On extension closure of $\mathcal{E}$-Gorenstein flat modules*†

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Abstract

Let R be an arbitrary ring and $\mathcal{E}$ an injectively resolving class of left R-modules. We prove that the class of $\mathcal{E}$-Gorenstein flat right R-modules is closed under extensions, and hence projectively resolving. This answers an open question in [Strongly $\mathcal{E}$-Gorenstein injective and flat modules, Rocky Mountain J. Math. 54 (2024), 143–160] affirmatively. As a consequence, we get that this class is covering. In addition, we introduce the notion of $\mathcal{E}$-projectively coresolved Gorenstein flat modules, and prove that the class of $\mathcal{E}$-projectively coresolved Gorenstein flat right R-modules is projectively resolving and closed under transfinite extensions.

1 Introduction

In Gorenstein homological algebra, Gorenstein injective and flat modules are important and fundamental research objects, which were introduced by Enochs, Jenda and Torrecillas in [7, 9]. It is well-known that the extension closure of Gorenstein flat modules is an important topic in Gorenstein homological algebra, which has been studied by many authors, see [1, 5, 26, 28, 29]. Recently, Šaroch and Šťovíček [26] proved that the class of Gorenstein flat modules is always closed under extensions and covering, regardless of the ring R.

On the other hand, the homological theory of various generalizations of these Gorenstein modules has become a vigorously active area of research in the recent years, see [2, 3, 4, 8, 16, 23, 25, 27] and references therein. In particular, Mao and Ding [25] introduced the notion of Gorenstein FP-injective modules. It was shown that such modules over coherent rings possess many nice properties analogous to those of Gorenstein injective modules over Noetherian rings [19, 25]. Along the same lines, Bravo, Gillespie and Hovey [4] introduced the notion of Gorenstein AC-injective modules in terms of the so-called absolutely clean modules, and many homological properties of Gorenstein AC-injective were obtained in [3, 4, 20]. Later,

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Bravo, Estrada and Iacob [2] introduced Gorenstein AC-flat modules by replacing injective modules with absolutely clean modules in the definition of Gorenstein flat modules. In [12], Estrada, Iacob and Pérez introduced and studied Gorenstein $\mathcal{B}$-flat modules, where $\mathcal{B}$ is a class of right R-modules. It was shown in [12] that if $\mathcal{B}$ is a semi-definable class (that is, $\mathcal{B}$ is closed under products and contains an elementary cogenerator of its definable closure), then the class of Gorenstein $\mathcal{B}$-flat modules is a left orthogonal class, and hence it is closed under extensions. In [2, 12], there are many nice results provided that the class of Gorenstein AC-flat modules (respectively, Gorenstein $\mathcal{B}$-flat modules) is closed under extensions. So it is important question to study the extension closure of generalized Gorenstein flat modules.

Let $\mathcal{E}$ be an injectively resolving class of left R-modules. Certain nice generalizations of Gorenstein injective and flat modules are $\mathcal{E}$-Gorenstein injective and flat modules, respectively, which were introduced in [15, 16], see Definition 2.3 for details. Note that the notion of $\mathcal{E}$-Gorenstein injective modules unifies the following notions: Gorenstein injective modules [7], Gorenstein FP-injective modules [25] and Gorenstein AC-injective modules [3], and that the notion of $\mathcal{E}$-Gorenstein flat modules unifies some known modules such as Gorenstein flat modules [9] and Gorenstein AC-flat modules [2]. In [15, 16, 17], some basic homological properties of $\mathcal{E}$-Gorenstein injective and flat modules have been obtained. Following the above philosophy, Gao and Zhong [17, p.151] raised naturally an open question: whether is the class of $\mathcal{E}$-Gorenstein flat modules closed under extensions for any ring? Furthermore, one can ask: whether is the class of $\mathcal{E}$-Gorenstein flat modules covering for any ring? Our aim is to answer these two questions affirmatively.

This paper is organized as follows. In Section 2, we will give some notions and notations needed in the sequel. Let $\mathcal{E}$ be an injectively resolving class of left R-modules. In Section 3, we prove the following result.

Theorem 1.1. (Theorems 3.1 and 3.4) *It holds that*

(1) *The class of $\mathcal{E}$-Gorenstein flat right R-modules is closed under extensions, and further it is projectively resolving.*

(2) *The class of $\mathcal{E}$-Gorenstein flat right R-modules is covering.*

In Section 4, we introduce the notion of $\mathcal{E}$-projectively coresolved Gorenstein flat modules. Note that the class of $\mathcal{E}$-projectively coresolved Gorenstein flat right R-modules is a subclass of two classes consisting of projectively coresolved Gorenstein flat right R-modules and $\mathcal{E}$-Gorenstein flat right R-modules respectively. We prove the following result.

Theorem 1.2. (Theorem 4.5) *The class of $\mathcal{E}$-projectively coresolved Gorenstein flat right R-modules is projectively resolving and closed under transfinite extensions.*

2 Preliminaries

Throughout this paper, R is an arbitrary associative ring with identity, and all modules are unitary. In this section, we collect some basic concepts and facts which will be useful in the sequel.

Definition 2.1. ([22])

(1) A class $\mathcal{G}$ of right R-modules is called *projectively resolving* if $\mathcal{G}$ contains all projective right R-modules, and $\mathcal{G}$ is closed under extensions and kernels of epimorphisms.

(2) A class $\mathcal{E}$ of right R-modules is called *injectively resolving* if $\mathcal{E}$ contains all injective right R-modules, and $\mathcal{E}$ is closed under extensions and cokernels of monomorphisms.

Let $\mathcal{X}$ be a class of left or right R-modules. A sequence of left or right R-modules is called $\mathrm{Hom}_R(\mathcal{X},-)$-*exact* (respectively, $\mathrm{Hom}_R(-,\mathcal{X})$-*exact*) if it is exact after applying the functor $\mathrm{Hom}_R(X,-)$ (respectively, $\mathrm{Hom}_R(-,X)$) for any $X \in \mathcal{X}$. Let $\mathcal{X}$ be a class of left (respectively, right) R-modules. A sequence of right (respectively, left) R-modules is called $(-\otimes_R \mathcal{X})$-*exact* (respectively, $(\mathcal{X}\otimes_R -)$-*exact*) if it is exact after applying the functor $-\otimes_R X$ (respectively, $X \otimes_R -$) for any $X \in \mathcal{X}$,

Definition 2.2. ([15, 16]) Let $\mathcal{E}$ be an injectively resolving class of left R-modules.

(1) A left R-module M is called $\mathcal{E}$-*Gorenstein injective* if there exists a $\mathrm{Hom}_R(\mathcal{E},-)$-exact exact sequence of injective left R-modules

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M \cong \mathrm{Ker}(E^0 \to E^1)$.

(2) A right R-module M is called $\mathcal{E}$-*Gorenstein flat* if there exists a $(-\otimes_R \mathcal{E})$-exact exact sequence of flat right R-modules

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Ker}(F^0 \to F^1)$.

We use $\mathcal{GF}_{\mathcal{E}}(\mathcal{R})$ to denote the class of Gorenstein $\mathcal{E}$-flat right R-modules. When $\mathcal{E}$ is the class of injective left R-modules, an $\mathcal{E}$-Gorenstein injective module and an $\mathcal{E}$-Gorenstein flat module are exactly a Gorenstein injective module and a Gorenstein flat module module, respectively.

Definition 2.3. ([7]) A right R-module M is called *Gorenstein projective* if there exists a $\mathrm{Hom}_R(-,\mathcal{P}(R))$-exact exact sequence of projective right R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$, where $\mathcal{P}(R)$ is the class of projective right R-modules.

Definition 2.4. ([4, 14])

(1) A left R-module F is called *super finitely presented* if there exists an exact sequence of left R-modules

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

with all P_i finitely generated projective.

(2) A left R-module M is called *absolutely clean* (or *weak injective*) if $\mathrm{Ext}^1_R(F,M)=0$ for any super finitely presented left R-module F.

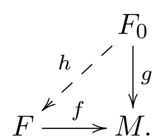
We use $\mathcal{A}$ to denote the class of absolutely clean left R-modules.

Definition 2.5. ([2, 4]) A right R-module M is called *Gorenstein AC-flat* if there exists a $(-\otimes_R \mathcal{A})$-exact exact sequence of flat right R-modules

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$.

Definition 2.6. ([6]) Let $\mathcal{F}$ be a class of right R-modules. A homomorphism $f: F \to M$ of right R-modules with $F \in \mathcal{F}$ is called an *$\mathcal{F}$-precover* of M if for any homomorphism $g: F_0 \to M$ of right R-modules with $F_0 \in \mathcal{F}$, there exists a homomorphism $h: F_0 \to F$ such that the following diagram commutes:



A homomorphism $f: F \to M$ is called *right minimal* if an endomorphism $h: F \to F$ is an automorphism whenever $f = fh$. An $\mathcal{F}$-precover $f: F \to M$ is called an *$\mathcal{F}$-cover* if f is right minimal. The class $\mathcal{F}$ is called *covering* if every right R-module has an $\mathcal{F}$-cover. Dually, the notions of an *$\mathcal{F}$-preenvelope*, a *left minimal homomorphism*, an *$\mathcal{F}$-envelope* and an *enveloping class* are defined.

3 Main results

In this section, assume that $\mathcal{E}$ is an injectively resolving class of left R-modules. We will prove that the class $\mathcal{GF}_{\mathcal{E}}(R)$ of $\mathcal{E}$-Gorenstein flat modules is closed under extensions, which gives an affirmative answer to the open question in [17, p.151]. As a consequence, we get that the class $\mathcal{GF}_{\mathcal{E}}(R)$ is covering.

We write $(-)^+ := \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Theorem 3.1. *The class $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under extensions, and further it is projectively resolving.*

Proof. Let

$$0 \to A \to B \to C \to 0 \tag{3.1}$$

be an exact sequence of right R-modules with $A, C \in \mathcal{GF}_{\mathcal{E}}(R)$. For any $E \in \mathcal{E}$, applying the functor $-\otimes_R E$ to the exact sequence (3.1), we have the following exact sequence

$$0 = \mathrm{Tor}_i^R(A,E) \to \mathrm{Tor}_i^R(B,E) \to \mathrm{Tor}_i^R(C,E) = 0$$

for any $i \geq 1$ by [15, Proposition 2.6]. Thus $\operatorname{Tor}_{\geq 1}^R(B, E) = 0$ for any $E \in \mathcal{E}$.

Since both A and C are Gorenstein flat by [15, Remark 2.2(1)], we get that B is Gorenstein flat since the class of Gorenstein flat modules is closed under extensions by [26, Theorem 4.11]. Then there is an exact sequence of right R-modules

$$0 \to B \to F^0 \to F^1 \to F^2 \to \cdots \tag{3.2}$$

with all F^i flat and all cycles Gorenstein flat.

In the following, we will show that the sequence (3.2) is $(- \otimes_R \mathcal{E})$-exact. Putting $B^1 := \operatorname{Im}(F^0 \to F^1)$, we obtain an exact sequence

$$0 \to B \to F^0 \to B^1 \to 0, \tag{3.3}$$

with F^0 flat and B^1 Gorenstein flat. This induces the following exact sequence

$$0 \to B^{1+} \to F^{0+} \to B^+ \to 0,$$

where F^{0+} is injective and B^{1+} is Gorenstein injective by [24, Theorem] and [22, Theorem 3.6], respectively. On the other hand, the exact sequence (3.1) gives rise to the exact sequence

$$0 \to C^+ \to B^+ \to A^+ \to 0,$$

with both A^+ and C^+ $\mathcal{E}$-Gorenstein injective by [15, Proposition 2.5]. Then B^+ is $\mathcal{E}$-Gorenstein injective by [16, Theorem 2.7]. It follows from [16, Proposition 2.6] that there exists an exact sequence of left R-modules

$$0 \to K \to I \to B^+ \to 0$$

with I injective and K $\mathcal{E}$-Gorenstein injective. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B^{1+} & \to & D & \to & I & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B^{1+} & \to & F^{0+} & \to & B^+ & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

In the middle column, both K and F^{0+} are $\mathcal{E}$-Gorenstein injective, then so is D by [16, Theorem 2.7]. Since B^{1+} is Gorenstein injective, we have $\operatorname{Ext}_R^{\geq 1}(I, B^{1+}) = 0$ by [22, Theorem 2.22], and thus the middle row splits. It follows that B^{1+} is $\mathcal{E}$-Gorenstein injective by [16, Theorem 2.7]. So for any $E \in \mathcal{E}$, we have

$$[\operatorname{Tor}_1^R(B^1, E)]^+ \cong \operatorname{Ext}_R^1(E, B^{1+}) = 0$$

by [21, Lemma 1.2.11(2)], and hence $\operatorname{Tor}_1^R(B^1, E) = 0$. It implies that the sequence (3.3) is $(- \otimes_R \mathcal{E})$-exact.

Now putting $B^2 := \operatorname{Ker}(F^2 \to F^3)$, we obtain the following exact sequence

$$0 \to B^1 \to F^1 \to B^2 \to 0 \tag{3.4}$$

with F^1 flat and B^2 Gorenstein flat. This gives the following exact sequence

$$0 \to B^{2^+} \to F^{1^+} \to B^{1^+} \to 0$$

with F^{1^+} injective and B^{2^+} Gorenstein injective. Notice that B^{1^+} is $\mathcal{E}$-Gorenstein injective by the foregoing proof, we have an exact sequence

$$0 \to K^1 \to I^1 \to B^{1^+} \to 0,$$

where I^1 is injective and K^1 is $\mathcal{E}$-Gorenstein injective by [16, Proposition 2.6]. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K^1 & = & K^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B^{2^+} & \to & D^1 & \to & I^1 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B^{2^+} & \to & F^{1^+} & \to & B^{1^+} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

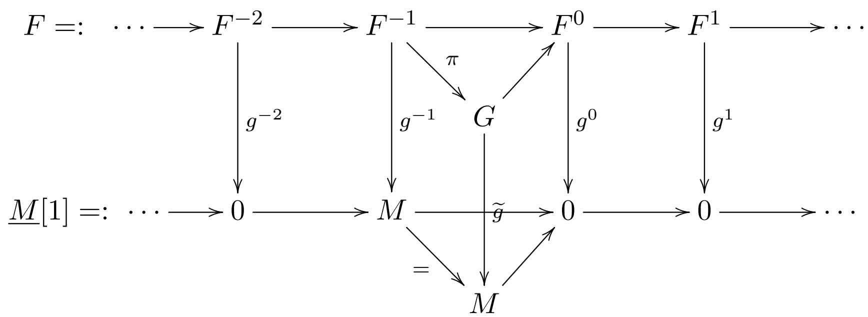
By repeating the argument as above, we get $\operatorname{Tor}_1^R(B^2, E) = 0$ for any $E \in \mathcal{E}$. It follows that the sequence (3.4) is $(- \otimes_R \mathcal{E})$-exact. Continuing this process, we get that the sequence (3.2) is $(- \otimes_R \mathcal{E})$-exact. Consequently, B is $\mathcal{E}$-Gorenstein flat by [15, Proposition 2.6]. This proves that the class $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under extensions, and then it is projectively resolving by [15, Theorem 2.7]. □

Now one naturally asks the following question: whether is the class $\mathcal{GF}_{\mathcal{E}}(R)$ covering for any ring? In the remainder of this section, we will show that, over any ring R, this class is covering. We first note that, the class $\mathcal{GF}_{\mathcal{E}}(R)$ is precovering for any ring R. General background materials of complexes are referred to [18, 29].

Proposition 3.2. *The class $\mathcal{GF}_{\mathcal{E}}(R)$ is precovering.*

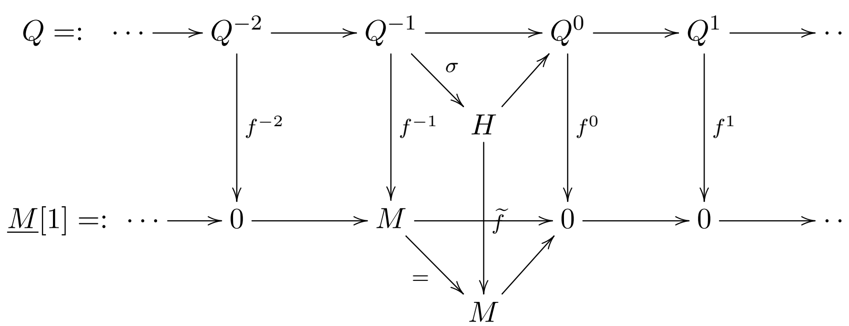
Proof. Let $\widetilde{\mathcal{F}}$ be the class of exact complexes of flat right R-modules $\mathbf{F}$ and $I \in \mathcal{E}$ such that $\mathbf{F} \otimes_R I$ is exact. The class $\widetilde{\mathcal{F}}$ is special precovering by [11, Theorem 3.7]. In the following, we use an argument similar to that in the proof of [30, Theorem A] to deduce that any right R-module has an $\mathcal{E}$-Gorenstein flat precover.

Let M be a right R-module and let $g : F \to \underline{M}[1]$ be an $\widetilde{\mathcal{F}}$-precover. Then we get the following commutative diagram

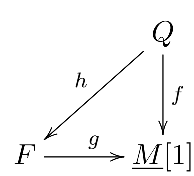


where $G = Z^0(F)$ is $\mathcal{E}$-Gorenstein flat. Next we will show that $\widetilde{g} : G \to M$ is an $\mathcal{GF}_{\mathcal{E}}(R)$-precover of M.

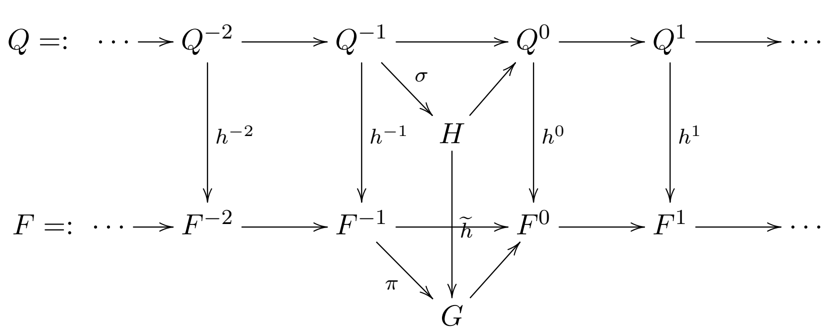
Let $\widetilde{f} : H \to M$ be a homomorphism with $H \in \mathcal{GF}_{\mathcal{E}}(R)$. Then there exists a complex $Q \in \widetilde{F}$ such that $H = Z^0(Q)$. Now one can extend $\widetilde{f}$ to a morphism $f : Q \to \underline{M}[1]$ of complexes as follows:



Notice that $g : F \to \underline{M}[1]$ is an $\widetilde{\mathcal{F}}$-precover, then there exists a morphism $h : Q \to F$ of complexes such that the following diagram



commutes. The morphism h gives rise to a homomorphism $\widetilde{h} : H \to G$ such that the following diagram



commutes. Since

$$\widetilde{f}\sigma = f^{-1} = g^{-1}h^{-1} = \widetilde{g}\pi h^{-1} = \widetilde{g}\widetilde{h}\sigma$$

and σ is an epimorphism, we have $\widetilde{f} = \widetilde{g}\widetilde{h}$. It follows that $\widetilde{g} : G \to M$ is an $\mathcal{GF}_{\mathcal{E}}(R)$-precover of M. ☐

Enochs and López-Ramos [10, Proposition 2.3] proved that if a class of R-modules $\mathcal{F}$ is closed under well-ordered direct limits, then it is closed under arbitrary direct limits. Holm [22, Theorem 3.7] showed that the class of Gorenstein flat left R-modules is closed under direct limits over a right coherent ring. Then, Yang and Liu [29] extended this result to left GF-closed rings. Here we have the following result.

Proposition 3.3. *The class $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under direct limits.*

Proof. The following argument is similar to that of [29, Lemma 3.1], we provide a complete proof here for readability. By Theorem 3.1, the class $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under extensions. Assume that $(M_\alpha)_{\alpha<\lambda}$ is a well-ordered direct system of modules in $\mathcal{GF}_{\mathcal{E}}(R)$. If $\lambda = n < \omega$, then $\varinjlim M_\alpha = M_{n-1}$ is $\mathcal{E}$-Gorenstein flat, as desired.

Now suppose $\lambda = \omega$. We first show that $\varinjlim M_n (n < \omega)$ is $\mathcal{E}$-Gorenstein flat. Since M_0 is $\mathcal{E}$-Gorenstein flat, there exists an exact sequence

$$G(0) =: 0 \to M_0 \to F_0^0 \to F_0^1 \to F_0^2 \to \cdots$$

with all F_0^i flat, such that $G(0) \otimes_R I$ is exact for any $I \in \mathcal{E}$. Set $K_0^i := \mathrm{Ker}(F_0^i \to F_0^{i+1})$ for $i \geq 0$ (where $K_0^0 = M_0$). Then each K_0^i is $\mathcal{E}$-Gorenstein flat by [15, Remark 2.2(1)].

Consider the following pushout diagram of $M_0 \to F_0^0$ and $M_0 \to M_1$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 & \longrightarrow & F_0^0 & \longrightarrow & K_0^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C & \longrightarrow & K_0^1 & \longrightarrow & 0.
\end{array}$$

Since $M_1, K_0^1 \in \mathcal{GF}_{\mathcal{E}}(R)$, we have $C \in \mathcal{GF}_{\mathcal{E}}(R)$ by Theorem 3.1. Then by [15, Proposition 2.6], there exists an exact sequence of right R-modules

$$0 \to C \to F_1^0 \to N \to 0$$

with F_1^0 flat and $N \in \mathcal{GF}_{\mathcal{E}}(R)$. Now consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C & \longrightarrow & K_0^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & F_1^0 & \longrightarrow & K_1^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & = & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

From the right column of the above diagram, one gets that $K_1^1 \in \mathcal{GF}_{\mathcal{E}}(R)$ by Theorem 3.1. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & F_0^0 & \longrightarrow & K_0^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & F_1^0 & \longrightarrow & K_1^1 & \longrightarrow & 0. \end{array}$$

Using the same method as above, one gets the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0^1 & \longrightarrow & F_0^1 & \longrightarrow & K_0^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_1^1 & \longrightarrow & F_1^1 & \longrightarrow & K_1^2 & \longrightarrow & 0, \end{array}$$

where F_1^1 is flat and $K_1^2 \in \mathcal{GF}_{\mathcal{E}}(R)$. Repeating the argument as above, we can deduce the following exact sequence:

$$G(1) =: 0 \to M_1 \to F_1^0 \to F_1^1 \to F_1^2 \to \cdots$$

with each F_1^i flat and each $K_1^i = \mathrm{Ker}(F_1^i \to F_1^{i+1}) \in \mathcal{GF}_{\mathcal{E}}(R)$ for $i \geq 0$. One checks readily that $G(1) \otimes_R I$ is exact for any $I \in \mathcal{E}$. Also, we get a morphism $G(0) \to G(1)$ induced by $M_0 \to M_1$. Continuing this process, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} G(0) =: 0 & \longrightarrow & M_0 & \longrightarrow & F_0^0 & \longrightarrow & F_0^1 & \longrightarrow & F_0^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ G(1) =: 0 & \longrightarrow & M_1 & \longrightarrow & F_1^0 & \longrightarrow & F_1^1 & \longrightarrow & F_1^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ G(2) =: 0 & \longrightarrow & M_2 & \longrightarrow & F_2^0 & \longrightarrow & F_2^1 & \longrightarrow & F_2^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

with each F_j^i flat and each $K_j^i := \mathrm{Ker}(F_j^i \to F_j^{i+1}) \in \mathcal{GF}_{\mathcal{E}}(R)$ for any $i \geq 0$ and $j \geq 0$ (where $K_j^0 = M_j$), and all sequences $G(n)$ are $(- \otimes_R \mathcal{E})$-exact. Now applying the exact functor $\varinjlim$ to the above commutative diagram, we obtain an exact sequence

$$\varinjlim G(n) := 0 \to \varinjlim M_n \to \varinjlim F_n^0 \to \varinjlim F_n^1 \to \varinjlim F_n^2 \to \cdots ,$$

such that all $\varinjlim F_n^i$ are flat. Notice that $\varinjlim$ commutes with the tensor product functor, then we have

$$\varinjlim G(n) \otimes_R I \cong \varinjlim (G(n) \otimes_R I)$$

is exact for any $I \in \mathcal{E}$. On the other hand, since all M_n are in $\mathcal{GF}_{\mathcal{E}}(R)$, one easily gets that

$$\mathrm{Tor}_i^R(\varinjlim M_n, I) \cong \varinjlim \mathrm{Tor}_i^R(M_n, I) = 0$$

for any $I \in \mathcal{E}$ and $i > 0$. Thus $\varinjlim M_n \in \mathcal{GF}_{\mathcal{E}}(R)$.

Finally, we reindex the modules

$$M_0, M_1, \cdots, M_\omega, M_{\omega+1} \cdots$$

such that $M_\omega = \varinjlim M_n$ and $M_{\omega+1}$ is the old of M_ω and so on. We may assume that the system $(M_\alpha)_{\alpha<\lambda}$ is continuous, i.e., $M_\beta = \varinjlim M_\alpha$ $(\alpha < \beta)$ if β is a limit ordinal with $\beta < \lambda$. Then using transfinite induction, we have that $\varinjlim M_\alpha$ $(\alpha < \lambda)$ is in $\mathcal{GF}_{\mathcal{E}}(R)$. The proof is finished. □

We are now in a position to state the following result.

Theorem 3.4. *The class $\mathcal{GF}_{\mathcal{E}}(R)$ is covering.*

Proof. By Propositions 3.2 and 3.3, the class $\mathcal{GF}_{\mathcal{E}}(R)$ is precovering and closed under direct limits, and then it is covering by [8, Corollray 5.2.7]. □

By Theorems 3.1, 3.4 and [13, Proposition 2.6(1)], we immediately get the following result.

Corollary 3.5. ([2, 12]) *The class of Gorenstein AC-flat right R-modules is projectively resolving and covering.*

4 A subclass of $\mathcal{GF}_{\mathcal{E}}(R)$

We introduce the following notion.

Definition 4.1. Let $\mathcal{E}$ be an injectively resolving class of left R-modules. A right R-module M is called *$\mathcal{E}$-projectively coresolved Gorenstein flat* if there exists a $(- \otimes_R \mathcal{E})$-exact exact sequence of projective right R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M \cong \operatorname{Ker}(P^0 \to P^1)$.

When $\mathcal{E}$ is the class of injective left R-modules, an $\mathcal{E}$-projectively coresolved Gorenstein flat module is exactly a projectively coresolved Gorenstein flat module introduced in [26]. We use $\mathcal{PGF}_{\mathcal{E}}(\mathcal{R})$ to denote the class consisting of all $\mathcal{E}$-projectively coresolved Gorenstein flat modules. It is clear that $\mathcal{PGF}_{\mathcal{E}}(\mathcal{R}) \subseteq \mathcal{GF}_{\mathcal{E}}(\mathcal{R})$.

In the following result, we give some equivalent characterizations of $\mathcal{E}$-projectively coresolved Gorenstein flat modules. Since the proof is similar to [15, Proposition 2.6], we omit it.

Proposition 4.2. *For any right R-module M, the following statements are equivalent.*

(1) *$M \in \mathcal{PGF}_{\mathcal{E}}(R)$.*

(2) *M satisfies the following two conditions:*

(2.1) $\mathrm{Tor}^R_{\geq 1}(M, E) = 0$ *for any* $E \in \mathcal{E}$.

(2.2) *There exists a* $(- \otimes_R \mathcal{E})$*-exact exact sequence of right R-modules*

$$0 \to M \to P^0 \to P^1 \to \cdots$$

with all P^i *projective.*

(3) *There exists an exact sequence of right R-modules*

$$0 \to M \to P \to N \to 0$$

with P projective and $N \in \mathcal{PGF}_{\mathcal{E}}(R)$.

We use $\mathcal{PGF}(R)$ and $\mathcal{GP}(R)$ to denote the classes consisting of all projectively coresolved Gorenstein flat right R-modules and Gorenstein projective modules, respectively.

Lemma 4.3. $\mathcal{PGF}_{\mathcal{E}}(R) \subseteq \mathcal{PGF}(R) \subseteq \mathcal{GP}(R)$,

Proof. The first inclusion is clear, and the second one follows from [26, Theorem 4.4]. □

Definition 4.4. Let $\mathcal{C}$ be a class of right R-modules and σ an ordinal. A right R-module M is called *$\mathcal{C}$-filtered* if there exists a well-ordered chain $(M_\alpha)_{\alpha<\sigma}$:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots ,$$

such that $M = \varinjlim_{\alpha<\sigma} M_i = \cup M_i$ and $M_{\alpha+1}/M_\alpha \in \mathcal{C}$ for any $\alpha < \sigma$. The class $\mathcal{C}$ is said to be *closed under transfinite extensions* if any $\mathcal{C}$-filtered module is in $\mathcal{C}$.

By the adjoint isomorphism theorem, we have

$$(- \otimes_R E)^+ \cong \mathrm{Hom}_R(-, E^+)$$

for any $E \in \mathcal{E}$, which yields that a sequence of right R-modules is $(- \otimes_R \mathcal{E})$-exact if and only if it is $\mathrm{Hom}_R(-, \mathcal{E}^+)$-exact, where $\mathcal{E}^+ = \{E^+ \mid E \in \mathcal{E}\}$.

Theorem 4.5. *It holds that*

(1) *The class* $\mathcal{PGF}_{\mathcal{E}}(R)$ *is closed under extensions and kernels of epimorphisms.*

(2) *The class* $\mathcal{PGF}_{\mathcal{E}}(R)$ *is projectively resolving.*

(3) *The class* $\mathcal{PGF}_{\mathcal{E}}(R)$ *is closed under transfinite extensions.*

Proof. (1) Let

$$0 \to A \to B \to C \to 0 \tag{4.1}$$

be an exact sequence of right R-modules with $C \in \mathcal{PGF}_{\mathcal{E}}(R)$. By Proposition 4.2, we have $\mathrm{Tor}^R_{\geq 1}(C, E) = 0$ for any $E \in \mathcal{E}$, and so the sequence (4.1) is $(- \otimes_R \mathcal{E})$-exact (equivalently, $\mathrm{Hom}_R(-, \mathcal{E}^+)$-exact). Also from Proposition 4.2, we get a $(- \otimes_R \mathcal{E})$-exact (equivalently, $\mathrm{Hom}_R(-, \mathcal{E}^+)$-exact) exact sequence

$$\cdots \to Q_1 \to Q_0 \to Q^0 \to Q^1 \to \cdots \tag{4.2}$$

with all Q_i and Q^i projective, such that $C \cong \mathrm{Im}(Q_0 \to Q^0)$.

Suppose $A \in \mathcal{PGF}_{\mathcal{E}}(R)$. By Proposition 4.2, for any $E \in \mathcal{E}$, we have $\mathrm{Tor}^R_{\geq 1}(A, E) = 0$ and there exist a $(- \otimes_R \mathcal{E})$-exact exact sequence

$$0 \to A \to G^0 \to G^1 \to \cdots \to G^i \to \cdots \tag{4.3}$$

with all G^i projective. Thus

$$\mathrm{Tor}^R_{\geq 1}(B, E) = 0$$

for any $E \in \mathcal{E}$. Since $C \in \mathcal{GP}(R)$ by Lemma 4.3, the sequence (4.1) is $\mathrm{Hom}_R(-, \mathcal{P}(R))$-exact. By [23, Lemma 3.1(2)], we get from (4.2) and (4.3) the following exact sequence:

$$0 \to B \to G^0 \oplus Q^0 \to G^1 \oplus Q^1 \to \cdots \to G^i \oplus Q^i \to \cdots .$$

It is easy to see that this sequence is $(- \otimes_R \mathcal{E})$-exact. Thus $B \in \mathcal{PGF}_{\mathcal{E}}(R)$ and the class $\mathcal{PGF}_{\mathcal{E}}(R)$ is closed under extensions.

Now suppose $B \in \mathcal{PGF}_{\mathcal{E}}(R)$. It follows from Proposition 4.2 that there exists a $(- \otimes_R \mathcal{E})$-exact (equivalently, $\mathrm{Hom}_R(-, \mathcal{E}^+)$-exact) exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots \tag{4.4}$$

with all P_i and P^i projective, such that $B \cong \mathrm{Im}(P_0 \to P^0)$. Then by [23, Theorem 3.2(1)], we get the following two exact sequences:

$$\cdots \to Q_{i+1} \oplus P_i \to \cdots \to Q_2 \oplus P_1 \to G \to A \to 0, \tag{4.5}$$

$$0 \to G \to Q_1 \oplus P_0 \to Q_0 \to 0. \tag{4.6}$$

By (4.5), we have that G is projective. Since both (4.2) and (4.4) are $\mathrm{Hom}_R(-, \mathcal{E}^+)$-exact, it follows from [23, Theorem 3.2(2)] that (4.5) is $\mathrm{Hom}_R(-, \mathcal{E}^+)$-exact (equivalently, $(- \otimes_R \mathcal{E})$-exact), and thus

$$\mathrm{Tor}^R_{\geq 1}(A, E) = 0 \tag{4.7}$$

for any $E \in \mathcal{E}$.

Since the image of each homomorphism in (4.4) is in $\mathcal{PGF}_{\mathcal{E}}(R)$, all these images are in $\mathcal{GP}(R)$ by Lemma 4.3, and thus (4.4) is $\mathrm{Hom}_R(-, P(R))$-exact. Then by [23, Theorem 3.8(1)], we get the following exact sequence:

$$0 \to A \to P^0 \to P^1 \oplus Q^0 \to P^1 \oplus Q^1 \to \cdots \to P^{i+1} \oplus Q^i \to \cdots . \tag{4.8}$$

Since all of (4.1), (4.2) and (4.4) are $\mathrm{Hom}_R(-,\mathcal{E}^+)$-exact, it follows from [23, Theorem 3.8(3)] that (4.8) is $\mathrm{Hom}_R(-,\mathcal{E}^+)$-exact (equivalently, $(-\otimes_R\mathcal{E})$-exact). In view of (4.7) and (4.8), we have $B\in\mathcal{PGF}_{\mathcal{E}}(R)$ by Proposition 4.2, and thus the class $\mathcal{PGF}_{\mathcal{E}}(R)$ is closed under kernels of epimorphisms.

(2) It is trivial that $\mathcal{P}(R)\subseteq\mathcal{PGF}_{\mathcal{E}}(R)$, thus the assertion follows from (1).

(3) Let

$$0=A_0\subseteq A_1\subseteq A_2\subseteq\cdots$$

be a well-ordered chain with $A_{\alpha+1}/A_\alpha\in\mathcal{PGF}_{\mathcal{E}}(R)$ for any $0\leq\alpha\leq\sigma$, where σ is an ordinal. Putting $A':=\cup_{\alpha<\sigma}A_\alpha$, we will use the transfinite induction on σ to prove $A'\in\mathcal{PGF}_{\mathcal{E}}(R)$. For the successor case, since

$$0\to A_\alpha\to A_{\alpha+1}\to A_{\alpha+1}/A_\alpha\to 0$$

is exact with $A_\alpha, A_{\alpha+1}/A_\alpha\in\mathcal{PGF}_{\mathcal{E}}(R)$, we have $A_{\alpha+1}\in\mathcal{PGF}_{\mathcal{E}}(R)$ by (1). For the limit case, suppose that β is a limit ordinal. By the construction of projective coresolution in the proof of (1), let

$$\mathbb{S}_\alpha: 0\to A_\alpha\to P_\alpha^0\to P_\alpha^1\to\cdots$$

be a $(-\otimes_R\mathcal{E})$-exact (equivalently, $\mathrm{Hom}_R(-,\mathcal{E}^+)$)-exact projective coresolution of A_α such that the following diagram

$$\begin{array}{ccccccccccccc}
\mathbb{S}_\alpha: & & 0 & \longrightarrow & A_\alpha & \longrightarrow & P_\alpha^0 & \longrightarrow & P_\alpha^1 & \longrightarrow\cdots\longrightarrow & P_\alpha^n & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle f^0_{\alpha+1,\alpha}} & & \downarrow{\scriptstyle f^1_{\alpha+1,\alpha}} & & \downarrow{\scriptstyle f^n_{\alpha+1,\alpha}} & & \\
\mathbb{S}_{\alpha+1}: & & 0 & \longrightarrow & A_{\alpha+1} & \longrightarrow & P_{\alpha+1}^0 & \longrightarrow & P_{\alpha+1}^1 & \longrightarrow\cdots\longrightarrow & P_{\alpha+1}^n & \longrightarrow & \cdots .
\end{array}$$

commutes with each $f^n_{\alpha+1,\alpha}$ a split monomorphism. It follows that $A_\beta=\varinjlim_{\alpha<\beta}A_\alpha$ admits a $\mathrm{Hom}_R(-,\mathcal{E}^+)$-exact (equivalently, $(-\otimes_R\mathcal{E})$-exact) projective coresolution:

$$0\to A_\beta\to\varinjlim_{\alpha<\beta}P_\alpha^0\to\varinjlim_{\alpha<\beta}P_\alpha^1\to\cdots\to\varinjlim_{\alpha<\beta}P_\alpha^n\to\cdots .$$

Since $\varinjlim$ commutes with the tensor product functor, we obtain $A'\in\mathcal{PGF}_{\mathcal{E}}(R)$. □

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