lambda)$ is a support τ-tilting pair in $\operatorname{mod}\Lambda$. We can construct many support τ-tilting R-modules by Theorem 4.3.

(1) Considering the support τ-tilting pair $(P_1P_2, 0)$ over Λ, we have $\tau(P_1P_2) = 0$ and $e\Lambda = 0$. Hence, we get the following support τ-tilting R-modules:

$$(P_1,0)(P_2,0)(P_1,P_3)(P_2,P_4)(0,P_5),\ (P_1,0)(P_2,0)(P_1,P_3)(P_2,P_4)(P_2,S_4),$$
$$(P_1,0)(P_2,0)(P_1,P_3)(S_1,S_3)(0,P_5),\ (P_1,0)(P_2,0)(P_2,P_4)(0,P_5),$$
$$(P_1,0)(P_2,0)(P_1,P_3)(P_2,S_4),\ (P_1,0)(P_2,0)(S_1,S_3)(0,P_5),$$
$$(P_1,0)(P_2,0)(P_1,P_3)(S_1,S_3),\ (P_1,0)(P_2,0)(P_2,P_4)(P_2,S_4),$$
$$(P_1,0)(P_2,0)(S_1,S_3),\ (P_1,0)(P_2,0)(P_2,S_4),$$
$$(P_1,0)(P_2,0)(0,P_5),\ (P_1,0)(P_2,0).$$

(2) Considering the support τ-tilting pair $(P_1S_1, 0)$ over Λ, we have $\tau(P_1S_1) = P_2$ and $e\Lambda = 0$. Hence, those R-modules

$$(P_1,0)(S_1,0)(P_1,P_3)(S_1,S_3)(0,P_5),\ (P_1,0)(S_1,0)(S_1,S_3)(0,P_5),$$
$$(P_1,0)(S_1,0)(P_1,P_3)(S_1,S_3),\ (P_1,0)(S_1,0)(S_1,S_3),$$
$$(P_1,0)(S_1,0)(0,P_5),\ (P_1,0)(S_1,0)$$

are support τ-tilting.

(3) Considering the support τ-tilting pair (P_2, P_1) over Λ, we have $\tau P_2 = 0$ and $e\Lambda = P_1$. Hence, we get the following support τ-tilting R-modules:

$$(P_2,0)(P_2,P_4)(0,P_5),\ (P_2,0)(P_2,P_4)(P_2,S_4),\ (P_2,0)(P_2,S_4),\ (P_2,0)(0,P_5),\ (P_2,0).$$

(4) Considering the support τ-tilting pair (S_1, P_2) over Λ, we have $\tau S_1 = P_2$ and $e\Lambda = P_2$. Hence, we get the following support τ-tilting R-modules:

$$(S_1,0)(S_1,S_3)(0,P_5),\ (S_1,0)(S_1,S_3),\ (S_1,0)(0,P_5),\ (S_1,0).$$

(5) Considering the support τ-tilting pair $(0, P_1P_2)$ over Λ, we have $e\Lambda = P_1P_2$. Hence, we get support τ-tilting R-modules: $(0, P_5)$ and 0.

(6) Considering the tilting Γ-module $P_3P_4S_4$ which is a silting module with respect to

$$\sigma\colon P_5 \to P_3P_4P_4,$$

we have that $\sigma \otimes_\Gamma M : 0 \to P_1P_2P_2$ is monic. Thus $(P_1, 0)(P_2, 0)(P_1, P_3)(P_2, P_4)(P_2, S_4)$ is a tilting R-module by Corollary 3.7, though ${}_\Gamma M \cong P_3S_4$ is not flat.

(7) Unfortunately, we cannot get all support τ-tilting R-modules by Theorem 4.3. For example, the module $(0, P_3)(P_2, 0)(P_1, P_3)(P_2, P_4)(0, P_5)$ is a support τ-tilting R-module, but it does not appear in (1)–(5).

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Hanpeng Gao and Zhaoyong Huang
Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, China
E-mail address: hpgao07@163.com, huangzy@nju.edu.cn