# TWO FILTRATION RESULTS FOR MODULES WITH APPLICATIONS TO THE AUSLANDER CONDITION

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Abstract. Under some strong cograde conditions, we obtain two different filtrations of modules in terms of the properties of cotransposes of modules with respect to a semidualizing bimodule. Then we apply these two filtrations of modules to investigate the relationship between artin algebras satisfying the Auslander condition and Gorenstein algebras, which is related to a conjecture of Auslander and Reiten.

## 1. Introduction

Let $R$ be a left and right noetherian ring and $M$ a finitely generated left $R$-module. In [Au], Auslander devised the so-called *Auslander sequence* as follows.

$$0 \to \mathrm{Ext}^1_{R^{op}}(\mathrm{Tr}\, M, R) \to M \to \mathrm{Hom}_{R^{op}}(\mathrm{Hom}_R(M, R), R) \to \mathrm{Ext}^2_{R^{op}}(\mathrm{Tr}\, M, R) \to 0,$$

where $\mathrm{Tr}\, M$ denotes the transpose of $M$. This sequence has already proved to be very valuable for the homological study of noetherian rings. Huang in [H1, Theorem 2.3] established a semidualizing version of this sequence. Under the Auslander condition, Hoshino and Nishida generalized in [HN, Theorem 2.2] the Auslander sequence by using a certain filtration of modules. Iyama and Jasso extended in [IJ, Proposition 2.7] this sequence to a dualizing $R$-variety. Recently, over arbitrary associative rings $R$ and $S$, we introduced in [TH1] the cotranspose $\mathrm{cTr}_\omega\, M$ of $M$ with respect to a semidualizing bimodule ${}_R\omega_S$, and used it as a tool to provide the *dual Auslander sequence* as follows.

$$0 \to \mathrm{Tor}^S_2(\omega, \mathrm{cTr}_\omega\, M) \to \omega \otimes_S \mathrm{Hom}_R(\omega, M) \to M \to \mathrm{Tor}^S_1(\omega, \mathrm{cTr}_\omega\, M) \to 0.$$

In analogy with the philosophy of Hoshino and Nishida in the filtration of modules, one of our aims in this paper is to look for a special filtration of modules to generalize the dual Auslander sequence.

On the other hand, the grade condition of modules is bound up with some interesting homological properties; see for example, [AB, AR1, DR, I, HI] and so on. In particular, Auslander and Bridger showed in [AB, Theorem 2.37] that if $R$ satisfies some grade condition, then for any finitely generated left $R$-module $M$ there exists a spherical filtration

$$M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M \oplus P$$

with $P$ a finitely generated projective left $R$-module. Furthermore, Huang presented in [H2] a different filtration result for modules over right quasi $k$-Gorenstein rings.

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Along this direction, the other aim of this paper is to see how the cograde condition induces some filtrations of modules.

The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let $R$ and $S$ be rings and ${}_R\omega_S$ a semidualizing bimodule. In Section 3, we particularly describe a certain filtration of submodules of a left noetherian $R$-module $M$ with finite Ext-cograde with respect to $\omega$ in case that $\omega$ satisfies the $\infty$-cograde condition (Theorem 3.12). It is a dual version of [HN, Theorem 2.2]. In Section 4, we prove that if $\omega$ satisfies the $n$-cograde condition with $n \geqslant 1$, then for any left $R$-module $M$, there exists an injective left $R$-module $I$ and a chain of monomorphisms

$$M_n \rightarrowtail M_{n-1} \rightarrowtail \cdots \rightarrowtail M_1 \rightarrowtail M_0 = M \oplus I$$

satisfying some interesting homological properties (Theorem 4.5).

Recall that an artin algebra $R$ is said to satisfy the *Auslander condition* if the projective dimension of the $i$-th term in a minimal injective resolution of ${}_RR$ is at most $i-1$ for any $i \geqslant 1$. Auslander and Reiten conjectured in [AR1] that an artin algebra $R$ satisfying the Auslander condition is Gorenstein (that is, the left and right self-injective dimensions of $R$ are finite). In Section 5, we apply the two filtrations of modules obtained in Sections 3 and 4 to give some necessary (and sufficient) conditions for an artin algebra satisfying the Auslander condition being Gorenstein (Theorems 5.2 and 5.4). Finally, we introduce the notion of dual Evans-Griffith presentations of modules. We prove that if $\omega$ satisfies the $n$-cograde condition with $n \geqslant 1$, then for any $0 \leqslant i \leqslant n-1$, each $i$-Bass-cosyzygy module admits a dual Evans-Griffith presentation (Proposition 5.6).

## 2. **Preliminaries**

Throughout this paper, $R$ and $S$ are fixed associative rings with unites. We use $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) to denote the class of left $R$-modules (resp. finitely generated left $R$-modules). Let $M \in \operatorname{Mod} R$. We use $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the projective and injective dimensions of $M$ respectively, and use $\operatorname{Add}_R M$ to denote the subclass of $\operatorname{Mod} R$ consisting of all direct summands of direct sums of copies of $M$.

**Definition 2.1.** ([HW]). An ($R$-$S$)-bimodule ${}_R\omega_S$ is called *semidualizing* if the following conditions are satisfied.

(1) ${}_R\omega$ admits a degreewise finite $R$-projective resolution.
(2) $\omega_S$ admits a degreewise finite $S$-projective resolution.
(3) The homothety map ${}_RR_R \stackrel{{}_R\gamma}{\to} \operatorname{Hom}_{S^{op}}(\omega,\omega)$ is an isomorphism.
(4) The homothety map ${}_SS_S \stackrel{\gamma_S}{\to} \operatorname{Hom}_R(\omega,\omega)$ is an isomorphism.
(5) $\operatorname{Ext}_R^{\geqslant 1}(\omega,\omega) = 0$.
(6) $\operatorname{Ext}_{S^{op}}^{\geqslant 1}(\omega,\omega) = 0$.

From now on, ${}_R\omega_S$ denotes a semidualizing bimodule. We write $(-)_* := \operatorname{Hom}(\omega,-)$. Following [HW], set

$$\mathcal{P}_\omega(R) := \{\omega \otimes_S P \mid P \text{ is projective in } \operatorname{Mod} S\},$$

$$\mathcal{I}_\omega(S) := \{I_* \mid I \text{ is injective in } \operatorname{Mod} R\}.$$

The modules in $\mathcal{P}_\omega(R)$ and $\mathcal{I}_\omega(S)$ are called *$\omega$-projective* and *$\omega$-injective* respectively. We use $\mathcal{I}(R)$ to denote the subclass of $\operatorname{Mod} R$ consisting of injective modules, and use $\mathcal{P}(S)$ to denote the subclasses of $\operatorname{Mod} S$ consisting of projective modules. Let $M \in \operatorname{Mod} R$ and $N \in \operatorname{Mod} S$. Then we have the following two canonical valuation homomorphisms:

$$\theta_M : \omega \otimes_S M_* \to M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_*$; and

$$\mu_N : N \to (\omega \otimes_S N)_*$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$.

**Definition 2.2.** ([HW]). The *Bass class* $\mathcal{B}_\omega(R)$ with respect to $\omega$ consists of all left $R$-modules $M$ satisfying the following conditions.

(1) $\operatorname{Ext}_R^{\geqslant 1}(\omega, M) = 0$.
(2) $\operatorname{Tor}^S_{\geqslant 1}(\omega, M_*) = 0$.
(3) $\theta_M$ is an isomorphism in $\operatorname{Mod} R$.

The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to $\omega$ consists of all left $S$-modules $N$ satisfying the following conditions.

(1) $\operatorname{Tor}^S_{i\geqslant 1}(\omega, N) = 0$.
(2) $\operatorname{Ext}_R^{\geqslant 1}(\omega, \omega \otimes_S N) = 0$.
(3) $\mu_N$ is an isomorphism in $\operatorname{Mod} S$.

Note that $\mathcal{I}_\omega(S) \cup \mathcal{P}(S) \subseteq \mathcal{A}_\omega(S)$ and $\mathcal{P}_\omega(R) \cup \mathcal{I}(R) \subseteq \mathcal{B}_\omega(R)$ ([HW, Lemma 4.1 and Corollary 6.1]).

Let $M \in \operatorname{Mod} R$. We use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} I^i(M) \xrightarrow{f^i} \cdots \tag{2.1}$$

to denote a minimal injective resolution of $M$. For any $n \geqslant 1$, $\operatorname{co}\Omega^n(M) := \operatorname{Im} f^{n-1}$ is called the *$n$-th cosyzygy* of $M$, and in particular, $\operatorname{co}\Omega^0(M) = M$.

**Definition 2.3.** ([TH1]). Let $M \in \operatorname{Mod} R$ and $n \geqslant 1$.

(1) $\operatorname{cTr}_\omega M := \operatorname{Coker} {f^0}_*$ is called the *cotranspose* of $M$ with respect to ${}_R\omega_S$, where $f^0$ is as in (2.1).
(2) $M$ is called *$n$-$\omega$-cotorsionfree* if $\operatorname{Tor}^S_{1\leqslant i\leqslant n}(\omega, \operatorname{cTr}_\omega M) = 0$. In particular, every module in $\operatorname{Mod} R$ is 0-$\omega$-cotorsionfree.

We use $\mathrm{c}\mathcal{T}^n_\omega(R)$ to denote the subcategory of $\operatorname{Mod} S$ consisting of $n$-$\omega$-cotorsionfree modules.

Dually, let $N \in \operatorname{Mod} S$. We use

$$\cdots \xrightarrow{f_i} F_i(N) \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} F_1(N) \xrightarrow{f_0} F_0(N) \xrightarrow{f_{-1}} N \to 0 \tag{2.2}$$

to denote a minimal flat resolution of $N$ in $\operatorname{Mod} S$, where each $F_i(N) \twoheadrightarrow \operatorname{Coker} f_i$ is the flat cover of $\operatorname{Coker} f_i$. The existence of such a resolution is guaranteed by the fact that any module has a flat cover (see [BBE]). Note that $(\omega \otimes_S -, \operatorname{Hom}_R(\omega, -))$ is an adjoint pair. So, it is reasonable to introduce the adjoint counterparts of the notions in Definition 2.3, which were given in [TH3] as follows.

**Definition 2.4.** ([TH3]). Let $N \in \operatorname{Mod} S$ and $n \geqslant 1$.

(1) $\operatorname{acTr}_\omega N := \operatorname{Ker}(1_\omega \otimes f_0)$ is called the *adjoint cotranspose* of $N$ with respect to ${}_R\omega_S$, where $f_0$ is as in (2.2).

(2) $N$ is called *adjoint $n$-$\omega$-cotorsionfree* if $\operatorname{Ext}_R^{1\leqslant i\leqslant n}(\omega, \operatorname{acTr}_\omega N) = 0$; and $N$ is called *adjoint $\infty$-$\omega$-cotorsionfree* if it is adjoint $n$-$\omega$-cotorsionfree for all $n$. In particular, every module in $\operatorname{Mod} S$ is adjoint 0-$\omega$-cotorsionfree.

We use $\operatorname{ac}\mathcal{T}(S)$ to denote the subcategory of $\operatorname{Mod} S$ consisting of adjoint $\infty$-$\omega$-cotorsionfree modules.

**Definition 2.5.** ([TH2, Definition 6.2]) Let $M \in \operatorname{Mod} R$, $N \in \operatorname{Mod} S$ and $n \geqslant 0$.

(1) The Ext-*cograde* of $M$ with respect to $\omega$ is defined as $\operatorname{E-cograde}_\omega M := \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(\omega, M) \neq 0\}$; and the *strong* Ext-*cograde* of $M$ with respect to $\omega$, denoted by $\operatorname{s.E-cograde}_\omega M$, is said to be at least $n$ if $\operatorname{E-cograde}_\omega X \geqslant n$ for any quotient module $X$ of $M$.
(2) The Tor-*cograde* of $N$ with respect to $\omega$ is defined as $\operatorname{T-cograde}_\omega N := \inf\{i \geqslant 0 \mid \operatorname{Tor}_i^S(\omega, N) \neq 0\}$; and the *strong* Tor-*cograde* of $N$ with respect to $\omega$, denoted by $\operatorname{s.T-cograde}_\omega N$, is said to be at least $n$ if $\operatorname{T-cograde}_\omega Y \geqslant n$ for any submodule $Y$ of $N$.

Let $M \in \operatorname{Mod} R$. An exact sequence (of finite or infinite length):

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ is called an *$\mathcal{X}$-resolution* of $M$ if all $X_i$ are in $\mathcal{X}$. The *$\mathcal{X}$-projective dimension* $\mathcal{X}\text{-}\operatorname{pd}_R M$ of $M$ is defined as $\inf\{n \mid$ there exists an $\mathcal{X}$-resolution

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

of $M$ in $\operatorname{Mod} R\}$. We always take $\mathcal{X}\text{-}\operatorname{pd}_R 0 = -1$. Dually, the notions of an *$\mathcal{X}$-coresolution* and the *$\mathcal{X}$-injective dimension* $\mathcal{X}\text{-}\operatorname{id}_R M$ of $M$ are defined.

## 3. A filtration of modules with finite Ext-cograde

In this section, given $M \in \operatorname{Mod} R$ and let $i, j$ be integers such that $i, j \geqslant 1$ or $i = j = 0$. Set

$$M_i^j := \operatorname{Tor}_i^S(\omega, \operatorname{cTr}_\omega \operatorname{co}\Omega^j(M)),$$

$$M_0^{-1} := M,\ M_1^{-1} := 0,\ M_i^0 := \operatorname{Tor}_i^S(\omega, \operatorname{cTr}_\omega M).$$

The following result is a generalization of the dual Auslander formula demonstrated in Section 1.

**Proposition 3.1.** *Let $i, j$ be integers such that $i, j \geqslant 1$ or $i = j = 0$ and let $M \in \operatorname{Mod} R$ with $\operatorname{Tor}_{i-1}^S(\omega, \operatorname{Ext}_R^j(\omega, M)) = 0$. Then there exists an exact sequence*

$$M_{i+1}^{j-1} \to M_{i+2}^j \to \operatorname{Tor}_i^S(\omega, \operatorname{Ext}_R^j(\omega, M)) \to M_i^{j-1} \to M_{i+1}^j \to 0.$$

*Proof.* The case for $i = j = 0$ has been proved in [TH1, Proposition 3.2].

Now suppose $i, j \geqslant 1$. Applying the functor $(-)_*$ to the minimal injective resolution of $M$ as in (2.1), we get a complex

$$0 \to M_* \to I^0(M)_* \to I^1(M)_* \to \cdots \xrightarrow{f^{j-1}_*} I^j(M)_* \xrightarrow{f^j_*} I^{j+1}(M)_* \to \cdots.$$

Consider the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} {f^j}_* & \longrightarrow & I^j(M)_* & \xrightarrow{{f^j}_*} & I^{j+1}(M)_* & \xrightarrow{\pi^{j+1}} & \operatorname{Coker} {f^j}_* & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi^j} & & & & & & \\
 & & & & \operatorname{Coker} {f^{j-1}}_* & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

with $\pi^j$ and $\pi^{j+1}$ natural epimorphisms. Because $\operatorname{Ker} \pi^j = \operatorname{Im} {f^{j-1}}_* \subseteq \operatorname{Ker} {f^j}_*$, there exists $\alpha^j : \operatorname{Coker} {f^{j-1}}_* \to I^{j+1}(M)_*$ such that ${f^j}_* = \alpha^j \cdot \pi^j$ by [AF, Theorem 3.6(1)]. So we get the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \operatorname{Ker} g & \xrightarrow{h} & \operatorname{Ker} \pi^j & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & \operatorname{Ker} {f^j}_* & \longrightarrow & I^j(M)_* & \xrightarrow{{f^j}_*} & I^{j+1}(M)_* & \xrightarrow{\pi^{j+1}} & \operatorname{Coker} {f^j}_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi^j} & & \| & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \operatorname{Ker} \alpha^j & \longrightarrow & \operatorname{Coker} {f^{j-1}}_* & \xrightarrow{\alpha^j} & I^{j+1}(M)_* & \longrightarrow & \operatorname{Coker} \alpha^j & \longrightarrow & 0 \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

with $f, g, h$ induced homomorphisms. By the diagram chasing, it is easy to see that $f$ is an isomorphism and

$$\operatorname{Coker} \alpha^j \cong \operatorname{Coker} {f^j}_* = \operatorname{cTr}_\omega \operatorname{co}\Omega^j(M).$$

Then $g$ is an epimorphism and $h$ is an isomorphism by the snake lemma. So

$$\operatorname{Ker} \alpha^j \cong \operatorname{Ker} {f^j}_* / \operatorname{Ker} g \cong \operatorname{Ker} {f^j}_* / \operatorname{Ker} \pi^j \cong \operatorname{Ker} {f^j}_* / \operatorname{Im} {f^{j-1}}_* \cong \operatorname{Ext}_R^j(\omega, M),$$

and hence we get the following exact sequence

$$0 \to \operatorname{Ext}_R^j(\omega, M) \to \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \xrightarrow{\alpha^j} I^{j+1}(M)_* \to \operatorname{cTr}_\omega \operatorname{co}\Omega^j(M) \to 0,$$

which induces two an exact sequences

$$0 \to \operatorname{Ext}_R^j(\omega, M) \to \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \to \operatorname{Im} \alpha^j \to 0, \tag{3.1}$$

and

$$0 \to \operatorname{Im} \alpha^j \to I^{j+1}(M)_* \to \operatorname{cTr}_\omega \operatorname{co}\Omega^j(M) \to 0. \tag{3.2}$$

Note that $\operatorname{Tor}_{\geqslant 1}^S(\omega, I^{j+1}(M)_*) = 0$ by [HW, Lemma 4.1]. So, applying the functor $\omega \otimes_S -$ to (3.2) yields

$$\operatorname{Tor}_i^S(\omega, \operatorname{Im} \alpha^j) \cong \operatorname{Tor}_{i+1}^S(\omega, \operatorname{cTr}_\omega \operatorname{co}\Omega^j(M)) = M_{i+1}^j$$

for any $i \geqslant 1$. Now applying the functor $\omega \otimes_S -$ to (3.1) yields the following exact sequence

$$M_{i+1}^{j-1} \to \operatorname{Tor}_{i+1}^S(\omega, \operatorname{Im}\alpha^j)(\cong M_{i+2}^j) \to \operatorname{Tor}_i^S(\omega, \operatorname{Ext}_R^j(\omega, M)) \to M_i^{j-1}$$
$$\to \operatorname{Tor}_i^S(\omega, \operatorname{Im}\alpha^j)(\cong M_{i+1}^j) \to \operatorname{Tor}_{i-1}^S(\omega, \operatorname{Ext}_R^j(\omega, M)). \tag{3.3}$$

Because $\operatorname{Tor}_{i-1}^S(\omega, \operatorname{Ext}_R^j(\omega, M)) = 0$ by assumption, the assertion follows. □

The following proposition is useful in the sequel.

**Proposition 3.2.** *Let* $M \in \operatorname{Mod} R$ *with* $\text{E-cograde}_\omega M = n \geqslant 1$. *Then we have* $\mathcal{I}_\omega(S)\text{-pd}_S \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \leqslant j$ *for any* $1 \leqslant j \leqslant n$ *and* $M \cong M_n^{n-1}$.

*Proof.* Since $\text{E-cograde}_\omega M = n$ by assumption, we get the following exact sequence

$$0 \to I^0(M)_* \to I^1(M)_* \to \cdots \to I^j(M)_* \to \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \to 0 \tag{3.4($j$)}$$

for any $1 \leqslant j \leqslant n$. It implies that $\mathcal{I}_\omega(S)\text{-pd}_S \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \leqslant j$. Note that $\operatorname{Tor}_{\geqslant 1}^S(\omega, I^i(M)_*) = 0$ for any $i \geqslant 0$ by [HW, Lemma 4.1]. So, applying the functor $\omega \otimes_S -$ to (3.4($n$)) gives the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_n^{n-1} & \longrightarrow & \omega \otimes_S I^0(M)_* & \longrightarrow & \omega \otimes_S I^1(M)_* \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \theta_{I^0(M)}} & & \downarrow{\scriptstyle \theta_{I^1(M)}} \\
0 & \longrightarrow & M & \longrightarrow & I^0(M) & \longrightarrow & I^1(M).
\end{array}$$

Since $\theta_{I^0(M)}$ and $\theta_{I^1(M)}$ are isomorphisms, the induced map $f$ is also an isomorphism and $M \cong M_n^{n-1}$. □

The following result establishes a relation between the strong Ext-cograde and the strong Tor-cograde of modules.

**Lemma 3.3.** ([TH2, Theorem 6.9]) *For any* $n \geqslant 1$, *the following statements are equivalent.*

(1) $\text{s.E-cograde}_\omega \operatorname{Tor}_i^S(\omega, N) \geqslant i$ *for any* $N \in \operatorname{Mod} S$ *and* $1 \leqslant i \leqslant n$.
(2) $\text{s.T-cograde}_\omega \operatorname{Ext}_R^i(\omega, M) \geqslant i$ *for any* $M \in \operatorname{Mod} R$ *and* $1 \leqslant i \leqslant n$.

Based on this lemma, we introduce the following

**Definition 3.4.** For an integer $n \geqslant 1$, we say that $\omega$ satisfies the *$n$-cograde condition* if one of the equivalent conditions in Lemma 3.3 is satisfied; and $\omega$ satisfies the *$\infty$-cograde condition* if it satisfies the $n$-cograde condition for any $n \geqslant 1$.

Let $R$ be an artin algebra and $D$ the usual duality between $\operatorname{mod} R$ and $\operatorname{mod} R^{op}$. Then $D(R)$ is a typical semidualizing $(R, R)$-bimodule. Recall from [FGR] that $R$ is said to satisfy the *Auslander condition* if $\operatorname{pd}_R I^i(R) \leqslant i$ for any $i \geqslant 0$; equivalently, $\operatorname{id}_{R^{op}} \operatorname{Hom}_R(P_i(R), D(R)) \leqslant i$ for any $i \geqslant 0$, where $P_i(R)$ is the $(i+1)$-st term in the minimal projective resolution of ${}_RR$. Note that an Artin algebra $R$ satisfies the Auslander condition if and only if $DR$ satisfies the $\infty$-cograde condition by [TH4, Proposition 7.7].

From Propositions 3.1 and 3.2, we get the following two corollaries.

**Corollary 3.5.** *Assume that* $\omega$ *satisfies the* $(n+1)$*-cograde condition with* $n \geqslant 0$. *If* $M \in \operatorname{Mod} R$ *with* $\text{E-cograde}_\omega M = n$, *then* $\text{T-cograde}_\omega \operatorname{Ext}_R^n(\omega, M) = n$.

*Proof.* If $n=0$, then $M_* \neq 0$. It follows from [TH2, Lemma 6.1(1)] that $\omega \otimes_S M_* \neq 0$ and $\text{T-cograde}_\omega M_* = 0$.

Now suppose $n \geqslant 1$. Then by Proposition 3.1, we have an exact sequence

$$\operatorname{Tor}_n^S(\omega, \operatorname{Ext}_R^n(\omega, M)) \to M_n^{n-1} \to M_{n+1}^n \to 0.$$

By Lemma 3.3, we have $\text{T-cograde}_\omega \operatorname{Ext}_R^n(\omega, M) \geqslant n$. If $\text{T-cograde}_\omega \operatorname{Ext}_R^n(\omega, M) > n$, then the above exact sequence implies $M_n^{n-1} \cong M_{n+1}^n$. So by Proposition 3.2 we have

$$M \cong M_{n+1}^n = \operatorname{Tor}_{n+1}^S(\omega, \operatorname{cTr}_\omega \operatorname{co}\Omega^n(M)).$$

Then it follows from Lemma 3.3 that $\text{E-cograde}_\omega M \geqslant n+1$, which contradicts the assumption $\text{E-cograde}_\omega M = n$. □

**Corollary 3.6.** *Let $M \in \operatorname{Mod} R$ with $\text{E-cograde}_\omega M = n \geqslant 1$. Then we have $\operatorname{Tor}_i^S(\omega, \operatorname{Ext}_R^n(\omega, M)) \cong M_{i+2}^n$ for any $i \geqslant n+1$.*

*Proof.* By the proof of Proposition 3.1, we have an exact sequence

$$M_{i+1}^{n-1} \to M_{i+2}^n \to \operatorname{Tor}_i^S(\omega, \operatorname{Ext}_R^n(\omega, M)) \to M_i^{n-1}.$$

Because $i \geqslant n+1$, it follows from Proposition 3.2 that $M_{i+1}^{n-1} = M_i^{n-1} = 0$ and the assertion follows. □

Applying Corollary 3.5, we get the following lemma which shows how the Ext-cograde and the Tor-cograde of modules behave in short exact sequences. Because the argument is standard, we omit it.

**Lemma 3.7.** *Assume that $\omega$ satisfies the $(n+1)$-cograde condition with $n \geqslant 0$.*

(1) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

*be an exact sequence in $\operatorname{Mod} R$ with $n_i = \text{E-cograde}_\omega M_i$ for $i = 1, 2, 3$ and $\max\{n_1, n_2, n_3\} \leqslant n$. Then $n_2 = \min\{n_1, n_3\}$.*

(2) *Let*

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

*be an exact sequence in $\operatorname{Mod} S$ with $n_i = \text{T-cograde}_\omega N_i$ for $i = 1, 2, 3$ and $\max\{n_1, n_2, n_3\} \leqslant n$. Then $n_2 = \min\{n_1, n_3\}$.*

We say that a module $M \in \operatorname{Mod} R$ is *pure of* Ext-*cograde* $k$ if $\text{E-cograde}_\omega M = \text{E-cograde}_\omega M/M' = k$ for any proper $R$-submodule $M'$ of $M$; dually, a module $N \in \operatorname{Mod} S$ is *pure of* Tor-*cograde* $l$ if $\text{T-cograde}_\omega N = \text{T-cograde}_\omega N' = l$ for any non-zero $S$-submodule $N'$ of $N$.

**Example 3.8.** Let $R$ be a finite-dimensional algebra over an algebraically closed field given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3.$$

Then $\omega := I(1) \oplus I(2) \oplus I(3)$ is a semidualizing ($R$-$R$)-bimodule. Set $M := S(2)$. It is easy to see that $M_* = 0$. By [ASS, IV.2 Theorem 2.13] and [ARS, VII.1 Example], we have that $\operatorname{Ext}_R^1(I(1), M) \cong D\overline{\operatorname{Hom}}_R(M, M) \neq 0$ and $\text{E-cograde}_\omega M = 1$. Because $M$ is simple, it follows that $M$ is pure of Ext-cograde of 1 and $D(M)$ is pure of Tor-cograde of 1.

On the other hand, because $N := I(3)$ is a direct summand of $\omega$, it follows that $\text{E-cograde}_\omega N = 0$. Thus $\text{E-cograde}_\omega M \oplus N = 0$ and $M \oplus N$ is not pure of Ext-cograde of 0. Because $\operatorname{Tor}_i^R(D(M), \omega) \cong D(\operatorname{Ext}_R^i(\omega, M))$ and $\operatorname{Tor}_i^R(D(M \oplus$

$N),\omega) \cong D(\mathrm{Ext}_R^i(\omega, M\oplus N))$ for any $i \geqslant 0$, we have that T-cograde$_\omega$ $D(M\oplus N) = 0$ and $M \oplus N$ is not pure of Tor-cograde of 0.

**Proposition 3.9.** *Assume that $\omega$ satisfies the $\infty$-cograde condition.*

(1) *If $M \in \mathrm{Mod}\, R$ with* E-cograde$_\omega$ $M = k$ *such that* $\mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^i(\omega, M)) = 0$ *for any $i \geqslant k+1$, then $M$ is pure of* Ext-*cograde $k$.*

(2) *If $N \in \mathrm{Mod}\, S$ with* T-cograde$_\omega$ $N = l$ *such that* $\mathrm{Ext}_R^i(\omega, \mathrm{Tor}_i^S(\omega, N)) = 0$ *for any $i \geqslant l+1$, then $N$ is pure of* Tor-*cograde $l$.*

*Proof.* (1) Let $M'$ be a proper $R$-submodule of $M$ and E-cograde$_\omega$ $M/M' = t$. Then T-cograde$_\omega$ $\mathrm{Ext}_R^t(\omega, M/M') = t$ by Corollary 3.5.

We claim that $t \leqslant k$. If $t > k$, then by assumption, T-cograde$_\omega$ $\mathrm{Ext}_R^t(\omega, M) \geqslant t$ and $\mathrm{Tor}_t^S(\omega, \mathrm{Ext}_R^t(\omega, M)) = 0$. So we have T-cograde$_\omega$ $\mathrm{Ext}_R^t(\omega, M) \geqslant t+1$. Consider the following exact sequence

$$\mathrm{Ext}_R^t(\omega, M) \xrightarrow{f} \mathrm{Ext}_R^t(\omega, M/M') \xrightarrow{g} \mathrm{Ext}_R^{t+1}(\omega, M').$$

By Lemma 3.7(2), we have

$$\text{T-cograde}_\omega \operatorname{Im} f \geqslant \text{T-cograde}_\omega \mathrm{Ext}_R^t(\omega, M) \geqslant t+1,$$

$$\text{T-cograde}_\omega \operatorname{Im} g \geqslant \text{T-cograde}_\omega \mathrm{Ext}_R^{t+1}(\omega, M') \geqslant t+1.$$

Thus T-cograde$_\omega$ $\mathrm{Ext}_R^t(\omega, M/M') \geqslant t+1$, which is a contradiction. The claim follows. Then by Lemma 3.7(1), we have E-cograde$_\omega$ $M =$ E-cograde$_\omega$ $M/M'$.

(2) It is dual to (1). □

As a consequence, we get the follwoing

**Corollary 3.10.** *Assume that $\omega$ satisfies the $\infty$-cograde condition. Then we have*

(1) $\mathrm{Ext}_R^k(\omega, M)$ *is pure of* Tor-*cograde $k$ for any $M \in \mathrm{Mod}\, R$ with* E-cograde$_\omega$ $M = k$.

(2) $\mathrm{Tor}_l^S(\omega, N)$ *is pure of* Ext-*cograde $l$ for any $N \in \mathrm{Mod}\, S$ with* T-cograde$_\omega$ $N = l$.

*Proof.* (1) Let $M \in \mathrm{Mod}\, R$ with E-cograde$_\omega$ $M = k$. It follows from Corollary 3.5 that T-cograde$_\omega$ $\mathrm{Ext}_R^k(\omega, M) = k$.

We claim that $\mathrm{Ext}_R^i(\omega, \mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^k(\omega, M))) = 0$ for any $i \geqslant k+1$. If $k = 0$, then E-cograde$_\omega$ $\mathrm{Tor}_i^S(\omega, M_*) =$ E-cograde$_\omega$ $\mathrm{Tor}_{i+2}^S(\omega, \mathrm{cTr}_\omega M) \geqslant i+2$ for any $i \geqslant 1$. If $k \geqslant 1$, then E-cograde$_\omega$ $\mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^k(\omega, M)) =$ E-cograde$_\omega$ $M_{i+2}^k \geqslant i+2$ for any $i \geqslant k+1$ by Corollary 3.6. The claim follows. Thus $\mathrm{Ext}_R^k(\omega, M)$ is pure of Tor-cograde $k$ by Proposition 3.9(2).

(2) It is dual to (1). □

Recall that a sequence

$$\cdots \to M_1 \to M_2 \to M_3 \to \cdots$$

in $\mathrm{Mod}\, R$ is called $\mathrm{Hom}_R(\omega, -)$-*exact* if it is exact after applying the functor $\mathrm{Hom}_R(\omega, -)$.

**Lemma 3.11.** *Let $M \in \mathcal{B}_\omega(R)$, then $\mathrm{cTr}_\omega M \in \mathcal{A}_\omega(S)$.*

*Proof.* Let $M \in \mathcal{B}_\omega(R)$. Then by [TH1, Proposition 3.7 and Theorem 3.9], there exists a $\mathrm{Hom}_R(\omega, -)$-exact exact sequence

$$\cdots \to W_1 \to W_0 \to I^0(M) \to I^1(M) \to \cdots \tag{3.5}$$

in $\operatorname{Mod} R$ with all $W_i$ in $\operatorname{Add}_R \omega$ such that $M \cong \operatorname{Im}(W_0 \to I^0(M))$. Applying the functor $(-)_*$ to (3.5) yields an exact sequence

$$\cdots \to W_{1*} \to W_{0*} \to I^0(M)_* \to I^1(M)_* \to \cdots . \tag{3.6}$$

Applying the functor $\omega_S \otimes -$ to (3.6), it is easy to verify that it remains exact. This implies that $\operatorname{Tor}^S_{\geqslant 1}(\omega, \operatorname{cTr}_\omega M) = 0$ and $\operatorname{cTr}_\omega M \in \operatorname{ac}\mathcal{T}(S)$ by Corollary [TH3, Corollary 3.9]. It follows from [TH3, Theroem 3.11(1)] that $\operatorname{cTr}_\omega M \in \mathcal{A}_\omega(S)$. □

Now we are ready to present the main theorem in this section, which is useful in providing an information about noetherian modules with finite Ext-cograde.

**Theorem 3.12.** *Assume that $\omega$ satisfies the $\infty$-cograde condition. If $M$ is a noetherian left $R$-module with* $\text{E-cograde}_\omega M = k < \infty$, *then there exists a filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots \tag{3.7}$$

*of $R$-submodules of $M$ such that*

(1) *$M_1 = \cdots = M_k = 0$ and there exists the following exact sequence*

$$0 \to \operatorname{Tor}^S_{k+2}(\omega, \operatorname{cTr}_\omega \operatorname{co}\Omega^k(M)) \to \operatorname{Tor}^S_k(\omega, \operatorname{Ext}^k_R(\omega, M)) \to M/M_k \to M/M_{k+1} \to 0.$$

(2) *If $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) \neq 0$, then* $\text{E-cograde}_\omega M/M_i = i$, *$M_i \neq M_{i+1}$ and $M_{i+1}/M_i$ is pure of* Ext*-cograde $i$.*

(3) *If $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) = 0$, then $M_i = M_{i+1}$.*

(4) *If $\mathcal{B}_\omega(R)\text{-id}_R M = d < \infty$, then*

$$M = M_{d+1} \text{ and } M/M_d \cong \operatorname{Tor}^S_d(\omega, \operatorname{Ext}^d_R(\omega, M)).$$

(5) *If $\mathcal{B}_\omega(R)\text{-id}_R M = d < \infty$, then $\operatorname{fil}(M) \leqslant d - k + 1$, and the equality holds whenever* $\text{T-cograde}_\omega \operatorname{Ext}^i_R(\omega, M) = i$ *for any $k \leqslant i \leqslant d$, where $\operatorname{fil}(M)$ is the number of strict inclusions in (3.7).*

*Proof.* By Proposition 3.1, there exists a chain of epimorphisms

$$M_0^{-1}(= M) \to M_1^0 \to \cdots \to M_i^{i-1} \to \cdots .$$

Then we get a filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

of $R$-submodules of $M$ with $M/M_i = M_i^{i-1}$.

(1) The case for $k = 0$ is trivial. If $k \geqslant 1$, then it follows from Proposition 3.2 that $M \cong M_k^{k-1}$. Since there exists an exact sequence

$$0 \to M_k \to M \to M_k^{k-1} (\cong M) \to 0$$

and $M$ is noetherian, we get from [L, Proposition 1.14] that $M_k = 0$, and hence $M_1 = \cdots = M_k = 0$. Since $\mathcal{I}_\omega(S)\text{-pd}_R \operatorname{cTr}_\omega \operatorname{co}\Omega^{k-1}(M) \leqslant k$ by Proposition 3.2 again, we have $M_{k+1}^{k-1} = 0$ by the dimension shifting. Now we get the desired exact sequence from Proposition 3.1.

(2) If $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) \neq 0$, then $\text{T-cograde}_\omega \operatorname{Ext}^i_R(\omega, M) = i$ by assumption. It follows from Corollary 3.10(2) that $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M))$ is pure of Ext-cograde $i$. By Proposition 3.1 we have the following exact sequence

$$M_{i+2}^i \xrightarrow{f} \operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) \to M_i^{i-1} \to M_{i+1}^i \to 0. \tag{3.8}$$

If $M_i = M_{i+1}$, then $M_i^{i-1} = M_{i+1}^i$ and $f$ is an epimorphism. So $\text{E-cograde}_\omega M_{i+2}^i \leqslant i$ by Lemma 3.7(1), a contradiction. Thus $M_i \neq M_{i+1}$. Since $M_{i+1}/M_i \cong \operatorname{Coker} f$

is a quotient module of $\mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^i(\omega, M))$, we have that $M_{i+1}/M_i$ is pure of Ext-cograde $i$. Notice that E-cograde$_\omega\, M_{i+1}^i \geqslant i+1$ by assumption, so we have E-cograde$_\omega\, M/M_i = i$ by Lemma 3.7(1).

(3) It is induced directly from the exact sequence (3.8).

(4) If $\mathcal{B}_\omega(R)$-id$_R\, M = d$, then $\mathrm{co}\Omega^d(M) \in \mathcal{B}_\omega(R)$ by [TH2, Theorem 4.2]. It follows from Lemma 3.11 that $\mathrm{cTr}_\omega\, \mathrm{co}\Omega^d(M) \in \mathcal{A}_\omega(S)$ and $M_{d+1}^d = M_{d+2}^d = 0$. Thus we have that $M = M_{d+1}$ and $M/M_d \cong \mathrm{Tor}_d^S(\omega, \mathrm{Ext}_R^d(\omega, M))$ by the exact sequence (3.8).

(5) It is a consequence of the former assertions. □

## 4. **Another filtration of modules**

We begin with the following

**Definition 4.1.** Let $n \geqslant 1$. A module $M$ in $\mathrm{Mod}\, R$ is called *n-Bass-cosyzygy* if there exists an exact sequence

$$B^{-(n-1)} \to \cdots \to B^{-1} \to B^0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all $B^i$ in $\mathcal{B}_\omega(R)$.

We use $\mathrm{co}\Omega^n_{\mathcal{B}}(R)$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of $n$-Bass-cosyzygy modules.

**Lemma 4.2.** *Let $n \geqslant 1$. If* T-cograde$_\omega\, \mathrm{Ext}_R^i(\omega, M) \geqslant i-1$ *for any $M \in \mathrm{Mod}\, R$ and $1 \leqslant i \leqslant n$, then $\mathrm{co}\Omega^i_{\mathcal{B}}(R) = \mathrm{c}\mathcal{T}^i_\omega(R)$ for any $1 \leqslant i \leqslant n$.*

*Proof.* Because $\mathcal{P}_\omega(R) \subseteq \mathcal{B}_\omega(R)$ by [HW, Corollary 6.1], we have $\mathrm{c}\mathcal{T}^i_\omega(R) \subseteq \mathrm{co}\Omega^i_{\mathcal{B}}(R)$ by [TH1, Proposition 3.7].

Assume that T-cograde$_\omega\, \mathrm{Ext}_R^i(\omega, M) \geqslant i-1$ for any $M \in \mathrm{Mod}\, R$ and $1 \leqslant i \leqslant n$. In the following, we proceed by induction on $n$ to show that $\mathrm{co}\Omega^i_{\mathcal{B}}(R) \subseteq \mathrm{c}\mathcal{T}^i_\omega(R)$ for any $1 \leqslant i \leqslant n$. Let $M \in \mathrm{co}\Omega^1_{\mathcal{B}}(R)$. Then there exists an exact sequence $B^0 \xrightarrow{f^0} M \to 0$ in $\mathrm{Mod}\, R$ with $B^0 \in \mathcal{B}_\omega(R)$, and we get the following commutative diagram with the bottom row exact

$$\begin{array}{ccccc} \omega \otimes_S {B^0}_* & \xrightarrow{1_\omega \otimes {f^0}_*} & \omega \otimes_S M_* & & \\ \downarrow{\scriptstyle \theta_{B^0}} & & \downarrow{\scriptstyle \theta_M} & & \\ B^0 & \xrightarrow{f^0} & M & \longrightarrow & 0. \end{array}$$

Since $\theta_{B^0}$ is an isomorphism, we have that $\theta_M$ is an epimorphism and $M \in \mathrm{c}\mathcal{T}^1_\omega(R)$. The case for $n=1$ is proved.

Now let $M \in \mathrm{co}\Omega^n_{\mathcal{B}}(R)$ with $n \geqslant 2$. Then there exists an exact sequence

$$B^{-(n-1)} \xrightarrow{f^{n-1}} \cdots \to B^{-1} \xrightarrow{f^1} B^0 \xrightarrow{f^0} M \to 0 \tag{4.1}$$

in $\mathrm{Mod}\, R$ with all $B^i$ in $\mathcal{B}_\omega(R)$. By the induction hypothesis, we have $\mathrm{Im}\, f^1 \in \mathrm{c}\mathcal{T}^{n-1}_\omega(R)$. Applying the functor $(-)_*$ to (4.1) gives an exact sequence

$$0 \to (\mathrm{Im}\, f^1)_* \to {B^0}_* \xrightarrow{{f^0}_*} M_* \to \mathrm{Ext}_R^n(\omega, \mathrm{Ker}\, f^{n-1}) \to 0. \tag{4.2}$$

Set $N := \mathrm{Im}\, {f^0}_*$ and let ${f^0}_* := \alpha \cdot \pi$ be the natural epic-monic decompositions of ${f^0}_*$ with $\pi : {B^0}_* \twoheadrightarrow N$ and $\alpha : N \hookrightarrow M_*$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Tor}_1^S(\omega, N) & \longrightarrow & \omega \otimes_S (\operatorname{Im} f^1)_* & \longrightarrow & \omega \otimes_S {B^0}_* & \xrightarrow{1_\omega \otimes \pi} & \omega \otimes_S N & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \theta_{\operatorname{Im} f^1}} & & \downarrow{\scriptstyle \theta_{B^0}} & & \downarrow{\scriptstyle g} & & \\
 & & 0 & \longrightarrow & \operatorname{Im} f^1 & \longrightarrow & B^0 & \xrightarrow{f^0} & M & \longrightarrow & 0.
\end{array}$$

Diagram (4.3)

So we have $\theta_M \cdot (1_\omega \otimes \alpha) \cdot (1_\omega \otimes \pi) = \theta_M \cdot (1_\omega \otimes {f^0}_*) = f^0 \cdot \theta_{B^0} = g \cdot (1_\omega \otimes \pi)$. Because $1_\omega \otimes \pi$ is epic, we have $\theta_M \cdot (1_\omega \otimes \alpha) = g$ and the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\omega \otimes_S N & \xrightarrow{1_\omega \otimes \alpha} & \omega \otimes_S M_* & \longrightarrow & \omega \otimes_S \operatorname{Ext}_R^n(\omega, \operatorname{Ker} f^{n-1}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \theta_M} & & & & \\
M & = & M. & & & &
\end{array}$$

Diagram (4.4)

Since $\theta_{\operatorname{Im} f^1}$ is an epimorphism by the above argument, it follows from the snake lemma that $g$ is a monomorphism. Because $\omega \otimes_S \operatorname{Ext}_R^n(\omega, \operatorname{Ker} f^{n-1}) = 0$ by assumption, we have that $\theta_M$ is an isomorphism and $M \in \mathrm{c}\mathcal{T}_\omega^2(R)$ by the diagram (4.4). It means that the assertion holds true for $n = 2$. If $n \geqslant 3$, then the fact that $\operatorname{Im} f^1 \in \mathrm{c}\mathcal{T}_\omega^{n-1}(R)$ implies $\theta_{\operatorname{Im} f^1}$ is an isomorphism. So $\operatorname{Tor}_1^S(\omega, N) = 0$ by the diagram (4.3). In addition, we have $\operatorname{Tor}^S_{1 \leqslant i \leqslant n-3}(\omega, (\operatorname{Im} f^1)_*) = 0$ by [TH1, Corollary 3.4(3)]. Because $\text{T-cograde}_\omega \operatorname{Ext}_R^n(\omega, \operatorname{Ker} f^{n-1}) \geqslant n-1$ by assumption, applying the dimension shifting to (4.2) we obtain $\operatorname{Tor}^S_{1 \leqslant i \leqslant n-2}(\omega, M_*) = 0$. Therefore we conclude that $M \in \mathrm{c}\mathcal{T}_\omega^n(R)$ by [TH1, Corollary 3.4(3)] again. □

The following result shows how the strong Tor-cograde conditions on modules affect the extension closure of $\mathrm{c}\mathcal{T}_\omega^n(R)$. It is a dual version of [AR2, Theorem 1.1].

**Lemma 4.3.** *Let* $n \geqslant 1$ *and*

$$0 \to A \to B \to C \to 0 \tag{4.5}$$

*be an exact sequence in* $\operatorname{Mod} R$ *with* $A, C \in \mathrm{c}\mathcal{T}_\omega^n(R)$. *If* $\text{s.T-cograde}_\omega \operatorname{Ext}_R^1(\omega, A) \geqslant n$, *then* $B \in \mathrm{c}\mathcal{T}_\omega^n(R)$.

*Proof.* Applying the functor $(-)_*$ to the exact sequence (4.5) gives rise to the following exact sequence

$$0 \to A_* \to B_* \to C_* \to \operatorname{Ext}_R^1(\omega, A).$$

Set $L = \operatorname{Coker}(B_* \to C_*)$ and $K := \operatorname{Im}(B_* \to C_*)$.

Let $n = 1$. Since $\text{s.T-cograde}_\omega \operatorname{Ext}_R^1(\omega, A) \geq 1$ and $L \subseteq \operatorname{Ext}_R^1(\omega, A)$, we have $\omega \otimes_S L = 0$. It yields an epimorphism $\omega \otimes_S B_* \to \omega \otimes_S C_*$ and the following commutative diagram with the bottom row exact

$$\begin{array}{ccccccccc}
 & & \omega\otimes_S A_* & \longrightarrow & \omega\otimes_S B_* & \longrightarrow & \omega\otimes_S C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_C} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

Because $A, C \in \mathrm{c}\mathcal{T}_\omega^1(R)$ by assumption, we have that $\theta_A$ and $\theta_C$ are epimorphisms. Then by the diagram chasing, we have that $\theta_B$ is also an epimorphism and $B \in \mathrm{c}\mathcal{T}_\omega^1(R)$.

Let $n = 2$. Since s.T-cograde$_\omega \operatorname{Ext}_R^1(\omega, A) \geqslant 2$ and $L \subseteq \operatorname{Ext}_R^1(\omega, A)$, we obtain an isomorphism $\omega \otimes_S K \to \omega \otimes_S C_*$. It yields the following exact sequence

$$\omega \otimes_S A_* \to \omega \otimes_S B_* \to \omega \otimes_S C_* \to 0$$

and the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \omega\otimes_S A_* & \longrightarrow & \omega\otimes_S B_* & \longrightarrow & \omega\otimes_S C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_C} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

Because $A, C \in \mathrm{c}\mathcal{T}_\omega^2(R)$ by assumption, we have that $\theta_A$ and $\theta_C$ are isomorphisms. So $\theta_B$ is also an isomorphism and $B \in \mathrm{c}\mathcal{T}_\omega^2(R)$.

Let $n \geqslant 3$. Since s.T-cograde$_\omega \operatorname{Ext}_R^1(\omega, A) \geqslant n \geqslant 3$, we have $B \in \mathrm{c}\mathcal{T}_\omega^2(R)$ by the above argument. Consider the following exact sequence

$$0 \to K \to C_* \to L \to 0.$$

Since $L \subseteq \operatorname{Ext}_R^1(\omega, A)$, we have $\operatorname{Tor}^S_{0\leqslant i\leqslant n-1}(\omega, L) = 0$. Then we have $\operatorname{Tor}_i^S(\omega, K) \cong \operatorname{Tor}_i^S(\omega, C_*)$ for any $0 \leqslant i \leqslant n-2$. Because $A, C \in \mathrm{c}\mathcal{T}_\omega^n(R)$ by assumption, we have $\operatorname{Tor}^S_{1\leqslant i\leqslant n-2}(\omega, A_*) = 0 = \operatorname{Tor}^S_{1\leqslant i\leqslant n-2}(\omega, C_*) = 0$ by [TH1, Corollory 3.4]. Now applying the functor $\omega \otimes_S -$ to the exact sequence

$$0 \to A_* \to B_* \to K \to 0$$

yields $\operatorname{Tor}^S_{1\leqslant i\leqslant n-2}(\omega, B_*) = 0$. Therefore $B \in \mathrm{c}\mathcal{T}_\omega^n(R)$ by [TH1, Corollory 3.4] again. □

The following proposition is crucial in proving the main result in this section.

**Proposition 4.4.** *Assume that $\omega$ satisfies the $n$-cograde condition with $n \geqslant 1$ and $M \in \mathrm{co}\Omega^i_{\mathcal{B}}(R)$ with $0 \leqslant i \leqslant n-1$. Then there exists a $\operatorname{Hom}_R(\omega, -)$-exact exact sequence*

$$0 \to A \to M \oplus I \to B \to 0$$

*in $\operatorname{Mod} R$ satisfying the following conditions.*

(1) *$A \in \mathrm{co}\Omega^{i+1}_{\mathcal{B}}(R)$, $I \in \mathcal{I}(R)$ and $B \cong \mathrm{co}\Omega^i(\operatorname{Tor}^S_{i+1}(\omega, \mathrm{cTr}_\omega M))$.*

(2) *$\mathcal{I}_\omega(S)$-$\mathrm{pd}_S B_* \leqslant i-1$.*

*Proof.* Let $i = 0$. Set $A := \operatorname{Im}\theta_M$, $I := 0$ and $B := \operatorname{Tor}_1^S(\omega, \mathrm{cTr}_\omega M)$. Then by [TH1, Proposition 3.2], we have an exact sequence

$$0 \to A \to M \to B \to 0. \tag{4.6}$$

Since $\theta_{\omega\otimes_S M_*}$ is an epimorphism by [TH2, Lemma 6.1], we have $\omega \otimes_S M_* \in \mathrm{c}\mathcal{T}_\omega^1(R)$. Note that $A$ is a quotient module of $\omega \otimes_S M_*$. So $A \in \mathrm{c}\mathcal{T}_\omega^1(R)$ by [TH1, Lemma 3.6], and hence $A \in \mathrm{co}\Omega^1_{\mathcal{B}}(R)$ by Lemma 4.2. On the other hand, since E-cograde$_\omega$ $B \geqslant 1$ by assumption, we have $B_* = 0$. So (4.6) is the desired exact sequence.

Let $i = 1$. Consider the exact sequence

$$0 \to \mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M) \to I \to B \to 0$$

in $\mathrm{Mod}\, R$ with $I = I^0(\mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M)) \in \mathcal{I}(R)$ and $B = \mathrm{co}\Omega^1(\mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M))$. Then by [TH1, Proposition 3.2], we have the following push-out diagram with the middle column splitting

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M) & \longrightarrow & I & \xrightarrow{f} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \omega \otimes_S M_* & \longrightarrow & M \oplus I & \xrightarrow{g} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Because E-cograde$_\omega(\mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M)) \geqslant 2$ by assumption, we have that $f_*$ is an isomorphism. So $B_*(\cong I_*) \in \mathcal{I}_\omega(S)$ and $g_*$ is an epimorphism. Now let

$$Q_1 \to Q_0 \to M_* \to 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with $Q_0, Q_1 \in \mathcal{P}(S)$. Then

$$\omega \otimes_S Q_1 \to \omega \otimes_S Q_0 \to \omega \otimes_S M_* \to 0$$

is exact in $\mathrm{Mod}\, R$ and $\omega \otimes_S M_* \in \mathrm{co}\Omega^2_{\mathcal{B}}(R)$. Thus the middle row in the above diagram is the desired exact sequence.

Now suppose $i \geqslant 2$. By Lemma 4.2, we have $M \in \mathrm{co}\Omega^i_{\mathcal{B}}(R) = \mathrm{c}\mathcal{T}^i_\omega(R)$. Then by [TH1, Proposition 3.7], there exists a $\mathrm{Hom}_R(\omega, -)$-exact exact sequence

$$0 \to N \to W_{i-1} \xrightarrow{f} W_{i-2} \to \cdots \to W_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all $W_j$ in $\mathrm{Add}_R\, \omega$ and $N = \mathrm{Ker}\, f$. Note that $\mathrm{Tor}_{i+1}^S(\omega, \mathrm{cTr}_\omega M) \cong \mathrm{Tor}_1^S(\omega, \mathrm{Coker}\, f_*)$ and $\mathrm{Tor}_{i+2}^S(\omega, \mathrm{cTr}_\omega M) \cong \mathrm{Tor}_2^S(\omega, \mathrm{Coker}\, f_*)$. Then by [TH4, Proposition 5.1], we have the following exact sequence

$$0 \to \mathrm{Tor}_{i+2}^S(\omega, \mathrm{cTr}_\omega M) \to \omega \otimes_S N_* \xrightarrow{\theta_N} N \to \mathrm{Tor}_{i+1}^S(\omega, \mathrm{cTr}_\omega M) \to 0.$$

Set $X := \operatorname{Im}\theta_N$ and $L := \operatorname{Tor}^S_{i+1}(\omega, \operatorname{cTr}_\omega M)$. Consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^0(X) & \longrightarrow & I^0(X)\oplus I^0(L) & \longrightarrow & I^0(L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^{i-1}(X) & \longrightarrow & I^{i-1}(X)\oplus I^{i-1}(L) & \longrightarrow & I^{i-1}(L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Since $X$ is a quotient module of $\omega\otimes_S N_*$, we have $X \in \operatorname{co}\Omega^1_{\mathcal{B}}(R)$ and $D \in \operatorname{co}\Omega^{i+1}_{\mathcal{B}}(R)$. Because $\text{E-cograde}_\omega L \geqslant i+1$ by assumption, we get the following exact sequence

$$0 \to I^0(L)_* \to I^1(L)_* \to \cdots \to I^{i-1}(L)_* \to B_* \to 0.$$

Thus $\mathcal{I}_\omega(S)\text{-pd}_S B_* \leqslant i-1$ and $\beta_*$ is an epimorphism. Next we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & W_{i-1} & \longrightarrow & W_{i-2} & \longrightarrow & \cdots & \longrightarrow & W_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g_{i-1}} & & \downarrow{\scriptstyle g_{i-2}} & & & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_{-1}} & & \\
0 & \longrightarrow & N & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots & \longrightarrow & E^{i-1} & \longrightarrow & H & \longrightarrow & 0,
\end{array}$$

where $E^j = I^j(X)\oplus I^j(L)$ for any $0 \leqslant j \leqslant i-1$. The injectivity of $E^j$ guarantees the existence of all $g_j$. Now we view the sequence $(g_{i-1}, g_{i-2}, \cdots, g_{-1})$ as a quasi-isomorphism between the following two complexes

$$0 \to W_{i-1} \to W_{i-2} \to \cdots \to W_0 \to M \to 0$$

and

$$0 \to E^0 \to E^1 \to \cdots \to E^{i-1} \to H \to 0.$$

We then obtain an exact sequence

$$0 \to W_{i-1} \to W_{i-2}\oplus E^0 \to W_{i-3}\oplus E^1 \to \cdots \to W_0\oplus E^{i-2} \to M\oplus E^{i-1} \to H \to 0.$$

Set $K := \mathrm{Im}(W_0 \oplus E^{i-2} \to M \oplus E^{i-1})$. It is not hard to see that $\mathrm{Ext}_R^{\geqslant 1}(\omega, K) = 0$ and $K \in \mathrm{co}\Omega_{\mathcal{B}}^{j}(R)$ for $j \geqslant 1$. Consider the following pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & D & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & K & \longrightarrow & M \oplus E^{i-1} & \xrightarrow{\gamma} & H & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\
 & & & & B & = & B & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since $K, D \in \mathrm{c}\mathcal{T}_{\omega}^{i+1}(R)$, we have $A \in \mathrm{c}\mathcal{T}_{\omega}^{i+1}(R)$ by Lemma 4.3. Thus $A \in \mathrm{co}\Omega_{\mathcal{B}}^{i+1}(R)$ by Lemma 4.2. It follows from the fact $\mathrm{Ext}_R^{\geqslant 1}(\omega, K) = 0$ that $\gamma_*$ is an epimorphism. Notice that $\beta_*$ is also an epimorphism, so

$$0 \to A_* \to (M \oplus E^{i-1})_* \to B_* \to 0$$

is exact. The proof is finished. □

We are now in a position to give the main result in this section.

**Theorem 4.5.** *Assume that $\omega$ satisfies the $n$-cograde condition with $n \geqslant 1$. Then for any $M \in \mathrm{Mod}\, R$, there exists an injective left $R$-module $I$ and a chain of monomorphisms*

$$M_n \rightarrowtail M_{n-1} \rightarrowtail \cdots \rightarrowtail M_1 \to M_0 = M \oplus I$$

*in $\mathrm{Mod}\, R$ such that for any $0 \leqslant i \leqslant n-1$, we have*

(1) $B_i = \mathrm{Coker}(M_{i+1} \to M_i) \cong \mathrm{co}\Omega^i(\mathrm{Tor}_{i+1}^S(\omega, \mathrm{cTr}_\omega M))$.
(2) $M_i \in \mathrm{co}\Omega_{\mathcal{B}}^i(R)$.
(3) $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\, B_{i*} \leqslant i-1$.
(4) *The exact sequence $0 \to M_{i+1} \to M_i \to B_i \to 0$ in $\mathrm{Mod}\, R$ is $\mathrm{Hom}_R(\omega, -)$-exact.*

*Proof.* From the proof of Proposition 4.4, we get a $\mathrm{Hom}_R(\omega, -)$-exact exact sequence

$$0 \to A_{i+1} \to A_i \oplus I_i \to B_i \to 0$$

in $\mathrm{Mod}\, R$ such that $A_0 = M$, $A_i \in \mathrm{co}\Omega_{\mathcal{B}}^i(R)$, $I_i \in \mathcal{I}(R)$, $B_i \cong \mathrm{co}\Omega^i(\mathrm{Tor}_{i+1}^S(\omega, \mathrm{cTr}_\omega M))$ with $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S B_{i*} \leqslant i-1$ for $0 \leqslant i \leqslant n-1$. Set $I := \oplus_{i=0}^{n-1} I_i$, $M_0 := M \oplus I$, $M_n := A_n$ and $M_i := A_i \oplus (\oplus_{j=i}^{n-1} I_j)$ for any $1 \leqslant i \leqslant n-1$. Now the assertion follows easily. □

As a consequence of Theorem 4.5, we have the following

**Corollary 4.6.** *Assume that $\omega$ satisfies the $n$-cograde condition with $n \geqslant 2$. Then for any $N \in \operatorname{Mod} S$, there exists an injective left $R$-module $I$ and a chain of monomorphisms*

$$N_n \rightarrowtail N_{n-1} \rightarrowtail \cdots \rightarrowtail N_2 \rightarrowtail (\omega \otimes_S N)_* \oplus I_*$$

*in $\operatorname{Mod} R$ such that*

(1) *$\mathcal{I}_\omega(S)$-$\operatorname{pd}_S Y_i \leqslant i$, where $Y_i = ((\omega \otimes_S N)_* \oplus I_*)/N_{i+2}$ for any $0 \leqslant i \leqslant n-2$.*
(2) *$0 \to \omega \otimes_S N_{i+2} \to \omega \otimes_S ((\omega \otimes_S N)_* \oplus I_*) \to \omega \otimes_S Y_i \to 0$ in $\operatorname{Mod} R$ is exact for any $0 \leqslant i \leqslant n-2$.*
(3) *For $1 \leqslant i \leqslant n-2$, the natural epimorphism $(\omega \otimes_S N)_* \oplus I_* \twoheadrightarrow Y_i$ in $\operatorname{Mod} S$ induces an isomorphism*

$$\operatorname{Tor}_j^S(\omega, (\omega \otimes_S N)_*) \xrightarrow{\cong} \operatorname{Tor}_j^S(\omega, Y_i)$$

*for any $1 \leqslant j \leqslant n-2$.*

*Proof.* Let $M = \omega \otimes_S N$. By Theorem 4.5, there exists a $\operatorname{Hom}_R(\omega, -)$-exact exact sequence

$$0 \to M_1 \to M_0 (\cong M \oplus I) \to B_0 \to 0$$

in $\operatorname{Mod} R$ such that $B_0 \cong \operatorname{Tor}_1^S(\omega, \operatorname{cTr}_\omega M)$, $M_1 \in \operatorname{co}\Omega^1_{\mathcal{B}}(R)$ and $B_{0*}(=0) \in \mathcal{I}_\omega(S)$. By Theorem 4.5 again, we further have the following two $\operatorname{Hom}_R(\omega, -)$-exact exact sequences

$$0 \to M_2 \to M_1 \to B_1 \to 0,$$

$$0 \to M_3 \to M_2 \to B_2 \to 0$$

in $\operatorname{Mod} R$ such that $M_2 \in \operatorname{co}\Omega^2_{\mathcal{B}}(R)$, $M_3 \in \operatorname{co}\Omega^3_{\mathcal{B}}(R)$, $B_{1*} \in \mathcal{I}_\omega(S)$ and $\mathcal{I}_\omega(S)$-$\operatorname{pd}_S B_{2*} \leqslant 1$. Now consider the following push-out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_{3*} & = & M_{3*} & & & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \\
0 & \to & M_{2*} & \xrightarrow{\delta} & M_{1*} & \to & B_{1*} & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \| & & \\
0 & \to & B_{2*} & \to & Y_1 & \to & B_{1*} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

By [HW, Theorem 6.4], we have $\operatorname{Ext}_R^{\geqslant 1}(V, B_{2*}) = 0$ for any $V \in \mathcal{I}_\omega(S)$. So $\mathcal{I}_\omega(S)$-$\operatorname{pd}_S Y_1 \leqslant 1$ by [EJ, Lemma 8.2.1]. Moreover, there exists the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \omega \otimes_S M_{3*} & \xrightarrow{1_\omega \otimes \alpha} & \omega \otimes_S M_{2*} & \to & \omega \otimes_S B_{2*} & \to & 0 \\
 & & \downarrow{\scriptstyle \theta_{M_3}} & & \downarrow{\scriptstyle \theta_{M_2}} & & \downarrow{\scriptstyle \theta_{B_2}} & & \\
0 & \to & M_3 & \to & M_2 & \to & B_2 & \to & 0.
\end{array}$$

Because $M_3 \in \mathrm{co}\Omega^3_{\mathcal{B}}(R) = \mathrm{c}\mathcal{T}^3_{\omega}(R)$ by Lemma 4.2, we have that $\theta_{M_3}$ is an isomorphism and $1_{\omega}\otimes\alpha$ is a monomorphism. Similarly $1_{\omega}\otimes\delta$ is also a monomorphism, and hence $1_{\omega}\otimes\beta$ is also a monomorphism. Since $\mathrm{Tor}_1^S(\omega, M_{3*}) = 0$ by [TH1, Corollary 3.4], the sequence

$$0 \to \omega \otimes_S M_{3*} \to \omega \otimes_S M_{1*}(\cong \omega \otimes_S ((\omega \otimes_S N)_* \oplus I_*)) \to \omega \otimes_S Y_1 \to 0$$

is exact and $\gamma$ induces an isomorphism $\mathrm{Tor}_1^S(\omega, (\omega \otimes_S N)_*) \xrightarrow{\cong} \mathrm{Tor}_1^S(\omega, Y_1)$. Now put $Y_0 := B_{1*}$ and $N_i := M_{i*}$ for $i = 2, 3$. Continuing this process, we may construct a submodule chain of $(\omega\otimes_S N)_*\oplus I_*$ satisfying the desired properties. □

## 5. **Applications**

In this section, we apply the two filtrations of modules obtained in Sections 3 and 4 to study mainly the relationship between artin algebras satisfying the Auslander condition and Gorenstein algebras.

Following [EJ, Definition 10.1.1], a module $M$ in $\mathrm{Mod}\, R$ is called *Gorenstein injective* if there exists an exact sequence

$$\mathbf{I} := \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in $\mathrm{Mod}\, R$ with all $I_i, I^i \in \mathcal{I}(R)$ such that $\mathrm{Hom}_R(E, \mathbf{I})$ is exact for any $E \in \mathcal{I}(R)$ and $M \cong \mathrm{Im}(I_0 \to I^0)$. We use $\mathcal{GI}$ to denote the class of Gorenstein injective modules and use $\mathrm{Gid}_R\, M$ to denote the $\mathcal{GI}$-injective dimension (that is, the Gorenstein injective dimension) of $M$.

Note that a module $M$ in $\mathrm{mod}\, R$ belongs to $\mathcal{B}_{D(R)}(R)$ if and only if $M$ is in $\mathcal{GI}$ by [TH1, Theorem 3.9 and Corollary 5.2]. So, if putting $\omega := D(R)$ in Theorem 3.12, then we get the following

**Corollary 5.1.** *Let $R$ be an artin algebra satisfying the Auslander condition. If $M \in \mathrm{mod}\, R$ with $\text{E-cograde}_{D(R)}\, M = k < \infty$, then there exists a filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

*of $R$-submodules of $M$ such that*

(1) *$M_1 = \cdots = M_k = 0$ and there exists the following exact sequence*

$$0 \to \mathrm{Tor}^R_{k+2}(D(R), \mathrm{cTr}_{D(R)}\, \mathrm{co}\Omega^k(M)) \to \mathrm{Tor}^R_k(D(R), \mathrm{Ext}^k_R(D(R), M)) \to M/M_k \to M/M_{k+1} \to 0.$$

(2) *If $\mathrm{Tor}^R_i(D(R), \mathrm{Ext}^i_R(D(R), M)) \neq 0$, then $\text{E-cograde}_{D(R)}\, M/M_i = i$, $M_i \neq M_{i+1}$ and $M_{i+1}/M_i$ is pure of $\text{Ext-cograde}\ i$.*

(3) *If $\mathrm{Tor}^R_i(D(R), \mathrm{Ext}^i_R(D(R), M)) = 0$, then $M_i = M_{i+1}$.*

(4) *If $\mathrm{Gid}_R\, M = d < \infty$, then*

$$M = M_{d+1} \text{ and } M/M_d \cong \mathrm{Tor}^R_d(D(R), \mathrm{Ext}^d_R(D(R), M)).$$

(5) *If $\mathrm{Gid}_R\, M = d < \infty$, then $\mathrm{fil}(M) \leqslant d - k + 1$, and the equality holds whenever $\text{T-cograde}_{D(R)}\, \mathrm{Ext}^i_R(D(R), M) = i$ for any $k \leqslant i \leqslant d$.*

Auslander and Reiten conjuctured in [AR1] that any artin algebra $R$ satisfying the Auslander condition is Gorenstein.

**Theorem 5.2.** *Let $R$ be an artin algebra satisfying the Auslander condition. If $R$ is Gorenstein with $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R = n$, then $\mathrm{fil}(\mathrm{co}\Omega^2(R/J)) \leqslant n - 1$, and the equality holds provided that $\mathrm{Tor}^R_i(D(R), \mathrm{Ext}^i_R(D(R), \mathrm{co}\Omega^2(R/J))) \neq 0$ for any $0 \leqslant i \leqslant n - 2$ or that $\mathrm{co}\Omega^2(R/J)$ is Gorenstein injective.*

*Proof.* Since $\mathrm{id}_R R = n$, it follows from [EJ, Theorem 12.3.1] and [T, Theorem 2.1] that $\mathrm{Gid}_R R/J = n$.

If $n \geqslant 2$, we get from [EJ, Theorem 12.3.1] that $\mathrm{co}\Omega^n(R/J)$ is Gorenstein injective. Thus $\mathrm{Gid}_R \mathrm{co}\Omega^2(R/J) \leqslant n-2$. Because $\mathrm{Gid}_R R/J = n$, we have $\mathrm{Gid}_R \mathrm{co}\Omega^2(R/J) = n-2$. So $\mathrm{co}\Omega^2(R/J) \neq 0$ and $D(\mathrm{co}\Omega^2(R/J)) \neq 0$. Because $D(\mathrm{co}\Omega^2(R/J))$ is 2-syzygy, it follows from [AR2, Theorems 1.7 and 4.7] that $\mathrm{Hom}_R(\mathrm{Hom}_{R^{op}}(D(\mathrm{co}\Omega^2(R/J)), R), R) \cong D(\mathrm{co}\Omega^2(R/J)) \neq 0$. Take $\omega := D(R)$. Then we have that

$$\begin{aligned} &\mathrm{Hom}_R(D(R), \mathrm{co}\Omega^2(R/J)) \\ \cong\ &\mathrm{Hom}_{R^{op}}(D(\mathrm{co}\Omega^2(R/J)), DD(R)) \\ \cong\ &\mathrm{Hom}_{R^{op}}(D(\mathrm{co}\Omega^2(R/J)), R) \\ \neq\ &0. \end{aligned}$$

and $\text{E-cograde}_{D(R)} \mathrm{co}\Omega^2(R/J) = 0$. Now the first assertion follows from Corollary 5.1.

If $n < 2$, then $\mathrm{co}\Omega^1(R/J)$ is Gorenstein injective. So $\mathrm{co}\Omega^2(R/J)$ is also Gorenstein injective by [EJ, Theorem 10.1.4]. □

Secondly, we turn to give an application of the filtration of modules obtained in Section 4. Inspired by [K, Definition 2.15], we give its dual version as follows.

**Definition 5.3.**

(1) Two homomorphisms $f : A \to B$ and $f' : A' \to B'$ in $\mathrm{Mod}\, R$ are said to be *isomorphic up to a direct sum of injective modules* if there exist injective modules $I$, $E$, $U$, $I'$, $E'$ and $U'$ such that
$$A \oplus I \oplus E \xrightarrow{g} B \oplus E \oplus U$$
and
$$A' \oplus I' \oplus E' \xrightarrow{h} B' \oplus E' \oplus U'$$
are isomorphic, where $g$ and $h$ are given by the following matrices
$$g = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} f' & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2) For an integer $k \geqslant 0$, a module $M \in \mathrm{Mod}\, R$ is called *injectively stationary of type $k$* if for any $i > k$, the inclusions $\lambda_i : M_i \to M_0$ and $\lambda_k : M_k \to M_0$ are isomorphic up to a direct sum of injective modules, where all $M_i$ are the modules as in Theorem 4.5.

We use $\underline{\mathrm{mod}}\, R$ to denote the stable category of $\mathrm{mod}\, R$ modulo projectives.

**Theorem 5.4.** *Let $R$ be an artin algebra satisfying the Auslander condition. Then the following statements are equivalent.*

(1) *$R$ is Gorenstein.*
(2) *For some $k \geqslant 0$, any 2-$D(R)$-cotorsionfree left $R$-module is injectively stationary of type $k$.*
(3) *For some $k \geqslant 0$, any finitely generated 2-$D(R)$-cotorsionfree left $R$-module is injectively stationary of type $k$.*

*Proof.* (1) $\Rightarrow$ (2) Let $M \in \operatorname{mod} R$. Since $R$ is Gorenstein, we have $\operatorname{Gid}_R M < \infty$ by [EJ, Theorem 12.3.1]. Then $M \in \mathcal{B}_{D(R)}(R)$ by [CFH, Theorem 4.4]. So $\operatorname{cTr}_{D(R)} M \in \mathcal{A}_{D(R)}(R)$ by Lemma 3.11. It implies

$$B_i \cong \operatorname{co}\Omega^i(\operatorname{Tor}^S_{i+1}(D(R), \operatorname{cTr}_{D(R)} M)) = 0$$

for any $i \geqslant 0$. Thus all filtration submodules $M_i$ equal $M_0$ and the assertion follows.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) By [G, Theorem 4.1], it only needs to show that $\operatorname{pd}_R M \leqslant k+2$ for any $M \in \operatorname{mod} R$ with $\operatorname{pd}_R M < \infty$. Let $M \in \operatorname{mod} R$ with $\operatorname{pd}_R M = l < \infty$ and

$$0 \to Q_l \to \cdots \to Q_1 \to Q_0 \to M \to 0$$

be a minimal projective resolution of $M$ in $\operatorname{mod} R$. Take $M' := \operatorname{Ker}(Q_1 \to Q_0)$ and $\omega := D(R)$. Because $M'$ is 2-syzygy, we have

$$\begin{aligned}
&(D(R) \otimes_R M')_* \\
\cong\ &(D(\operatorname{Hom}_R(M', R)))_* \text{ (by [EJ, Theorem 3.2.13])} \\
\cong\ &\operatorname{Hom}_{R^{op}}(\operatorname{Hom}_R(M', R), R) \\
\cong\ &M' \text{ (by [AR2, Theorems 1.7 and 4.7]).}
\end{aligned}$$

So $M'$ is adjoint 2-$D(R)$-cotorsionfree and $(D(R) \otimes_R M')_* \cong M'$. Note that $I_*$ is projective left $R$-module for any injective left $R$-module $I$ by [EJ, Theorem 3.2.9]. So, putting $N = M'$ in Corollary 4.6, from the proof of this corollary, we get that there exists an exact sequence

$$0 \to N_{i+2} \to M' \oplus P \xrightarrow{f_i} Y_i \to 0$$

in $\operatorname{mod} R$ with $P \in \mathcal{P}(R)$ and $\operatorname{pd}_R Y_i \leqslant i$ for any $i \geqslant 0$, and the homomorphism $f_i$ also induces an isomorphism

$$\operatorname{Tor}^R_j(D(R), M' \oplus P) \xrightarrow{\cong} \operatorname{Tor}^R_j(D(R), Y_i)$$

for any $j \geqslant 1$. By [EJ, Theorem 3.2.13], we have

$\operatorname{Tor}^R_j(D(R), M' \oplus P) \cong D(\operatorname{Ext}^j_R(M' \oplus P, R))$ and $\operatorname{Tor}^R_j(D(R), Y_i) \cong D(\operatorname{Ext}^j_R(Y_i, R))$.

Then by [AB, Lemma 2.42], any homomorphism $\underline{M'} \to \underline{L}$ in $\underline{\operatorname{mod}}\, R$ with $\operatorname{pd}_R L \leqslant i$ factors through $\underline{f_i}$. Because $D(R) \otimes_R (D(R) \otimes_R M')_* \cong D(R) \otimes_R M'$, we have that $D(R) \otimes_R M' \in \operatorname{mod} R$ is 2-$D(R)$-cotorsionfree. By the construction of $N_i$ and [K, Lemma 2.16], we have $\underline{Y_k} \cong \underline{Y_i}$ for any $i > k$ by the assumption of (3). We immediately have a homomorphism $g : Y_{l-2} \to M'$ of left $R$-modules such that $\underline{1_{M'}} = \underline{g} \cdot \underline{f_{l-2}}$. It follows that there exists a projective left $R$-module $Q$ such that $M'$ is isomorphic a direct summand of $Y_{l-2} \oplus Q$. Since $\underline{Y_k} \cong \underline{Y_{l-2}}$ in $\underline{\operatorname{mod}}\, R$, by [FLM, Proposition 3.1] there exist projective left $R$-modules $P_1$ and $P_2$ such that $Y_{l-2} \oplus P_1 \cong Y_k \oplus P_2$. Thus $\operatorname{pd}_R M' \leqslant k$ and $\operatorname{pd}_R M \leqslant k+2$. □

For a commutative noetherian ring $R$ and an $n$-syzygy module $M$ in $\operatorname{mod} R$ with $n \geqslant 0$, an *Evans-Griffith presentation* of $M$ is defined as such an exact sequence

$$0 \to S \to B \to M \to 0$$

in $\operatorname{mod} R$ with $B$ an $n$-th syzygy of $\operatorname{Ext}^{n+1}_{R^{op}}(\operatorname{Tr} M, R)$ and $S$ an $(n+2)$-syzygy module ([EG, M]). We introduce the dual version of this notion as follows.

**Definition 5.5.** Let $n \geqslant 0$ and $M \in \mathrm{co}\Omega^n_{\mathcal{B}}(R)$. A *dual Evans-Griffith presentation* of $M$ is an exact sequence

$$0 \to M \to B \to C \to 0$$

in $\mathrm{Mod}\, R$ with $B$ an $n$-th cosyzygy of $\mathrm{Tor}^S_{n+1}(\omega, \mathrm{cTr}_\omega M)$ and $C \in \mathrm{co}\Omega^{n+2}_{\mathcal{B}}(R)$.

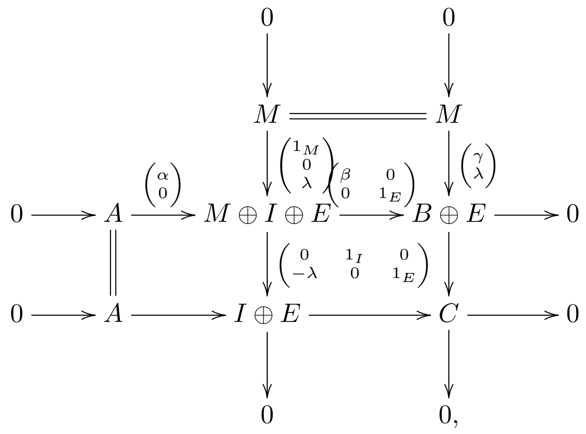
As an application of Proposition 4.4, we have the following

**Proposition 5.6.** *Assume that $\omega$ satisfies the $n$-cograde condition with $n \geqslant 1$. Then for any $0 \leqslant i \leqslant n-1$, each module in $\mathrm{co}\Omega^i_{\mathcal{B}}(R)$ admits a dual Evans-Griffith presentation.*

*Proof.* Let $M \in \mathrm{co}\Omega^i_{\mathcal{B}}(R)$ with $0 \leqslant i \leqslant n-1$. Then by Proposition 4.4, there exists an exact sequence

$$0 \to A \xrightarrow{\alpha} M \oplus I \xrightarrow{\beta} B \to 0$$

in $\mathrm{Mod}\, R$ with $A \in \mathrm{co}\Omega^{i+1}_{\mathcal{B}}(R)$, $I \in \mathcal{I}(R)$ and $B \cong \mathrm{co}\Omega^i(\mathrm{Tor}^S_{i+1}(\omega, \mathrm{cTr}_\omega M))$. Let $\gamma := \beta \begin{pmatrix} 1_M \\ 0 \end{pmatrix}$ and $\lambda : M \hookrightarrow E$ be an embedding in $\mathrm{Mod}\, R$ with $E$ injective. Then we have the following commutative diagram with exact columns and rows



where $C = \mathrm{Coker} \begin{pmatrix} \gamma \\ \lambda \end{pmatrix}$. It yields from the bottom row in the above diagram that $C \in \mathrm{co}\Omega^{i+2}_{\mathcal{B}}(R)$. Thus the rightmost column is a dual Evans-Griffith presentation of $M$. □

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