Gorenstein Cophantom Objects and Morphisms

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Abstract

We first introduce and study Gorenstein cophantom objects. Let \( \mathcal{C} \) be a full and additive subcategory of an abelian category \( A \) which is self-orthogonal and closed under isomorphisms. Then the subcategory \( \mathcal{C}G(\mathcal{C}) \) of \( A \) consisting of Gorenstein cophantom objects is both projectively resolving and injectively coresolving. Let \( \mathcal{G}(\mathcal{C}) \) be the Gorenstein subcategory of \( \mathcal{A} \). Then the subcategory \( \text{SPC}(\mathcal{G}(\mathcal{C})) \) of \( A \) consisting of objects admitting special \( \mathcal{G}(\mathcal{C}) \)-precovers is closed under extensions and \( \mathcal{C} \)-free direct summands, and with respect to this property, \( \text{SPC}(\mathcal{G}(\mathcal{C})) \) is the minimal subcategory of \( A \) containing \( \mathcal{C}G(\mathcal{C}) \cup \mathcal{G}(\mathcal{C}) \). We next introduce the notions of Gorenstein cophantom and projective morphisms. By giving some criteria for identifying Gorenstein cophantom and projective morphisms, we get that Gorenstein cophantom and projective morphisms are the morphism versions of Gorenstein cophantom and projective objects respectively. Moreover, if \( \mathcal{G}(\mathcal{C}) \) is special precovering in \( \mathcal{A} \), then the class of Gorenstein projective morphisms and that of Gorenstein cophantom morphisms forms a complete ideal cotorsion pair.

1 Introduction

Approximation theory is the main part of relative homological algebra and its starting point is to approximate arbitrary objects by a class of suitable subcategories. In these subcategories, a special kind of subcategories provides a fruitful context to the development of relative homological algebra, that is, the subcategories \( \mathcal{C} \) for which every object admits a (special) \( \mathcal{C} \)-precover. In this aspect, a lot of achievements have been scored, for example, the existence of projective precovers, flat covers and pure projective precovers, and so on. Recently, it has become an active topic about the study of the existence of (special) Gorenstein projective precovers, see [3, 5, 11, 13, 29, 30] and references therein.

On the other hand, the phantom ideal plays an important role in the aspect of providing a certain ideal cotorsion pair, and it has been investigated in algebraic topology [26], stable homotopy categories of spectra [1], triangulated categories [27], and stable categories of finite group rings [6, 7, 8]. In particular, Herzog generalized in [19] the phantom morphism to the category of left \( R \)-modules of arbitrary associative ring \( R \) in the following way: a morphism \( f : M \to N \) of left \( R \)-modules is called a phantom morphism if the natural transformation \( \text{Tor}_1^R(\mathcal{C}, \mathcal{C}) : \text{Tor}_1^R(\mathcal{C}, M) \to \text{Tor}_1^R(\mathcal{C}, N) \) is zero, or equivalently, the pullback of any short exact sequence along \( f \) is pure exact. Then he showed that every module has a phantom cover. As a generalization of (classical) approximation theory for subcategories, Fu, Guil Asensio, Herzog and Torrecillas developed in [17] approximation theory of an exact category \( \mathcal{A} \) for ideal cotorsion pairs. Like the classical case, the main concern in ideal approximation theory is to study the existence of (pre)covers or
(pre)envelopes with respect to a certain kind of ideals. It was showed in [17] that every complete cotorsion pair \((\mathcal{A}, \mathcal{B})\) of subcategories induces a complete ideal cotorsion pair \((< \mathcal{A} >, < \mathcal{B} >)\), where \(< \mathcal{A} >\) denotes the ideal of those morphisms that factor through an object of the additive subcategory \(\mathcal{A} \subseteq \mathcal{A}\).

This result provides a valid bridge between the special precovers of subcategories and that of ideals. Also in [17], Fu, Guil Asensio, Herzog and Torrecillas introduced a relative version of (co)phantom morphisms, that is, given an exact category \(\mathcal{A}\) and an additive subfunctor \(\mathcal{F} \subseteq \text{Ext}\), a morphism \(\phi : X \to Y\) in \(\mathcal{A}\) is an \(\mathcal{F}\)-phantom (resp. \(\mathcal{F}\)-cphantom) morphism if the pullback (resp. pushout) along \(\phi\) of any conflation (that is, any short exact sequence) in \(\mathcal{A}\) belongs to the subfunctor \(\mathcal{F}\). Then they gave a classification for special precovering ideals. They showed that, for a suitable exact category, every special precovering ideal can be represented as an ideal of \(\mathcal{F}\)-phantom morphisms for an additive subfunctor \(\mathcal{F} \subseteq \text{Ext}\) with enough injective morphisms.

Based on these mentioned above, in this paper we introduce and study Gorenstein cphantom objects and morphisms, which are defined by a subfunctor \(G\text{Ext}\) (Gorenstein derived functor) of \(\text{Ext}\). Then we study the existence of special precovers and special preenvelopes of a class of cphantom ideals. This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we introduce the notion of Gorenstein cphantom objects. Let \(\mathcal{C}\) be a full and additive category of an abelian category \(\mathcal{A}\) which is self-orthogonal and closed under isomorphisms. We prove that the subcategory \(CG(\mathcal{C})\) of \(\mathcal{A}\) consisting of Gorenstein cphantom objects is both projectively resolving and injectively coresolving. Let \(G(\mathcal{C})\) be the Gorenstein subcategory of \(\mathcal{A}\). We prove that \(G(\mathcal{C})\) is special precovering in \(\mathcal{A}\) if and only if \((G(\mathcal{C}), CG(\mathcal{C}))\) is a complete cotorsion pair. Motivated by this result, we call \(\mathcal{A}\) Gorenstein complete if \(G(\mathcal{C})\) is special precovering in \(\mathcal{A}\). Furthermore, we prove that the subcategory SPC\((G(\mathcal{C}))\) of \(\mathcal{A}\) consisting of objects admitting special \(G(\mathcal{C})\)-precovers is closed under extensions and \(\mathcal{C}\)-free direct summands (*). Furthermore, if \(CG(\mathcal{C})\) has enough projective objects in \(\mathcal{A}\), then we get the following two results: (1) SPC\((G(\mathcal{C}))\) is the minimal subcategory of \(\mathcal{A}\) containing \(CG(\mathcal{C}) \cup G(\mathcal{C})\) with respect to the property (*); and (2) SPC\((G(\mathcal{C}))\) is \(\mathcal{C}\)-resolving with a \(\mathcal{C}\)-proper generator \(\mathcal{C}\).

In Section 4, we first introduce the notion of Gorenstein cphantom and projective morphisms. By using the so-called pushout-pullback factorization of a morphism of short exact sequences, we prove that a morphism \(\psi\) is Gorenstein cphantom if and only if \(\text{Ext}^1_{\mathcal{A}}(G, \psi) = 0\) for any \(G \in G(\mathcal{C})\). It induces that Gorenstein cphantom morphisms are the morphism version of Gorenstein cphantom objects. Then we give some applications of this result. For an additive subcategory \(\mathcal{A}'\) of \(\mathcal{A}\), we use \(< \mathcal{A}' >\) to denote the ideal of \(\mathcal{A}\) generated by morphisms of the form \(1_X\) with \(X \in \mathcal{A}'\). Let \(\mathcal{A}\) be a Gorenstein complete category. We get that the ideal cotorsion pair \((< G(\mathcal{C}) >, < CG(\mathcal{C}) >)\) is complete. We also get that a morphism is Gorenstein cphantom if and only if it factors through some object in \(CG(\mathcal{C})\), which induces that the ideal \(G(\mathcal{C})\)-Mor consisting of Gorenstein projective morphisms of \(\mathcal{A}\) is an object ideal, and the ideal cotorsion pair \((G(\mathcal{C})\)-Mor, Coph-\(G(\mathcal{C})\)) is complete.

In Section 5, we give two examples to demonstrate Gorenstein cphantom objects and morphisms.
2 Preliminaries

Throughout this paper, $\mathcal{A}$ is an abelian category and all subcategories of $\mathcal{A}$ are full, additive and closed under isomorphisms. We use $\mathcal{P}(\mathcal{A})$ (resp. $\mathcal{I}(\mathcal{A})$) to denote the subcategory of $\mathcal{A}$ consisting of projective (resp. injective) objects. For a non-negative integer $n$, we use $\mathcal{P}(\mathcal{A})_{\leq n}$ (resp. $\mathcal{I}(\mathcal{A})_{\leq n}$) to denote the subcategory of $\mathcal{A}$ consisting of objects with projective (resp. injective) dimension at most $n$, and use $\mathcal{P}(\mathcal{A})_{<\infty}$ (resp. $\mathcal{I}(\mathcal{A})_{<\infty}$) to denote the subcategory of $\mathcal{A}$ consisting of objects with finite projective (resp. injective) dimension.

Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Recall that a sequence in $\mathcal{A}$ is called $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact if it is exact after applying the functor $\text{Hom}_{\mathcal{A}}(X, -)$ for any $X \in \mathcal{X}$. Dually, the notion of a $\text{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact sequence is defined. Set

\[
\mathcal{X}^\perp := \{ M \mid \text{Ext}^1_{\mathcal{A}}(X, M) = 0 \text{ for any } X \in \mathcal{X} \},
\]
\[
\perp \mathcal{X} := \{ M \mid \text{Ext}^1_{\mathcal{A}}(M, X) = 0 \text{ for any } X \in \mathcal{X} \}
\]

and

\[
\mathcal{X}^{\perp n} := \{ M \mid \text{Ext}^n_{\mathcal{A}}(X, M) = 0 \text{ for any } X \in \mathcal{X} \},
\]
\[
\perp n \mathcal{X} := \{ M \mid \text{Ext}^n_{\mathcal{A}}(M, X) = 0 \text{ for any } X \in \mathcal{X} \}
\]

for a fixed positive integer $n$. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{A}$. We write $\mathcal{X} \perp \mathcal{Y}$ if $\text{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

**Definition 2.1.** (cf. [12]) Let $\mathcal{X} \subseteq \mathcal{Y}$ be subcategories of $\mathcal{A}$. The morphism $f : X \to Y$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ is called an $\mathcal{X}$-precover of $Y$ if the sequence $X \xrightarrow{f} Y \to 0$ is $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact. An $\mathcal{X}$-precover $f : X \to Y$ is called special if $f$ is epic and $\text{Ker } f \in \mathcal{X}^{\perp 1}$, and is called an $\mathcal{X}$-cover if a morphism $h : X \to Y$ is an automorphism whenever $f = fh$. $\mathcal{X}$ is called (pre)covering in $\mathcal{Y}$ if any object in $\mathcal{Y}$ admits an $\mathcal{X}$-(pre)cover, and is called special precovering in $\mathcal{Y}$ if any object in $\mathcal{Y}$ admits a special $\mathcal{X}$-precover. Dually, the notions of an $\mathcal{X}$-(pre)envelope, a special $\mathcal{X}$-preenvelope, a (pre)enveloping subcategory and a special preenveloping subcategory are defined.

**Definition 2.2.** (cf. [20]) A subcategory of $\mathcal{A}$ is called projectively resolving if it contains $\mathcal{P}(\mathcal{A})$ and is closed under extensions and under kernels of epimorphisms. Dually, the notion of injectively coresolving subcategories is defined.

From now on, assume that $\mathcal{C}$ is a given subcategory of $\mathcal{A}$.

**Definition 2.3.** (cf. [28]) The Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of $\mathcal{A}$ is defined as $\mathcal{G}(\mathcal{C}) = \{ M \mid \text{ there exists an exact sequence: } \cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots \}$ ($2.1$) in $\mathcal{C}$, which is both $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$-exact, such that $M \cong \text{Im}(C_0 \to C^0)$; in this case, $(2.1)$ is called a complete $\mathcal{C}$-resolution of $M$.

In what follows, $R$ is an associative ring with identity, $\text{Mod } R$ is the category of left $R$-modules and $\text{mod } R$ is the category of finitely generated left $R$-modules.
Remark 2.4.

(1) Let $R$ be a left and right Noetherian ring. Then $G(\mathcal{P}(\mod R))$ coincides with the subcategory of $\mod R$ consisting of modules with Gorenstein dimension zero ([4]).

(2) $G(\mathcal{P}(\Mod R))$ (resp. $G(\mathcal{I}(\Mod R))$) coincides with the subcategory of $\Mod R$ consisting of Gorenstein projective (resp. injective) modules ([14]).

(3) Let $R$ be a left Noetherian ring, $S$ a right Noetherian ring and $RVS$ a dualizing bimodule. Put $\mathcal{W} = \{ V \otimes_S P \mid P \in \mathcal{P}(\Mod S) \}$ and $\mathcal{U} = \{ \Hom_S(V, E) \mid E \in \mathcal{I}(\Mod S^{op}) \}$. Then $G(\mathcal{W})$ (resp. $G(\mathcal{U})$) coincides with the subcategory of $\Mod R$ consisting of $V$-Gorenstein projective (resp. injective) modules ([16]).

Definition 2.5. (cf. [28]) Let $\mathcal{X} \subseteq \mathcal{I}$ be subcategories of $\mathcal{A}$. Then $\mathcal{X}$ is called a generator (resp. cogenerator) for $\mathcal{I}$ if for any $T \in \mathcal{I}$, there exists an exact sequence $0 \rightarrow T' \rightarrow X \rightarrow T \rightarrow 0$ (resp. $0 \rightarrow T \rightarrow X \rightarrow T' \rightarrow 0$) in $\mathcal{I}$ with $X \in \mathcal{X}$; and $\mathcal{X}$ is called a projective generator (resp. an injective cogenerator) for $\mathcal{I}$ if $\mathcal{X}$ is a generator (resp. cogenerator) for $\mathcal{I}$ and $\mathcal{X} \perp \mathcal{I}$ (resp. $\mathcal{I} \perp \mathcal{X}$).

We have the following easy observation.

Lemma 2.6. Assume that $\mathcal{C} \perp \mathcal{C}$ and $\mathcal{P}(\mathcal{A})$ is a generator for $\mathcal{C}$. Then for any $G \in G(\mathcal{C})$, there exists a $\Hom_{\mathcal{A}}(\mathcal{C}, -)$-exact and $\Hom_{\mathcal{A}}(-, \mathcal{C})$-exact exact sequence

$$0 \rightarrow G' \rightarrow P \rightarrow G \rightarrow 0$$

in $\mathcal{A}$ with $P \in \mathcal{P}(\mathcal{A})$ and $G' \in G(\mathcal{C})$.

Proof. Let $G \in G(\mathcal{C})$. Then there exists a $\Hom_{\mathcal{A}}(\mathcal{C}, -)$-exact and $\Hom_{\mathcal{A}}(-, \mathcal{C})$-exact exact sequence

$$0 \rightarrow G_1 \rightarrow C_0 \rightarrow G \rightarrow 0$$

in $\mathcal{A}$ with $C_0 \in \mathcal{C}$ and $G_1 \in G(\mathcal{C})$. Because $\mathcal{P}(\mathcal{A})$ is a generator for $\mathcal{C}$ by assumption, there exists an exact sequence

$$0 \rightarrow C' \rightarrow P \rightarrow C_0 \rightarrow 0$$

in $\mathcal{A}$ with $P \in \mathcal{P}(\mathcal{A})$ and $C' \in \mathcal{C}$. Consider the following pullback diagram

$$\begin{array}{c}
0 \\
\downarrow \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}$$

$$\begin{array}{c}
C' \\
\downarrow \\
\downarrow \\
G' \\
\downarrow \\
G_1
\end{array}$$

$$\begin{array}{c}
P \\
\downarrow \\
\downarrow \\
G \\
\downarrow \\
C_0
\end{array}$$

$$\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}$$

By [21, Lemma 2.5], the middle row is both $\Hom_{\mathcal{A}}(\mathcal{C}, -)$-exact and $\Hom_{\mathcal{A}}(-, \mathcal{C})$-exact, and hence $G' \in G(\mathcal{C})$ by [21, Proposition 4.7], that is, the middle row is the desired sequence. \qed
The following result is useful in the sequel.

**Proposition 2.7.** Assume that \( \mathcal{C} \perp \mathcal{C} \) and \( \mathcal{P}(\mathcal{A}) \) is a generator for \( \mathcal{C} \). Then

1. \( \mathcal{G}(\mathcal{C})^{\perp_1} = \mathcal{G}(\mathcal{C})^{\perp} \).
2. \( \mathcal{G}(\mathcal{C}) \subseteq \perp \mathcal{C} \cap \mathcal{C}^{\perp} \).

**Proof.** (1) It suffices to prove that \( \mathcal{G}(\mathcal{C})^{\perp_1} \subseteq \mathcal{G}(\mathcal{C})^{\perp} \). Let \( M \in \mathcal{G}(\mathcal{C})^{\perp_1} \) and \( G \in \mathcal{G}(\mathcal{C}) \). By Lemma 2.6, we have an exact sequence

\[
0 \to G' \to P \to G \to 0
\]

in \( \mathcal{A} \) with \( P \in \mathcal{P}(\mathcal{A}) \) and \( G' \in \mathcal{G}(\mathcal{C}) \). It induces \( \text{Ext}^2_{\mathcal{A}}(G, M) \cong \text{Ext}^1_{\mathcal{A}}(G', M) = 0 \), and hence \( \text{Ext}^2_{\mathcal{A}}(G', M) = 0 \) and \( \text{Ext}^1_{\mathcal{A}}(G, M) \cong \text{Ext}^2_{\mathcal{A}}(G', M) = 0 \). Repeating this process, we get \( \text{Ext}^{\geq 1}_{\mathcal{A}}(G, M) = 0 \).

(2) See [21, Lemma 5.7]. \( \square \)

Note that if \( \mathcal{A} \) has enough projective objects, and if \( \mathcal{P}(\mathcal{A}) \subseteq \mathcal{C} \) and \( \mathcal{C} \) is closed under kernels of epimorphisms, then \( \mathcal{P}(\mathcal{A}) \) is a generator for \( \mathcal{C} \).

**Definition 2.8.** (cf. [18, Definition 5.15]) A pair \( (\mathcal{X}, \mathcal{Y}) \) of subcategories of \( \mathcal{A} \) is called a cotorsion pair if \( \mathcal{X} = \mathcal{X}^{\perp_1} \mathcal{Y} \) and \( \mathcal{Y} = \mathcal{X}^{\perp_1} \); in this case \( \mathcal{X} \cap \mathcal{Y} \) is called the kernel of \( (\mathcal{X}, \mathcal{Y}) \). A cotorsion pair \( (\mathcal{X}, \mathcal{Y}) \) is called complete if \( \mathcal{X} \) is special precovering and \( \mathcal{Y} \) is special preenveloping in \( \mathcal{A} \), and is called perfect if \( \mathcal{X} \) is covering and \( \mathcal{Y} \) is enveloping in \( \mathcal{A} \). A cotorsion pair \( (\mathcal{X}, \mathcal{Y}) \) is called hereditary if \( \mathcal{X} \) is projectively resolving and \( \mathcal{Y} \) is injectively coresolving.

The following is the Salce lemma.

**Lemma 2.9.** (cf. [18, Lemma 5.20]) For a cotorsion pair \( (\mathcal{X}, \mathcal{Y}) \) in \( \mathcal{A} \), the following statements are equivalent.

1. \( (\mathcal{X}, \mathcal{Y}) \) is complete.
2. \( \mathcal{X} \) is special precovering in \( \mathcal{A} \).
3. \( \mathcal{Y} \) is special preenveloping in \( \mathcal{A} \).

**Definition 2.10.** (cf. [24]) An ideal \( \mathcal{E} \) of \( \mathcal{A} \) is a collection \( \mathcal{E}(A, B)_{A, B \in \mathcal{A}} \) of abelian subgroups \( \mathcal{E}(A, B) \subseteq \text{Hom}_{\mathcal{A}}(A, B) \) indexed by pairs of objects of \( \mathcal{A} \), which is closed under compositions with morphisms whenever defined.

In fact, the ideals of \( \mathcal{A} \) have a close relation with the subfunctors of Hom: *Ideals are in a one-to-one correspondence with subfunctors of Hom* (cf. [25, p.36]). Let \( \mathcal{E} \) be an ideal of \( \mathcal{A} \). Set

\[
\mathcal{E}^{\perp_1} := \{ j \mid \text{Ext}^1_{\mathcal{A}}(i, j) = 0 \text{ for any } i \in \mathcal{E} \},
\]

\[
^{\perp_1} \mathcal{E} := \{ j \mid \text{Ext}^1_{\mathcal{A}}(j, i) = 0 \text{ for any } i \in \mathcal{E} \}.
\]
Definition 2.11. (cf. [17, p.751]) Let $\mathcal{E}$ be an ideal of $\mathcal{A}$. The morphism $i : X \to A$ in $\mathcal{A}$ with $i \in \mathcal{E}$ is called an $\mathcal{E}$-precover of $A$ if the induced sequence $\text{Hom}_\mathcal{A}(A', X) \to \mathcal{E}(A', A) \to 0$ is exact for any $A' \in \mathcal{A}$. An $\mathcal{E}$-precover $i : X \to A$ is called special if it is obtained as the pushout of a short exact sequence $\eta$ along a morphism $j \in \mathcal{E}^\perp$ as follows.

\[
\begin{array}{cccccccc}
\eta : 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0.
\end{array}
\]

The ideal $\mathcal{E}$ is called (special) precovering in $\mathcal{A}$ if any object in $\mathcal{A}$ admits a (special) $\mathcal{E}$-precover. Dually, the notions of a (special) $\mathcal{E}$-preenvelope and a (special) preenveloping ideal are defined.

Definition 2.12. (cf. [17, Definition 12]) A pair $(\mathcal{E}, \mathcal{F})$ of ideals of $\mathcal{A}$ is called an ideal cotorsion pair if $\mathcal{E} = \mathcal{F}^\perp$ and $\mathcal{F} = \mathcal{E}^\perp$. An ideal cotorsion pair $(\mathcal{E}, \mathcal{F})$ is called complete if $\mathcal{E}$ is special precovering and $\mathcal{F}$ is special preenveloping in $\mathcal{A}$.

Let $\mathcal{X}$ be an additive subcategory of $\mathcal{A}$. We use $\langle \mathcal{X} \rangle$ to denote the ideal of $\mathcal{A}$ generated by morphisms of the form $1_X$, where $X \in \mathcal{X}$. It is the ideal of morphisms factoring through an object in $\mathcal{X}$. Note that a cotorsion pair and its ideal cotorsion pair generated by the original objects have a nice relation as follows.

Lemma 2.13. (cf. [17, Theorem 28]) If $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair in $\mathcal{A}$, then the pair $(\langle \mathcal{X} \rangle, \langle \mathcal{Y} \rangle)$ is a complete ideal cotorsion pair in $\mathcal{A}$.

3 Gorenstein cophantom objects

In the rest of this section, assume that $\mathcal{C}$ is self-orthogonal (that is, $\mathcal{C} \perp \mathcal{C}$) and $\mathcal{P}(\mathcal{A})$ is a generator for $\mathcal{C}$ except for Section 3.2.

For an object $A$ in $\mathcal{A}$, the $\mathcal{C}$-dimension of $A$, denoted by $\mathcal{C}$-dim $A$, is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to A \to 0$ in $\mathcal{A}$ with all $C_i$ in $\mathcal{C}\}$. Set $\mathcal{C}$-dim $A = \infty$ if no such integer exists (cf. [22]). For a non-negative integer $n$, we use $\mathcal{C} \leq n$ (resp. $\mathcal{C} < \infty$) to denote the subcategory of $\mathcal{A}$ consisting of objects with $\mathcal{C}$-dimension at most $n$ (resp. finite $\mathcal{C}$-dimension).

3.1 Gorenstein cophantom objects

In this subsection, we introduce Gorenstein cophantom objects and give some basic properties of them.

Definition 3.1. An object $M \in \mathcal{A}$ is called Gorenstein cophantom relative to $\mathcal{C}$ if $M \in \mathcal{G}(\mathcal{C})^{\perp}$.

We use $\mathcal{G}(\mathcal{C})$ to denote the subcategory of $\mathcal{A}$ consisting of Gorenstein cophantom objects relative to $\mathcal{C}$. The reason why we call objects in $\mathcal{G}(\mathcal{C})^{\perp}$ Gorenstein cophantom relative to $\mathcal{C}$ is that they are exactly the object version of Gorenstein cophantom morphisms relative to $\mathcal{C}$, see Theorem 4.4.

Example 3.2.
By Proposition 2.7 and [21, Theorem 5.8], we have $\mathcal{P}(A) \subseteq \mathcal{C} \subseteq \mathcal{C}^\leq \subseteq \mathcal{C}G(\mathcal{C})$.

(2) $\mathcal{P}(A)^{\prec\leq} \cup \mathcal{I}(A)^{\prec\leq} \subseteq \mathcal{C}G(\mathcal{C})$.

(3) If the global dimension of $R$ is finite, then $\mathcal{C}G(\mathcal{P}(\text{Mod } R)) = \text{Mod } R$.

(4) By [15, Theorem 11.5.1] and [2, Theorem 31.9], we have that $R$ is quasi-Frobenius if and only if $\mathcal{C}G(\mathcal{P}(\text{Mod } R)) = \mathcal{P}(\text{Mod } R) = \mathcal{I}(\text{Mod } R)$.

For a non-negative integer $n$, recall that a left and right noetherian ring $R$ is called $n$-Gorenstein if the left and right self-injective dimensions of $R$ are at most $n$. The following result is a generalization of Example 3.2(4).

**Example 3.3.** If $R$ is $n$-Gorenstein, then

$$\mathcal{C}G(\mathcal{P}(\text{Mod } R)) = \mathcal{P}(\text{Mod } R)^{\leq n} = \mathcal{P}(\text{Mod } R)^{\prec\leq} = \mathcal{I}(\text{Mod } R)^{\leq n} = \mathcal{I}(\text{Mod } R)^{\prec\leq} \subseteq \mathcal{C}G(\mathcal{P}(\text{Mod } R))$$

**Proof.** By [23, Theorem 2] and Example 3.2(2), we have $\mathcal{P}(\text{Mod } R)^{\leq n} = \mathcal{P}(\text{Mod } R)^{\prec\leq} = \mathcal{I}(\text{Mod } R)^{\leq n} = \mathcal{I}(\text{Mod } R)^{\prec\leq} \subseteq \mathcal{C}G(\mathcal{P}(\text{Mod } R))$.

Now let $M \in \mathcal{C}G(\mathcal{P}(\text{Mod } R))$ and $N \in \text{Mod } R$. Since $R$ is $n$-Gorenstein, there exists an exact sequence

$$0 \to G_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

in $\text{Mod } R$ with all $P_i$ in $\mathcal{P}(\text{Mod } R)$ and $G_n \in \mathcal{G}(\mathcal{P}(\text{Mod } R))$ by [15, Theorem 11.5.1]. Then we have $\text{Ext}^{n+1}_R(N, M) \cong \text{Ext}^1_R(G_n, M) = 0$ and $M \in \mathcal{I}(\text{Mod } R)^{\leq n}$, and thus $\mathcal{C}G(\mathcal{P}(\text{Mod } R)) \subseteq \mathcal{I}(\text{Mod } R)^{\leq n}$.

**Proposition 3.4.**

(1) $\mathcal{C}G(\mathcal{C})$ is closed under direct products, direct summands and extensions.

(2) $\mathcal{C}G(\mathcal{C})$ is projectively resolving.

(3) $\mathcal{C}G(\mathcal{C})$ is injectively coresolving.

**Proof.** (1) It is trivial.

(2) By Example 3.2(1), $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}G(\mathcal{C})$. Let $G \in \mathcal{G}(\mathcal{C})$ and

$$0 \to L \to M \to N \to 0$$

be an exact sequence in $\mathcal{A}$ with $M, N \in \mathcal{C}G(\mathcal{C})$. By Proposition 2.7(1), we have $\text{Ext}^1_\mathcal{A}(G, M) = 0 = \text{Ext}^1_\mathcal{A}(G, N)$. Then $\text{Ext}^2_\mathcal{A}(G, L) = 0$. Because $G \in \mathcal{G}(\mathcal{C})$, we have an exact sequence

$$0 \to G \to C^0 \to G^1 \to 0$$

in $\mathcal{A}$ with $C^0 \in \mathcal{C}$ and $G^1 \in \mathcal{G}(\mathcal{C})$. For $C^0$, there exists an exact sequence

$$0 \to C^{-1} \to P^0 \to C^0 \to 0$$


in \( \mathcal{A} \) with \( P^0 \in \mathcal{P}(\mathcal{A}) \) and \( C^{-1} \in \mathcal{C} \). Consider the following pullback diagram

\[
\begin{array}{c}
  0 \\
  \downarrow \\
  C^{-1} \cong C^{-1} \\
  \downarrow \\
  0 \rightarrow G^0 \rightarrow P^0 \rightarrow 0 \\
  \downarrow \\
  0 \rightarrow G \rightarrow C^0 \rightarrow G^1 \rightarrow 0 \\
  \downarrow \\
  0 \\
\end{array}
\]

By the above argument, we have \( \text{Ext}^1_{\mathcal{A}}(G^0, L) \cong \text{Ext}^2_{\mathcal{A}}(G^1, L) = 0 \). Because the leftmost column splits by Proposition 2.7(2), \( G \) is isomorphic to a direct summand of \( G^0 \) and \( \text{Ext}^1_{\mathcal{A}}(G, L) = 0 \), which shows that \( L \in \text{CG}(\mathcal{C}) \).

(3) It is trivial that \( \mathcal{I}(\mathcal{A}) \subseteq \text{CG}(\mathcal{C}) \). By Proposition 2.7, we have that \( \text{CG}(\mathcal{C}) \) is closed under cokernels of monomorphisms. Thus \( \text{CG}(\mathcal{C}) \) is injectively coresolving.

Before giving some applications of Proposition 3.4(2), consider the following example.

**Example 3.5.** Let \( Q \) be a quiver:

\[
\begin{array}{c}
  2 \\
  \downarrow \\
  1 \hspace{1cm} 3 \\
\end{array}
\]

and \( I = \langle a_1a_3a_2, a_2a_1a_3 \rangle \). Let \( R = kQ/I \) with \( k \) a field. Then the Auslander-Reiten quiver \( \Gamma(\text{mod } R) \) of \( \text{mod } R \) is as follows.

\[
\begin{array}{c}
  1 \\
  \downarrow \\
  2 \hspace{1cm} 3 \\
\end{array}
\]

By a direct computation, we have that

\[
\begin{array}{c}
  1 \\
  \downarrow \\
  2 \hspace{1cm} 3 \\
\end{array}
\]

\[
\begin{array}{c}
  2 \\
  \downarrow \\
  3 \hspace{1cm} 1 \\
\end{array}
\]

\[
\begin{array}{c}
  3 \\
  \downarrow \\
  1 \hspace{1cm} 2 \\
\end{array}
\]

where the terms marked by a box are indecomposable projective modules in \( \text{mod } R \). Then we get that the intersection of \( \mathcal{G}(\mathcal{P}(\text{mod } R)) \) and \( \text{CG}(\mathcal{P}(\text{mod } R)) \) precisely equals to the subcategory \( \mathcal{P}(\text{mod } R) \) of \( \text{mod } R \).
In general, we have the following

**Corollary 3.6.** If \( \mathcal{C} \) is closed under direct summands, then for any \( n \geq 0 \), we have

\[
\mathcal{G}(\mathcal{C})^{\leq n} \cap \mathcal{C} \mathcal{G}(\mathcal{C}) = \mathcal{C}^{\leq n}.
\]

**Proof.** By Example 3.2(1), we have \( \mathcal{C}^{\leq n} \subseteq \mathcal{G}(\mathcal{C})^{\leq n} \cap \mathcal{C} \mathcal{G}(\mathcal{C}) \).

Now let \( M \in \mathcal{G}(\mathcal{C})^{\leq n} \cap \mathcal{C} \mathcal{G}(\mathcal{C}) \). By [21, Theorem 5.8], there exists an exact sequence

\[
0 \to K_n \to C_n \to \cdots \to C_0 \to M \to 0
\]

in \( \mathcal{A} \) with all \( C_i \) in \( \mathcal{C} \) and \( K_n \in \mathcal{G}(\mathcal{C}) \). Because \( \mathcal{C} \) is closed under direct summands by assumption, it follows easily from the definition of \( \mathcal{G}(\mathcal{C}) \) that \( K_n \in \mathcal{C} \) and \( M \in \mathcal{C}^{\leq n} \).

**Proposition 3.7.** For any \( M \in \mathcal{A} \), the following statements are equivalent.

1. \( M \in \mathcal{C} \mathcal{G}(\mathcal{C}) \).
2. The functor \( \text{Hom}_\mathcal{A}(-,M) \) is exact with respect to any short exact sequence in \( \mathcal{A} \) ending with an object in \( \mathcal{G}(\mathcal{C}) \).
3. Every short exact sequence starting with \( M \) is \( \text{Hom}_\mathcal{A}(\mathcal{G}(\mathcal{C}),-) \)-exact.

If, moreover, \( R \) is a commutative ring, \( \mathcal{A} = \text{Mod} R \) and \( \mathcal{C} = \mathcal{P}(\text{Mod} R) \), then the above conditions are equivalent to the following

4. \( \text{Hom}_R(Q,M) \in \mathcal{C} \mathcal{G}(\mathcal{P}(\text{Mod} R)) \) for any \( Q \in \mathcal{P}(\text{Mod} R) \).

**Proof.** (1) \( \iff \) (2) \( \iff \) (3) It is easy.

Now let \( R \) be a commutative ring.

(1) \( \Rightarrow \) (4) For any \( G \in \mathcal{G}(\mathcal{P}(\text{Mod} R)) \), we have an exact sequence

\[
0 \to K \overset{f}{\to} P \to G \to 0
\]

in \( \text{Mod} R \) with \( P \in \mathcal{P}(\text{Mod} R) \). Let \( Q \in \mathcal{P}(\text{Mod} R) \). Then

\[
0 \to Q \otimes_R K \overset{1_Q \otimes f}{\to} Q \otimes_R P \to Q \otimes_R G \to 0
\]

is exact. It is easy to check that \( Q \otimes_R G \in \mathcal{G}(\mathcal{P}(\text{Mod} R)) \). Then \( \text{Ext}^1_R(\mathcal{G}(\mathcal{P}(\text{Mod} R)), M) = 0 \) by (1), and so \( \text{Hom}_R(1_Q \otimes f, M) \) is epic. By the adjoint isomorphism, we have that \( \text{Hom}_R(f, \text{Hom}_R(Q, M)) \) is also epic. So applying the functor \( \text{Hom}_R(-, \text{Hom}_R(Q, M)) \) to (3.1) we get \( \text{Ext}^1_R(G, \text{Hom}_R(Q, M)) = 0 \), and hence \( \text{Hom}_R(Q, M) \in \mathcal{C} \mathcal{G}(\mathcal{P}(\text{Mod} R)) \).

(4) \( \Rightarrow \) (1) It is trivial by setting \( Q = R \).

In the following result, we characterize categories over which all objects are in \( \mathcal{C} \mathcal{G}(\mathcal{C}) \).

**Proposition 3.8.** Assume that \( \mathcal{C} \) is closed under direct summands. Consider the following conditions.

1. \( \mathcal{C} \mathcal{G}(\mathcal{C}) = \mathcal{A} \).
(2) $\mathcal{G}(\mathcal{C}) \subseteq \mathcal{CG}(\mathcal{C})$.

(3) $\mathcal{G}(\mathcal{C}) = \mathcal{C}$.

Then we have $(1) \Rightarrow (2) \Rightarrow (3)$. If $\mathcal{C}$ is a projective generator for $\mathcal{A}$, then all of them are equivalent.

Proof. The implication $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (3)$ Let $G \in \mathcal{G}(\mathcal{C})$. Then there exists an exact sequence

$$0 \to G_1 \to C_0 \to G \to 0$$

in $\mathcal{A}$ with $C_0 \in \mathcal{C}$ and $G_1 \in \mathcal{G}(\mathcal{C})$. By $(2)$, we have that $G_1 \in \mathcal{CG}(\mathcal{C})$ and the above exact sequence splits. Thus as a direct summand of $C_0$, $G \in \mathcal{C}$ by assumption.

If $\mathcal{C}$ is a projective generator for $\mathcal{A}$, then the implication $(3) \Rightarrow (1)$ follows directly. \qed

### 3.2 The pair $(\mathcal{G}(\mathcal{C}), \mathcal{CG}(\mathcal{C}))$

The following result is useful in the next section.

**Proposition 3.9.** The subcategory $\mathcal{G}(\mathcal{C})$ is special precovering in $\mathcal{A}$ if and only if $(\mathcal{G}(\mathcal{C}), \mathcal{CG}(\mathcal{C}))$ is a complete cotorsion pair.

Proof. The necessity is trivial. For the sufficiency, we first prove that $(\mathcal{G}(\mathcal{C}), \mathcal{CG}(\mathcal{C}))$ is a cotorsion pair. Because $\mathcal{G}(\mathcal{C}) \perp_1 = \mathcal{CG}(\mathcal{C})$, we have $\mathcal{G}(\mathcal{C}) \subseteq \perp_1 \mathcal{CG}(\mathcal{C})$. Now let $M \in \perp_1 \mathcal{CG}(\mathcal{C})$. Because $\mathcal{G}(\mathcal{C})$ is special precovering in $\mathcal{A}$ by assumption, there exists an exact sequence

$$0 \to K \to G \to M \to 0$$

in $\mathcal{A}$ with $G, M \in \mathcal{G}(\mathcal{C})$ and $K \in \mathcal{CG}(\mathcal{C})$. Then it splits. So as a direct summand of $G$, $M \in \mathcal{G}(\mathcal{C})$ by [21, Theorem 4.6(2)]. Thus $\perp_1 \mathcal{CG}(\mathcal{C}) \subseteq \mathcal{G}(\mathcal{C})$ and $\mathcal{G}(\mathcal{C}) = \perp_1 \mathcal{CG}(\mathcal{C})$, and therefore $(\mathcal{G}(\mathcal{C}), \mathcal{CG}(\mathcal{C}))$ is a cotorsion pair. By Lemma 2.9, $(\mathcal{G}(\mathcal{C}), \mathcal{CG}(\mathcal{C}))$ is complete. \qed

Motivated by Proposition 3.9, we introduce the following

**Definition 3.10.** $\mathcal{A}$ is called Gorenstein complete relative to $\mathcal{C}$ if $\mathcal{G}(\mathcal{C})$ is special precovering in $\mathcal{A}$.

A ring $R$ is called Gorenstein complete if $\text{Mod} R$ is Gorenstein complete relative to $\mathcal{P}(\text{Mod} R)$, that is, $\mathcal{G}(\mathcal{P}(\text{Mod} R))$ is special precovering in $\text{Mod} R$. The following list shows that the class of Gorenstein complete rings is rather large.

**Example 3.11.** The class of Gorenstein complete rings includes the following

(1) Commutative Noetherian rings of finite Krull dimension ([11, Remark 5.8]).

(2) Rings in which all projective left $R$-modules have finite injective dimension ([30, Corollary 4.3]); especially, Gorenstein rings (that is, $n$-Gorenstein rings for some $n \geq 0$).

(3) Right coherent rings in which all flat $R$-modules have finite projective dimension ([3, Theorem 3.5] and [9, Proposition 8.10]); especially, right coherent and left perfect rings, and right Artinian rings.
3.3 Characterizing special Gorenstein projective precovers

In this subsection, we investigate the structure of a certain subcategory of $\mathcal{A}$ consisting of objects admitting special $\mathcal{G}(\mathcal{C})$-precovers in terms of the properties of $\mathcal{C}\mathcal{G}(\mathcal{C})$. Note that $\mathcal{C}\mathcal{G}(\mathcal{C})$ has enough projective objects in $\mathcal{A}$ if and only if $\mathcal{P}(\mathcal{A})$ is a generator for $\mathcal{C}\mathcal{G}(\mathcal{C})$, and if and only if $\mathcal{C}$ is a generator for $\mathcal{C}\mathcal{G}(\mathcal{C})$. We will use this fact in the sequel freely.

**Proposition 3.12.**

(1) Let $M \in \mathcal{C}\mathcal{G}(\mathcal{C})$ and $f : C \to M$ be an epimorphism in $\mathcal{A}$ with $C \in \mathcal{C}$. Then $\text{Ker } f \in \mathcal{C}\mathcal{G}(\mathcal{C})$ and $f$ is a special $\mathcal{G}(\mathcal{C})$-precover of $M$.

(2) Consider an exact sequence

$$0 \to M' \to C \to M \to 0.$$  \hspace{1cm} (3.2)

If $M'$ admits special $\mathcal{G}(\mathcal{C})$-precover, then so is $M$. The converse is true if $\mathcal{C}\mathcal{G}(\mathcal{C})$ has enough projective objects in $\mathcal{A}$ and (3.2) is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact.

**Proof.** (1) The assertion follows from Example 3.2(1) and Proposition 3.4(2).

(2) Assume that $M'$ admits a special $\mathcal{G}(\mathcal{C})$-precover and

$$0 \to N \to G \to M' \to 0$$

is an exact sequence in $\mathcal{A}$ with $G \in \mathcal{G}(\mathcal{C})$ and $N \in \mathcal{C}\mathcal{G}(\mathcal{C})$. Combining it with the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact exact sequence

$$0 \to G \xrightarrow{i} C^0 \xrightarrow{p} G^1 \to 0$$

in $\mathcal{A}$ with $C^0 \in \mathcal{C}$ and $G^1 \in \mathcal{G}(\mathcal{C})$, we get the following commutative diagram with exact columns and rows

\[
\begin{array}{ccccccccc}
0 & \to & G & \xrightarrow{i} & C^0 & \xrightarrow{p} & G^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow g & & \downarrow h & & \\
0 & \to & M' & \to & C & \to & M & \to & 0 \\
\end{array}
\]

Adding the exact sequence $0 \to 0 \to C \xrightarrow{1_0} C \to 0$ to the middle row, we obtain the following com-
mutative diagram with exact columns and rows

\[
\begin{array}{ccc}
0 & \to & N \\
\downarrow & & \downarrow \\
0 & \to & G \\
\downarrow^{(g)} & & \downarrow \\
0 & \to & C_0 \\
\downarrow & & \downarrow \\
0 & \to & M' \\
\downarrow & & \downarrow \\
0 & \to & C \\
\downarrow & & \downarrow \\
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

which can be completed to a commutative diagram with exact columns and rows as follows.

\[
\begin{array}{ccc}
0 & \to & N \\
\downarrow & & \downarrow \\
0 & \to & G \\
\downarrow^{(g)} & & \downarrow \\
0 & \to & C'' \\
\downarrow & & \downarrow \\
0 & \to & M'' \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

Note that \( G^{1} \oplus C \in \mathcal{G}(\mathcal{C}) \). Moreover, since \( N \in \mathcal{C}G(\mathcal{C}) \), we have \( M'' \in \mathcal{C}G(\mathcal{C}) \) by Proposition 3.4(3). Thus the rightmost column in the above diagram is a special \( \mathcal{G}(\mathcal{C}) \)-precover of \( M \).

Now let \( \mathcal{C}G(\mathcal{C}) \) have enough projective objects in \( \mathcal{A} \) and (3.2) be \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact. Assume that \( M \) admits a special \( \mathcal{G}(\mathcal{C}) \)-precover and

\[
0 \to L \to G \to M \to 0,
\]

\[
0 \to L' \to C' \to L \to 0
\]

are exact sequences in \( \mathcal{A} \) with \( G \in \mathcal{G}(\mathcal{C}) \), \( L \in \mathcal{C}G(\mathcal{C}) \) and \( C' \in \mathcal{C} \). By [21, Lemma 3.1(1)], we get the following commutative diagram with exact columns and rows

\[
\begin{array}{ccc}
0 & \to & L' \\
\downarrow & & \downarrow \\
0 & \to & G' \\
\downarrow & & \downarrow \\
0 & \to & M' \\
\downarrow & & \downarrow \\
0 & \to & C' \\
\downarrow & & \downarrow \\
0 & \to & C \\
\downarrow & & \downarrow \\
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]
By Propositions 2.7(2) and 3.4(2), we have \( L' \in \mathcal{G}(\mathcal{C}) \) and the leftmost column is \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact. So the middle column is also \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact. On the other hand, the middle column is \( \text{Hom}_{\mathcal{A}}(-, \mathcal{C}) \)-exact by Proposition 2.7(2). So \( G' \in \mathcal{G}(\mathcal{C}) \) by [21, Proposition 4.7(5)], and hence the upper row is a special \( \mathcal{G}(\mathcal{C}) \)-precover of \( M' \).

We use \( \text{SPC}(\mathcal{G}(\mathcal{C})) \) to denote the subcategory of \( \mathcal{A} \) consisting of objects admitting special \( \mathcal{G}(\mathcal{C}) \)-precovers. For the sake of convenience, we say that a subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is closed under \( \mathcal{C} \)-free direct summands provided that the condition \( \mathcal{X} \oplus C \in \mathcal{X} \) with \( C \in \mathcal{C} \) implies \( X \in \mathcal{X} \).

**Theorem 3.13.**

(1) \( \text{SPC}(\mathcal{G}(\mathcal{C})) \) is closed under extensions.

(2) \( \text{SPC}(\mathcal{G}(\mathcal{C})) \) is closed under \( \mathcal{C} \)-free direct summands.

**Proof.** (1) Let

\[
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0
\]

be an exact sequence in \( \mathcal{A} \). Assume that \( L \) and \( N \) admit special \( \mathcal{G}(\mathcal{C}) \)-precovers and

\[
0 \rightarrow L' \rightarrow G_L \xrightarrow{f} L \rightarrow 0, \\
0 \rightarrow N' \rightarrow G_N \xrightarrow{g} N \rightarrow 0
\]

are exact sequences in \( \mathcal{A} \) with \( G_L, G_N \in \mathcal{G}(\mathcal{C}) \) and \( L', N' \in \mathcal{G}(\mathcal{C}) \). Consider the following pullback diagram

\[
\begin{array}{c}
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \\
\downarrow \alpha \downarrow \leftarrow \downarrow g \\
0 \rightarrow L' \rightarrow Q \rightarrow G_N \rightarrow 0 \\
\downarrow \beta \downarrow \leftarrow \downarrow f \\
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0.
\end{array}
\]

Since \( \text{Ext}_R^2(G_N, L') = 0 \) by Proposition 2.7(1), we get an epimorphism \( \text{Ext}_R^1(G_N, f) : \text{Ext}_R^1(G_N, G_L) \rightarrow \text{Ext}_R^1(G_N, L) \). It induces the following commutative diagram with exact rows

\[
\begin{array}{c}
0 \rightarrow G_L \rightarrow G_M \rightarrow G_N \rightarrow 0 \\
\downarrow f \downarrow \beta \downarrow \leftarrow \downarrow g \\
0 \rightarrow L \rightarrow Q \rightarrow G_N \rightarrow 0 \\
\downarrow \alpha \downarrow \leftarrow \downarrow \alpha \\
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0.
\end{array}
\]

Set \( M' := \text{Ker} \alpha \beta \). Then we get the following commutative diagram with exact columns and rows

\[
\begin{array}{c}
0 \rightarrow 0 \rightarrow 0 \\
\downarrow \downarrow \downarrow \\
0 \rightarrow L' \rightarrow M' \rightarrow N' \rightarrow 0 \\
\downarrow \downarrow \downarrow \downarrow \\
0 \rightarrow G_L \rightarrow G_M \rightarrow G_N \rightarrow 0 \\
\downarrow \downarrow \downarrow \downarrow \\
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \\
\downarrow \downarrow \downarrow \downarrow \\
0 \rightarrow 0 \rightarrow 0.
\end{array}
\]
Note that $G_M \in \mathcal{G}(\mathcal{C})$ (by [28, Corollary 4.5]) and $M' \in \mathcal{C}\mathcal{G}(\mathcal{C})$ (by Proposition 3.4(1)). Thus the middle column in the above diagram is a special $\mathcal{G}(\mathcal{C})$-precover of $M$. This proves that $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is closed under extensions.

(2) Let $M \in \text{SPC}(\mathcal{G}(\mathcal{C}))$ and 
$$0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$$
be an exact sequence in $\mathcal{A}$ with $G \in \mathcal{G}(\mathcal{C})$ and $K \in \mathcal{C}\mathcal{G}(\mathcal{C})$. Assume that $M \cong L \oplus C$ with $C \in \mathcal{C}$, we have an exact and split sequence 
$$0 \rightarrow C \rightarrow M \rightarrow L \rightarrow 0$$
in $\mathcal{A}$. Consider the following pullback diagram

\[
\begin{array}{c}
\begin{array}{cccccc}
0 & 0 \\
& & \\
& & \\
0 & \rightarrow & K & \rightarrow & L' & \rightarrow & C & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
0 & \rightarrow & K & \rightarrow & G & \rightarrow & M & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
L & = & = & L & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
0 & 0 \\
\end{array}
\end{array}
\]

Since $K, C \in \mathcal{C}\mathcal{G}(\mathcal{C})$, we have $L' \in \mathcal{C}\mathcal{G}(\mathcal{C})$ by Proposition 3.4(1). Thus the middle column in the above diagram is a special $\mathcal{G}(\mathcal{C})$-precover of $L$. \hfill \square

The following question seems to be interesting.

**Question 3.14.** Is $\text{SPC}(\mathcal{G}(\mathcal{C}))$ closed under direct summands?

The following result shows that $\text{SPC}(\mathcal{G}(\mathcal{C}))$ possesses certain minimality.

**Theorem 3.15.** Assume that $\mathcal{C}\mathcal{G}(\mathcal{C})$ has enough projective objects in $\mathcal{A}$. Then we have

(1) $\mathcal{C}\mathcal{G}(\mathcal{C}) \cup \mathcal{G}(\mathcal{C}) \subseteq \text{SPC}(\mathcal{G}(\mathcal{C}))$ and $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is closed under extensions and $\mathcal{C}$-free direct summands.

(2) $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is the minimal subcategory with respect to the property (1) as above.

To prove this theorem, we need the following

**Lemma 3.16.** Let 
$$0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$$
be an exact sequence in $\mathcal{A}$ with $K \in \mathcal{C}\mathcal{G}(\mathcal{C})$ and $G \in \mathcal{G}(\mathcal{C})$. Then there exists an exact sequence 
$$0 \rightarrow G \rightarrow M \oplus C \rightarrow K' \rightarrow 0$$
in $\mathcal{A}$ with $K' \in \mathcal{C}\mathcal{G}(\mathcal{C})$ and $C \in \mathcal{C}$. 
Proof. Let

\[ 0 \to K \to G \to M \to 0 \]

be an exact sequence in \( \mathcal{A} \) with \( K \in \mathcal{C}(\mathcal{C}) \) and \( G \in \mathcal{G}(\mathcal{C}) \). Since \( G \in \mathcal{G}(\mathcal{C}) \), there exists a \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact exact sequence

\[ 0 \to G \to C \to G' \to 0 \]

in \( \mathcal{A} \) with \( C \in \mathcal{C} \) and \( G' \in \mathcal{G}(\mathcal{C}) \). Consider the following pushout diagram

\[
\begin{array}{ccc}
0 & \to & K \\
\downarrow & & \downarrow \\
0 & \to & G \\
\downarrow & & \downarrow \\
0 & \to & M \\
\end{array}
\]

Since \( K, C \in \mathcal{C}(\mathcal{C}) \), we have \( K' \in \mathcal{C}(\mathcal{C}) \) by Proposition 3.4(3).

Consider the following pullback diagram

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \\
G = \mathcal{C} & \to & C \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

Since \( K, G \in \mathcal{C}(\mathcal{C}) \), we have \( K' \in \mathcal{C}(\mathcal{C}) \) by Proposition 3.4(3).

Proof of Theorem 3.15. (1) It follows from Proposition 3.12(1) and Theorem 3.13.

(2) Let \( \mathcal{X} \) be a subcategory of \( \mathcal{A} \) such that \( \mathcal{C}(\mathcal{C}) \cup \mathcal{G}(\mathcal{C}) \subseteq \mathcal{X} \) and \( \mathcal{X} \) is closed under extensions and \( \mathcal{C} \)-free direct summands. Let \( M \in \text{SPC}(\mathcal{G}(\mathcal{C})) \). Then by Lemma 3.16, we have an exact sequence

\[ 0 \to G \to M \oplus C \to K' \to 0 \]

in \( \mathcal{A} \) with \( K' \in \mathcal{C}(\mathcal{C}) \), \( G \in \mathcal{G}(\mathcal{C}) \) and \( C \in \mathcal{C} \). Because \( G, K' \in \mathcal{X} \), we have that \( M \oplus C \in \mathcal{X} \) and \( M \in \mathcal{X} \). It follows that \( \text{SPC}(\mathcal{G}(\mathcal{C})) \subseteq \mathcal{X} \).

As an immediate consequence of Theorem 3.15, we get the following
Corollary 3.17. Let $R$ be a Gorenstein complete ring and $\mathcal{X}$ a full subcategory of $\text{Mod } R$ closed under isomorphisms. If $\mathcal{C}(\mathcal{P}(\text{Mod } R)) \cup \mathcal{G}(\mathcal{P}(\text{Mod } R)) \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under extensions and $\mathcal{P}(\text{Mod } R)$-free direct summands, then $\mathcal{X} = \text{Mod } R$.

Proof. If $R$ is a Gorenstein complete ring, then $\text{SPC}(\mathcal{G}(\mathcal{P}(\text{Mod } R))) = \text{Mod } R$. Now the assertion follows from Theorem 3.15. \qed

We recall the following definition from [22].

Definition 3.18. Let $\mathcal{C}$, $\mathcal{T}$ and $\mathcal{E}$ be subcategories of $\mathcal{A}$ with $\mathcal{C} \subseteq \mathcal{T}$.

(1) ([22, Definition 2.5(b)]) $\mathcal{C}$ is called an $\mathcal{E}$-proper generator (resp. $\mathcal{E}$-coproper cogenerator) for $\mathcal{T}$ if for any object $T$ in $\mathcal{T}$, there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ (resp. $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$)-exact exact sequence $0 \to T' \to C \to T \to 0$ (resp. $0 \to T \to C \to T' \to 0$) in $\mathcal{A}$ such that $C$ is an object in $\mathcal{C}$ and $T'$ is an object in $\mathcal{T}$.

(2) ([22, Definition 3.1]) $\mathcal{T}$ is called $\mathcal{E}$-preresolving in $\mathcal{A}$ if the following conditions are satisfied.

(i) $\mathcal{T}$ admits an $\mathcal{E}$-proper generator.

(ii) $\mathcal{T}$ is closed under $\mathcal{E}$-proper extensions, that is, for any $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$, if both $A_1$ and $A_3$ are objects in $\mathcal{T}$, then $A_2$ is also an object in $\mathcal{T}$.

An $\mathcal{E}$-preresolving subcategory $\mathcal{T}$ of $\mathcal{A}$ is called $\mathcal{E}$-resolving if the following condition is satisfied.

(iii) $\mathcal{T}$ is closed under kernels of $\mathcal{E}$-proper epimorphisms, that is, for any $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$, if both $A_2$ and $A_3$ are objects in $\mathcal{T}$, then $A_1$ is also an object in $\mathcal{T}$.

In the following, we investigate when $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is $\mathcal{C}$-resolving. We need the following two lemmas.

Lemma 3.19. For any $M \in \text{SPC}(\mathcal{G}(\mathcal{C}))$, there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact exact sequence

$$0 \to K \to C \to M \to 0$$

in $\mathcal{A}$ with $C \in \mathcal{C}$.

Proof. Let $M \in \text{SPC}(\mathcal{G}(\mathcal{C}))$. Then there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact exact sequence

$$0 \to K' \to G \to M \to 0$$

in $\mathcal{A}$ with $G \in \mathcal{G}(\mathcal{C})$ and $K' \in \mathcal{C}\mathcal{G}(\mathcal{C})$. For $G$, there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact exact sequence

$$0 \to G' \to C \to G \to 0$$

in $\mathcal{A}$.
in \( \mathcal{A} \) with \( C \in \mathcal{C} \) and \( G' \in \mathcal{G}(\mathcal{C}) \). Consider the following pullback diagram

\[
\begin{array}{ccc}
0 & \rightarrow & L \\
\downarrow & & \downarrow \\
K & \rightarrow & C \\
\downarrow & & \downarrow \\
0 & \rightarrow & M
\end{array}
\]

By [21, Lemma 2.5], the middle row is \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact, as desired.

**Lemma 3.20.** Assume that \( \mathcal{G}(\mathcal{C}) \) has enough projective objects in \( \mathcal{A} \). Given a \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact exact sequence

\[
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0
\]

in \( \mathcal{A} \), we have

1. If \( M, N \in \text{SPC}(\mathcal{G}(\mathcal{C})) \), then \( L \in \text{SPC}(\mathcal{G}(\mathcal{C})) \).

2. If \( L, M \in \text{SPC}(\mathcal{G}(\mathcal{C})) \) and there exists a \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact exact sequence

\[
0 \rightarrow K \rightarrow C \rightarrow N \rightarrow 0
\]

in \( \mathcal{A} \) with \( C \in \mathcal{C} \), then \( N \in \text{SPC}(\mathcal{G}(\mathcal{C})) \).

**Proof.** Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be a \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact exact sequence in \( \mathcal{A} \).

1. Assume that \( M, N \in \text{SPC}(\mathcal{G}(\mathcal{C})) \). By Lemma 3.19, there exists a \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact exact sequence

\[
0 \rightarrow K \rightarrow C \rightarrow N \rightarrow 0
\]

in \( \mathcal{A} \) with \( C \in \mathcal{C} \). Consider the following pullback diagram

\[
\begin{array}{ccc}
0 & \rightarrow & L \\
\downarrow & & \downarrow \\
K & \rightarrow & C \\
\downarrow & & \downarrow \\
0 & \rightarrow & M \\
\downarrow & & \downarrow \\
0 & \rightarrow & N
\end{array}
\]

By Proposition 3.12(2), \( K \in \text{SPC}(\mathcal{G}(\mathcal{C})) \). Then it follows from Theorem 3.13(1) and the exactness of the middle column that \( T \in \text{SPC}(\mathcal{G}(\mathcal{C})) \). Notice that the middle row \( \text{Hom}_{\mathcal{A}}(\mathcal{C}, -) \)-exact by [21, Lemma 2.4(1)], so it splits and \( T \cong L \oplus C \). Thus \( L \in \text{SPC}(\mathcal{G}(\mathcal{C})) \) by Theorem 3.13(2).
(2) Assume $L, M \in \text{SPC}(\mathcal{G}(\mathcal{C}))$ and there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact exact sequence

$$0 \to K \to C \to N \to 0$$

in $\mathcal{A}$ with $C \in \mathcal{C}$. As in the above diagram, since $L, C \in \text{SPC}(\mathcal{G}(\mathcal{C}))$, we have $T \in \text{SPC}(\mathcal{G}(\mathcal{C}))$ by Theorem 3.13(1). Moreover, the middle column is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact by [21, Lemma 2.4(1)]. So $K \in \text{SPC}(\mathcal{G}(\mathcal{C}))$ by (1), and hence $N \in \text{SPC}(\mathcal{G}(\mathcal{C}))$ by Proposition 3.12(2).

Now we are ready to prove the following

**Theorem 3.21.** Assume that $\mathcal{C}\mathcal{G}(\mathcal{C})$ has enough projective objects in $\mathcal{A}$. Then $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is $\mathcal{C}$-resolving with a $\mathcal{C}$-proper generator $\mathcal{C}$.

**Proof.** Following Theorem 3.13(1) and Lemma 3.20, we know that $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is closed under $\mathcal{C}$-proper extensions and kernels of $\mathcal{C}$-proper epimorphisms. Now let $M \in \text{SPC}(\mathcal{G}(\mathcal{C}))$. Then by Lemma 3.19, there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact exact sequence

$$0 \to K \to C \to M \to 0$$

in $\mathcal{A}$ with $C \in \mathcal{C}$. By Proposition 3.12(2), we have $K \in \text{SPC}(\mathcal{G}(\mathcal{C}))$. It follows that $\mathcal{C}$ is a $\mathcal{C}$-proper generator for $\text{SPC}(\mathcal{G}(\mathcal{C}))$ and $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is a $\mathcal{C}$-resolving. □

As a consequence, we get the following

**Corollary 3.22.** If $\mathcal{C}$ is a projective generator for $\mathcal{A}$, then $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is projectively resolving and injectively coresolving.

**Proof.** Let $\mathcal{C}$ be a projective generator for $\mathcal{A}$. Then $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is projectively resolving by Theorem 3.21. Now let $I$ be an injective object in $\mathcal{A}$ and

$$0 \to K \to P \xrightarrow{f} I \to 0$$

an exact sequence in $\mathcal{A}$ with $P \in \mathcal{C}$. Then it is easy to see that $K \in \mathcal{C}\mathcal{G}(\mathcal{C})$ by Example 3.2(1) and Proposition 3.4(2). So $f$ is a special $\mathcal{G}(\mathcal{C})$-precover of $I$ and $I \in \text{SPC}(\mathcal{G}(\mathcal{C}))$. On the other hand, by Lemma 3.20, we have that $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is closed under cokernels of monomorphisms. Thus we conclude that $\text{SPC}(\mathcal{G}(\mathcal{C}))$ is injectively coresolving. □

### 4 Gorenstein cophantom and projective morphisms

#### 4.1 Gorenstein cophantom morphisms

Recall that two extensions

$$\xi : 0 \to L \to M \to N \to 0$$

and

$$\xi' : 0 \to L \to M' \to N \to 0$$

of $L$ by $N$ in $\mathcal{A}$ are said to be *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc}
0 & \to & L \\
\downarrow & & \downarrow \\
M & \to & N \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}$$

and

$$\begin{array}{ccc}
0 & \to & L \\
\downarrow & & \downarrow \\
M' & \to & N \\
\downarrow & & \downarrow \\
0 & \to & 0.
\end{array}$$

This is a fundamental concept in the study of cophantom and projective morphisms.
For the sake of convenience, we denote by $[\xi]$ the equivalence class of $\xi$, and define

$$\text{GExt}^1_A(N, L) := \{[\xi] \mid \xi: 0 \to L \to M \to N \to 0 \text{ is a Hom}_A(\mathcal{C}, -)-exact extension}\}.$$ 

Accordingly, the symbol GExt stands for the class of all equivalence classes of Hom$_A(\mathcal{C}, -)$-exact extensions.

**Remark 4.1.** Let $R$ be a Gorenstein ring.

1. By [15, Theorem 11.5.1], any module in Mod $R$ admits a $\mathcal{G}(\mathcal{P}(\text{Mod} R))$-precover. So for any $M \in \text{Mod} R$, there exists a $\text{Hom}_R(\mathcal{G}(\mathcal{P}(\text{Mod} R)), -)$-exact exact sequence

$$\cdots \to G_1 \to G_0 \to M \to 0 \quad (4.1)$$

in Mod $R$ with all $G_i$ in $\mathcal{G}(\mathcal{P}(\text{Mod} R))$. On the other hand, any module in Mod $R$ admits a $\mathcal{G}(\mathcal{I}(\text{Mod} R))$-preenvelope by [15, Theorem 11.2.1]. So for any $N \in \text{Mod} R$, there exists a $\text{Hom}_R(-, \mathcal{G}(\mathcal{I}(\text{Mod} R)))$-exact exact sequence

$$0 \to N \to G^0 \to G^1 \to \cdots \quad (4.2)$$

in Mod $R$ with all $G^i$ in $\mathcal{G}(\mathcal{I}(\text{Mod} R))$. By [15, Theorem 12.1.4], $\text{Hom}_R(-, -)$ is right balanced on Mod $R \times$ Mod $R$ by $\mathcal{G}(\mathcal{P}(\text{Mod} R)) \times \mathcal{G}(\mathcal{I}(\text{Mod} R))$. Let Gext$_R^1(-, -)$ denote the $n$th right derived functor of $\text{Hom}_R(-, -)$ with respect to the pair $\mathcal{G}(\mathcal{P}(\text{Mod} R)) \times \mathcal{G}(\mathcal{I}(\text{Mod} R))$. Then for any $M, N \in \text{Mod} R$, Gext$_R^1(M, N)$ may be computed by using (4.1) or (4.2).

2. It is easy to check that GExt coincides with Gext, that is, an exact sequence

$$0 \to N \to L \to M \to 0$$

in Mod $R$ belongs to GExt$_R^1(M, N)$ if and only if it is $\text{Hom}_R(\mathcal{G}(\mathcal{P}(\text{Mod} R)), -)$-exact, and if and only if it is $\text{Hom}_R(-, \mathcal{G}(\mathcal{I}(\text{Mod} R)))$-exact.

By [21, Lemma 2.4], we immediately have the following

**Proposition 4.2.** GExt is closed under pullbacks and pushouts.

According to this proposition, if $h \in \text{Hom}_A(N', N)$, then we may define a map

$$\text{GExt}^1_A(h, L) : \text{GExt}^1_A(N, L) \to \text{GExt}^1_A(N', L)$$

which is induced by the pullback operation; dually, if $\psi \in \text{Hom}_A(L, L')$, then we may define a map

$$\text{GExt}^1_A(N, \psi) : \text{GExt}^1_A(N, L) \to \text{GExt}^1_A(N, L')$$

which is induced by the pushout operation.

**Definition 4.3.** (cf. [17, p.774]) Let $\psi \in \text{Hom}_A(L, L')$ and

$$0 \to L \to M \to N \to 0$$
be an exact sequence in \( \mathcal{A} \). Consider the following pushout diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & &  \\
0 & \rightarrow & L' & \rightarrow & M' & \rightarrow & N & \rightarrow & 0 \\
\end{array}
\]

If the bottom row of any such pushout is \( \text{Hom}_\mathcal{A}(\mathcal{G}(\mathcal{C}), -) \)-exact, then we call the morphism \( \varphi \) a \textit{Gorenstein cophantom morphism} relative to \( \mathcal{C} \).

We use \( \text{Coph} - \mathcal{G}(\mathcal{C}) \) to denote the class of Gorenstein cophantom morphisms relative to \( \mathcal{C} \), and use \( \text{Ob}(\text{Coph} - \mathcal{G}(\mathcal{C})) \) to denote the class of those objects \( L \) such that the identity morphism \( 1_L \) is a Gorenstein cophantom morphism relative to \( \mathcal{C} \). It is easy to see that \( \psi \in \text{Hom}_\mathcal{A}(L, L') \) is in \( \text{Coph} - \mathcal{G}(\mathcal{C}) \) if and only if \( \text{Im} \text{Ext}_\mathcal{A}^1(N, \psi) \subseteq \text{GExt}_\mathcal{A}^1(N, L') \) for any \( N \in \mathcal{A} \).

Given two short exact sequences and a commutative diagram

\[
\begin{array}{ccccccc}
\xi : & 0 & \rightarrow & L & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\
\downarrow & f & \downarrow & g & \downarrow & h & & & &  \\
\xi' : & 0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z & \rightarrow & 0 \\
\end{array}
\]

in \( \mathcal{A} \). This morphism of short exact sequences factors uniquely through the pushout of \( \xi \) along \( f \) or uniquely through the pullback of \( \xi' \) along \( h \) as

\[
\begin{array}{ccccccc}
\xi : & 0 & \rightarrow & L & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\
\downarrow & f & \downarrow & g & \downarrow & h & & & &  \\
0 & \rightarrow & X & \rightarrow & Q & \rightarrow & N & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & & &  \\
0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z & \rightarrow & 0 \\
\end{array}
\]

This is the so-called \textit{pushout-pullback factorization} of a morphism of short exact sequences; see [10, Proposition 3.1] (cf. [17, p.755]). The following result shows that Gorenstein cophantom morphisms are the morphism version of Gorenstein cophantom objects.

**Theorem 4.4.** For any \( \psi \in \text{Hom}_\mathcal{A}(L, L') \), the following statements are equivalent.

1. \( \psi \in \text{Coph} - \mathcal{G}(\mathcal{C}) \).
2. \( \text{Ext}_\mathcal{A}^1(G, \psi) = 0 \) for any \( G \in \mathcal{G}(\mathcal{C}) \).

Consequently, \( \text{Ob}(\text{Coph} - \mathcal{G}(\mathcal{C})) = \mathcal{C}\mathcal{G}(\mathcal{C}) \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( \psi \in \text{Coph} - \mathcal{G}(\mathcal{C}) \) and \( G \in \mathcal{G}(\mathcal{C}) \), and let

\[
0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} G \rightarrow 0
\]

be an exact sequence in \( \mathcal{A} \). Consider the following pushout diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & M & \rightarrow & G & \rightarrow & 0 \\
\downarrow & \psi & \downarrow & & \downarrow & & \downarrow & &  \\
0 & \rightarrow & L' & \rightarrow & M' & \rightarrow & G & \rightarrow & 0 \\
\end{array}
\]
Then the bottom row is $\text{Hom}_\mathcal{A}(G,\mathcal{C}),-)$-exact, and hence splits. It implies $\text{Ext}^1_\mathcal{A}(G,\psi) = 0$.

(2) $\Rightarrow$ (1) To prove $\psi \in \text{Coph} - G(\mathcal{C})$, it suffices to prove that for any exact sequence

$$\xi : 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in $\mathcal{A}$, the sequence $\xi' = \text{Ext}^1_\mathcal{A}(N,\psi)(\xi)$ obtained by the pushout of $\xi$ along $\psi$ is $\text{Hom}_\mathcal{A}(G,\mathcal{C}),-)$-exact.

Let $G \in G(\mathcal{C})$ and $\varphi \in \text{Hom}_\mathcal{A}(G,N)$. Consider the pullback of $\xi$ along $\varphi$ and the pushout of $\xi$ along $\psi$ respectively as follows.

$$\eta : 0 \to L' \xrightarrow{\psi} M' \xrightarrow{g'} N' \to 0$$

By the pushout-pullback factorization, we get the following commutative diagram

Then $\eta' = \text{Ext}^1_\mathcal{A}(G,\psi)(\eta)$ and there exists $q'' \in \text{Hom}_\mathcal{A}(G,Q')$ such that $q'q'' = 1_G$. Put $\theta := \varphi'q''$. Then $g'\theta = g'\varphi'q'' = \varphi'q'' = \varphi 1_G = \varphi$, which implies that $\xi'$ is $\text{Hom}_\mathcal{A}(G,\mathcal{C}),-)$-exact, as desired.

For example, $\text{Ob}(\text{Coph} - G(\mathcal{P}(\text{mod} R))) = \left\{ \frac{3}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, 1, \frac{2}{3} \right\}$ in Example 3.5.

**Proposition 4.5.** If $\mathcal{A}$ is a Gorenstein complete category, then the ideal cotorsion pair

$$(\perp, \perp_\mathcal{A}(\text{Coph}(\mathcal{C})) >, < \text{Ob}(\text{Coph} - G(\mathcal{C})) >) = (G(\mathcal{C}), < CG(\mathcal{C}) >)$$

is complete.

**Proof.** By Theorem 4.4 and Proposition 3.9, we have

$$\text{Ob}(\text{Coph} - G(\mathcal{C})) = CG(\mathcal{C}) \text{ and } \perp \text{Ob}(\text{Coph} - G(\mathcal{C})) = G(\mathcal{C}).$$

Now the assertion follows from Proposition 3.9 and Lemma 2.13.

Now we give a criterion for identifying Gorenstein cophantom morphisms, which implies that the ideal $\text{Coph} - G(\mathcal{C})$ is an object ideal, that is, $\text{Coph} - G(\mathcal{C}) = \langle \text{Ob}(\text{Coph} - G(\mathcal{C})) \rangle$.

**Theorem 4.6.** Let $\mathcal{A}$ be a Gorenstein complete category and $\psi \in \text{Hom}_\mathcal{A}(L,L')$. Then the following statements are equivalent.

1. $\psi \in \text{Coph} - G(\mathcal{C})$.
(2) $\psi$ factors through some object in $CG(\mathcal{C})$.

Consequently, $\text{Coph}\cdot G(\mathcal{C}) = \langle CG(\mathcal{C}) \rangle = \langle \text{Ob}(\text{Coph}\cdot G(\mathcal{C})) \rangle$.

Proof. (1) $\Rightarrow$ (2) Let $\psi \in \text{Coph}\cdot G(\mathcal{C})$. Since $CG(\mathcal{C})$ is special preenveloping by Proposition 3.9, there exists an exact sequence

$$0 \to L \to E \to G \to 0$$

in $\mathcal{A}$ with $E \in CG(\mathcal{C})$ and $G \in G(\mathcal{C})$. Consider the following pushout diagram

$$
\begin{array}{ccc}
0 & \to & L \\
\downarrow \psi & & \downarrow \\
0 & \to & L' \\
\end{array}
\begin{array}{ccc}
& \to & E \\
\downarrow f & & \downarrow f' \\
& \to & E' \\
\downarrow g & & \downarrow g' \\
& \to & G \\
0 & \to & 0 \\
\end{array}
\begin{array}{ll}
\uparrow \psi' & \uparrow h \\
\end{array}
\begin{array}{ll}
\uparrow & \\
\end{array}
\begin{array}{ll}
\end{array}
$$

By Theorem 4.4, we have that $\text{Ext}_A^1(G, \psi) = 0$ and the bottom row splits. Thus there exists $f'' \in \text{Hom}_A(E', L')$ such that $f''f' = 1_{L'}$. Put $h := f''\psi'$. Then $hf = f''\psi'f = f''f'\psi = 1_{L'}\psi = \psi$, that is, $\psi$ factors through $E$.

(2) $\Rightarrow$ (1) Assume that $\psi$ factors through some object $E$ in $CG(\mathcal{C})$, that is, $\psi = (L \xrightarrow{\psi_1} E \xrightarrow{\psi_2} L')$. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence in $\mathcal{A}$. Consider the following twice pushout operations

$$
\begin{array}{ccc}
0 & \to & L \\
\downarrow \psi_1 & & \downarrow \\
0 & \to & L' \\
\end{array}
\begin{array}{ccc}
& \to & M \\
\downarrow \psi_2 & & \downarrow \\
& \to & M' \\
\end{array}
\begin{array}{ccc}
& \to & N \\
\uparrow & & \uparrow \\
& \to & N' \\
\end{array}
\begin{array}{ll}
\uparrow & \uparrow \\
\end{array}
\begin{array}{ll}
0 & \to & 0 \\
\end{array}
$$

Since the middle row is $\text{Hom}_A(\mathcal{C}, -)$-exact, so is the bottom row by Proposition 4.2. It implies $\psi \in \text{Coph}\cdot G(\mathcal{C})$.

Finally, we have $\langle CG(\mathcal{C}) \rangle = \langle \text{Ob}(\text{Coph}\cdot G(\mathcal{C})) \rangle$ by Theorem 4.4. 

Consider the quotient category $\text{Mod} R/\text{Coph}\cdot G(\mathcal{P}(\text{Mod} R))$. By Examples 3.2 and 3.3 and Theorem 4.6, we have the following

Corollary 4.7.

(1) If $R$ is quasi-Frobenius, then $\text{Mod} R/\text{Coph}\cdot G(\mathcal{P}(\text{Mod} R)) \simeq \text{Mod} R$ (the stable category of $\text{Mod} R$).

(2) If $R$ is Gorenstein, then $\text{Mod} R/\text{Coph}\cdot G(\mathcal{P}(\text{Mod} R)) \simeq \text{Mod} R/ \langle \mathcal{P}(\text{Mod} R)^{<\infty} \rangle$ (which is called the singularly stable category in [31]).

(3) If $R$ has finite global dimension, then $\text{Mod} R/\text{Coph}\cdot G(\mathcal{P}(\text{Mod} R))$ is trivial.
Example 4.8. As in Example 3.5, we can identify the irreducible Gorenstein cophantom morphisms relative to \( \mathcal{P}(\text{mod } R) \) in \( \text{mod } R \) as follows.

\[
\Gamma(\text{mod } R):
\]

where the morphisms marked by the dashed arrows are irreducible Gorenstein cophantom morphisms and the objects marked in a box are indecomposable Gorenstein cophantom objects.

4.2 Gorenstein projective morphisms

Definition 4.9. Let \( \varphi \in \text{Hom}_\mathcal{A}(N', N) \) and

\[
0 \to L \to M \to N \to 0
\]

be a \( \text{Hom}_\mathcal{A}(\mathcal{G}(\mathcal{E}), -) \)-exact exact sequence in \( \mathcal{A} \). Consider the following pullback diagram

\[
\begin{array}{c}
0 \to L \to M \to N \to 0 \\
\| \| \| \| \|
0 \to L \to M \to N \to 0
\end{array}
\]

If the top row of any such pullback splits, then we call the morphism \( \varphi \) a Gorenstein projective morphism relative to \( \mathcal{E} \).

We use \( \mathcal{G}(\mathcal{E})\text{-Mor} \) to denote the ideal of Gorenstein projective morphisms relative to \( \mathcal{E} \), and use \( \text{Ob}(\mathcal{G}(\mathcal{E})\text{-Mor}) \) to denote the class of those objects \( L \) such that the identity morphism \( 1_L \) is a Gorenstein projective morphism relative to \( \mathcal{E} \). It is trivial that \( \varphi \in \text{Hom}_\mathcal{A}(N', N) \) is in \( \mathcal{G}(\mathcal{E})\text{-Mor} \) if and only if \( \text{Im} \text{GExt}^1_{\mathcal{A}}(\varphi, L) = 0 \) for any \( L \in \mathcal{A} \). The following result shows that Gorenstein projective morphisms are the morphism version of Gorenstein objects provided that \( \mathcal{A} \) is a Gorenstein complete category.

Proposition 4.10.

1. \( \mathcal{G}(\mathcal{E}) \subseteq \text{Ob}(\mathcal{G}(\mathcal{E})\text{-Mor}) \).

2. If \( \mathcal{A} \) is a Gorenstein complete category, then \( \mathcal{G}(\mathcal{E}) = \text{Ob}(\mathcal{G}(\mathcal{E})\text{-Mor}) \).

Proof. (1) Let \( G \in \mathcal{G}(\mathcal{E}) \). Then for any \( \text{Hom}_\mathcal{A}(\mathcal{G}(\mathcal{E}), -) \)-exact exact sequence

\[
\xi : 0 \to L \xrightarrow{f} M \xrightarrow{g} G \to 0
\]

in \( \mathcal{A} \), we have that \( \xi = \text{GExt}_\mathcal{A}^1(1_G, L)(\xi) \) splits. So \( 1_G \in \mathcal{G}(\mathcal{E})\text{-Mor} \) and \( G \in \text{Ob}(\mathcal{G}(\mathcal{E})\text{-Mor}) \).

(2) Let \( \mathcal{A} \) be a Gorenstein complete category and \( N \in \text{Ob}(\mathcal{G}(\mathcal{E})\text{-Mor}) \). Then there exists a \( \text{Hom}_\mathcal{A}(\mathcal{G}(\mathcal{E}), -) \)-exact exact sequence

\[
\xi' : 0 \to K \to G_0 \to N \to 0
\]
in \( \mathcal{A} \) with \( G_0 \in \mathcal{G}(\mathcal{C}) \). So \( \xi' = \text{GExt}^1_{\mathcal{A}}(1_N, K)(\xi') \) splits, and hence \( N \in \mathcal{G}(\mathcal{C}) \) as a direct summand of \( G_0 \).

The following result gives a criterion for identifying the elements of \( \mathcal{G}(\mathcal{C})\)-Mor, which implies that the ideal \( \mathcal{G}(\mathcal{C})\)-Mor is an object ideal.

**Theorem 4.11.** Let \( \mathcal{A} \) be a Gorenstein complete category and \( \varphi \in \text{Hom}_{\mathcal{A}}(N', N) \). Then the following statements are equivalent.

1. \( \varphi \in \mathcal{G}(\mathcal{C})\)-Mor.
2. \( \varphi \) factors through some object in \( \mathcal{G}(\mathcal{C}) \).

Consequently, \( \mathcal{G}(\mathcal{C})\)-Mor = \( \langle \mathcal{G}(\mathcal{C}) \rangle = \langle \text{Ob}(\mathcal{G}(\mathcal{C})\text{-Mor}) \rangle \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( \varphi \in \mathcal{G}(\mathcal{C})\)-Mor. Because \( \mathcal{A} \) is a Gorenstein complete category, there exists an exact sequence

\[
\xi: 0 \to K \xrightarrow{f} G \xrightarrow{g} N \to 0
\]

in \( \mathcal{A} \) with \( G \in \mathcal{G}(\mathcal{C}) \) and \( K \in \mathcal{C}\mathcal{G}(\mathcal{C}) \). Consider the following pullback diagram

\[
\begin{array}{ccc}
0 & \to & K \\
\uparrow & & \downarrow h \\
0 & \to & K \\
\end{array}
\quad \begin{array}{ccc}
0 & \to & Q \\
\uparrow & & \downarrow g' \\
0 & \to & M \\
\end{array}
\quad \begin{array}{ccc}
0 & \to & M' \\
\uparrow & & \downarrow g'' \\
0 & \to & N \\
\end{array}
\quad \begin{array}{ccc}
0 & \to & N' \\
\uparrow & & \downarrow g \\
0 & \to & 0 \\
\end{array}
\]

Then the upper row splits and there exists \( g'' \in \text{Hom}_{\mathcal{A}}(N', Q) \) such that \( g'g'' = i_{N'} \). Put \( \theta := hg'' \). Thus \( g\theta = ghg'' = \varphi g'g'' = \varphi \), that is, \( \varphi \) factors through \( G \).

(2) \( \Rightarrow \) (1) Assume that \( \varphi \) factors through some object \( G \) in \( \mathcal{G}(\mathcal{C}) \), that is, \( \varphi = (N' \xrightarrow{\xi_1} G \xrightarrow{\xi_2} N) \). Let

\[
0 \to L \to M \to N \to 0
\]

be a \( \text{Hom}_{\mathcal{A}}(\mathcal{G}(\mathcal{C}), -)\)-exact exact sequence in \( \mathcal{A} \). Consider the following twice pullback operations

\[
\begin{array}{ccc}
0 & \to & L \\
\uparrow & & \downarrow \varphi_1 \\
0 & \to & M'' \\
\uparrow & & \downarrow \varphi_2 \\
0 & \to & M' \\
\end{array}
\quad \begin{array}{ccc}
0 & \to & G \\
\uparrow & & \downarrow \varphi \\
0 & \to & N \\
\end{array}
\quad \begin{array}{ccc}
0 & \to & N' \\
\uparrow & & \downarrow \varphi \\
0 & \to & 0 \\
\end{array}
\]

By Proposition 4.2, the middle row is \( \text{Hom}_{\mathcal{A}}(\mathcal{G}(\mathcal{C}), -)\)-exact, and hence splits. Thus the top row splits too. It implies \( \varphi \in \mathcal{G}(\mathcal{C})\)-Mor.

Finally, we have \( \langle \mathcal{G}(\mathcal{C}) \rangle = \langle \text{Ob}(\mathcal{G}(\mathcal{C})\text{-Mor}) \rangle \) by Proposition 4.10.
**Example 4.12.** As in Example 3.5, we can identify the irreducible Gorenstein projective morphisms relative to $\mathcal{P}$(mod $R$) in mod $R$ as follows.

\[
\Gamma(\text{mod } R) : \quad \begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
2 & 3 & 1 \\
\downarrow & \downarrow & \downarrow \\
3 & 1 & 2 \\
\downarrow & \downarrow & \downarrow \\
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
2 & 3 & 1 \\
\downarrow & \downarrow & \downarrow \\
3 & 1 & 2 \\
\downarrow & \downarrow & \downarrow \\
1 & 2 & 3 \\
\end{array}
\]

where the morphisms marked by the dashed arrows are irreducible Gorenstein projective morphisms relative to $\mathcal{P}$(mod $R$) and the objects marked in a box are indecomposable Gorenstein projective objects relative to $\mathcal{P}$(mod $R$).

**Theorem 4.13.** If $\mathcal{A}$ be a Gorenstein complete category, then the ideal cotorsion pair

\[(\mathcal{G}(\mathcal{C})\text{-Mor}, \text{Coph-}\mathcal{G}(\mathcal{C}))\]

is complete.

**Proof.** By Theorems 4.11 and 4.6, we have

\[\mathcal{G}(\mathcal{C})\text{-Mor} = \langle \mathcal{G}(\mathcal{C}) \rangle \text{ and } \text{Coph-}\mathcal{G}(\mathcal{C}) = \langle \mathcal{C}\mathcal{G}(\mathcal{C}) \rangle .\]

Now the assertion follows from Proposition 4.5.

A natural question is the following

**Question 4.14.** Does the converse of Theorem 4.13 hold true?

Recall from [17] that a morphism $\varphi \in \text{Hom}_\mathcal{A}(X, A)$ is called a projective morphism if $\text{Ext}^1_\mathcal{A}(\varphi, B) = 0$ for any $B \in \mathcal{A}$, and an object $L \in \mathcal{A}$ is called a projective object if the identity morphism $1_L$ of $L$ is projective. We use $\mathcal{P}(\mathcal{A})\text{-Mor}$ to denote the ideal of projective morphisms, and use $\text{Ob}(\mathcal{P}(\mathcal{A})\text{-Mor})$ to denote the class of projective objects. Clearly, we have

\[\mathcal{P}(\mathcal{A})\text{-Mor} \subseteq \mathcal{G}(\mathcal{P}(\mathcal{A}))\text{-Mor} \text{ and } \langle \text{Ob}(\mathcal{P}(\mathcal{A})\text{-Mor}) \rangle \subseteq \langle \text{Ob}(\mathcal{G}(\mathcal{P}(\mathcal{A})) \rangle .\]

It is easy to see that $\varphi \in \mathcal{P}(\mathcal{A})\text{-Mor}$ if and only if for any $B \in \mathcal{A}$ and any exact sequence

\[0 \to B \to C \to A \to 0\]

in $\mathcal{A}$, the upper row in the following pullback diagram splits.

\[
\begin{array}{ccc}
0 & \to & B & \to & Y & \to & X & \to & 0 \\
\downarrow & & \downarrow & & \varphi & & \downarrow & & \downarrow \\
0 & \to & B & \to & C & \to & A & \to & 0.
\end{array}
\]

The following result is an analogy of Theorem 4.11, we omit its proof.
**Theorem 4.15.** Let $\mathcal{A}$ be a category with enough projective objects. Then for any $\varphi \in \text{Hom}_{\mathcal{A}}(X,A)$, the following statements are equivalent.

1. $\varphi \in \mathcal{P}(\mathcal{A})\text{-Mor}$.
2. $\varphi$ factors through some object in $\mathcal{P}$.

Consequently, $\mathcal{P}(\mathcal{A})\text{-Mor} = \langle \mathcal{P}(\mathcal{A}) \rangle = \langle \text{Ob}(\mathcal{P}(\mathcal{A})\text{-Mor}) \rangle$.

**Example 4.16.** In Example 3.5, the irreducible projective morphisms in $\text{mod } R$ are marked by the dashed arrows as follows.

$$
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}

Then the intersection of irreducible morphisms in $\mathcal{G}(\mathcal{P}(\text{mod } R))\text{-Mor}$ and ones in $\text{Coph-}\mathcal{G}(\mathcal{P}(\text{mod } R))$ is marked by the dashed arrows as follows.

$$
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}

Obviously, $\mathcal{P}(\text{mod } R)\text{-Mor} \subseteq \mathcal{G}(\mathcal{P}(\text{mod } R))\text{-Mor} \cap \text{Coph-}\mathcal{G}(\mathcal{P}(\text{mod } R))$.

Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in $\mathcal{A}$. Then it is trivial that $\langle A \cap B \rangle \subseteq \langle A \rangle \cap \langle B \rangle$. However, $\langle A \cap B \rangle \neq \langle A \rangle \cap \langle B \rangle$ in general even though the cotorsion pair $(\mathcal{A},\mathcal{B})$ is complete as shown in Example 4.16 (cf. Proposition 3.11 and Theorems 4.6, 4.11 and 4.15).

**Remark 4.17.** We may define Gorenstein phantom objects and morphisms relative to $\mathcal{C}$ as follows.

1. An object $M \in \mathcal{A}$ is called Gorenstein phantom relative to $\mathcal{C}$ if $M \in \perp_{-1} \mathcal{G}(\mathcal{C})$.
2. $\mathcal{A}$ is called Gorenstein cocomplete relative to $\mathcal{C}$ if $\mathcal{G}(\mathcal{C})$ is preenveloping in $\mathcal{A}$.
3. Let $\psi \in \text{Hom}_{\mathcal{A}}(N',N)$ and
   $$
   0 \to L \to M \to N \to 0
   $$
   be an exact sequence in $\mathcal{A}$. Consider the following pullback diagram
   $$
   \begin{array}{ccc}
   0 & \longrightarrow & L \\
   \downarrow & & \downarrow \psi \\
   0 & \longrightarrow & M \\
   \downarrow & & \downarrow \\
   0 & \longrightarrow & N
   \end{array}
   $$
   If the top row of any such pullback is $\text{Hom}_{\mathcal{A}}(-,\mathcal{G}(\mathcal{C}))$-exact, then we call the morphism $\psi$ a Gorenstein phantom morphism relative to $\mathcal{C}$.
(4) Let $\varphi \in \text{Hom}_A(L, L')$ and 

$$0 \to L \to M \to N \to 0$$

be a $\text{Hom}_A(\cdot, G(\mathcal{C}))$-exact exact sequence in $A$. Consider the following pushout diagram

\begin{align*}
0 & \to L \to M \to N \to 0 \\
& \downarrow \varphi \downarrow \downarrow \downarrow \\
0 & \to L' \to M' \to N' \to 0.
\end{align*}

If the bottom row of any such pushout splits, then we call $\varphi$ a Gorenstein injective morphism relative to $\mathcal{C}$.

Note that all results in Sections 3 and 4 have Gorenstein phantom counterparts.

5 Examples

In this section, we give two examples to further demonstrate Gorenstein cophantom objects and morphisms.

Example 5.1. Let $Q_1$ and $Q_2$ be the following two quivers

$Q_1: 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_3} 3 \xleftarrow{\alpha_4}$ \quad $Q_2: a \xrightarrow{\alpha_a} b \xrightarrow{\alpha_b} c \xrightarrow{\alpha_c} d,$

and let $I_1 = \langle \alpha_2 \alpha_1, \alpha_1 \alpha_2, \alpha_4 \alpha_3, \alpha_3^2 \rangle$ and $I_2 = \langle \alpha_b \alpha_a, \alpha_a \alpha_b \rangle$. Let $R_1 = \text{KQ}_1/I_1$ and $R_2 = \text{KQ}_2/I_2$. Note that $R_2$ is Gorenstein and $R_1$ is not Gorenstein. The Auslander-Reiten quivers of mod $R_1$ and mod $R_2$ are as follows.

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}
where the morphisms marked by the dashed arrows are irreducible Gorenstein cophantom morphisms relative to $\mathcal{P}(\text{mod } R_i)$ and the objects marked in a box are indecomposable Gorenstein cophantom objects relative to $\mathcal{P}(\text{mod } R_i)$ ($i = 1, 2$). Clearly,

$$\text{mod } R_1/\text{coph-} \mathcal{G}(\mathcal{P}(\text{mod } R_1)) \simeq \text{mod } R_2/\text{coph-} \mathcal{G}(\mathcal{P}(\text{mod } R_2)).$$

Moreover, it is easy to check that $\text{mod } R_1 \nsimeq \text{mod } R_2$.

**Example 5.2.** Let $Q_1$ and $Q_2$ be the following two quivers

$$Q_1: \quad \begin{array}{ccc}
1 & \alpha_1 & 2 \\
\alpha_2 & 3 & \alpha_4 \\
\alpha_3 & 4 & \end{array} \quad Q_2: \quad \begin{array}{ccc}
1 & \alpha_2 & 2 \\
\alpha_4 & 3 & \alpha_5 \\
\alpha_3 & 4 & \end{array}$$

and let $I_1 = \langle \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_2\alpha_4 \rangle$ and $I_2 = \langle \alpha_c\alpha_b, \alpha_d\alpha_c, \alpha_b\alpha_d \rangle$. Let $R_1 = KQ_1/I_1$ and $R_2 = KQ_2/I_2$. Then the Auslander-Reiten quivers of $\text{mod } R_1$ and $\text{mod } R_2$ are as follows.

$$\Gamma(\text{mod } R_1): \quad \begin{array}{ccc}
2 & \alpha_1 & 1 \\
\alpha_2 & 3 & \alpha_4 \\
\alpha_3 & 4 & \end{array} \quad \Gamma(\text{mod } R_2): \quad \begin{array}{ccc}
1 & \alpha_2 & 2 \\
\alpha_4 & 3 & \alpha_5 \\
\alpha_3 & 4 & \end{array}$$

where the morphisms marked by the dashed arrows are irreducible Gorenstein cophantom morphisms relative to $\mathcal{P}(\text{mod } R_i)$ and the objects marked in a box are indecomposable Gorenstein cophantom objects relative to $\mathcal{P}(\text{mod } R_i)$ ($i = 1, 2$). Clearly,

$$\text{mod } R_1/\text{coph-} \mathcal{G}(\mathcal{P}(\text{mod } R_1)) \simeq \text{mod } R_2/\text{coph-} \mathcal{G}(\mathcal{P}(\text{mod } R_2)).$$

However, it is easy to check that $\text{mod } R_1 \nsimeq \text{mod } R_2$.

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**References**

Gorenstein cophantom objects and morphisms


