Gorenstein Modules Induced by Foxby Equivalence

Zhaoyong Huang and Mingzhi Sheng

School of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P.R. China

E-mail: huangzy@nju.edu.cn (Huang), 1175529879@qq.com (Sheng)

Abstract

Let R, S be arbitrary associative rings and C a semidualizing (R, S)-bimodule. For a subcategory $\mathscr{H}$ (resp. $\mathscr{T}$) of the category of left R-modules (resp. left S-modules), we introduce $\mathscr{H}_C$-Gorenstein projective and flat modules (resp. $\mathscr{T}_C$-Gorenstein injective modules). Under certain conditions, we prove that the $\mathscr{H}_C$-Gorenstein projective dimension of any left R-module is at most n if and only if the projective dimension of any C-injective left S-module and the injective dimension of any module in $\mathscr{H}$ are at most n. The dual result about the $\mathscr{T}_C$-Gorenstein injective dimension of modules also holds true. As a consequence, we get that the supremum of the C-Gorenstein projective dimensions of all left R-modules and that of the C-Gorenstein injective dimensions of all left S-modules are identical; and the maximum of the common value of the quantities and its symmetric common value is at least the supremum of the C-Gorenstein flat dimensions of all left R-modules. Moreover, we obtain some equivalent characterizations for the finiteness of the left and right injective dimensions of ${}_RC_S$ in terms of the properties of the projective and injective dimensions of modules relative to various classes of C-Gorenstein modules. As an application, we provide some support for the Wakamatsu tilting conjecture.

Key Words: C-Gorenstein classes, $\mathscr{H}_C$-Gorenstein projective dimension, $\mathscr{H}_C$-Gorenstein flat dimension, $\mathscr{T}_C$-Gorenstein injective dimension, C-Gorenstein global dimension, Finite injective dimension.

2020 Mathematics Subject Classification: 18G25, 16E10.

1 Introduction

Semidualizing modules and related Auslander and Bass classes in commutative rings were introduced by Foxby [12] and by Golod [16]. Then Holm and White [19] extended them to arbitrary associative rings. Many authors have studied the properties of semidualizing modules and related modules, see [12, 14, 16, 18, 19, 22, 23, 26], [29]–[31], [34]–[42] and the references therein. Let R and S be arbitrary rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$ be the Auslander and Bass classes with respect to C respectively. It was shown in [19, Theorem 1] that there exists the following Foxby equivalence:

$$\mathcal{A}_C(S) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_S -}{\underset{\sim}{\rightleftarrows}}} \mathcal{B}_C(R).$$

For other Foxby equivalences between some subclasses of $\mathcal{A}_C(S)$ and that of $\mathcal{B}_C(R)$, the reader is referred to [19, Theorem 1] and [30, Theorem 4.6]. Among various research areas on semidualizing modules, one basic theme is to extend the "absolute" classical results in homological algebra to the "relative" setting with respect to semidualizing modules.

One of our motivations comes from the following Gorenstein versions of two classical results: for any ring R, the left Gorenstein weak global dimension of R is at most the maximum of its left and right Gorenstein global dimensions ([5, Corollary 1.2(1)]), and the Gorenstein weak global dimension of R is left and right symmetric ([7, Corollary 2.5]). On the other hand, as an extension of [2, Theorem 4.20], [20, Theorem] and [21, Theorem 1.4], the first-named author proved that a left and right Noetherian ring R is n-Gorenstein if and only if the Gorenstein projective (resp. injective, flat) dimension of any left R-module is at most n ([27, Theorem 1.2]). Another motivation for us comes from this work. We are interested in whether these results have relative counterparts with respect to semidualizing modules.

The paper is organized as follows. In Section 2, we give some terminology and some preliminary results. In Section 3, assume that R is an arbitrary ring and $\operatorname{Mod} R$ is the category of left R-modules. Let $\mathscr{D} \subseteq \mathscr{E}$ be subcategories of $\operatorname{Mod} R$ with $\mathscr{D}$ additive, and let

$$\cdots \to X_i \to \cdots \to X_1 \to X_0 \to Y^0 \to Y^1 \to \cdots \to Y^j \to \cdots$$

be an exact sequence in $\operatorname{Mod} R$. By using the $\mathscr{E}$-coproper $\mathscr{D}$-coresolutions of all X_i and the $\mathscr{E}$-proper $\mathscr{D}$-resolutions of all Y^j, we construct a grid-type commutative diagram (Theorem 3.3). This construction is crucial in studying the behavior of the projective and injective dimensions of modules relative to various classes of relative Gorenstein modules. As mentioned above, the symmetry of the Gorenstein weak global dimension of any ring was proved in [7, Corollary 2.5], which is a consequence of [9, Theorem 5.3]. Note that the latter one depends on the construction of projective resolutions of certain modules by using the horseshoe lemma (see the proof of [9, Lemma 5.2] for details). However, the horseshoe lemma is inapplicable in the relative case. The above construction of ours does not only overcome this difficulty, but also gives some wider applications in the sequel.

Let R, S be arbitrary rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathscr{H}$ (resp. $\mathscr{T}$) be a subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{Mod} S$-modules). In Section 4, we introduce $\mathscr{H}_C$-Gorenstein projective and flat modules (resp. $\mathscr{T}_C$-Gorenstein injective modules). In fact, our research will be conducted under this unified framework. Assume that $\mathscr{T}$ is a resolving subcategory of the Auslander class $\mathcal{A}_C(S)$ and $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$ is precovering in $\operatorname{Mod} R$ which is closed under finite direct sums and direct summands. Under certain conditions, we obtain some equivalent characterizations for the $\mathscr{H}$-Gorenstein flat dimension of any module being at most n (Proposition 4.5). Moreover, we prove the following result.

Theorem 1.1. (Theorem 4.6) *For any $n \geq 0$, the following statements are equivalent.*

(1) *The $\mathscr{H}_C$-Gorenstein projective dimension of any left R-module is at most n.*

(2) *The projective dimension of any C-injective left S-module and the injective dimension of any module in $\mathscr{H}$ are at most n.*

(3) *The C-projective dimension of any injective left R-module and the C-injective dimension of any module in $\mathscr{T}$ are at most n.*

Assume that $\mathscr{H}$ is a coresolving subcategory of the Bass class $\mathcal{B}_C(R)$ and $\mathscr{T} := \{\operatorname{Hom}_R(C, H) \mid H \in \mathscr{H}\}$ is preenveloping in $\operatorname{Mod} R$ which is closed under finite direct sums and direct summands. Then the dual of Theorem 1.1 about the $\mathscr{T}_C$-Gorenstein injective dimension of modules also holds true (Theorem 4.11). Note that under the assumption in either Theorem 4.6 or Theorem 4.11, there exists the following Foxby equivalence:

$$\mathscr{T} \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_S -}{\underset{\sim}{\rightleftarrows}}} \mathscr{H}.$$

In Section 5, we give some applications of the above results. Under certain conditions, we establish the left and right symmetry of the C-Gorenstein flat dimension of any module being at most n (Theorem 5.2). In addition, we prove the following theorem, which is the C-version of [9, Theorem 4.1].

Theorem 1.2. (Theorem 5.4) *For any $n \geq 0$, the following statements are equivalent.*

(1) *The C-Gorenstein projective dimension of any left R-module is at most n.*
(2) *The C-Gorenstein injective dimension of any left S-module is at most n.*
(3) *The projective dimension of any C-injective left S-module and the injective dimension of any C-projective left R-module are at most n.*
(4) *The C-projective dimension of any injective left R-module and the C-injective dimension of any projective left S-module are at most n.*

As an immediate consequence of Theorem 1.2, we get that the supremum of the C-Gorenstein projective dimensions of all left R-modules and that of the C-Gorenstein injective dimensions of all left S-modules are identical, and call the common value of these two quantities the *left C-Gorenstein global dimension* G_C-gldim of R and S. Symmetrically, the *right C-Gorenstein global dimension* G_C-gldim^{op} of R and S is defined. We prove that if either the flat dimension of any C-injective right R-module or G_C-gldim^{op} is finite, then any C-projective left R-module is C-flat (Theorem 5.14).

For a module $M \in \operatorname{Mod} R$, we use $\mathrm{G}_C\text{-fd}_R\, M$ to denote the C-Gorenstein flat dimension of M. Set $\operatorname{spclfc} R := \sup\{$the C-projective dimensions of all C-flat left R-modules$\}$. By using Theorem 5.14 and the relationship between $\operatorname{spclfc} R$ and the C-Gorenstein projective dimension of any C-Gorenstein flat module (Lemma 5.17), we obtain the following result, which is the C-version of [5, Corollary 1.2(1)] and part of [7, Theorem 3.3].

Theorem 1.3. (Theorem 5.18) *It holds that*

(1) $\sup\{\mathrm{G}_C\text{-fd}_R\, M \mid M \in \operatorname{Mod} R\} \leq \max\{\mathrm{G}_C\text{-gldim},\ \mathrm{G}_C\text{-gldim}^{op}\}$.
(2) *If S is a right Noetherian ring, then*

$$\mathrm{G}_C\text{-gldim} \leq \sup\{\mathrm{G}_C\text{-fd}_R\, M \mid M \in \operatorname{Mod} R\} + \operatorname{spclfc} R.$$

We give some equivalent characterizations for the finiteness of the left and right injective dimensions of ${}_RC_S$ in terms of the properties of the projective and injective dimensions of modules relative to some classes of C-Gorenstein modules as follows. It is the C-version of [27, Theorem 1.2], but the proof here is essentially not parallel to that of [27].

Theorem 1.4. (Theorem 5.20) *Let R be a left and right Noetherian ring and $n \geq 0$. Then the following statements are equivalent.*

(1) *The left and right injective dimensions of ${}_RC_S$ are at most n.*
(2) *The C-Gorenstein projective dimension of any left R-module is at most n.*
(3) *The C-Gorenstein injective dimension of any left R-module is at most n.*
(4) *The C-Gorenstein flat dimension of any left R-module is at most n.*
(5) *The C-strongly Gorenstein flat dimension of any left R-module is at most n.*
(6) *The C-projectively coresolved Gorenstein flat dimension of any left R-module is at most n.*

$(i)^{op}$ *Opposite side version of (i) $(2 \leq i \leq 6)$.*

The Wakamatsu tilting conjecture states that if R and S are artin algebras, then the left and right injective dimensions of ${}_RC_S$ are identical ([5]). It still remains open. Recall that a left and right Noetherian ring R is called *Gorenstein* if its left and right self-injective dimensions are finite. As an application of Theorem 1.4, we prove that if R and S are Gorenstein rings, then the left and right injective dimensions of ${}_RC_S$ are identical (Theorem 5.22(3)).

2 Preliminaries

Throughout this paper, all rings are arbitrary associative rings. Let R be a ring. We use $\operatorname{Mod} R$ to denote the category of left R-modules, and all subcategories of $\operatorname{Mod} R$ involved are full and closed under isomorphisms. We use $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ to denote the subcategories of $\operatorname{Mod} R$ consisting of projective, flat and injective modules respectively. For a module $M \in \operatorname{Mod} R$, we use $\operatorname{Add}_R M$ to denote the subcategory of $\operatorname{Mod} R$ consisting of direct summands of direct sums of copies of M, and use $\operatorname{pd}_R M$, $\operatorname{fd}_R M$ and $\operatorname{id}_R M$ to denote the projective, flat and injective dimensions of M respectively.

Definition 2.1. ([10, 11]) Let $\mathscr{X}$ be a subcategory of $\operatorname{Mod} R$.

(1) A homomorphism $f : X \to Y$ in $\operatorname{Mod} R$ with $X \in \mathscr{X}$ is called an *$\mathscr{X}$-precover* of Y if $\operatorname{Hom}_R(X', f)$ is epic for any $X' \in \mathscr{X}$; and an *$\mathscr{X}$-precover* $f : X \to Y$ is called an *$\mathscr{X}$-cover* of Y if any endomorphism $h : X \to X$ is an automorphism whenever $f = fh$. The subcategory $\mathscr{X}$ is called *(pre)covering* in $\operatorname{Mod} R$ if any module in $\operatorname{Mod} R$ admits an $\mathscr{X}$-(pre)cover. Dually, the notions of an *$\mathscr{X}$-(pre)envelope* and a *(pre)enveloping subcategory* are defined.

(2) The subcategory $\mathscr{X}$ is called *resolving* if $\mathcal{P}(R) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and kernels of epimorphisms. Dually, the notion of *coresolving subcategories* is defined.

Let $\mathscr{X}$ be a subcategory of $\operatorname{Mod} R$. We write

$$^{\perp}\mathscr{X} := \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(A, X) = 0 \text{ for any } X \in \mathscr{X}\},$$

$$\mathscr{X}^{\perp} := \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(X, A) = 0 \text{ for any } X \in \mathscr{X}\}.$$

Let $\mathscr{B}$ be a subcategory of $\operatorname{Mod} R^{op}$. We write

$$\mathscr{B}^{\top} := \{M \in \operatorname{Mod} R \mid \operatorname{Tor}^R_{\geq 1}(B, M) = 0 \text{ for any } B \in \mathscr{B}\}.$$

Let $M \in \operatorname{Mod} R$. The *$\mathscr{X}$-projective dimension* $\mathscr{X}\text{-}\operatorname{pd} M$ of M is defined as

$$\inf\{n \mid \text{there exists an exact sequence } 0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$
$$\text{in } \operatorname{Mod} R \text{ with all } X_i \in \mathscr{X}\},$$

and set $\mathscr{X}\text{-}\operatorname{pd} M = \infty$ if no such integer exists. Dually, the *$\mathscr{X}$-injective dimension* $\mathscr{X}\text{-}\operatorname{id} M$ of M is defined as

$$\inf\{n \mid \text{there exists an exact sequence } 0 \to M \to X^0 \to X^1 \to \cdots \to X^n \to 0$$
$$\text{in } \operatorname{Mod} R \text{ with all } X^i \in \mathscr{X}\},$$

and set $\mathscr{X}\text{-}\operatorname{id} M = \infty$ if no such integer exists. For any $n \geq 0$, we use $\mathscr{X}\text{-}\operatorname{pd}^{\leq n}$ (resp. $\mathscr{X}\text{-}\operatorname{id}^{\leq n}$) to denote the subcategory of $\operatorname{Mod} R$ consisting of modules with $\mathscr{X}$-projective (resp. $\mathscr{X}$-injective) dimension at most n.

2.1 Relative preresolving and precoresolving subcategories

Let $\mathscr{E}$ be a subcategory of $\operatorname{Mod} R$. Recall from [11] that a sequence

$$\cdots \to S_1 \to S_2 \to S_3 \to \cdots$$

in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(\mathscr{E},-)$-*exact* (resp. $\operatorname{Hom}_R(-,\mathscr{E})$-*exact*) if it is exact after applying the functor $\operatorname{Hom}_R(E,-)$ (resp. $\operatorname{Hom}_R(-,E)$) for any $E \in \mathscr{E}$.

Let $\mathscr{D} \subseteq \mathscr{X}$ be subcategories of $\operatorname{Mod} R$. We recall some notions from [25]. The subcategory $\mathscr{D}$ is called an *$\mathscr{E}$-proper generator* (resp. *$\mathscr{E}$-coproper cogenerator*) for $\mathscr{X}$ if for any $X \in \mathscr{X}$, there exists a $\operatorname{Hom}_R(\mathscr{E},-)$ (resp. $\operatorname{Hom}_R(-,\mathscr{E})$)-exact exact sequence

$$0 \to X' \to D \to X \to 0 \text{ (resp. } 0 \to X \to D \to X' \to 0)$$

in $\operatorname{Mod} R$ with $D \in \mathscr{D}$ and $X' \in \mathscr{X}$. The subcategory $\mathscr{X}$ is called *$\mathscr{E}$-preresolving* (resp. *$\mathscr{E}$-precoresolving*) in $\operatorname{Mod} R$ if the following conditions are satisfied.

(a) $\mathscr{X}$ admits an $\mathscr{E}$-proper generator (resp. $\mathscr{E}$-coproper cogenerator).

(b) $\mathscr{X}$ is *closed under $\mathscr{E}$-proper* (resp. *$\mathscr{E}$-coproper*) *extensions*, that is, for any $\operatorname{Hom}_R(\mathscr{E},-)$-exact (resp. $\operatorname{Hom}_R(-,\mathscr{E})$-exact) exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\operatorname{Mod} R$, if $A_1, A_3 \in \mathscr{X}$, then $A_2 \in \mathscr{X}$.

An $\mathscr{E}$-preresolving (resp. $\mathscr{E}$-precoresolving) subcategory $\mathscr{X}$ is called *$\mathscr{E}$-resolving* (resp. *$\mathscr{E}$-coresolving*) if the following condition is satisfied.

(c) For any $\operatorname{Hom}_R(\mathscr{E},-)$-exact (resp. $\operatorname{Hom}_R(-,\mathscr{E})$-exact) exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\operatorname{Mod} R$, if both $A_2, A_3 \in \mathscr{X}$ (resp. $A_1, A_2 \in \mathscr{X}$), then $A_1 \in \mathscr{X}$ (resp. $A_3 \in \mathscr{X}$).

If $\mathscr{E} = \mathcal{P}(R)$ (resp. $\mathcal{I}(R)$)), then $\mathscr{E}$-resolving (resp. $\mathscr{E}$-coresolving) subcategories are exactly resolving (resp. coresolving) subcategories.

Let $\mathscr{E}$ and $\mathscr{D}$ be subcategories of $\operatorname{Mod} R$. We define

$$\widetilde{\operatorname{res}_{\mathscr{E}}} \mathscr{D} := \{M \in \operatorname{Mod} R \mid \text{there exists a } \operatorname{Hom}_R(\mathscr{E},-)\text{-exact exact sequence}$$

$$\cdots \to D_i \to \cdots \to D_1 \to D_0 \to M \to 0 \text{ in } \operatorname{Mod} R \text{ with all } D_i \text{ in } \mathscr{D}\}.$$

Dually, we define

$$\widetilde{\operatorname{cores}_{\mathscr{E}}} \mathscr{D} := \{M \in \operatorname{Mod} R \mid \text{there exists a } \operatorname{Hom}_R(-,\mathscr{E})\text{-exact exact sequence}$$

$$0 \to M \to D^0 \to D^1 \to \cdots \to D^i \to \cdots \text{ in } \operatorname{Mod} R \text{ with all } D^i \text{ in } \mathscr{D}\}.$$

For later use, we need the following two lemmas.

Lemma 2.2. *Let $\mathscr{X}$ and $\mathscr{E}$ be subcategories of* $\operatorname{Mod} R$.

(1) *Assume that $\mathscr{X}$ is $\mathscr{E}$-precoresolving in* $\operatorname{Mod} R$ *admitting an $\mathscr{E}$-coproper cogenerator $\mathscr{D}$. If $\mathscr{D}$-$\operatorname{pd}^{\leq n}$ is closed under direct summands for any $n \geq 0$, then we have*

$$\mathscr{X}\text{-}\operatorname{pd} A = \mathscr{D}\text{-}\operatorname{pd} A$$

for any $A \in \mathscr{X}^{\perp}$.

(2) *Assume that $\mathscr{X}$ is $\mathscr{E}$-preresolving in* $\operatorname{Mod} R$ *admitting an $\mathscr{E}$-proper generator $\mathscr{D}$. If $\mathscr{D}$-$\operatorname{id}^{\leq n}$ is closed under direct summands for any $n \geq 0$, then we have*

$$\mathscr{X}\text{-}\operatorname{id} A = \mathscr{D}\text{-}\operatorname{id} A$$

for any $A \in {}^{\perp}\mathscr{X}$.

Proof. (1) It is clear that $\mathscr{X}\text{-}\operatorname{pd} A \leq \mathscr{D}\text{-}\operatorname{pd} A$ for any $A \in \operatorname{Mod} R$. Now suppose $A \in \mathscr{X}^{\perp}$ and $\mathscr{X}\text{-}\operatorname{pd} A = n < \infty$. By [25, Theorem 4.7], there exists an exact sequence

$$0 \to Y \to X \to A \to 0 \tag{2.1}$$

in $\operatorname{Mod} R$ with $X \in \mathscr{X}$ and $\mathscr{D}\text{-}\operatorname{pd} Y \leq n-1$. In view that the proof of [25, Theorem 4.7] was not presented there, we prove the existence of the exact sequence (2.1) for the reader's convenience. We proceed by induction on n. The case for $n = 0$ is trivial. If $n = 1$, then there exists an exact sequence

$$0 \to X_1 \to X_0 \to A \to 0$$

in $\operatorname{Mod} R$ with $X_0, X_1 \in \mathscr{X}$. Since $\mathscr{D}$ is an $\mathscr{E}$-coproper cogenerator for $\mathscr{X}$, there exists a $\operatorname{Hom}_R(-, \mathscr{E})$-exact exact sequence

$$0 \to X_1 \to Y \to X_1' \to 0$$

in $\operatorname{Mod} R$ with $X_1' \in \mathscr{X}$ and $Y \in \mathscr{D}$. Consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dashed] & Y \arrow[r, dashed] \arrow[d] & X \arrow[r, dashed] \arrow[d, dashed] & A \arrow[r, dashed] & 0 \\
 & X_1' \arrow[r, equal] \arrow[d] & X_1' \arrow[d, dashed] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

By [24, Lemma 2.4(2)], the middle column in this diagram is $\operatorname{Hom}_R(-, \mathscr{E})$-exact. Since $\mathscr{X}$ is $\mathscr{E}$-precoresolving, we have $X \in \mathscr{X}$, and thus the middle row in the above diagram is as desired. Now suppose $n \geq 2$. Then there exists an exact sequence

$$0 \to Y_0 \to X_0 \to A \to 0 \tag{2.2}$$

in $\operatorname{Mod} R$ with $X_0 \in \mathscr{X}$ and $\mathscr{X}\text{-}\operatorname{pd} Y_0 \leq n-1$. By the induction hypothesis, there exists an exact sequence

$$0 \to Y_1 \to X_0' \to Y_0 \to 0$$

in $\operatorname{Mod} R$ with $X_0' \in \mathscr{X}$ and $\mathscr{D}\text{-}\operatorname{pd} Y_1 \leq n-2$. Since there exists a $\operatorname{Hom}_R(-, \mathscr{E})$-exact exact sequence

$$0 \to X_0' \to D \to X_0'' \to 0$$

in $\operatorname{Mod} R$ with $X_0'' \in \mathscr{X}$ and $D \in \mathscr{D}$, we obtain the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d, dashed] & \\
0 \arrow[r] & Y_1 \arrow[r] \arrow[d, equal] & X_0' \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r, dashed] & Y_1 \arrow[r, dashed] & D \arrow[r, dashed] \arrow[d] & Y \arrow[r, dashed] \arrow[d, dashed] & 0 \\
 & & X_0'' \arrow[r, equal] \arrow[d] & X_0'' \arrow[d, dashed] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Diagram (2.1)

The middle row in this diagram implies $\mathscr{D}\text{-}\operatorname{pd} Y \leq n-1$. From the exact sequence (2.2) and the rightmost column in the above diagram we obtain the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & & \\
0 \arrow[r] & Y_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dashed] & Y \arrow[r, dashed] \arrow[d] & X \arrow[r, dashed] \arrow[d, dashed] & A \arrow[r, dashed] & 0 \\
 & X_0'' \arrow[r, equal] \arrow[d] & X_0'' \arrow[d, dashed] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Diagram (2.2)

By [24, Lemma 2.4(2)], the rightmost column in Diagram (2.1), and hence the middle column in Diagram (2.2), is $\operatorname{Hom}_R(-, \mathscr{E})$-exact. Since $\mathscr{X}$ is $\mathscr{E}$-precoresolving, we have $X \in \mathscr{X}$, and thus the middle row in Diagram (2.2) is the desired exact sequence.

Since there exists an exact sequence

$$0 \to X \to D \to X' \to 0$$

in $\operatorname{Mod} R$ with $D \in \mathscr{D}$ and $X' \in \mathscr{X}$, we obtain the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equals] & X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r, dashed] & Y \arrow[r, dashed] & D \arrow[r, dashed] \arrow[d] & Y' \arrow[r, dashed] \arrow[d, dashed] & 0 \\
 & & X' \arrow[r, equals] \arrow[d] & X' \arrow[d, dashed] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By the middle row in this diagram, we have $\mathscr{D}\text{-pd}\, Y' \leq n$. Since $A \in \mathscr{X}^{\perp}$, the rightmost column in the above diagram splits and A is a direct summand of Y'. Furthermore, since $\mathscr{D}\text{-pd}^{\leq n}$ is closed under direct summands, we have $\mathscr{D}\text{-}\operatorname{pd} A \leq n$.

(2) It is dual to (1). $\square$

Lemma 2.3. *Let $\mathscr{D}$ and $\mathscr{E}$ be subcategories of* $\operatorname{Mod} R$.

(1) *If $\mathscr{D} \subseteq {}^{\perp}\mathscr{E}$, then $\mathscr{D}$ is an $\mathscr{E}$-coproper cogenerator for ${}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$. Furthermore, if $\mathscr{D}$ is additive and $\mathscr{D} \subseteq {}^{\perp}\mathscr{E} \cap \mathscr{E}$, then ${}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$ is $\mathscr{E}$-precoresolving in* $\operatorname{Mod} R$.

(2) *If $\mathscr{D} \subseteq \mathscr{E}^{\perp}$, then $\mathscr{D}$ is an $\mathscr{E}$-proper generator for $\mathscr{E}^{\perp} \cap \widetilde{\mathrm{res}}_{\mathscr{E}}\, \mathscr{D}$. Furthermore, if $\mathscr{D}$ is additive and $\mathscr{D} \subseteq \mathscr{E}^{\perp} \cap \mathscr{E}$, then $\mathscr{E}^{\perp} \cap \widetilde{\mathrm{res}}_{\mathscr{E}}\, \mathscr{D}$ is $\mathscr{E}$-preresolving in* $\operatorname{Mod} R$.

Proof. (1) Set $\mathscr{X} =: {}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$. Let $X \in \mathscr{X}$. Then there exists a $\operatorname{Hom}_R(-, \mathscr{E})$-exact exact sequence

$$0 \to X \to D \to X' \to 0$$

in $\operatorname{Mod} R$ with $D \in \mathscr{D}$ and $X' \in \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$. Since $\mathscr{D} \subseteq {}^{\perp}\mathscr{E}$, we have $\mathscr{D} \subseteq \mathscr{X}$ and $X' \in {}^{\perp}\mathscr{E}$. Thus $X' \in {}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$ and $\mathscr{D}$ is an $\mathscr{E}$-coproper cogenerator for $\mathscr{X}$.

If $\mathscr{D} \subseteq \mathscr{E}$, then it is easy to see that $\widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$ is closed under $\mathscr{E}$-coproper extensions by [24, Lemma 3.1(2)]. Thus, if $\mathscr{D} \subseteq {}^{\perp}\mathscr{E} \cap \mathscr{E}$, then ${}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D}$ is $\mathscr{E}$-precoresolving in $\operatorname{Mod} R$.

(2) It is dual to (1). $\square$

2.2 Semidualizing bimodules and related module classes

We say that a module $M \in \operatorname{Mod} R$ admits a *degreewise finite R-projective resolution* if there exists an exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all P_i finitely generated projective.

Definition 2.4. ([1, 19]). Let R and S be arbitrary rings. An (R-S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied.

(a1) ${}_RC$ admits a degreewise finite R-projective resolution.

(a2) C_S admits a degreewise finite S^{op}-projective resolution.

(b1) The homothety map ${}_RR_R \overset{{}_R\gamma}{\to} \operatorname{Hom}_{S^{op}}(C, C)$ is an isomorphism.

(b2) The homothety map ${}_SS_S \stackrel{\gamma_S}{\to} \mathrm{Hom}_R(C,C)$ is an isomorphism.
(c1) $\mathrm{Ext}_R^{\geq 1}(C,C)=0$.
(c2) $\mathrm{Ext}_{S^{op}}^{\geq 1}(C,C)=0$.

Recall from [40] that a module $T \in \mathrm{Mod}\, R$ is called *generalized tilting* if the following conditions are satisfied: (1) ${}_RT$ admits a degreewise finite R-projective resolution; (2) $\mathrm{Ext}_R^{\geq 1}(T,T)=0$; and (3) ${}_RR \in \mathrm{cores}_{\mathrm{add}_R T}\,\mathrm{add}_R T$, where $\mathrm{add}_R T$ is the subcategory of $\mathrm{Mod}\, R$ consisting of direct summands of finite direct sums of ${}_RT$. Generalized tilting modules are usually called *Wakamatsu tilting modules*, see [4, 31]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if ${}_RC$ is Wakamatsu tilting with $S=\mathrm{End}({}_RC)$, and if and only if C_S is Wakamatsu tilting with $R=\mathrm{End}(C_S)$ ([42, Corollary 3.2]). Typical examples of semidualizing bimodules include the free module of rank one and the dualizing module over a Cohen-Macaulay local ring. For more examples of semidualizing bimodules, the reader is referred to [19, 36, 41].

In the following, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. We write

$$(-)_* := \mathrm{Hom}(C,-),$$

and write

$$\mathcal{P}_C(R) := \{C\otimes_S P \mid P\in\mathcal{P}(S)\},\quad \mathcal{P}_C(S^{op}) := \{P'\otimes_R C \mid P'\in\mathcal{P}(R^{op})\},$$

$$\mathcal{F}_C(R) := \{C\otimes_S F \mid F\in\mathcal{F}(S)\},\quad \mathcal{F}_C(S^{op}) := \{F'\otimes_R C \mid F'\in\mathcal{F}(R^{op})\},$$

$$\mathcal{I}_C(S) := \{I_* \mid I\in\mathcal{I}(R)\},\quad \mathcal{I}_C(R^{op}) := \{I'_* \mid I'\in\mathcal{I}(S^{op})\}.$$

The modules in $\mathcal{P}_C(R)$ (resp. $\mathcal{P}_C(S^{op})$), $\mathcal{F}_C(R)$ (resp. $\mathcal{F}_C(S^{op})$) and $\mathcal{I}_C(S)$ (resp. $\mathcal{I}_C(R^{op})$) are called *C-projective*, *C-flat* and *C-injective* respectively. When ${}_RC_S={}_RR_R$, C-projective, C-flat and C-injective modules are exactly projective, flat and injective modules respectively.

Let $M\in\mathrm{Mod}\, R$. Then we have a canonical evaluation homomorphism

$$\theta_M : C\otimes_S M_* \to M$$

defined by $\theta_M(x\otimes f)=f(x)$ for any $x\in C$ and $f\in M_*$. The module M is called *C-coreflexive* if θ_M is an isomorphism (see [36]). We use $\mathrm{Cor}_C(R)$ to denote the subcategory of $\mathrm{Mod}\, R$ consisting of C-coreflexive modules.

Let $N\in\mathrm{Mod}\, S$. Then we have a canonical evaluation homomorphism

$$\mu_N : N \to (C\otimes_S N)_*$$

defined by $\mu_N(y)(x)=x\otimes y$ for any $y\in N$ and $x\in C$. The module N is called *adjoint C-coreflexive* if μ_N is an isomorphism. We use $\mathrm{Acot}_C(S)$ to denote the subcategory of $\mathrm{Mod}\, S$ consisting of adjoint C-coreflexive modules.

Definition 2.5. ([19])
(1) The *Auslander class* $\mathcal{A}_C(S)$ with respect to C consists of all left S-modules N satisfying
 (A1) $N\in {C_S}^{\top}$;
 (A2) $\mathrm{Ext}_R^{\geq 1}(C, C\otimes_S N)=0$;
 (A3) $N\in\mathrm{Acot}_C(S)$.
(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all left R-modules M satisfying
 (B1) $M\in {}_RC^{\perp}$;
 (B2) $\mathrm{Tor}^S_{\geq 1}(C, M_*)=0$;
 (B3) $M\in\mathrm{Cor}_C(R)$.

Symmetrically, the *Auslander class* $\mathcal{A}_C(R^{op})$ in $\operatorname{Mod} R^{op}$ and the *Bass class* $\mathcal{B}_C(S^{op})$ in $\operatorname{Mod} S^{op}$ are defined.

Lemma 2.6. *It holds that*
(1) $\operatorname{fd}_S I_* = \mathcal{F}_C(R)\text{-pd}\, I$ *and* $\operatorname{pd}_S I_* = \mathcal{P}_C(R)\text{-pd}\, I$ *for any* $I \in \mathcal{I}(R)$.
(2) $\operatorname{fd}_{R^{op}} I'_* = \mathcal{F}_C(S^{op})\text{-pd}\, I'$ *and* $\operatorname{pd}_{R^{op}} I'_* = \mathcal{P}_C(S^{op})\text{-pd}\, I'$ *for any* $I' \in \mathcal{I}(S^{op})$.

Proof. (1) Let I be an injective left R-module. Since $I \in \mathcal{B}_C(R)$ by [19, Lemma 4.1], we have $\mathcal{F}_C(R)\text{-pd}\, I = \operatorname{fd}_S I_*$ and and $\operatorname{pd}_S I_* = \mathcal{P}_C(R)\text{-pd}\, I$ by [38, Lemma 2.6(1)(2)].
(2) It is the symmetric version of (1). □

Recall from [13] that a module $N \in \operatorname{Mod} S$ is called *weak flat* if $\operatorname{Tor}_1^S(X, N) = 0$ for any right S-module X admitting a degreewise finite S^{op}-projective resolution; and a module $M \in \operatorname{Mod} R$ is called *weak injective* if $\operatorname{Ext}_R^1(X, M) = 0$ for any left R-module X admitting a degreewise finite R-projective resolution. Symmetrically, the notions of weak flat modules in $\operatorname{Mod} R^{op}$ and weak injective modules in $\operatorname{Mod} S^{op}$ are defined. In [6], weak flat modules and weak injective modules are called *level modules* and *absolutely clean modules* respectively.

We use $\mathcal{WF}(S)$ (resp. $\mathcal{WI}(R)$) to denote the subcategory of $\operatorname{Mod} S$ (resp. $\operatorname{Mod} R$) consisting of weak flat (resp. weak injective) modules, and use $\mathcal{WF}(R^{op})$ (resp. $\mathcal{WI}(S^{op})$) to denote the subcategory of $\operatorname{Mod} R^{op}$ (resp. $\operatorname{Mod} S^{op}$) consisting of weak flat (resp. weak injective) modules. We write

$$\mathcal{WF}_C(R) := \{C \otimes_S F \mid F \in \mathcal{WF}(S)\} \text{ and } \mathcal{WF}_C(S^{op}) := \{F' \otimes_R C \mid F' \in \mathcal{WF}(R^{op})\},$$

$$\mathcal{WI}_C(S) := \{I_* \mid I \in \mathcal{WI}(R)\} \text{ and } \mathcal{WI}_C(R^{op}) := \{I'_* \mid I' \in \mathcal{WI}(S^{op})\}.$$

Lemma 2.7. ([38, Lemma 2.5(1)], [37, Corollary 3.5(2)] and [14, Corollary 2.3])
(1) $\mathcal{P}(S) \cup \mathcal{I}_C(S) \subseteq \mathcal{F}(S) \cup \mathcal{I}_C(S) \subseteq \mathcal{WF}(S) \cup \mathcal{I}_C(S) \subseteq \mathcal{A}_C(S) \subseteq {}^{\perp}\mathcal{I}_C(S) \cap \operatorname{Acot}_C(S)$.
(2) $\mathcal{I}(R) \cup \mathcal{P}_C(R) \subseteq \mathcal{I}(R) \cup \mathcal{F}_C(R) \subseteq \mathcal{I}(R) \cup \mathcal{WF}_C(R) \subseteq \mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp} \cap \operatorname{Cor}_C(R)$.

Let $\mathscr{B}$ be a subcategory of $\operatorname{Mod} R^{op}$. Recall that a sequence in $\operatorname{Mod} R$ is called $(\mathscr{B} \otimes_R -)$-*exact* if it is exact after applying the functor $B \otimes_R -$ for any $B \in \mathscr{B}$. The following notions were introduced by Holm and Jørgensen [18] over commutative rings. The following are their non-commutative versions.

Definition 2.8. ([30, 34])
(1) A module $M \in \operatorname{Mod} R$ is called *C-Gorenstein projective* if

$$M \in {}^{\perp}\mathcal{P}_C(R) \cap \operatorname{cores}\widetilde{_{\mathcal{P}_C(R)}\,\mathcal{P}_C}(R).$$

Symmetrically, the notion of *C-Gorenstein projective modules* in $\operatorname{Mod} S^{op}$ is defined.
(2) A module $M \in \operatorname{Mod} R$ is called *C-Gorenstein flat* if $M \in \mathcal{I}_C(R^{op})^{\top}$ and there exists an $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact exact sequence

$$0 \to M \to Q^0 \to Q^1 \to \cdots \to Q^i \to \cdots$$

in $\operatorname{Mod} R$ with all Q^i in $\mathcal{F}_C(R)$. Symmetrically, the notion of *C-Gorenstein flat modules* in $\operatorname{Mod} S^{op}$ is defined.
(3) A module $N \in \operatorname{Mod} S$ is called *C-Gorenstein injective* if

$$N \in \mathcal{I}_C(S)^{\perp} \cap \operatorname{res}\widetilde{_{\mathcal{I}_C(S)}\,\mathcal{I}_C}(S).$$

Symmetrically, the notion of *C-Gorenstein injective modules* in $\operatorname{Mod} R^{op}$ is defined.

We use $\mathcal{GP}_C(R)$ (resp. $\mathcal{GF}_C(R)$) to denote the subcategory of $\operatorname{Mod} R$ consisting of C-Gorenstein projective (resp. flat) modules, and use $\mathcal{GI}_C(S)$ to denote the subcategory of $\operatorname{Mod} S$ consisting of C-Gorenstein injective modules. Symmetrically, we use $\mathcal{GP}_C(S^{op})$ (resp. $\mathcal{GF}_C(S^{op})$) to denote the subcategory of $\operatorname{Mod} S^{op}$ consisting of C-Gorenstein projective (resp. flat) modules, and use $\mathcal{GI}_C(R^{op})$ to denote the subcategory of $\operatorname{Mod} R^{op}$ consisting of C-Gorenstein injective modules. When ${}_RC_S = {}_RR_R$, C-Gorenstein projective, flat and injective modules are exactly Gorenstein projective, flat and injective modules respectively ([11, 17]).

For a subcategory $\mathscr{X}$ of $\operatorname{Mod} R$ (or $\operatorname{Mod} R^{op}$), we write

$$\mathscr{X}^+ := \{X^+ \mid X \in \mathscr{X}\},$$

where $(-)^+ = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ with $\mathbb{Z}$ the additive group of integers and $\mathbb{Q}$ the additive group of rational numbers.

Lemma 2.9. *It holds that*

(1) $\mathcal{F}_C(R) \subseteq {}^{\perp}[\mathcal{I}_C(R^{op})^+] = \mathcal{I}_C(R^{op})^{\top}$.

(2) $\mathcal{P}(R) \cup \mathcal{P}_C(R) \subseteq \mathcal{GP}_C(R)$ *and* $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R) \subseteq \mathcal{GF}_C(R)$.

Proof. (1) The former inclusion follows from [34, Lemma 4.13], and the latter equality follows from [15, Lemma 2.16(b)].

(2) Note that the former assertion has been proved in [43, Proposition 2.6] in the commutative case and the argument there is also valid in the non-commutative case. For the latter assertion, it is easy to that $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ and that the inclusion $\mathcal{F}_C(R) \subseteq \mathcal{GF}_C(R)$ follows from (1) and the definition of C-Gorenstein flat modules. $\square$

By Lemma 2.9(1) and [15, Lemma 2.16(a)], we have

$$\mathcal{GF}_C(R) = {}^{\perp}[\mathcal{I}_C(R^{op})^+] \cap \operatorname{cores}\widetilde{_{\mathcal{I}_C(R^{op})^+}} \mathcal{F}_C(R),$$

$$\mathcal{GF}_C(S^{op}) = {}^{\perp}[\mathcal{I}_C(S)^+] \cap \operatorname{cores}\widetilde{_{\mathcal{I}_C(S)^+}} \mathcal{F}_C(S^{op}).$$

3 A construction of a grid-type commutative diagram

In this section, R is an arbitrary ring. Let

$$\cdots \to X_i \to \cdots \to X_1 \to X_0 \to Y^0 \to Y^1 \to \cdots \to Y^j \to \cdots$$

be an exact sequence in $\operatorname{Mod} R$. By using special coresolutions of all X_i and special resolutions of all Y^j, we will construct a grid-type commutative diagram, which plays a crucial role in the sequel. We begin with the following observation.

Lemma 3.1. *Let* $\mathscr{D}$ *be a subcategory of* $\operatorname{Mod} R$*, and let*

$$0 \to X_1 \to D \xrightarrow{f} X_2 \to 0 \tag{3.1}$$

be an exact sequence in $\operatorname{Mod} R$ *with* $D \in \mathscr{D}$*.*

(1) *Assume that* (3.1) *is* $\operatorname{Hom}_R(\mathscr{D}, -)$*-exact and*

$$0 \to W^0 \xrightarrow{g^0} D^0 \xrightarrow{g^1} D^1 \xrightarrow{g^2} \cdots \xrightarrow{g^i} D^i \xrightarrow{g^{i+1}} \cdots \tag{3.2}$$

is an exact sequence in $\operatorname{Mod} R$ *with all* $D^i \in \mathscr{D}$*. If* (3.2) *is both* $\operatorname{Hom}_R(-, D)$*-exact and* $\operatorname{Hom}_R(-, X_2)$*-exact, then it also* $\operatorname{Hom}_R(-, X_1)$*-exact.*

(2) *Assume that* (3.1) *is* $\mathrm{Hom}_R(-,\mathscr{D})$*-exact and*

$$\cdots \to D_i \to \cdots \to D_1 \to D_0 \to W_0 \to 0 \tag{3.3}$$

is an exact sequence in $\mathrm{Mod}\, R$ *with all* $D_i \in \mathscr{D}$*. If* (3.3) *is both* $\mathrm{Hom}_R(D,-)$*-exact and* $\mathrm{Hom}_R(X_1,-)$*-exact, then it also* $\mathrm{Hom}_R(X_2,-)$*-exact.*

Proof. (1) For any $i \geq 1$, let $W^i = \mathrm{Im}\, g^i$ and let $g^i = \lambda^i\pi^i$ be the epic-monic decomposition of g^i with $\pi^i : D^{i-1} \twoheadrightarrow W^i$ and $\lambda^i : W^i \rightarrowtail D^i$. Note that (3.1) is $\mathrm{Hom}_R(\mathscr{D},-)$-exact and (3.2) is both $\mathrm{Hom}_R(-,D)$-exact and $\mathrm{Hom}_R(-,X_2)$-exact by assumption. So for any $i \geq 0$, we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(W^{i+1},X_1) \arrow[r] \arrow[d] & \mathrm{Hom}_R(D^i,X_1) \arrow[r, "\mathrm{Hom}_R(\lambda^i,X_1)"] \arrow[d] & \mathrm{Hom}_R(W^i,X_1) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(W^{i+1},D) \arrow[r] \arrow[d, "\mathrm{Hom}_R(W^{i+1},f)"] & \mathrm{Hom}_R(D^i,D) \arrow[r] \arrow[d] & \mathrm{Hom}_R(W^i,D) \arrow[r] \arrow[d, "\mathrm{Hom}_R(W^i,f)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(W^{i+1},X_2) \arrow[r] & \mathrm{Hom}_R(D^i,X_2) \arrow[r] \arrow[d] & \mathrm{Hom}_R(W^i,X_2) \arrow[r] & 0 \\
 & & 0, & &
\end{tikzcd}$$

where $\lambda^0 = g^0$. Then each $\mathrm{Hom}_R(W^i,f)$ is epic. Thus by the snake lemma, each $\mathrm{Hom}_R(\lambda^i,X_1)$ is epic and the assertion follows.

(2) It is dual to (1). $\square$

For the sake of simplicity, we introduce the following notions.

Definition 3.2. Let $\mathscr{D}$ and $\mathscr{E}$ be subcategories of $\mathrm{Mod}\, R$. Let $X \in \mathrm{Mod}\, R$. A module $B \in \mathrm{Mod}\, R$ is said to satisfy the $(X, \widetilde{\mathrm{cores}}_{\mathscr{E}}\, \mathscr{D})$*-coproper property* if any $\mathrm{Hom}_R(-,\mathscr{E})$-exact exact sequence

$$0 \to X \to D^0 \to D^1 \to \cdots \to D^i \to \cdots$$

in $\mathrm{Mod}\, R$ with all D^i in $\mathscr{D}$ is $\mathrm{Hom}_R(-,B)$-exact; dually, the module B is said to satisfy the $(\widetilde{\mathrm{res}}_{\mathscr{E}}\, \mathscr{D}, X)$*-proper property* if any $\mathrm{Hom}_R(\mathscr{E},-)$-exact exact sequence

$$\cdots \to D_i \to \cdots \to D_1 \to D_0 \to X \to 0$$

in $\mathrm{Mod}\, R$ with all D_i in $\mathscr{D}$ is $\mathrm{Hom}_R(B,-)$-exact.

In the following result, we construct certain (co)resolutions of modules, which form a grid-type commutative diagram. It is crucial in studying the behavior of the projective and injective dimensions of modules relative to various classes of C-Gorenstein modules.

Theorem 3.3. *Let* $\mathscr{D}, \mathscr{E}, \mathscr{E}'$ *be subcategories of* $\mathrm{Mod}\, R$ *such that* $\mathscr{D} \subseteq \mathscr{E} \cap \mathscr{E}'$ *and* $\mathscr{D}$ *is additive, and let*

$$\cdots \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{\delta} Y^0 \xrightarrow{\alpha^1} Y^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^j} Y^j \xrightarrow{\alpha^{j+1}} \cdots \tag{3.4}$$

be an exact sequence in $\operatorname{Mod} R$ *with* $X_i \in \widetilde{\mathrm{cores}_{\mathscr{E}}\,\mathscr{D}}$ *and* $Y^j \in \widetilde{\mathrm{res}_{\mathscr{E}'}\,\mathscr{D}}$ *for any* $i, j \geq 0$. *Set* $M := \operatorname{Im}\delta$ *and let* $\delta = \alpha^0 f_0$ *be the epic-monic decomposition of* δ *with* $f_0 : X^0 \twoheadrightarrow M$ *and* $\alpha^0 : M \rightarrowtail Y^0$. *If one of the following two conditions is satisfied:*

(1) Y^j *satisfies the* $(X_i, \widetilde{\mathrm{cores}_{\mathscr{E}}\,\mathscr{D}})$*-coproper property for any* $i, j \geq 0$,

(2) X_i *satisfies the* $(\widetilde{\mathrm{res}_{\mathscr{E}'}\,\mathscr{D}}, Y^j)$*-proper property for any* $i, j \geq 0$,

then there exists the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "f_{i+1}"'] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & X_i \arrow[r] \arrow[d, "f_i"'] & D_i^0 \arrow[r] \arrow[d] & D_i^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & D_i^j \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d, "f_2"'] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "f_1"'] & D_1^0 \arrow[r] \arrow[d] & D_1^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & D_1^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X_0 \arrow[r] \arrow[d, "f_0"'] & D_0^0 \arrow[r] \arrow[d] & D_0^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & D_0^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & M \arrow[r, "\alpha^0"] \arrow[d] & Y^0 \arrow[r, "\alpha^1"] \arrow[d] & Y^1 \arrow[r, "\alpha^2"] \arrow[d] & \cdots \arrow[r, "\alpha^j"] & Y^j \arrow[r, "\alpha^{j+1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

in $\operatorname{Mod} R$ *with all* D_i^j *in* $\mathscr{D}$, *such that all rows but the bottom one are* $\operatorname{Hom}_R(-, \mathscr{E})$*-exact and all columns but the leftmost one are* $\operatorname{Hom}_R(\mathscr{E}', -)$*-exact.*

Proof. (1) Set $K_i := \operatorname{Im} f_i$ and $M^i := \operatorname{Im}\alpha^i$ for any $i \geq 0$. Since $X_0 \in \widetilde{\mathrm{cores}_{\mathscr{E}}\,\mathscr{D}}$ and Y^j satisfies the $(X_0, \widetilde{\mathrm{cores}_{\mathscr{E}}\,\mathscr{D}})$-coproper property for any $j \geq 0$, there exists a $\operatorname{Hom}_R(-, \mathscr{E})$-exact exact sequence

$$0 \to X_0 \xrightarrow{e^0} D^0 \to X_0' \to 0 \tag{3.5}$$

in $\operatorname{Mod} R$ with $D^0 \in \mathscr{D}$ and $X_0' \in \widetilde{\mathrm{cores}_{\mathscr{E}}\,\mathscr{D}}$, which is also $\operatorname{Hom}_R(-, Y^j)$-exact for any $j \geq 0$. So there exists a homomorphism $h^0 \in \operatorname{Hom}_R(D^0, Y^0)$ such that the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & K_1 \arrow[d] & & & \\
0 \arrow[r] & X_0 \arrow[r, "e^0"] \arrow[d, "f_0"'] & D^0 \arrow[r] \arrow[d, dashed, "h^0"] & X'_0 \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha^0"] \arrow[d] & Y^0 \arrow[r] & M^1 \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

commutes. Since $Y^0 \in \widetilde{\operatorname{res}_{\mathscr{E}'} \mathscr{D}}$, there exists a $\operatorname{Hom}_R(\mathscr{E}', -)$-exact exact sequence

$$0 \to Y_0^0 \to D_0 \xrightarrow{g_0} Y^0 \to 0 \tag{3.6}$$

in $\operatorname{Mod} R$ with $D_0 \in \mathscr{D}$ and $Y_0^0 \in \widetilde{\operatorname{res}_{\mathscr{E}'} \mathscr{D}}$. Then we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & \\
0 \arrow[r, dashed] & K_1 \arrow[r, dashed] \arrow[d] & K_1^0 \arrow[r, dashed] \arrow[d] & K_1^{(1)} \arrow[r, dashed] \arrow[d, dashed] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "\binom{e^0}{0}"] \arrow[d, "f_0"'] & D_0^0 \arrow[r] \arrow[d, "(h^0{,}g_0)"] & X_0^{(1)} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha^0"] \arrow[d] & Y^0 \arrow[r] \arrow[d] & M^1 \arrow[r] \arrow[d, dashed] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

Diagram (3.1(1))

where $\utilde{D_0^0} = D^0 \oplus D_0 (\in \mathscr{D})$ and $\utilde{X_0^{(1)}} = X'_0 \oplus D_0$. By the exact sequence (3.5), we have $X'_0 \in \operatorname{cores}_{\mathscr{E}} \mathscr{D}$, and hence $X_0^{(1)} \in \operatorname{cores}_{\mathscr{E}} \mathscr{D}$. It is easy to see that the middle row in Diagram (3.1(1)) is $\operatorname{Hom}_R(-, \mathscr{E})$-exact and $\operatorname{Hom}_R(-, Y^j)$-exact for any $j \geq 0$ and that the middle column is $\operatorname{Hom}_R(\mathscr{E}', -)$-exact. Moreover, the middle column yields the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y_0^0 \arrow[r] \arrow[d, equal] & K_1^0 \arrow[r] \arrow[d] & D^0 \arrow[r] \arrow[d, "h^0"] & 0 \\
0 \arrow[r] & Y_0^0 \arrow[r] & D_0 \arrow[r, "g_0"] & Y^0 \arrow[r] & 0.
\end{tikzcd}$$

Since the lower row is $\mathrm{Hom}_R(\mathscr{E}', -)$-exact, it follows from [24, Lemma 2.4(1)] that the upper row is also $\mathrm{Hom}_R(\mathscr{E}', -)$-exact, and hence $\mathrm{Hom}_R(\mathscr{D}, -)$-exact as $\mathscr{D} \subseteq \mathscr{E}'$. It implies that the upper row splits and $K_1^0 \cong Y_0^0 \oplus D^0$, which yields $K_1^0 \in \widetilde{\mathrm{res}}_{\mathscr{E}'}\,\mathscr{D}$. Since $\mathscr{D} \subseteq \mathscr{E}$ and Y^0 satisfies the $(X_i, \widetilde{\mathrm{cores}}_{\mathscr{E}}\,\mathscr{D})$-coproper property for any $i \geq 0$, it follows from Lemma 3.1(1) and the exact sequence (3.6) that Y_0^0, and hence K_1^0, satisfies the $(X_i, \widetilde{\mathrm{cores}}_{\mathscr{E}}\,\mathscr{D})$-coproper property for any $i \geq 0$.

Since Y^1 satisfies the $(X_0, \widetilde{\mathrm{cores}}_{\mathscr{E}}\,\mathscr{D})$-coproper property, it follows from the middle row in Diagram (3.1(1)) that Y^1 also satisfies the $(X_0^{(1)}, \widetilde{\mathrm{cores}}_{\mathscr{E}}\,\mathscr{D})$-coproper property. Similar to the above argument, we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & \\
0 \arrow[r, dashed] & K_1^{(1)} \arrow[r, dashed] \arrow[d] & K_1^1 \arrow[r, dashed] \arrow[d] & K_1^{(2)} \arrow[r, dashed] \arrow[d, dashed] & 0 \\
0 \arrow[r] & X_0^{(1)} \arrow[r] \arrow[d] & D_0^1 \arrow[r] \arrow[d] & X_0^{(2)} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & M^1 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & M^2 \arrow[r] \arrow[d, dashed] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

Diagram (3.1(2))

such that the middle row is $\mathrm{Hom}_R(-, \mathscr{E})$-exact and the middle column is $\mathrm{Hom}_R(\mathscr{E}', -)$-exact, and such that $K_1^1 \in \widetilde{\mathrm{res}}_{\mathscr{E}'}\,\mathscr{D}$ satisfies the $(X_i, \widetilde{\mathrm{cores}}_{\mathscr{E}}\,\mathscr{D})$-coproper property for any $i \geq 0$.

Continuing this process and splicing Diagrams (3.1(1)), (3.1(2)), $\cdots$ from left to right, we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & K_1^0 \arrow[r] \arrow[d] & K_1^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & K_1^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X_0 \arrow[r] \arrow[d, "f_0"'] & D_0^0 \arrow[r] \arrow[d] & D_0^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & D_0^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & M \arrow[r, "\alpha^0"] \arrow[d] & Y^0 \arrow[r, "\alpha^1"] \arrow[d] & Y^1 \arrow[r, "\alpha^2"] \arrow[d] & \cdots \arrow[r, "\alpha^j"] & Y^j \arrow[r, "\alpha^{j+1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

Diagram (3.2(1))

in $\operatorname{Mod} R$ with all D_0^j in $\mathscr{D}$, such that the middle row is $\operatorname{Hom}_R(-,\mathscr{E})$-exact and all columns but the leftmost one are $\operatorname{Hom}_R(\mathscr{E}',-)$-exact, and such that $K_1^j \in \widetilde{\operatorname{res}_{\mathscr{E}'}\mathscr{D}}$ satisfies the $(X_i, \widetilde{\operatorname{cores}_{\mathscr{E}}\mathscr{D}})$-coproper property for any $i, j \geq 0$.

Similar to the above argument, we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d] & K_2^0 \arrow[r] \arrow[d] & K_2^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & K_2^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & D_1^0 \arrow[r] \arrow[d] & D_1^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & D_1^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & K_1^0 \arrow[r] \arrow[d] & K_1^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & K_1^j \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

Diagram (3.2(2))

in $\operatorname{Mod} R$ with all D_1^j in $\mathscr{D}$, such that the middle row is $\operatorname{Hom}_R(-,\mathscr{E})$-exact and all columns but the leftmost one are $\operatorname{Hom}_R(\mathscr{E}',-)$-exact, and such that $K_2^j \in \widetilde{\operatorname{res}_{\mathscr{E}'}\mathscr{D}}$ satisfies the $(X_i, \widetilde{\operatorname{cores}_{\mathscr{E}}\mathscr{D}})$-coproper property for any $i, j \geq 0$.

Continuing this process and splicing Diagrams (3.2(1)), (3.2(2)), $\cdots$ from bottom to top, we get the desired commutative diagram.

(2) It is similar to (1). $\square$

It is trivial that in the exact sequence (3.4), if

$$0 \to M \to Y^0 \to Y^1 \to \cdots \to Y^j \to \cdots$$

is an injective coresolution of M, then the condition (1) in Theorem 3.3 is satisfied; and if

$$\cdots \to X_i \to \cdots \to X_1 \to X_0 \to M \to 0$$

is a projective resolution of M, then the condition (2) in Theorem 3.3 is satisfied.

4 C-Gorenstein modules

From now on, assume that R and S are arbitrary rings and ${}_RC_S$ is semidualizing bimodule. We introduce the following notions, which are useful in providing unified proofs of related results.

Definition 4.1.

(1) Let $\mathscr{H}$ be a subcategory of $\operatorname{Mod} R$. A module $M \in \operatorname{Mod} R$ is called *$\mathscr{H}_C$-Gorenstein projective* if $M \in {}^{\perp}\mathscr{H}$ and there exists a $\operatorname{Hom}_R(-,\mathscr{H})$-exact exact sequence

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^i \to \cdots$$

in $\operatorname{Mod} R$ with all G^i in $\mathcal{P}_C(R)$. Symmetrically, the notion of *$\mathscr{H}_C$-Gorenstein projective modules* in $\operatorname{Mod} S^{op}$ is defined.

(2) Let $\mathscr{H}$ be a subcategory of $\operatorname{Mod} R$. A module $M \in \operatorname{Mod} R$ is called *$\mathscr{H}_C$-Gorenstein flat* if $M \in \mathcal{I}_C(R^{op})^{\top}$ and there exists an $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact exact sequence

$$0 \to M \to H^0 \to H^1 \to \cdots \to H^i \to \cdots$$

in $\operatorname{Mod} R$ with all H^i in $\mathscr{H}$. Symmetrically, the notion of *$\mathscr{H}_C$-Gorenstein flat modules* in $\operatorname{Mod} S^{op}$ is defined.

(3) Let $\mathscr{T}$ be a subcategory of $\operatorname{Mod} S$. A module $N \in \operatorname{Mod} S$ is called *$\mathscr{T}_C$-Gorenstein injective* if $N \in \mathscr{T}^{\perp}$ and there exists a $\operatorname{Hom}_R(\mathscr{T}, -)$-exact exact sequence

$$\cdots \to E_i \to \cdots \to E_1 \to E_0 \to N \to 0$$

in $\operatorname{Mod} S$ with all E_i in $\mathcal{I}_C(S)$. Symmetrically, the notion of *$\mathscr{T}_C$-Gorenstein injective modules* in $\operatorname{Mod} R^{op}$ is defined.

Let $\mathscr{H}$ be a subcategory of $\operatorname{Mod} R$. We use $\mathcal{GP}_C(\mathscr{H})$ and $\mathcal{GF}_C(\mathscr{H})$ to denote the subcategories of $\operatorname{Mod} R$ consisting of $\mathscr{H}_C$-Gorenstein projective modules and $\mathscr{H}_C$-Gorenstein flat modules respectively. Then we have

$$\mathcal{GP}_C(\mathscr{H}) = {}^{\perp}\mathscr{H} \cap \operatorname{cores}\widetilde{_{\mathscr{H}}\, \mathcal{P}_C}(R).$$

By Lemma 2.9(1) and [15, Lemma 2.16(a)], we have

$$\mathcal{GF}_C(\mathscr{H}) = {}^{\perp}[\mathcal{I}_C(R^{op})^{+}] \cap \operatorname{cores}\widetilde{_{\mathcal{I}_C(R^{op})^{+}}}\, \mathscr{H}.$$

Let $\mathscr{T}$ be a subcategory of $\operatorname{Mod} S$. We use $\mathcal{GI}_C(\mathscr{T})$ to denote the subcategory of $\operatorname{Mod} S$ consisting of $\mathscr{T}_C$-Gorenstein injective modules. We have

$$\mathcal{GI}_C(\mathscr{T}) = \mathscr{T}^{\perp} \cap \operatorname{res}\widetilde{_{\mathscr{T}}\, \mathcal{I}_C}(S).$$

4.1 $\mathscr{H}_C$-Gorenstein flat and projective dimensions

In this subsection, assume that $\mathscr{T}$ is a resolving subcategory of $\mathcal{A}_C(S)$ and

$$\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$$

which is closed under finite direct sums and direct summands. By [39, Lemma 2.4(3)], there exists the following Foxby equivalence:

$$\operatorname{Acot}_C(S) \underset{(-)_*}{\overset{C\otimes_S -}{\rightleftarrows}} \operatorname{Cor}_C(R),$$

which induces the following Foxby equivalence:

$$\mathscr{T} \underset{(-)_*}{\overset{C\otimes_S -}{\rightleftarrows}} \mathscr{H}.$$

Lemma 4.2. *It holds that*

(1) $\mathcal{P}_C(R) \subseteq \mathscr{H} \subseteq \mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp}$.

(2) $\mathscr{H}$ *is a* $\mathcal{P}_C(R)$*-resolving subcategory of* $\operatorname{Mod} R$ *with a* $\mathcal{P}_C(R)$*-proper generator* $\mathcal{P}_C(R)$*.*

(3) *The subcategory* $\mathscr{H}\text{-pd}^{\leq n}$ *is closed under direct summands for any* $n \geq 0$*.*

Proof. (1) By Lemma 2.7(2), we have $\mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp}$. Since $\mathcal{P}(S) \subseteq \mathscr{T} \subseteq \mathcal{A}_C(S)$, the assertion follows easily.

(2) It follows from [35, Proposition 3.7 and Theorem 3.9] and [29, Lemma 3.5(1)].

(3) Since $\mathscr{H}$ is closed under finite direct sums and direct summands, the assertion follows from the former two assertions and [25, Corollary 3.9]. □

A module $M \in \operatorname{Mod} R$ is said to *admit an infinite* $\mathscr{D}$*-coproper coresolution* if there exists a $\operatorname{Hom}_R(-, \mathscr{D})$-exact exact sequence

$$0 \to M \to D^0 \to D^1 \to \cdots \to D^i \to \cdots$$

in $\operatorname{Mod} R$ with all $D^i \in \mathscr{D}$; dually, the module M is said to *admit an infinite* $\mathscr{D}$*-proper resolution* if there exists a $\operatorname{Hom}_R(\mathscr{D}, -)$-exact exact sequence

$$\cdots \to D_i \to \cdots \to D_1 \to D_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all $D_i \in \mathscr{D}$.

Lemma 4.3. *If* $M \in \operatorname{Mod} R$ *with* $\mathscr{H}\text{-pd}\, M \leq n$ *with* $n \geq 0$*, then* M *admits an infinite* $\mathcal{P}_C(R)$*-proper resolution*

$$\cdots \to G_i \to \cdots \to G_1 \to G_0 \to M \to 0$$

in $\operatorname{Mod} R$*, such that* $\operatorname{Im}(G_n \to G_{n-1}) \in \mathscr{H}$*.*

Proof. It follows from Lemma 4.2(2) and [25, Theorem 3.6]. □

The following lemma is a consequence of Theorem 3.3, which plays a crucial role in the sequel.

Lemma 4.4. *Let* $M \in \operatorname{Mod} R$*, and let*

$$\cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

be a projective resolution of M *in* $\operatorname{Mod} R$*. If* $\mathscr{H}\text{-pd}\, I \leq n$ *for any* $I \in \mathcal{I}(R)$*, then there exists an exact sequence*

$$\cdots \xrightarrow{f_{n+2}} P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_0} K_n^0 \xrightarrow{f^1} K_n^1 \xrightarrow{f^2} \cdots \xrightarrow{f^i} K_n^i \xrightarrow{f^{i+1}} \cdots \tag{4.1}$$

in $\operatorname{Mod} R$ *with all* K_n^i *in* $\mathscr{H}$*.*

Proof. Let $M \in \operatorname{Mod} R$ and let

$$\cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{\delta} I^0 \xrightarrow{\alpha^1} I^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^i} I^i \xrightarrow{\alpha^{i+1}} \cdots \tag{4.2}$$

be an exact sequence in $\operatorname{Mod} R$ with all P_i in $\mathcal{P}(R)$ and all I^i in $\mathcal{I}(R)$, such that $M = \operatorname{Im} \delta$. By Lemma 2.9(2), all P_i are in $\mathcal{GP}_C(R) \subseteq \operatorname{cores}_{\mathcal{P}_C(R)} \mathcal{P}_C(R)$.

By assumption, we have $\mathscr{H}\text{-pd}\, I^i \leq n$ for any $i \geq 0$. The assertion for the case $n = 0$ is trivial. Now suppose $n \geq 1$. It follows from Lemma 4.3 that all I^i are in $\operatorname{res}_{\mathcal{P}_C(R)} \mathcal{P}_C(R)$. Then by Theorem 3.3(1), there exists the following commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & K_n \arrow[r] \arrow[d] & K_n^0 \arrow[r] \arrow[d] & K_n^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & K_n^i \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & Q_{n-1}^0 \arrow[r] \arrow[d] & Q_{n-1}^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_{n-1}^i \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d, "f_2"'] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & Q_1^0 \arrow[r] \arrow[d] & Q_1^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_1^i \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & Q_0^0 \arrow[r] \arrow[d] & Q_0^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_0^i \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & M \arrow[r] \arrow[d] & I^0 \arrow[r, "\alpha^1"] \arrow[d] & I^1 \arrow[r, "\alpha^2"] \arrow[d] & \cdots \arrow[r, "\alpha^i"] & I^i \arrow[r, "\alpha^{i+1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

in $\operatorname{Mod} R$ with all Q_j^i in $\mathcal{P}_C(R)$ and $K_n = \operatorname{Im} f_n$, such that all columns but the leftmost one are $\operatorname{Hom}_R(\mathcal{P}_C(R), -)$-exact. It follows from Lemma 4.2(1)(2) and [25, Theorem 3.8(1)] that all K_n^i are in $\mathscr{H}$. From the exact sequence (4.2) and the top row in the above diagram, we get the desired exact sequence (4.1) such that $K_n = \operatorname{Im} f_n$. $\square$

Under certain conditions, we obtain some equivalent characterizations for the $\mathscr{H}_C$-Gorenstein flat dimension of any module being at most n.

Proposition 4.5. *For any $n \geq 0$, consider the following conditions.*

(1) *$\mathcal{GF}_C(\mathscr{H})$-pd $M \leq n$ for any $M \in \operatorname{Mod} R$.*

(2) *$\mathscr{H}$-pd $I \leq n$ for any $I \in \mathcal{I}(R)$, and $\operatorname{fd}_{R^{op}} E' \leq n$ for any $E' \in \mathcal{I}_C(R^{op})$.*

(3) *$\mathscr{T}$-pd $E \leq n$ for any $E \in \mathcal{I}_C(S)$, and $\mathcal{F}_C(S^{op})$-pd $I' \leq n$ for any $I' \in \mathcal{I}(S^{op})$.*

We have (2) $\Longleftrightarrow$ (3).

If $\mathscr{H} \subseteq \mathcal{I}_C(R^{op})^{\top}$ (equivalently $\mathscr{H} \subseteq {}^{\perp}[\mathcal{I}_C(R^{op})^{+}]$), then (2) $\Longrightarrow$ (1); and if further $\mathcal{GF}_C(\mathscr{H})$ is closed under $\mathcal{I}_C(R^{op})^{+}$-coproper extensions, then the above three conditions are equivalent.

Proof. Because $\mathcal{I}(R) \subseteq \mathcal{B}_C(R)$ and $\mathcal{I}(S^{op}) \subseteq \mathcal{B}_C(S^{op})$, we get (2) $\Longleftrightarrow$ (3) by [29, Theorem 3.2].

In the case where $\mathscr{H} \subseteq \mathcal{I}_C(R^{op})^{\top}$, we will prove (2) $\Longrightarrow$ (1). Let $M \in \operatorname{Mod} R$ and let

$$\cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

be a projective resolution of M in $\operatorname{Mod} R$. By (2), we have $\mathscr{H}$-pd $I \leq n$ for any $I \in \mathcal{I}(R)$. Then from Lemma 4.4 we get an exact sequence

$$\cdots \xrightarrow{f_{n+2}} P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_0} K_n^0 \xrightarrow{f^1} K_n^1 \xrightarrow{f^2} \cdots \xrightarrow{f^i} K_n^i \xrightarrow{f^{i+1}} \cdots \tag{4.3}$$

in $\operatorname{Mod} R$ with all K_n^i in $\mathscr{H}$, such that $\operatorname{Im} f_0 \cong \operatorname{Im} f_n$.

Let $E' \in \mathcal{I}_C(R^{op})$. By (2), we have $\operatorname{fd}_{R^{op}} E' \leq n$. Since $\mathscr{H} \subseteq \mathcal{I}_C(R^{op})^{\top}$, applying the functor $E' \otimes_R -$ to the exact sequence (4.3), it is easy to see that each $\operatorname{Im} f_i$ and each $\operatorname{Im} f^i$ are in $E'^{\top}$. It follows that the exact sequence (4.3) is $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact. So $\operatorname{Im} f_0$, and hence $\operatorname{Im} f_n$, is in $\mathcal{GF}_C(\mathscr{H})$. This yields $\mathcal{GF}_C(\mathscr{H})\text{-pd}\, M \leq n$.

Finally, suppose that $\mathscr{H} \subseteq \mathcal{I}_C(R^{op})^{\top}$ (equivalently $\mathscr{H} \subseteq {}^{\perp}[\mathcal{I}_C(R^{op})^{+}]$) and $\mathcal{GF}_C(\mathscr{H})$ is closed under $\mathcal{I}_C(R^{op})^{+}$-coproper extensions, then $\mathcal{GF}_C(\mathscr{H})$ is $\mathcal{I}_C(R^{op})^{+}$-precoresolving in $\operatorname{Mod} R$ admitting an $\mathcal{I}_C(R^{op})^{+}$-coproper cogenerator $\mathscr{H}$. We will prove (1) $\Longrightarrow$ (2).

Let $E' \in \mathcal{I}_C(R^{op})$. Since $\mathcal{GF}_C(\mathscr{H}) \subseteq \mathcal{I}_C(R^{op})^{\top}$, it follows from (1) and dimension shifting that $\operatorname{Tor}^R_{\geq n+1}(E', M) = 0$ for any $M \in \operatorname{Mod} R$, and so $\operatorname{fd}_{R^{op}} E' \leq n$. On the other hand, let $I \in \mathcal{I}(R)$. Then $\mathcal{GF}_C(\mathscr{H})\text{-pd}\, I \leq n$ by (1). Since the class $\mathscr{H}\text{-pd}^{\leq m}$ is closed under direct summands for any $m \geq 0$ by Lemma 4.2(3), it follows from Lemma 2.2(1) that $\mathscr{H}\text{-}\operatorname{pd} I = \mathcal{GF}_C(\mathscr{H})\text{-}\operatorname{pd} I \leq n$. $\square$

In the following result, we give some equivalent characterizations for the $\mathscr{H}_C$-Gorenstein projective dimension of any module being at most n.

Theorem 4.6. *For any $n \geq 0$, the following statements are equivalent.*

(1) *$\mathcal{GP}_C(\mathscr{H})\text{-pd}\, M \leq n$ for any $M \in \operatorname{Mod} R$.*

(2) *$\mathcal{P}_C(R)\text{-pd}\, I \leq n$ for any $I \in \mathcal{I}(R)$, and $\operatorname{id}_R H \leq n$ for any $H \in \mathscr{H}$.*

(3) *$\operatorname{pd}_S E \leq n$ for any $E \in \mathcal{I}_C(S)$, and $\mathcal{I}_C(S)\text{-id}\, T \leq n$ for any $T \in \mathscr{T}$.*

Proof. Because $\mathcal{I}(R) \subseteq \mathcal{B}_C(R)$ and $\mathscr{H} \subseteq \mathcal{B}_C(R)$, we get (2) $\Longleftrightarrow$ (3) by [29, Proposition 4.1 and Corollary 4.4].

(2) $\Longrightarrow$ (1) Let $M \in \operatorname{Mod} R$ and let

$$\cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

be a projective resolution of M in $\operatorname{Mod} R$. By (2), we have $\mathcal{P}_C(R)\text{-pd}\, I \leq n$ for any $I \in \mathcal{I}(R)$. Then from Lemma 4.4(1) we get an exact sequence

$$\cdots \xrightarrow{f_{n+2}} P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_0} K_n^0 \xrightarrow{f^1} K_n^1 \xrightarrow{f^2} \cdots \xrightarrow{f^i} K_n^i \xrightarrow{f^{i+1}} \cdots \tag{4.4}$$

in $\operatorname{Mod} R$ with all K_n^i in $\mathcal{P}_C(R)$, such that $\operatorname{Im} f_0 \cong \operatorname{Im} f_n$.

Let $H \in \mathscr{H}$. By (2), we have $\operatorname{id}_R H \leq n$. Since

$$\mathscr{H} \subseteq \mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp}$$

by Lemma 2.7(2), applying the functor $\operatorname{Hom}_R(-, H)$ to the exact sequence (4.4), it is easy to see that each $\operatorname{Im} f_i$ and each $\operatorname{Im} f^i$ are in ${}^{\perp}H$. It follows that the exact sequence (4.4) is $\operatorname{Hom}_R(-, \mathscr{H})$-exact. So $\operatorname{Im} f_0$, and hence $\operatorname{Im} f_n$, is in $\mathcal{GP}_C(\mathscr{H})$. This yields $\mathcal{GP}_C(\mathscr{H})\text{-pd}\, M \leq n$.

(1) $\Longrightarrow$ (2) Let $H \in \mathscr{H}$. Since $\mathcal{GP}_C(\mathscr{H}) \subseteq {}^{\perp}\mathscr{H}$, it follows from (1) and dimension shifting that $\operatorname{Ext}_R^{\geq n+1}(M, H) = 0$ for any $M \in \operatorname{Mod} R$, and so $\operatorname{id}_R H \leq n$.

It is trivial that $\mathcal{P}_C(R) \subseteq \mathscr{H}$. Since $\mathcal{P}_C(R) \subseteq {}^{\perp}\mathscr{H}$ by [38, Lemma 2.5(1)], we have that $\mathcal{GP}_C(\mathscr{H})$ is $\mathscr{H}$-precoresolving in $\operatorname{Mod} R$ admitting a $\mathscr{H}$-coproper cogenerator $\mathcal{P}_C(R)$ by Lemma 2.3(1). Since the class $\mathcal{P}_C(R)\text{-pd}^{\leq m}$ is closed under direct summands for any $m \geq 0$ by Lemma 4.2(3), it follows from (1) and Lemma 2.2(1) that $\mathcal{P}_C(R)\text{-pd}\, I = \mathcal{GP}_C(\mathscr{H})\text{-pd}\, I \leq n$ for any $I \in \mathcal{I}(R)$. $\square$

We give a sufficient condition for a module in ${}^{\perp}\mathscr{H}$ to be also in $\mathcal{GP}_C(\mathscr{H})$.

Proposition 4.7. *If $M \in {}^{\perp}\mathscr{H}$ with $\mathcal{GP}_C(\mathscr{H})$-pd $M < \infty$, then $M \in \mathcal{GP}_C(\mathscr{H})$.*

Proof. Let $M \in {}^{\perp}\mathscr{H}$ with $\mathcal{GP}_C(\mathscr{H})$-pd $M = n < \infty$. Then there exists an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all G_i in $\mathcal{GP}_C(\mathscr{H})(\subseteq {}^{\perp}\mathscr{H})$. Since $M \in {}^{\perp}\mathscr{H}$, this exact sequence is $\operatorname{Hom}_R(-,\mathscr{H})$-exact. For any $0 \leq i \leq n$, there exists a $\operatorname{Hom}_R(-,\mathscr{H})$-exact exact sequence

$$0 \to G_i \to Q_i^0 \to Q_i^1 \to \cdots \to Q_i^j \to \cdots$$

in $\operatorname{Mod} R$ with all Q_i^j in $\mathcal{P}_C(R)(\subseteq \mathscr{H})$. By [24, Theorem 3.4], we get the following two $\operatorname{Hom}_R(-,\mathscr{H})$-exact exact sequences

$$0 \to M \to Q \to \oplus_{i=0}^n Q_i^{i+1} \to \oplus_{i=0}^n Q_i^{i+2} \to \oplus_{i=0}^n Q_i^{i+3} \to \cdots, \tag{4.5}$$

$$0 \to Q_n^0 \to Q_{n-1}^0 \oplus Q_n^1 \to \cdots \to \oplus_{i=2}^n Q_i^{i-2} \to \oplus_{i=1}^n Q_i^{i-1} \to \oplus_{i=0}^n Q_i^i \to Q \to 0. \tag{4.6}$$

Since $\mathcal{P}_C(R) \subseteq \mathscr{H}$, the exact sequence (4.6) splits, and hence $Q \in \mathcal{P}_C(R)$. It follows from the exact sequence (4.5) that $M \in \widetilde{\operatorname{cores}}_{\mathscr{H}}\, \mathcal{P}_C(R)$, and thus $M \in \mathcal{GP}_C(\mathscr{H})$. □

4.2 $\mathscr{T}_C$-Gorenstein injective dimension

In this subsection, assume that $\mathscr{H}$ is a coresolving subcategory of $\mathcal{B}_C(R)$ and

$$\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$$

which is closed under finite direct sums and direct summands. As in the beginning of Subsection 4.1, there exists the following Foxby equivalence:

$$\mathscr{T} \underset{(-)_*}{\overset{C\otimes_S -}{\rightleftarrows}} \mathscr{H}.$$

The proofs of the following three results are completely dual to those of Lemmas 4.2–4.4 respectively, so we omit them.

Lemma 4.8. *It holds that*

(1) $\mathcal{I}_C(S) \subseteq \mathscr{T} \subseteq \mathcal{A}_C(S) \subseteq {}^{\perp}\mathcal{I}_C(S)$.
(2) *$\mathscr{T}$ is an $\mathcal{I}_C(S)$-coresolving subcategory of $\operatorname{Mod} S$ with an $\mathcal{I}_C(S)$-coproper cogenerator $\mathcal{I}_C(S)$.*
(3) *The subcategory $\mathscr{T}$-id$^{\leq n}$ is closed under direct summands for any $n \geq 0$.*

The proof of Lemma 4.10 needs to use the following lemma.

Lemma 4.9. *If $N \in \operatorname{Mod} S$ with $\mathscr{T}$-id $N \leq n$ with $n \geq 0$, then N admits an infinite $\mathcal{I}_C(S)$-coproper coresolution*

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$$

in $\operatorname{Mod} S$, such that $\operatorname{Im}(E^{n-1} \to E^n) \in \mathscr{T}$.

The following result is a consequence of Theorem 3.3.

Lemma 4.10. *Let $N \in \operatorname{Mod} S$, and let*

$$0 \to N \to I^0 \xrightarrow{g^1} I^1 \xrightarrow{g^2} \cdots \xrightarrow{g^i} I^i \xrightarrow{g^{i+1}} \cdots$$

be an injective coresolution of N in $\operatorname{Mod} S$. If $\mathscr{T}$-id $P \leq n$ for any $P \in \mathcal{P}(S)$, then there exists an exact sequence

$$\cdots \xrightarrow{g_{i+1}} T_i^n \xrightarrow{g_i} \cdots \xrightarrow{g_2} T_1^n \xrightarrow{g_1} T_0^n \xrightarrow{g^0} I^n \xrightarrow{g^{n+1}} I^{n+1} \xrightarrow{g^{n+2}} \cdots$$

in $\operatorname{Mod} S$ with all T_i^n in $\mathscr{T}$.

In the following result, we give some equivalent characterizations for the $\mathscr{T}_C$-Gorenstein injective dimension of any module being at most n. It is dual to Theorem 4.6, but we still give the proof for the reader's convenience.

Theorem 4.11. *For any $n \geq 0$, the following statements are equivalent.*
(1) *$\mathcal{GI}_C(\mathscr{T})$-id $N \leq n$ for any $N \in \operatorname{Mod} S$.*
(2) *$\operatorname{id}_R Q \leq n$ for any $Q \in \mathcal{P}_C(R)$, and $\mathcal{P}_C(R)$-pd $H \leq n$ for any $H \in \mathscr{H}$.*
(3) *$\mathcal{I}_C(S)$-id $P \leq n$ for any $P \in \mathcal{P}(S)$, and $\operatorname{pd}_S T \leq n$ for any $T \in \mathscr{T}$.*

Proof. Because $\mathcal{P}(S) \subseteq \mathcal{A}_C(S)$ and $\mathscr{H} \subseteq \mathcal{B}_C(R)$, we get (2) $\Longleftrightarrow$ (3) by [29, Propositions 4.1 and 4.3].

(3) $\Longrightarrow$ (1) Let $N \in \operatorname{Mod} S$ and let

$$0 \to N \to I^0 \xrightarrow{g^1} I^1 \xrightarrow{g^2} \cdots \xrightarrow{g^i} I^i \xrightarrow{g^{i+1}} \cdots$$

be an injective coresolution of N in $\operatorname{Mod} S$. By (3), we have $\mathcal{I}_C(S)$-id $P \leq n$ for any $P \in \mathcal{P}(S)$. Then from Lemma 4.10 we get an exact sequence

$$\cdots \xrightarrow{g_{i+1}} T_i^n \xrightarrow{g_i} \cdots \xrightarrow{g_2} T_1^n \xrightarrow{g_1} T_0^n \xrightarrow{g^0} I^n \xrightarrow{g^{n+1}} I^{n+1} \xrightarrow{g^{n+2}} \cdots \tag{4.7}$$

in $\operatorname{Mod} S$ with all T_i^n in $\mathcal{I}_C(S)$, such that $\operatorname{Im} g^0 \cong \operatorname{Im} g^n$.

Let $T \in \mathscr{T}$. By (3), we have $\operatorname{pd}_S T \leq n$. Since

$$\mathscr{T} \subseteq \mathcal{A}_C(S) \subseteq {}^{\perp}\mathcal{I}_C(S)$$

by Lemma 2.7(1), applying the functor $\operatorname{Hom}_S(T, -)$ to the exact sequence (4.7), it is easy to see that each image in this exact sequence is in $T^{\perp}$. It follows that the exact sequence (4.7) is $\operatorname{Hom}_R(\mathscr{T}, -)$-exact. So $\operatorname{Im} g^0$, and hence $\operatorname{Im} g^n$, is in $\mathcal{GI}_C(\mathscr{T})$. This yields $\mathcal{GI}_C(\mathscr{T})$-id $N \leq n$.

(1) $\Longrightarrow$ (3) Let $T \in \mathscr{T}$. Since $\mathcal{GI}_C(\mathscr{T}) \subseteq \mathscr{T}^{\perp}$, it follows from (1) and dimension shifting that $\operatorname{Ext}_S^{\geq n+1}(T, N) = 0$ for any $N \in \operatorname{Mod} S$, and so $\operatorname{pd}_S T \leq n$.

It is trivial that $\mathcal{I}_C(S) \subseteq \mathscr{T}$. Since $\mathcal{I}_C(S) \subseteq \mathscr{T}^{\perp}$ by Lemma 2.7(1), we have that $\mathcal{GI}_C(\mathscr{T})$ is $\mathscr{T}$-preresolving in $\operatorname{Mod} S$ admitting a $\mathscr{T}$-proper generator $\mathcal{I}_C(S)$ by Lemma 2.3(2). Since the class $\mathcal{I}_C(S)$-id$^{\leq m}$ is closed under direct summands for any $m \geq 0$ by Lemma 4.8(3), it follows from (1) and Lemma 2.2(2) that $\mathcal{I}_C(S)$-id $P = \mathcal{GI}_C(\mathscr{T})$-id $P \leq n$ for any $P \in \mathcal{P}(S)$. $\square$

We give a sufficient condition for a module in $\mathscr{T}^{\perp}$ to be also in $\mathcal{GI}_C(\mathscr{T})$ as follows. It is dual to Proposition 4.7.

Proposition 4.12. *If $N \in \mathscr{T}^{\perp}$ with $\mathcal{GI}_C(\mathscr{T})$-id $N < \infty$, then $N \in \mathcal{GI}_C(\mathscr{T})$.*

5 Applications

5.1 Usual C-Gorenstein modules

Following the usual customary notation, we write

$$\mathrm{G}_C\text{-pd}_R\, M := \mathcal{GP}_C(R)\text{-}\operatorname{pd} M,\ \ \mathrm{G}_C\text{-fd}_R\, M := \mathcal{GF}_C(R)\text{-}\operatorname{pd} M,\ \ \mathrm{G}_C\text{-id}_{R^{op}}\, M := \mathcal{GI}_C(R^{op})\text{-}\operatorname{id} M.$$

$$\mathrm{G}_C\text{-pd}_{S^{op}}\, N := \mathcal{GP}_C(S^{op})\text{-}\operatorname{pd} N,\ \ \mathrm{G}_C\text{-fd}_{S^{op}}\, N := \mathcal{GF}_C(S^{op})\text{-}\operatorname{pd} N,\ \ \mathrm{G}_C\text{-id}_S\, N := \mathcal{GI}_C(S)\text{-}\operatorname{id} N.$$

Following the notations below Definition 4.1, we have

$$\mathcal{GF}_C(R) = \mathcal{GF}_C(\mathcal{F}_C(R)) \text{ and } \mathcal{GF}_C(S^{op}) = \mathcal{GF}_C(\mathcal{F}_C(S^{op})).$$

Lemma 5.1. *It holds that*
(1) *If S is a right coherent ring, then $\mathcal{GF}_C(R)$ is closed under extensions.*
(2) *If R is a left coherent ring, then $\mathcal{GF}_C(S^{op})$ is closed under extensions.*

Proof. (1) Let S be a right coherent ring, and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with $M_1, M_3 \in \mathcal{GF}_C(R)$. Then

$$0 \to M_3^+ \to M_2^+ \to M_1^+ \to 0$$

is an exact sequence in $\operatorname{Mod} R^{op}$. By [27, Theorem 4.17(2)], we have $M_1^+, M_3^+ \in \mathcal{GI}_C(R^{op})$. Then $M_2^+ \in \mathcal{GI}_C(R^{op})$ by [27, Remark 4.4(3)(b)], which implies $M_2 \in \mathcal{GF}_C(R)$ by [27, Theorem 4.17(2)] again.

(2) It is the symmetric version of (1). $\square$

Under certain conditions, we establish the left and right symmetry of the C-Gorenstein flat dimension of any module being at most n.

Theorem 5.2. *For any $n \geq 0$, consider the following conditions.*
(1) G_C-$\mathrm{fd}_R\, M \leq n$ *for any* $M \in \operatorname{Mod} R$.
(2) G_C-$\mathrm{fd}_{S^{op}}\, N \leq n$ *for any* $N \in \operatorname{Mod} S^{op}$.
(3) $\mathcal{F}_C(R)$-$\mathrm{pd}\, I \leq n$ *for any* $I \in \mathcal{I}(R)$, *and* $\mathrm{fd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$.
(4) $\mathrm{fd}_S\, E \leq n$ *for any* $E \in \mathcal{I}_C(S)$, *and* $\mathcal{F}_C(S^{op})$-$\mathrm{pd}\, I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$.

We have (1) $\Longleftarrow$ (3) $\Longleftrightarrow$ (4) $\Longrightarrow$ (2). *Furthermore, it holds that*
(*a*) *If $\mathcal{GF}_C(R)$ is closed under $\mathcal{I}_C(R^{op})^+$-coproper extensions, then* (1) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).
(*b*) *If $\mathcal{GF}_C(S^{op})$ is closed under $\mathcal{I}_C(S)^+$-coproper extensions, then* (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).
(*c*) *If R is a left coherent ring and S is a right coherent ring, then the conditions* (1)–(4) *are equivalent.*

Proof. By Lemma 2.9(1), we have $\mathcal{F}_C(R) \subseteq {}^{\perp}[\mathcal{I}_C(R^{op})^+] = \mathcal{I}_C(R^{op})^{\top}$. It is trivial that $\mathcal{F}(S)$ is resolving, and note that $\mathcal{F}_C(R)$ is closed under finite direct sums and direct summands by [19, Proposition 5.1(a)]. Then the assertions (1) $\Longleftarrow$ (3) $\Longleftrightarrow$ (4) and (a) follow from Proposition 4.5 by setting $\mathscr{T} = \mathcal{F}(S)$ and $\mathscr{H} = \mathcal{F}_C(R)$. Symmetrically, we get the assertions (3) $\Longleftrightarrow$ (4) $\Longrightarrow$ (2) and (b). The assertion (c) follows from the assertions (a), (b) and Lemma 5.1. $\square$

When ${}_RC_S = {}_RR_R$, we write

$$\text{G-fd}_R\, M := \mathrm{G}_C\text{-fd}_R\, M \quad \text{and} \quad \text{G-fd}_{R^{op}}\, N := \mathrm{G}_C\text{-fd}_{S^{op}}\, N.$$

Corollary 5.3. ([7, Theorem 2.4]) *For any $n \geq 0$, the following statements are equivalent.*

(1) G-$\mathrm{fd}_R M \leq n$ *for any* $M \in \mathrm{Mod}\, R$.

(2) G-$\mathrm{fd}_{R^{op}} N \leq n$ *for any* $N \in \mathrm{Mod}\, R^{op}$.

(3) $\mathrm{fd}_R E \leq n$ *for any* $E \in \mathcal{I}(R)$, *and* $\mathrm{fd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}(R^{op})$.

Proof. Since the class of Gorenstein flat left (resp. right) R-modules is closed under extensions by [33, Theorem 4.11], the assertion follows from Theorem 5.2 by putting ${}_RC_S = {}_RR_R$. □

It is clear that

$$\mathcal{GP}_C(R) = \mathcal{GP}_C(\mathcal{P}_C(R)) \text{ and } \mathcal{GP}_C(S^{op}) = \mathcal{GP}_C(\mathcal{P}_C(S^{op})),$$

$$\mathcal{GI}_C(S) = \mathcal{GI}_C(\mathcal{I}_C(S)) \text{ and } \mathcal{GI}_C(R^{op}) = \mathcal{GI}_C(\mathcal{I}_C(R^{op})).$$

In the following result, we show that the C-Gorenstein projective dimension of any left R-module is at most n if and only if the C-Gorenstein injective dimension of any left S-module is at most n.

Theorem 5.4. *For any $n \geq 0$, it holds that*

(1) *The following statements are equivalent.*

(1.1) G_C-$\mathrm{pd}_R M \leq n$ *for any* $M \in \mathrm{Mod}\, R$.

(1.2) G_C-$\mathrm{id}_S N \leq n$ *for any* $N \in \mathrm{Mod}\, S$.

(1.3) $\mathcal{P}_C(R)$-$\mathrm{pd}\, I \leq n$ *for any* $I \in \mathcal{I}(R)$, *and* $\mathrm{id}_R H \leq n$ *for any* $H \in \mathcal{P}_C(R)$.

(1.4) $\mathrm{pd}_S E \leq n$ *for any* $E \in \mathcal{I}_C(S)$, *and* $\mathcal{I}_C(S)$-$\mathrm{id}\, T \leq n$ *for any* $T \in \mathcal{P}(S)$.

(2) *The following statements are equivalent.*

(2.1) G_C-$\mathrm{pd}_{S^{op}} N' \leq n$ *for any* $N' \in \mathrm{Mod}\, S^{op}$.

(2.2) G_C-$\mathrm{id}_{R^{op}} M' \leq n$ *for any* $M' \in \mathrm{Mod}\, R^{op}$.

(2.3) $\mathcal{P}_C(S^{op})$-$\mathrm{pd}\, I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$, *and* $\mathrm{id}_{S^{op}} H' \leq n$ *for any* $H' \in \mathcal{P}_C(S^{op})$.

(2.4) $\mathrm{pd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$, *and* $\mathcal{I}_C(R^{op})$-$\mathrm{id}\, T' \leq n$ *for any* $T' \in \mathcal{P}(R^{op})$.

Proof. (1) It is trivial that $\mathcal{P}(S)$ is resolving, and note that $\mathcal{P}_C(R)$ is closed under finite direct sums and direct summands by [19, Proposition 5.1(b)]. Then the assertion (1.1) $\Longleftrightarrow$ (1.3) $\Longleftrightarrow$ (1.4) follows from Theorem 4.6 by setting $\mathscr{T} = \mathcal{P}(S)$ and $\mathscr{H} = \mathcal{P}_C(R)$. On the other hand, it is trivial that $\mathcal{I}(R)$ is coresolving, and note that $\mathcal{I}_C(S)$ is closed under finite direct sums and direct summands by [19, Proposition 5.1(c)]. Then the assertion (1.2) $\Longleftrightarrow$ (1.3) $\Longleftrightarrow$ (1.4) follows from Theorem 4.11 by setting $\mathscr{H} = \mathcal{I}(R)$ and $\mathscr{T} = \mathcal{I}_C(S)$.

(2) It is the symmetric version of (1). □

We introduce the C-versions of strongly Gorenstein flat modules and projectively coresolved Gorenstein flat modules as follows.

Definition 5.5.

(1) A module $M \in \mathrm{Mod}\, R$ is called *C-strongly Gorenstein flat* if

$$M \in {}^{\perp}\mathcal{F}_C(R) \cap \widetilde{\mathrm{cores}_{\mathcal{F}_C(R)}}\, \mathcal{P}_C(R).$$

Symmetrically, the notion of *C-strongly Gorenstein flat modules* in $\mathrm{Mod}\, S^{op}$ is defined.

(2) A module $M \in \mathrm{Mod}\, R$ is called *C-projectively coresolved Gorenstein flat* if $M \in \mathcal{I}_C(R^{op})^{\top}$ and there exists an $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact exact sequence

$$0 \to M \to Q^0 \to Q^1 \to \cdots \to Q^i \to \cdots$$

in $\mathrm{Mod}\, R$ with all Q^i in $\mathcal{P}_C(R)$. Symmetrically, the notion of *C-projectively coresolved Gorenstein flat modules* in $\mathrm{Mod}\, S^{op}$ is defined.

We use $\mathcal{SGF}_C(R)$ (resp. $\mathcal{PGF}_C(R)$) to denote the subcategory of $\operatorname{Mod} R$ consisting of C-strongly Gorenstein flat modules (resp. C-projectively coresolved Gorenstein flat modules). Symmetrically, we use $\mathcal{SGF}_C(S^{op})$ (resp. $\mathcal{PGF}_C(S^{op})$) to denote the subcategory of $\operatorname{Mod} S^{op}$ consisting of C-strongly Gorenstein flat modules (resp. C-projectively coresolved Gorenstein flat modules). When ${}_RC_S = {}_RR_R$, C-strongly Gorenstein flat modules and C-projectively coresolved Gorenstein flat modules are exactly strongly Gorenstein flat modules [8] and projectively coresolved Gorenstein flat modules [33] respectively. Following the notations below Definition 4.1, we have

$$\mathcal{SGF}_C(R) = \mathcal{GP}_C(\mathcal{F}_C(R)) \text{ and } \mathcal{SGF}_C(S^{op}) = \mathcal{GP}_C(\mathcal{F}_C(S^{op})),$$

$$\mathcal{PGF}_C(R) = \mathcal{GF}_C(\mathcal{P}_C(R)) \text{ and } \mathcal{PGF}_C(S^{op}) = \mathcal{GF}_C(\mathcal{P}_C(S^{op})).$$

Proposition 5.6.

(1) *For any $n \geq 0$, consider the following conditions.*

(1.1) $\mathcal{PGF}_C(R)$-$\operatorname{pd} M \leq n$ *for any* $M \in \operatorname{Mod} R$.

(1.2) $\mathcal{P}_C(R)$-$\operatorname{pd} I \leq n$ *for any* $I \in \mathcal{I}(R)$, *and* $\operatorname{fd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$.

(1.3) $\operatorname{pd}_S E \leq n$ *for any* $E \in \mathcal{I}_C(S)$, *and* $\mathcal{F}_C(S^{op})$-$\operatorname{pd} I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$.

We have (1.1) $\Longleftarrow$ (1.2) $\Longleftrightarrow$ (1.3). Furthermore, if $\mathcal{PGF}_C(R)$ is closed under $\mathcal{I}_C(R^{op})^+$-coproper extensions, then all these three conditions are equivalent.

(2) *For any $n \geq 0$, consider the following conditions.*

(1.1) $\mathcal{PGF}_C(S^{op})$-$\operatorname{pd} N \leq n$ *for any* $N \in \operatorname{Mod} S^{op}$.

(1.2) $\mathcal{P}_C(S^{op})$-$\operatorname{pd} I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$, *and* $\operatorname{fd}_S E \leq n$ *for any* $E \in \mathcal{I}_C(S)$.

(1.3) $\operatorname{pd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$, *and* $\mathcal{F}_C(R)$-$\operatorname{pd} I \leq n$ *for any* $I \in \mathcal{I}(R)$.

We have (2.1) $\Longleftarrow$ (2.2) $\Longleftrightarrow$ (2.3). Furthermore, if $\mathcal{PGF}_C(S^{op})$ is closed under $\mathcal{I}_C(S)^+$-coproper extensions, then all these three conditions are equivalent.

Proof. (1) By Lemma 2.9, we have

$$\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R) \subseteq {}^{\perp}[\mathcal{I}_C(R^{op})^+] = \mathcal{I}_C(R^{op})^{\top}.$$

Then the assertion follows from Proposition 4.5 by setting $\mathscr{T} = \mathcal{P}(S)$ and $\mathscr{H} = \mathcal{P}_C(R)$.

(2) It is the symmetric version of (1). □

Proposition 5.7. *For any $n \geq 0$, it holds that*

(1) *The following statements are equivalent.*

(1.1) $\mathcal{SGF}_C(R)$-$\operatorname{pd} M \leq n$ *for any* $M \in \operatorname{Mod} R$.

(1.2) $\mathcal{P}_C(R)$-$\operatorname{pd} I \leq n$ *for any* $I \in \mathcal{I}(R)$, *and* $\operatorname{id}_R H \leq n$ *for any* $H \in \mathcal{F}_C(R)$.

(1.3) $\operatorname{pd}_S E \leq n$ *for any* $E \in \mathcal{I}_C(S)$, *and* $\mathcal{I}_C(S)$-$\operatorname{id} T \leq n$ *for any* $T \in \mathcal{F}(S)$.

(2) *The following statements are equivalent.*

(2.1) $\mathcal{SGF}_C(S^{op})$-$\operatorname{pd} N \leq n$ *for any* $N \in \operatorname{Mod} S^{op}$.

(2.2) $\mathcal{P}_C(S^{op})$-$\operatorname{pd} I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$, *and* $\operatorname{id}_{S^{op}} H' \leq n$ *for any* $H' \in \mathcal{F}_C(S^{op})$.

(2.3) $\operatorname{pd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$, *and* $\mathcal{I}_C(R^{op})$-$\operatorname{id} T' \leq n$ *for any* $T' \in \mathcal{F}(R^{op})$.

Proof. It follows from Theorem 4.6 by setting $\mathscr{T} = \mathcal{F}(S)$ and $\mathscr{H} = \mathcal{F}_C(R)$.

(2) It is the symmetric version of (1). □

5.2 Other C-Gorenstein modules

In the following result, we show that the $\mathcal{GP}_C(\mathcal{B}_C(R))$-projective dimension of any left R-module is at most n if and only if the $\mathcal{GI}_C(\mathcal{A}_C(S))$-injective dimension of any left S-module is at most n.

Theorem 5.8. *For any $n \geq 0$, it holds that*

(1) *The following statements are equivalent.*

(1.1) $\mathcal{GP}_C(\mathcal{B}_C(R))$-pd $M \leq n$ *for any* $M \in \operatorname{Mod} R$.

(1.2) $\mathcal{GI}_C(\mathcal{A}_C(S))$-id $N \leq n$ *for any* $N \in \operatorname{Mod} S$.

(1.3) $\mathcal{P}_C(R)$-pd $I \leq n$ *for any* $I \in \mathcal{I}(R)$, *and* $\operatorname{id}_R B \leq n$ *for any* $B \in \mathcal{B}_C(R)$.

(1.4) $\operatorname{pd}_S E \leq n$ *for any* $E \in \mathcal{I}_C(S)$, *and* $\mathcal{I}_C(S)$-id $A \leq n$ *for any* $A \in \mathcal{A}_C(S)$.

(1.5) $\mathcal{P}_C(R)$-pd $B \leq n$ *for any* $B \in \mathcal{B}_C(R)$, *and* $\operatorname{id}_R Q \leq n$ *for any* $Q \in \mathcal{P}_C(R)$.

(1.6) $\operatorname{pd}_S A \leq n$ *for any* $A \in \mathcal{A}_C(S)$, *and* $\mathcal{I}_C(S)$-id $P \leq n$ *for any* $P \in \mathcal{P}(S)$.

(2) *The following statements are equivalent.*

(2.1) $\mathcal{GP}_C(\mathcal{B}_C(S^{op}))$-pd $M' \leq n$ *for any* $M' \in \operatorname{Mod} S^{op}$.

(2.2) $\mathcal{GI}_C(\mathcal{A}_C(R^{op}))$-id $N' \leq n$ *for any* $N' \in \operatorname{Mod} R^{op}$.

(2.3) $\mathcal{P}_C(S^{op})$-pd $I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$, *and* $\operatorname{id}_{S^{op}} B' \leq n$ *for any* $B' \in \mathcal{B}_C(S^{op})$.

(2.4) $\operatorname{pd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$, *and* $\mathcal{I}_C(R^{op})$-id $A' \leq n$ *for any* $A' \in \mathcal{A}_C(R^{op})$.

(2.5) $\mathcal{P}_C(S^{op})$-pd $B' \leq n$ *for any* $B' \in \mathcal{B}_C(S^{op})$, *and* $\operatorname{id}_{S^{op}} Q' \leq n$ *for any* $Q' \in \mathcal{P}_C(S^{op})$.

(2.6) $\operatorname{pd}_{R^{op}} A' \leq n$ *for any* $A' \in \mathcal{A}_C(R^{op})$, *and* $\mathcal{I}_C(R^{op})$-id $P' \leq n$ *for any* $P' \in \mathcal{P}(R^{op})$.

Proof. By [19, Theorem 6.2], we have that $\mathcal{A}_C(S)$ is resolving. By [26, Theorem 3.3(2)] and [19, Proposition 4.2(a)], we have that $\mathcal{B}_C(R)$ is covering in $\operatorname{Mod} R$ and closed under finite direct sums and direct summands. Now the assertion (1.1) $\Longleftrightarrow$ (1.3) $\Longleftrightarrow$ (1.4) follows from Theorem 4.6 by setting $\mathscr{T} = \mathcal{A}_C(S)$ and $\mathscr{H} = \mathcal{B}_C(R)$.

By [19, Theorem 6.2], we have that $\mathcal{B}_C(R)$ is coresolving. By [26, Theorem 3.5(1)] and [19, Proposition 4.2(a)], we have that $\mathcal{A}_C(S)$ is preenveloping in $\operatorname{Mod} S$ and closed under finite direct sums and direct summands. Now the assertion (1.2) $\Longleftrightarrow$ (1.5) $\Longleftrightarrow$ (1.6) follows from Theorem 4.11 by setting $\mathscr{H} = \mathcal{B}_C(R)$ and $\mathscr{T} = \mathcal{A}_C(S)$.

(1.3) $\Longrightarrow$ (1.5) Since $\mathcal{P}_C(R) \subseteq \mathcal{B}_C(R)$ by Lemma 2.7(2), we have $\operatorname{id}_R Q \leq n$ for any $Q \in \mathcal{P}_C(R)$ by (1.3). Now let $B \in \mathcal{B}_C(R)$. Then $\operatorname{id}_R B \leq n$ by (1.3), and thus there exists an exact sequence

$$0 \to B \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

in $\operatorname{Mod} R$ with all I^i in $\mathcal{I}(R)$. Since $B \in \mathcal{P}_C(R)^{\perp}$ by Lemma 2.7(2), applying the functor $(-)_*$ to the above exact sequence yields the following exact sequence

$$0 \to B_* \to {I^0}_* \to {I^1}_* \to \cdots \to {I^n}_* \to 0 \tag{5.1}$$

in $\operatorname{Mod} S$. By (1.3), we have $\mathcal{P}_C(R)$-pd $I^i \leq n$ for any $0 \leq i \leq n$. Since all I_i are in $\mathcal{B}_C(R)$ by Lemma 2.7(2), it follows from [29, Proposition 4.1] that $\operatorname{pd}_S {I^i}_* \leq n$ for any $0 \leq i \leq n$. By the exact sequence (5.1), we have $\operatorname{pd}_S B_* \leq n$. Thus $\mathcal{P}_C(R)$-pd $B \leq n$ by [29, Proposition 4.1] again.

(1.5) $\Longrightarrow$ (1.3) Since $\mathcal{I}(R) \subseteq \mathcal{B}_C(R)$ by Lemma 2.7(2), we have $\mathcal{P}_C(R)$-pd $I \leq n$ for any $I \in \mathcal{I}(R)$ by (1.5). Now let $B \in \mathcal{B}_C(R)$. Then $\mathcal{P}_C(R)$-pd $B \leq n$ by (1.5), and hence there exists an exact sequence

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to B \to 0$$

in $\operatorname{Mod} R$ with all Q_i in $\mathcal{P}_C(R)$. By (1.5), we have $\operatorname{id}_R Q_i \leq n$ for any $0 \leq i \leq n$, and hence $\operatorname{id}_R B \leq n$.

(2) It is the symmetric version of (1). $\square$

When ${}_RC_S = {}_RR_R$, it is easy to see that $\mathcal{B}_C(R) = \operatorname{Mod} R = \mathcal{A}_C(S)$, and hence

$$\mathcal{GP}_C(\mathcal{B}_C(R)) = \mathcal{P}(R) \text{ and } \mathcal{GI}_C(\mathcal{A}_C(S)) = \mathcal{I}(R).$$

It yields

$$\mathcal{GP}_C(\mathcal{B}_C(R))\text{-}\operatorname{pd} M = \operatorname{pd}_R M \text{ and } \mathcal{GI}_C(\mathcal{A}_C(S))\text{-}\operatorname{id} M = \operatorname{id}_R M$$

for any $M \in \operatorname{Mod} R$. Thus, putting ${}_RC_S = {}_RR_R$ in Theorem 5.8, from the equivalence (1.1) $\Longleftrightarrow$ (1.2) we get the following well-known classical result (cf. [32, Theorem 8.14]).

Corollary 5.9. *For any ring R, we have*

$$\sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R\} = \sup\{\operatorname{id}_R M \mid M \in \operatorname{Mod} R\}.$$

The common value of the quantities is known as the left global dimension of R.

Proposition 5.10. *For any $n \geq 0$, it holds that*

(1) *The following statements are equivalent.*
  (1.1) $\mathcal{GP}_C(\mathcal{WF}_C(R))$-pd $M \leq n$ *for any* $M \in \operatorname{Mod} R$.
  (1.2) $\mathcal{P}_C(R)$-pd $I \leq n$ *for any* $I \in \mathcal{I}(R)$, *and* $\operatorname{id}_R H \leq n$ *for any* $H \in \mathcal{WF}_C(R)$.
  (1.3) $\operatorname{pd}_S E \leq n$ *for any* $E \in \mathcal{I}_C(S)$, *and* $\mathcal{I}_C(S)$-id $T \leq n$ *for any* $T \in \mathcal{WF}(S)$.

(2) *The following statements are equivalent.*
  (2.1) $\mathcal{GP}_C(\mathcal{WF}_C(S^{op}))$-pd $N \leq n$ *for any* $N \in \operatorname{Mod} S^{op}$.
  (2.2) $\mathcal{P}_C(S^{op})$-pd $I' \leq n$ *for any* $I' \in \mathcal{I}(S^{op})$, *and* $\operatorname{id}_{S^{op}} H' \leq n$ *for any* $H' \in \mathcal{WF}_C(S^{op})$.
  (2.3) $\operatorname{pd}_{R^{op}} E' \leq n$ *for any* $E' \in \mathcal{I}_C(R^{op})$, *and* $\mathcal{I}_C(R^{op})$-id $T' \leq n$ *for any* $T' \in \mathcal{WF}(R^{op})$.

Proof. It follows from [13, Proposition 2.6(2)] that $\mathcal{WF}(S)$ is resolving, and note that $\mathcal{WF}_C(R)$ is closed under finite direct sums and direct summands by [14, Proposition 2.8]. Then the assertion follows from Theorem 4.6 by setting $\mathscr{T} = \mathcal{WF}(S)$ and $\mathscr{H} = \mathcal{WF}_C(R)$.

(2) It is the symmetric version of (1). □

Proposition 5.11. *For any $n \geq 0$, it holds that*

(1) *The following statements are equivalent.*
  (1.1) $\mathcal{GI}_C(\mathcal{WI}_C(S))$-id $N \leq n$ *for any* $N \in \operatorname{Mod} S$.
  (1.2) $\operatorname{id}_R Q \leq n$ *for any* $Q \in \mathcal{P}_C(R)$, *and* $\mathcal{P}_C(R)$-pd $H \leq n$ *for any* $H \in \mathcal{WI}(R)$.
  (1.3) $\mathcal{I}_C(S)$-id $P \leq n$ *for any* $P \in \mathcal{P}(S)$, *and* $\operatorname{pd}_S T \leq n$ *for any* $T \in \mathcal{WI}_C(S)$.

(2) *The following statements are equivalent.*
  (2.1) $\mathcal{GI}_C(\mathcal{WI}_C(R^{op}))$-id $M' \leq n$ *for any* $M' \in \operatorname{Mod} R^{op}$.
  (2.2) $\operatorname{id}_{S^{op}} Q' \leq n$ *for any* $Q' \in \mathcal{P}_C(S^{op})$, *and* $\mathcal{P}_C(S^{op})$-pd $H' \leq n$ *for any* $H' \in \mathcal{WI}(S^{op})$.
  (2.3) $\mathcal{I}_C(R^{op})$-id $P' \leq n$ *for any* $P' \in \mathcal{P}(R^{op})$, *and* $\operatorname{pd}_{R^{op}} T' \leq n$ *for any* $T' \in \mathcal{WI}_C(R^{op})$.

Proof. (1) It follows from [13, Proposition 2.6(1)] that $\mathcal{WI}(R)$ is coresolving, and note that $\mathcal{WI}_C(S)$ is closed under finite direct sums and direct summands by [14, Proposition 2.8]. Then the assertion follows from Theorem 4.11 by setting $\mathscr{H} = \mathcal{WI}(R)$ and $\mathscr{T} = \mathcal{WI}_C(S)$.

(2) It is the symmetric version of (1). □

5.3 C-Gorenstein global dimension

In the following result, the assertion (1) follows from Theorem 5.2, and the assertion (2) follows from Corollary 5.3.

Corollary 5.12. *It holds that*

(1) *If R is a left coherent ring and S is a right coherent ring, then*

$$\sup\{\mathrm{G}_C\text{-}\mathrm{fd}_R M \mid M \in \operatorname{Mod} R\} = \sup\{\mathrm{G}_C\text{-}\mathrm{fd}_{S^{op}} N \mid N \in \operatorname{Mod} S^{op}\}.$$

(2) ([7, Corollary 2.5]) *We have*

$$\sup\{\text{G-fd}_R M \mid M \in \operatorname{Mod} R\} = \sup\{\text{G-fd}_{R^{op}} N \mid N \in \operatorname{Mod} R^{op}\}.$$

As an immediate consequence of Theorem 5.4, we get the following corollary, which is the C-version of [5, Theorem 1.1].

Corollary 5.13. *It holds that*

(1) $\sup\{\mathrm{G}_C\text{-pd}_R M \mid M \in \operatorname{Mod} R\} = \sup\{\mathrm{G}_C\text{-id}_S N \mid N \in \operatorname{Mod} S\}$.

(2) $\sup\{\mathrm{G}_C\text{-pd}_R M \mid M \in \operatorname{Mod} S^{op}\} = \sup\{\mathrm{G}_C\text{-id}_S N \mid N \in \operatorname{Mod} R^{op}\}$.

We call the common value of the quantities in Corollary 5.13 (1) and (2) the *left C-Gorenstein global dimension* and *right C-Gorenstein global dimension* of R and S respectively, and denote them by G_C-gldim and $\mathrm{G}_C\text{-gldim}^{op}$ respectively.

A well-known open question is: whether or when is a Gorenstein projective module Gorenstein flat? It also makes sense for the C-version of this question. As an application of Theorem 5.4, we get the following result.

Theorem 5.14. *If one of the following conditions is satisfied, then $\mathcal{GP}_C(R) \subseteq \mathcal{GF}_C(R)$.*

(1) $\mathrm{fd}_{R^{op}} E' < \infty$ *for any* $E' \in \mathcal{I}_C(R^{op})$.

(2) $\mathrm{G}_C\text{-gldim}^{op} < \infty$.

Proof. (1) Let $G \in \mathcal{GP}_C(R)$ and let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to G^0 \to G^1 \to \cdots \to G^i \to \cdots \tag{5.2}$$

be a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence in $\operatorname{Mod} R$ with all P_i projective and all G^i in $\mathcal{P}_C(R)$, such that $G \cong \operatorname{Im}(P_0 \to G^0)$. Let $E' \in \mathcal{I}_C(R^{op})$. By Lemma 2.9, we have that each G^i is in ${E'}^{\top}$. Since $\mathrm{fd}_{R^{op}} E' < \infty$ by assumption, using dimension shifting it is easy to see that the image of each homomorphism in the exact sequence (5.2) is also in ${E'}^{\top}$. It follows that (5.2) is $(E' \otimes_R -)$-exact, and thus $G \in \mathcal{GF}_C(R)$.

(2) If $\mathrm{G}_C\text{-gldim}^{op} < \infty$, then by Theorem 5.4(2), we have $\mathrm{fd}_{R^{op}} E' \leq \mathrm{pd}_{R^{op}} E' < \infty$ for any $E' \in \mathcal{I}_C(R^{op})$, and thus the assertion follows from (1). □

We need the following easy observation.

Lemma 5.15. *It holds that*

(1) *A module $M \in \mathcal{F}_C(R)$ if and only if $M^+ \in \mathcal{I}_C(R^{op})$.*

(2) *If S is a right Noetherian ring, then a module $N \in \mathcal{I}_C(R^{op})$ if and only if $N^+ \in \mathcal{F}_C(R)$.*

Proof. (1) It follows from [27, Theorem 4.17(1)].

(2) Let $N \in \operatorname{Mod} R^{op}$. If $N \in \mathcal{I}_C(R^{op})$, then $N^+ \in \mathcal{F}_C(R)$ by [38, Lemma 2.3(2)]. Conversely, if $N^+ \in \mathcal{F}_C(R)$, then $N^{++} \in \mathcal{I}_C(R^{op})$ by (1). Since N is a pure submodule of N^{++} by [15, Corollary 2.21(b)], it follows from [19, Lemma 5.2(b)] that $N \in \mathcal{I}_C(R^{op})$. □

In the following result, we establish the relationship among some kinds of C-Gorenstein modules, in which the first assertion is the C-version of [28, Theorem 2].

Lemma 5.16. *It holds that*

(1) *If S is a right Noetherian ring, then $\mathcal{SGF}_C(R) = \mathcal{PGF}_C(R)$.*

(2) *$\mathcal{SGF}_C(R) \subseteq \mathcal{GP}_C(R)$, with equality when $\mathcal{P}_C(R)$-pd $X < \infty$ for any $X \in \mathcal{F}_C(R)$.*

(3) *Assume that one of the following conditions is satisfied:*

(3.1) *S is a right Noetherian ring and $\mathcal{P}_C(R)$-pd $X < \infty$ for any $X \in \mathcal{F}_C(R)$;*
(3.2) *R is a left Noetherian ring and S is a right Noetherian ring with $\mathrm{id}_R\, C < \infty$.*
Then

$$\mathcal{SGF}_C(R) = \mathcal{PGF}_C(R) = \mathcal{GP}_C(R) \subseteq \mathcal{GF}_C(R).$$

Proof. Let $M \in \mathrm{Mod}\, R$, and let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to Q^0 \to Q^1 \to \cdots \to Q^i \to \cdots \tag{5.3}$$

be an exact sequence in $\mathrm{Mod}\, R$ with all P_i projective and all Q^i in $\mathcal{P}_C(R)$, such that $M \cong \mathrm{Im}(P_0 \to Q^0)$. Set $M^i := \mathrm{Im}(Q^i \to Q^{i+1})$ for any $i \geq 0$.

(1) Suppose $M \in \mathcal{SGF}_C(R)$. In this case, the exact sequence (5.3) may be assumed to be $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact. Since $\mathcal{I}_C(R^{op})^+ \subseteq \mathcal{F}_C(R)$ by Lemma 5.15(2), the exact sequence (5.3) is $\mathrm{Hom}_R(-, \mathcal{I}_C(R^{op})^+)$-exact, and hence $(\mathcal{I}_C(R^{op}) \otimes -)$-exact. This yields $M \in \mathcal{PGF}_C(R)$.

Conversely, suppose $M \in \mathcal{PGF}_C(R)$. In this case, the exact sequence (5.3) may be assumed to be $(\mathcal{I}_C(R^{op}) \otimes -)$-exact. By Lemma 5.15 and [6, Theorem A.6], as part of (5.3), the complex

$$\cdots \to P_i \to \cdots \to P_1 \to P_0$$

is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, which implies that the exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

is also $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact. Thus $M \in {}^{\perp}\mathcal{F}_C(R)$.

Consider the $(\mathcal{I}_C(R^{op}) \otimes -)$-exact (equivalently $\mathrm{Hom}_R(-, \mathcal{I}_C(R^{op})^+)$-exact) exact sequence

$$0 \to M \to Q^0 \to M^0 \to 0.$$

Let

$$\cdots \to P_i^0 \to \cdots \to P_1^0 \to P_0^0 \to Q^0 \to 0$$

be a projective resolution of Q^0 in $\mathrm{Mod}\, R$. It is $\mathrm{Hom}_R(-, \mathcal{I}_C(R^{op})^+)$-exact by [34, Lemma 4.13]. Then, according to [24, Theorem 3.6], we get the following $\mathrm{Hom}_R(-, \mathcal{I}_C(R^{op})^+)$-exact (equivalently $(\mathcal{I}_C(R^{op}) \otimes -)$-exact) exact sequence

$$\cdots \to P_i \oplus P_{i+1}^0 \to \cdots \to P_0 \oplus P_1^0 \to P_0^0 \to M^0 \to 0,$$

which is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact by Lemma 5.15 and [6, Theorem A.6] again. This yields $M^0 \in {}^{\perp}\mathcal{F}_C(R)$. Similarly, we get $M^i \in {}^{\perp}\mathcal{F}_C(R)$ for any $i \geq 1$. It follows that the exact sequence (5.3) is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, and thus $M \in \mathcal{SGF}_C(R)$.

(2) It is trivial that $\mathcal{SGF}_C(R) \subseteq \mathcal{GP}_C(R)$. Conversely, let $M \in \mathcal{GP}_C(R)$. In this case, the exact sequence (5.3) may be assumed to be $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact. Then $M \in {}^{\perp}\mathcal{P}_C(R)$. Suppose $\mathcal{P}_C(R)$-pd $X < \infty$ for any $X \in \mathcal{F}_C(R)$. Then $M \in {}^{\perp}\mathcal{F}_C(R)$ by dimension shifting. Note that all M^i are in $\mathcal{GP}_C(R)$ by [30, Corollary 2.10]. Then, similarly, we get $M^i \in {}^{\perp}\mathcal{F}_C(R)$ for any $i \geq 0$. It follows that the exact sequence (5.3) is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, and thus $M \in \mathcal{SGF}_C(R)$.

(3) It is trivial that $\mathcal{PGF}_C(R) \subseteq \mathcal{GF}_C(R)$. So, the case for (3.1) follows immediately from (1) and (2). On the other hand, when R is a left Noetherian ring with $\mathrm{id}_R\, C < \infty$, it follows from [38, Corollary 3.2] and [3, Theorem 1.1] that $\mathcal{P}_C(R)$-pd $X < \infty$ for any $X \in \mathcal{F}_C(R)$, and thus the case for (3.2) follows from the former assertion. $\square$

We write

$$\mathrm{spclfc}\, R := \sup\{\mathcal{P}_C(R)\text{-}\,\mathrm{pd}\, M \mid M \in \mathcal{F}_C(R)\}.$$

Lemma 5.17. *If S is a right Noetherian ring, then*

$$\mathrm{G}_C\text{-}\mathrm{pd}_R\, M \leq \mathrm{spclfc}\, R$$

for any $M \in \mathcal{GF}_C(R)$.

Proof. Let $M \in \mathcal{GF}_C(R)$. Then there exists an $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to F^0 \to F^1 \to \cdots \to F^j \to \cdots \tag{5.4}$$

in $\mathrm{Mod}\, R$ with all P_i projective and all F^j in $\mathcal{F}_C(R)$, such that $M \cong \mathrm{Im}(P_0 \to F^0)$.

Suppose $\mathrm{spclfc}\, R = t < \infty$. Then for any $j \geq 0$, we have $\mathcal{P}_C(R)$-pd $F^j \leq t$, and hence $F^j \in \mathrm{res}_{\widetilde{\mathcal{P}_C(R)}}\, \mathcal{P}_C(R)$ by Lemma 4.3. On the other hand, by Lemma 5.16(3.1), we have

$$P_i \in \mathcal{GP}_C(R) = \mathcal{SGF}_C(R) \subseteq \mathrm{cores}_{\widetilde{\mathcal{F}_C(R)}}\, \mathcal{P}_C(R)$$

for any $i \geq 0$. Then by Theorem 3.3(2), we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & K_t \arrow[r] \arrow[d] & K_t^0 \arrow[r] \arrow[d] & K_t^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & K_t^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & P_{t-1} \arrow[r] \arrow[d] & G_{t-1}^0 \arrow[r] \arrow[d] & G_{t-1}^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{t-1}^j \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & G_1^0 \arrow[r] \arrow[d] & G_1^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G_1^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & G_0^0 \arrow[r] \arrow[d] & G_0^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G_0^j \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & M \arrow[r] \arrow[d] & F^0 \arrow[r] \arrow[d] & F^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & F^j \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

in $\mathrm{Mod}\, R$ with all G_i^j in $\mathcal{P}_C(R)$, such that the middle t rows are $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact and all columns but the leftmost one are $\mathrm{Hom}_R(\mathcal{P}_C(R), -)$-exact. By Lemma 5.15(2), the middle t rows are $\mathrm{Hom}_R(-, \mathcal{I}_C(R^{op})^+)$-exact, equivalently $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact. The exact sequence (5.4) implies that the leftmost column and the bottom row in this diagram are $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact. Notice that all G_i^j and F^j are in $\mathcal{I}_C(R^{op})^\top$ by Lemma 2.9, so all columns starting with the second column in the above diagram are $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact by dimension shifting. Thus we conclude that all columns in this diagram are $(\mathcal{I}_C(R^{op})\otimes_R -)$-exact, and hence the top row is also $(\mathcal{I}_C(R^{op})\otimes_R -)$-exact.

Since $M\in\mathcal{GF}_C(R)\subseteq\mathcal{I}_C(R^{op})^{\top}$, we have $K_t\in\mathcal{I}_C(R^{op})^{\top}$. Since $\mathcal{P}_C(R)\text{-pd}\,F^j\leq t$ for any $j\geq 0$, it follows from Lemma 4.2(2) (with $\mathscr{H}=\mathcal{P}_C(R)$) and [25, Theorem 3.8(1)] that all K_t^j are in $\mathcal{P}_C(R)$, and thus $K_t\in\mathcal{PGF}_C(R)$. Thus by Lemma 5.16(3.1), we have $K_t\in\mathcal{GP}_C(R)$ and $\mathrm{G}_C\text{-pd}_R\,M\leq t$. □

In the following result, assertions (1) and (2) are the C-versions of [5, Corollary 1.2(1)] and part of [7, Theorem 3.3] respectively.

Theorem 5.18. *It holds that*

(1) $\sup\{\mathrm{G}_C\text{-fd}_R\,M\mid M\in\operatorname{Mod}R\}\leq\max\{\mathrm{G}_C\text{-gldim},\ \mathrm{G}_C\text{-gldim}^{op}\}$.

(2) *If S is a right Noetherian ring, then*

$$\mathrm{G}_C\text{-gldim}\leq\sup\{\mathrm{G}_C\text{-fd}_R\,M\mid M\in\operatorname{Mod}R\}+\operatorname{spclfc}R.$$

Proof. (1) Suppose $\max\{\mathrm{G}_C\text{-gldim},\ \mathrm{G}_C\text{-gldim}^{op}\}=n<\infty$. Let $M\in\operatorname{Mod}R$. Then $\mathrm{G}_C\text{-pd}_R\,M\leq n$ and there exists an exact sequence

$$0\to G_n\to\cdots\to G_1\to G_0\to M\to 0$$

in $\operatorname{Mod}R$ with all G_i in $\mathcal{GP}_C(R)$. By Theorem 5.14, we have that all G_i are in $\mathcal{GF}_C(R)$, and thus $\mathrm{G}_C\text{-fd}_R\,M\leq n$. The assertion follows.

(2) Suppose $\sup\{\mathrm{G}_C\text{-fd}_R\,M\mid M\in\operatorname{Mod}R\}=s<\infty$ and $\operatorname{spclfc}R=t<\infty$. Let $M\in\operatorname{Mod}R$ and let

$$0\to G_s\to\cdots\to G_1\to G_0\to M\to 0$$

be an exact sequence in $\operatorname{Mod}R$ with all G_i in $\mathcal{GF}_C(R)$. By Lemma 5.17, we have $\mathrm{G}_C\text{-pd}_R\,G_i\leq t$ for any $0\leq i\leq s$. By [27, Theorem 3.2 and Remark 4.4(3)(a)], it is easy to get $\mathrm{G}_C\text{-pd}_R\,M\leq s+t$, and thus $\mathrm{G}_C\text{-gldim}\leq s+t$. □

5.4 Finite injective dimension

Lemma 5.19. *It holds that*

(1) *Let R be a left Noetherian ring. Then we have*

  (1.1) $\mathrm{id}_R\,C=\sup\{\mathrm{fd}_{R^{op}}\,E'\mid E'\in\mathcal{I}_C(R^{op})\}=\sup\{\mathcal{F}_C(S^{op})\text{-pd}\,I'\mid I'\in\mathcal{I}(S^{op})\}$.

  (1.2) *If $\mathrm{id}_R\,C=n<\infty$ and $M\in\operatorname{Mod}R$ with $\mathcal{F}_C(R)\text{-pd}\,M<\infty$, then $\mathcal{P}_C(R)\text{-pd}\,M\leq n$.*

(2) *Let S be a right Noetherian ring. Then we have*

  (2.1) $\mathrm{id}_{S^{op}}\,C=\sup\{\mathrm{fd}_S\,E\mid E\in\mathcal{I}_C(S)\}=\sup\{\mathcal{F}_C(R)\text{-pd}\,I\mid I\in\mathcal{I}(R)\}$.

  (2.2) *If $\mathrm{id}_{S^{op}}\,C=n<\infty$ and $N\in\operatorname{Mod}S^{op}$ with $\mathcal{F}_C(S^{op})\text{-pd}\,N<\infty$, then $\mathcal{P}_C(S^{op})\text{-pd}\,N\leq n$.*

Proof. (1) In (1.1), the first equality follows from [22, Lemma 17.2.4(2)], and the second one follows from Lemma 2.6(2).

(1.2) Let $M\in\operatorname{Mod}R$ with $\mathcal{F}_C(R)\text{-pd}\,M=m<\infty$. By Lemma 4.3, there exists an exact sequence

$$\cdots\xrightarrow{f_{i+1}}G_i\xrightarrow{f_i}\cdots\xrightarrow{f_2}G_1\xrightarrow{f_1}G_0\xrightarrow{f_0}M\to 0\tag{5.5}$$

in $\operatorname{Mod}R$ with all G_i in $\mathcal{P}_C(R)$, such that $\operatorname{Im}f_m\in\mathcal{F}_C(R)$. By [36, Proposition 3.4(1)], we have that $\operatorname{Im}f_m$ is isomorphic to a direct summand of a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_RC$. Then $\mathrm{id}_R\operatorname{Im}f_m\leq\mathrm{id}_R\,C=n$ by [3, Theorem 1.1].

We claim that $m \le n$. Otherwise, if $m > n$, then $\operatorname{Ext}_R^m(M, \operatorname{Im} f_m) = 0$. Notice that $\operatorname{Ext}_R^{\ge 1}(G_i, \operatorname{Im} f_m) = 0$ for any $i \ge 0$ by Lemma 2.7(2), so applying the functor $\operatorname{Hom}_R(-, \operatorname{Im} f_m)$ to the exact sequence (5.5) yields

$$\operatorname{Ext}_R^1(\operatorname{Im} f_{m-1}, \operatorname{Im} f_m) \cong \operatorname{Ext}_R^2(\operatorname{Im} f_{m-2}, \operatorname{Im} f_m) \cong \cdots$$

$$\cong \operatorname{Ext}_R^{m-1}(\operatorname{Im} f_1, \operatorname{Im} f_m) \cong \operatorname{Ext}_R^m(M, \operatorname{Im} f_m) = 0.$$

It implies that the exact sequence

$$0 \to \operatorname{Im} f_m \to G_{m-1} \to \operatorname{Im} f_{m-1} \to 0$$

splits and $G_{m-1} \cong \operatorname{Im} f_m \oplus \operatorname{Im} f_{m-1}$. Then $\operatorname{Im} f_{m-1} \in \mathcal{P}_C(R)$ by [19, Proposition 5.1(b)], and thus $\mathcal{F}_C(R)$-pd $M \le m-1$, which is a contradiction. The claim is proved. Then $\operatorname{Im} f_{n+1} \in \mathcal{F}_C(R)$. By using an argument similar to above, we get $\operatorname{Im} f_n \in \mathcal{P}_C(R)$ and $\mathcal{P}_C(R)$-pd $M \le n$.

(2) It is the symmetric version of (1). □

Let R be a left Noetherian ring and S a right Noetherian ring. By [23, Theorem 2.7], we have that $\operatorname{id}_R C = \operatorname{id}_{S^{op}} C$ if both of them are finite. In the following result, we give some equivalent characterizations for the finiteness of $\operatorname{id}_R C$ and $\operatorname{id}_{S^{op}} C$ in terms of the properties of the projective and injective dimensions of modules relative to some classes of C-Gorenstein modules. It is the C-version of [27, Theorem 1.2].

Theorem 5.20. *Let R be a left Noetherian ring and S a right Noetherian ring. Then for any $n \ge 0$, the following statements are equivalent.*

(1) $\operatorname{id}_R C = \operatorname{id}_{S^{op}} C \le n$.
(2) G_C-$\mathrm{pd}_R M \le n$ *for any* $M \in \operatorname{Mod} R$.
(3) G_C-$\mathrm{id}_S N \le n$ *for any* $N \in \operatorname{Mod} S$.
(4) G_C-$\mathrm{fd}_R M \le n$ *for any* $M \in \operatorname{Mod} R$.
(5) $\mathcal{PGF}_C(R)$-pd $M \le n$ *for any* $M \in \operatorname{Mod} R$.
(6) $\mathcal{SGF}_C(R)$-pd $M \le n$ *for any* $M \in \operatorname{Mod} R$.
$(i)^{op}$ *Symmetric version of* (i) *with* $2 \le i \le 6$.

Proof. By Theorem 5.2 and Lemma 5.19(1.1)(2.1), we have (1) $\Longleftrightarrow$ (4). By Theorem 5.4(1), we have (2) $\Longleftrightarrow$ (3). By Lemma 5.16(1)(2), we have (6) $\Longleftrightarrow$ (5) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1) By (2) and Theorem 5.4(1), we have $\operatorname{id}_R C \le n$ and $\operatorname{fd}_S E \le \operatorname{pd}_S E \le n$ for any $E \in \mathcal{I}_C(S)$. By Lemma 5.19(2.1), we have $\operatorname{id}_{S^{op}} C \le n$.

(1) + (4) $\Longrightarrow$ (5) By (4) and Theorem 5.2, we have that $\mathcal{F}_C(R)$-pd $I \le n$ for any $I \in \mathcal{I}(R)$ and that $\mathcal{F}_C(S^{op})$-pd $I' \le n$ for any $I' \in \mathcal{I}(S^{op})$. It follows from (1) and Lemma 5.19(1.2) that $\mathcal{P}_C(R)$-pd $I \le n$ for any $I \in \mathcal{I}(R)$. Now the assertion follows from Proposition 5.6(1).

By symmetry, the proof is finished. □

The following result is a consequence of Theorem 5.20.

Corollary 5.21. *Let R be a left Noetherian ring and S a right Noetherian ring with $\operatorname{id}_R C = \operatorname{id}_{S^{op}} C < \infty$. Then the following assertions hold.*

(1) $\mathcal{SGF}_C(R) = \mathcal{PGF}_C(R) = \mathcal{GP}_C(R) = {}^{\perp}\mathcal{P}_C(R) = {}^{\perp}\mathcal{F}_C(R)$.
(2) $\mathcal{GI}_C(S) = \mathcal{I}_C(S)^{\perp}$.
(3) $\mathcal{GF}_C(R) = \mathcal{I}_C(R^{op})^{\top}$.

Proof. (1) By Lemma 5.16(3.2), we have $\mathcal{SGF}_C(R) = \mathcal{PGF}_C(R) = \mathcal{GP}_C(R)$. By Theorem 5.20, we have $\mathrm{G}_C\text{-pd}_R\, M < \infty$ and $\mathcal{SGF}_C(R)\text{-pd}\, M < \infty$ for any $M \in \mathrm{Mod}\, R$. It follows from Proposition 4.7 that $\mathcal{GP}_C(R) = {}^{\perp}\mathcal{P}_C(R)$ and $\mathcal{SGF}_C(R) = {}^{\perp}\mathcal{F}_C(R)$.

(2) By Theorem 5.20, we have $\mathrm{G}_C\text{-id}_S\, N < \infty$ for any $N \in \mathrm{Mod}\, S$. Now the assertion follows from Proposition 4.12.

(3) Let $\mathrm{id}_R\, C = \mathrm{id}_{S^{op}}\, C = n < \infty$ and let $M \in \mathcal{I}_C(R^{op})^{\top}$. By Theorem 5.20, we have $\mathrm{G}_C\text{-id}_{R^{op}}\, M^+ \leq n$. Let $E \in \mathcal{I}_C(R^{op})$. It follows from [15, Lemma 2.16(b)] that

$$\mathrm{Ext}^i_{R^{op}}(E, M^+) \cong [\mathrm{Tor}^R_i(E, M)]^+ = 0$$

for any $i \geq 1$, that is, $M^+ \in \mathcal{I}_C(R^{op})^{\perp}$. Then $M^+ \in \mathcal{GI}_C(R^{op})$ by the symmetric version of (2), and thus $M \in \mathcal{GF}_C(R)$ by [27, Theorem 4.17(2)]. □

Recall that a left and right Noetherian ring R is called *n-Gorenstein* with $n \geq 0$ if $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R \leq n$. The Wakamatsu tilting conjecture states that if R and S are artin algebras, then the left and right injective dimensions of ${}_RC_S$ are identical ([5]). It still remains open. The following result provides some support for this conjecture.

Theorem 5.22. *It holds that*

1. *If R is an n-Gorenstein ring, then $\mathrm{id}_R\, C \leq n$ if and only if $\mathrm{G}_C\text{-pd}_R\, M \leq n$ for any $M \in \mathrm{Mod}\, R$.*
2. *If S is an n-Gorenstein ring, then $\mathrm{id}_{S^{op}}\, C \leq n$ if and only if $\mathrm{G}_C\text{-pd}_{S^{op}}\, N \leq n$ for any $N \in \mathrm{Mod}\, S^{op}$.*
3. *If R and S are Gorenstein rings, then $\mathrm{id}_R\, C = \mathrm{id}_{S^{op}}\, C$.*

Proof. (1) We first prove the sufficiency. Let $M \in \mathrm{Mod}\, R$. Then $\mathrm{G}_C\text{-pd}_R\, M \leq n$ and there exists an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all G_i in $\mathcal{GP}_C(R)$. Applying the functor $\mathrm{Hom}_R(-, C)$ to it yields

$$\mathrm{Ext}^{n+i}_R(M, C) \cong \mathrm{Ext}^i_R(G_n, C) = 0$$

for any $i \geq 1$, and thus $\mathrm{id}_R\, C \leq n$.

In the following, we prove the necessity. Let $M \in \mathrm{Mod}\, R$ and let

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be an exact sequence in $\mathrm{Mod}\, R$ with all P_i projective. By dimension shifting, we have

$$\mathrm{Ext}^i_R(K_n, X) \cong \mathrm{Ext}^{n+i}_R(M, X) = 0$$

for any $X \in \mathrm{Mod}\, R$ with $\mathrm{id}_R\, X \leq n$ and $i \geq 1$. Since $\mathrm{id}_R\, C \leq n$, we have $\mathrm{id}_R\, Q \leq n$ for any $Q \in \mathcal{P}_C(R)$ by [3, Theorem 1.1], and so $K_n \in {}^{\perp}\mathcal{P}_C(R)$.

Since R is an n-Gorenstein ring, the Gorenstein projective dimension of M is at most n by [27, Theorem 1.2], and hence K_n is Gorenstein projective. Thus there exists an exact sequence

$$0 \to K_n \to P^0 \to P^1 \to \cdots \to P^i \to \cdots$$

in $\mathrm{Mod}\, R$ with all P^i in $\mathcal{P}(R)$. Since $\mathcal{P}(R) \subseteq \mathcal{GP}_C(R)$, there exists a $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence

$$0 \to P^i \to Q^i_0 \to Q^i_1 \to \cdots \to Q^i_j \to \cdots$$

in $\operatorname{Mod} R$ with all Q_j^i in $\mathcal{P}_C(R)$ for any $i, j \geq 0$. By [24, Theorem 3.8], we get the following exact sequence

$$0 \to K_n \to Q_0^0 \xrightarrow{f^1} Q_1^0 \oplus Q_0^1 \xrightarrow{f^2} \cdots \xrightarrow{f^m} \oplus_{i=0}^m Q_{m-i}^i \xrightarrow{f^{m+1}} \cdots \tag{5.6}$$

in $\operatorname{Mod} R$. By [19, Proposition 5.1(b)], we have all $\oplus_{i=0}^m Q_{m-i}^i$ are in $\mathcal{P}_C(R)$. Then we have

$$\operatorname{Ext}_R^1(\operatorname{Im} f^m, Q) \cong \operatorname{Ext}_R^{n+1}(\operatorname{Im} f^{m+n}, Q) = 0$$

for any $Q \in \mathcal{P}_C(R)$ and $m \geq 1$, which implies that the exact sequence (5.6) is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact. Thus $K_n \in \mathcal{GP}_C(R)$ and $\mathrm{G}_C\text{-pd}_R\, M \leq n$.

(2) It is the symmetric version of (1).

(3) Suppose $\operatorname{id}_R C < \infty$. In this case, we may suppose that R is an n-Gorenstein ring and $\operatorname{id}_R C \leq n$ for some $n \geq 0$. By (1) and Theorem 5.20, we have $\operatorname{id}_{S^{op}} C = \operatorname{id}_R C \leq n$. Symmetrically, if $\operatorname{id}_{S^{op}} C < \infty$, then $\operatorname{id}_R C = \operatorname{id}_{S^{op}} C$ by (2) and Theorem 5.20. □

Acknowledgements. The research was partially supported by National Natural Science Foundation of China (Grant Nos. 12371038, 12171207). The authors thank the referee for very detailed and useful suggestions.

References

[1] T. Araya, R. Takahashi and Y. Yoshino, *Homological invariants associated to semi-dualizing bimodules*, J. Math. Kyoto Univ. **45** (2005), 287–306.

[2] M. Auslander and M. Bridger, Stable Module Theory, Memoirs Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969.

[3] H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102**(1962),18–29.

[4] A. Beligiannis and I. Reiten, Homological and Homotopical Aspects of Torsion Theories, Memoirs Amer. Math. Soc. **188** (883), Amer. Math. Soc., Providence, RI, 2007.

[5] D. Bennis and N. Mahdou, *Global Gorenstein dimensions*, Proc. Amer. Math. Soc. **138** (2010), 461–465.

[6] D. Bravo, J. Gillespie and M. Hovey, *The stable module category of a general ring*, Preprint is available at: arXiv:1210.0196.

[7] L. W. Christensen, S. Estrada and P. Thompson, *Gorenstein weak global dimension is symmetric*, Math. Nach. **294** (2021), 2121–2128.

[8] N. Q. Ding, Y. L. Li and L. X. Mao, *Strongly Gorenstein flat modules*, J. Aust. Math Soc. **86** (2009), 323–338.

[9] I. Emmanouil, *On the finiteness of Gorenstein homological dimensions*, J. Algebra **372** (2012), 376–396.

[10] E. E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.

[11] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, Vol. 1, the second revised and extended edition, de Gruyter Expositions in Math. **30**, Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

[12] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267-284.

[13] Z. H. Gao and Z. Y. Huang, *Weak injective covers and dimension of modules*, Acta Math. Hungar. **147** (2015), 135–157.

[14] Z. H. Gao and T. W. Zhao, *Foxby equivalence relative to C-weak injective and C-weak flat modules*, J. Korean Math. Soc. **54** (2017), 1457–1482.

[15] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, Vol. 1, Approimations, Second revised and extended edition, de Gruyter Exp. in Math. **41**, Walter de Gruyter GmbH & Co. KG, Berlin, 2012.

[16] E. S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66.

[17] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), 167–193.

[18] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), 423–445.

[19] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), 781–808.

[20] M. Hoshino, *Algebras of finite self-injective dimension*, Proc. Amer. Math. Soc. **112** (1991), 619–622.

[21] C. H. Huang and Z. Y. Huang, *Torsionfree dimension of modules and self-injective dimension of rings*, Osaka J. Math. **49** (2012), 21–35.

[22] Z. Y. Huang, *Wakamatsu tilting modules, U-dominant dimension and k-Gorenstein modules*, Abelian Groups, Rings, Modules, and Homological Algebra, Eds. P. Goeters and O. M. G. Jenda, Lect. Notes Pure Appl. Math. **249**, Chapman and Hall/CRC, Taylor and Francis Group, New York, 2006, pp.183–202.

[23] Z. Y. Huang, *Generalized titling modules with finite injective dimension*, J. Algebra **311** (2007), 619–634.

[24] Z. Y. Huang, *Proper resolutions and Gorenstein categories*, J. Algebra, **393** (2013), 142–169.

[25] Z. Y. Huang, *Homological dimensions relative to preresolving subcategories*, Kyoto J. Math. **54** (2014), 727–757.

[26] Z. Y. Huang, *Duality pairs induced by Auslander and Bass classes*, Georgian Math. J. **28** (2021), 867–882.

[27] Z. Y. Huang, *Homological dimensions relative to preresolving subcategories II*, Forum Math. **34** (2022), 507–530.

[28] A. Iacob, *Projectively coresolved Gorenstein flat and ding projective modules*, Comm. Algebra **48** (2020), 2883–2893.

[29] Y. N. Li and Z. Y. Huang, Homological dimensions under Foxby equivalence, Kodai Math. J. **48** (2025), 11–33.

[30] Z. F. Liu, Z. Y. Huang and A. M. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra **41** (2013), 1–18.

[31] F. Mantese and I. Reiten, *Wakamatsu tilting modules*, J. Algebra **278** (2004), 532–552.

[32] J. J. Rotman, An Introduction to Homological Algebra, Second edition, Universitext, Springer, New York, 2009.

[33] J. Šaroch and J. Šťovíček, *Singular compactness and definability for* Σ*-cotorsion and Gorenstein modules*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 23, 40 pp.

[34] W. L. Song and T. W. Zhao and Z. Y. Huang, *Duality pairs induced by one-sided Gorenstein subcategories*, Bull. Malays. Math. Sci. Soc. **43** (2020), 1989–2007.

[35] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose*, Forum Math. **27** (2015), 3717–3743.

[36] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose II*, Kyoto J. Math. **57** (2017), 17–53.

[37] X. Tang and Z. Y. Huang, *Homological aspects of the adjoint cotranspose*, Colloq. Math. **150** (2017), 293–311.

[38] X. Tang and Z. Y. Huang, *Homological invariants related to semidualizing bimodules*, Colloq. Math. **156** (2019), 135–151.

[39] X. Tang and Z. Y. Huang, *Coreflexive modules and semidualizing modules with finite projective dimension*, Taiwanese J. Math. **21** (2017), 1283–1324.

[40] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra **114** (1988), 106–114.

[41] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298–325.

[42] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra **275** (2004), 3–39.

[43] D. White, *Gorensten projective dimension with respect to a semidualizing module*, J. Comm. Algebra **2** (2010), 111–137.