GORENSTEIN PROJECTIVE SUPPORT τ-TILTING MODULES OVER GENTLE ALGEBRAS

YU-ZHE LIU, YAFENG ZHANG, AND ZHAOYONG HUANG

ABSTRACT. We provide a description of the endomorphism algebra of any Gorenstein projective support τ-tilting module over gentle algebras via geometric models. By using it, we show that a gentle algebra A is representation-finite if and only if the endomorphism algebra of any Gorenstein projective support τ-tilting A-module is representation-finite.

1. Introduction

Tilting theory plays a central role in the representation theory of algebras, in which tilting modules are fundamental. On the other hand, mutation is an operation for a certain class of objects in a fixed category to construct a new object from a given one by replacing a summand, which is possible only when the given object has two complements. Happel and Unger [15] gave some necessary and sufficient conditions under which mutation of tilting modules is possible; however, mutation of tilting modules is not always possible. As a generalization of tilting theory, τ-tilting theory was introduced by Adachi, Iyama and Reiten [1]. In τ-tilting theory, support τ-tilting modules are important since the mutations of support τ-tilting modules always exist. Recently, Gorenstein projective support τ-tilting modules, introduced by Xie and Zhang [24], were used to study functorially finite Gorenstein torsion pairs over finite dimensional algebras. For a finite dimensional algebra A, there is a bijection between the set of all isomorphism classes of Gorenstein projective support τ-tilting A-modules and the set of all functorially finite Gorenstein torsion pairs of A-modules; and the CM-τ-tilting finiteness is left and right symmetric, see [24, Theorems 1.2 and 1.3].

Gentle algebras were introduced by Assem and Skowroński [4] as appropriate context in the study of algebras derived equivalent to hereditary algebras of type $\widetilde{\mathbb{A}}$. Their module categories and derived categories were studied by many authors, see [2, 8, 9, 11, 18, 22, 23] and references therein. The geometric models of gentle algebras were introduced by Haiden, Katzarkov and Kontsevich [14]. After that, Baur-Coelho-Simões [7] and Opper-Plamondon-Schroll [20] provided the axiomatic definitions of geometric models of module categories and derived categories for gentle algebras respectively. To be more precise, all indecomposable objects in module category (respectively, derived categories) are described by special curves in surfaces; all irreducible morphisms between two indecomposable modules in module categories are described by the pivot elementary moves of curves; and all morphisms between two indecomposable complexes in derived categories are described by the intersections of two curves respectively corresponding to the above indecomposable objects. He, Zhou and Zhu [16] investigated the geometric models of module categories over skew-gentle algebras (a generalization of gentle algebras), and they provided another geometric characterization of irreducible morphisms between two indecomposable modules by the intersections of curves respectively corresponding to the above indecomposable modules. Furthermore, they pointed out that any support τ-tilting module corresponds to a generalized dissection (see Definition 4.1) of marked ribbon surfaces.

In the representation theory of algebras, determining whether an algebra is representation-finite is fundamental and important. By using the "brick-τ-tilting correspondence" [12],

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Plamondon proved that a gentle algebra is representation-finite if and only if it is τ-tilting finite [21, Theorem 1.1]. In this paper, we will study Gorenstein projective support τ-tilting modules over gentle algebras via geometric models, and then show that a gentle algebra A is representation-finite if and only if the endomorphism algebra of any Gorenstein projective support τ-tilting A-module is representation-finite.

The paper is organized as follows. In Section 2, we recall some terminologies and some preliminary results needed in this paper. In particular, we give the definition of gentle algebras and some related notions related to their geometric models. In Section 3, we observe that a gentle algebra is representation-finite if and only if its quiver has no band (Proposition 3.6).

Let A be a gentle algebra and $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$ its marked ribbon surface. In Section 4, we prove that the marked ribbon surface of the Cohen-Macaulay-Auslander algebra A^{CMA} of A can be given as follows: If any full formal $\bullet$-arc system of $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$ provides no ∞-elementary $\bullet$-polygon (see Figure 2.1), then the marked ribbon surface of A^{CMA} is homotopic to $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$; otherwise, for each ∞-elementary $\bullet$-polygon $\mathcal{P}_i$ with ℓ_i vertices $p_{i1}, p_{i2}, \ldots, p_{i\ell_i}$ in clockwise order, we add ℓ_i new marked points $q_{i1}, q_{i2}, \ldots, q_{i\ell_i}$ clockwise arranging on the unmarked boundary component b_i corresponding to $\mathcal{P}_i$ and add ℓ_i new $\bullet$-arcs w_{ij} whose endpoints are p_{ij} and q_{ij} ($1 \le j \le \ell_i$) (Theorem 4.9).

In Section 5, we prove the main results in this paper.

Theorem 1.1. *Let A be a finite dimensional basic algebra over an algebraically closed field. If A is gentle, then it holds that*

(1) (Theorem 5.2) *The category* $\mathsf{mod}A$ *of finitely generated right A-modules contains at least one non-trivial G-projective τ-tilting module if and only if some full relational oriented cycle $C = \alpha_1 \cdots \alpha_\ell$ $(s(\alpha_1) =: v_1)$ of the quiver of A has at least one vertex v_i $(1 \le i \le \ell)$ which is not a target of any arrow except α_i.*

(2) (Theorem 5.5) *The algebra A is representation-finite if and only if the endomorphism algebra* $\mathrm{End}_A T$ *is representation-finite for any Gorenstein projective support τ-tilting module A-module T.*

In Section 6, we give an example to illustrate that the Gorenstein projective condition in Theorem 1.1(2) is necessary (Example 6.1). It is known that any tilting right A-module T over an arbitrary finite dimensional algebra A is a tilting left B-module ${}_BT$ with $B = \mathrm{End}(T_A)$ and $\mathrm{End}({}_BT) \cong A$. We give some examples to show that this property cannot be generalized to G-projective τ-tilting modules even over gentle algebras (Examples 6.5 and 6.6).

2. The geometric models of module categories for gentle algebras

In this paper, assume that $A = \Bbbk\mathcal{Q}/\mathcal{I}$ is a finite dimensional basic $\Bbbk$-algebra over an algebraically closed field $\Bbbk$, where $\mathcal{I}$ is an admissible ideal of $\Bbbk\mathcal{Q}$ and $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a finite quiver with $\mathcal{Q}_0, \mathcal{Q}_1$ the sets of all vertices and arrows, respectively. We use s, t to denote two functions from Q_1 to Q_0 which send each arrow to its source and target respectively. The multiplication $\alpha_1\alpha_2$ of two arrows α_1 and α_2 in $\mathcal{Q}_1$ is defined by the concatenation if $t(\alpha_1) = s(\alpha_2)$ or zero if $t(\alpha_1) \ne s(\alpha_2)$. All A-modules considered are basic right A-modules. We use $\mathsf{mod}A$ to denote the category of finitely generated right A-modules. For a set S, we use $\sharp S$ to denote the cardinality of S.

Definition 2.1. [4] We say $A \cong \Bbbk\mathcal{Q}/\mathcal{I}$ is a *gentle algebra* if it satisfies (G1)–(G4) as follows:

(G1) Any vertex in $\mathcal{Q}_0$ is the source of at most two arrows and the target of at most two arrows.

(G2) For any $\alpha \in \mathcal{Q}_1$, there is at most one arrow β whose source (respectively, target) is $t(\alpha)$ (respectively, $s(\alpha)$) such that $\alpha\beta \in \mathcal{I}$ (respectively, $\beta\alpha \in \mathcal{I}$).

(G3) For each arrow $\alpha \in \mathcal{Q}_1$, there is at most one arrow β whose source (respectively, target) is $t(\alpha)$ (respectively, $s(\alpha)$) such that $\alpha\beta \notin \mathcal{I}$ (respectively, $\beta\alpha \notin \mathcal{I}$).

(G4) $\mathcal{I}$ is generated by paths of length 2.

2.1. **Marked surfaces.** A *marked surface* is a pair $(\mathcal{S}, \mathcal{M})$ defined by a smooth surface $\mathcal{S}$ and a finite subset $\mathcal{M}$ of the boundary $\partial\mathcal{S}$ of $\mathcal{S}$, where each element in $\mathcal{M}$, denoted by $\bullet$, is called a *marked point*. A *formal* $\bullet$*-arc system* (=$\bullet$-FAS) of marked surface is a collection of curves with endpoints belonging to $\mathcal{M}$ such that:

- $a_1 \cap a_2 \cap (\mathcal{S}\backslash\partial\mathcal{S}) = \varnothing$ holds for any $a_1, a_2 \in \Delta_\bullet$; and
- each *elementary* $\bullet$*-polygon*, the polygon obtained by $\Delta_\bullet$ cutting $\mathcal{S}$, has at least one edge which does not belong to $\Delta_\bullet$. Especially, an ∞*-elementary* $\bullet$*-polygon* $\mathcal{P}$ is an elementary $\bullet$-polygon such that $\mathcal{P}$ has unique edge $b \subseteq \partial\mathcal{S}$ which is a boundary component of $\mathcal{S}$ without marked points (FIGURE 2.1).

The elements in $\Delta_\bullet$ are called $\bullet$*-arcs*. Furthermore, $\Delta_\bullet$ is called a *full formal* $\bullet$*-arc system* (=$\bullet$-FFAS) if every elementary $\bullet$-polygon has a unique edge which does not belong to $\Delta_\bullet$.

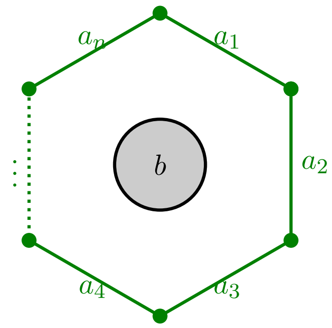


FIGURE 2.1. ∞-elementary $\bullet$-polygon.

Definition 2.2. [7] A *marked ribbon surface* is a triple $\mathbf{S}_\bullet = (\mathcal{S}, \mathcal{M}, \Delta_\bullet)$, where $\Delta_\bullet$ is a $\bullet$-FFAS.

Remark 2.3.

(1) Let $\mathbf{S}_\bullet$ be a marked ribbon surface and let be $\mathcal{Y}$ the finite subset of $\partial\mathcal{S}$ whose elements are represented by symbols $\circ$ and called *dull marked points* such that marked points in $\mathcal{M}$ and $\mathcal{Y}$ are alternate in every boundary component. Then we can define the *formal* $\circ$*-arc system* (=$\circ$-FAS) (respectively, *full formal* $\bullet$*-arc system* (=$\circ$-FFAS)) $\Delta_\circ$ which is dual of $\bullet$-FAS (respectively, $\bullet$-FFAS) such that
  - for any two curves $c_1, c_2 \in \Delta_\circ$, we have $c_1 \cap c_2 \cap (\mathcal{S}\backslash\partial\mathcal{S}) = \varnothing$;
  - each *elementary* $\circ$*-polygon*, the polygon obtained by $\Delta_\circ$ cutting $\mathcal{S}$, has at least one edge (respectively, has a unique edge) which does not belong to $\Delta_\circ$.

(2) For any $\bullet$-FFAS $\Delta_\bullet$ (respectively, $\circ$-FFAS $\Delta_\circ$), there is a unique $\circ$-FFAS $\Delta_\circ$ (respectively, $\bullet$-FFAS $\Delta_\bullet$) such that
  - for any $\bullet$-arc a in $\Delta_\bullet$, there is a unique $\circ$ arc $a^\star$ such that $a \cap a^\star \neq \varnothing$ and $\sharp(a \cap a^\star) = 1$ hold;
  - for any $\circ$-arc a in $\Delta_\bullet$, there is a unique $\bullet$ arc $a^\star$ such that $a^\star \cap a \neq \varnothing$ and $\sharp(a^\star \cap a) = 1$ hold;

  We say one of $\Delta_\bullet$ and $\Delta_\circ$ is the *dual* of the other, and denote by $\Delta_\bullet = \Delta_\circ^\star$ or $\Delta_\circ = \Delta_\bullet^\star$.

(3) In [20], any $\bullet$-FFAS $\Delta_\bullet$ is called a *ribbon graph* of the marked surface, and the $\circ$-FFAS $\Delta_\circ = \Delta_\bullet^\star$ is called the *lamination* corresponding $\Delta_\bullet$.

Notation 2.4.

(1) We use $\mathrm{EP}_\bullet(\mathbf{S}_\bullet)$ (respectively, $\mathrm{EP}_\circ(\mathbf{S}_\bullet)$) to denote the set of all elementary $\bullet$-polygons (respectively, elementary $\circ$-polygons) of $\mathbf{S}_\bullet$.

(2) For any digon in $\mathrm{EP}_{\bullet}(\mathbf{S}_{\bullet})$, we add a point $\diamond$ on the side which is the subset of $\partial\mathcal{S}$, and say it is a *extra marked point*. We use $\mathcal{E}$ to denote the set of all extra marked points and use $\mathbf{S}_{\bullet}^{\mathcal{E}}$ to denote the marked ribbon surface with extra marked points. For simplicity, $\mathbf{S}_{\bullet}^{\mathcal{E}}$ is still called a marked ribbon surface in the sequel.

Remark 2.5. *$\mathcal{E}$ can be seen as a subset of $\mathcal{Y}$.*

Every marked ribbon surface induces a gentle algebra by the following construction.

Construction 2.6. Let $\mathbf{S}_{\bullet}^{\mathcal{E}} = (\mathcal{S}, \mathcal{M}, \Delta_{\bullet})^{\mathcal{E}}$ be a marked ribbon surface. Then $\mathbf{S}_{\bullet}^{\mathcal{E}}$ is the marked ribbon surface of $A(\mathbf{S}_{\bullet}^{\mathcal{E}}) = \Bbbk\mathcal{Q}/\mathcal{I}$, where

Step 1: there is a bijection $\mathfrak{v} : \Delta_{\bullet} \to \mathcal{Q}_0$, that is, each arc can be viewed as a vertex in $\mathcal{Q}_0$;
Step 2: for each elementary polygon $\mathcal{P}$ obtained by $\Delta_{\bullet}$ cutting $\mathcal{S}$, there is an arrow $\alpha : \mathfrak{v}(a_1) \to \mathfrak{v}(a_2)$ if $a_1, a_2 \in \Delta_{\bullet}$ are two edges of $\mathcal{P}$ with common endpoints $p \in \mathcal{Y}$ and a_1 precedes a_2 in the counterclockwise order arounding p;
Step 3: the ideal $\mathcal{I}$ is generated by such composition $\alpha\beta$ where $\mathfrak{v}^{-1}(s(\alpha))$, $\mathfrak{v}^{-1}(s(\beta))$, $\mathfrak{v}^{-1}(t(\beta))$ are edges of the same elementary polygon.

Remark 2.7. The corresponding $\mathbf{S}_{\bullet}^{\mathcal{E}} \mapsto A(\mathbf{S}_{\bullet}^{\mathcal{E}})$ given in Construction 2.6 induces a bijection between the set of all homotopy classes of marked ribbon surfaces and that of isoclasses of gentle algebras. Therefore, up to homotopy equivalence, for any gentle algebra A, there is a unique marked ribbon surface $\mathbf{S}_{\bullet}^{\mathcal{E}}$ such that $A \cong A(\mathbf{S}_{\bullet}^{\mathcal{E}})$. We say that $\mathbf{S}_{\bullet}^{\mathcal{E}}$ is the *marked ribbon surface* of A and denote it by $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$. For simplicity, we use "$\sim$" to denote the homotopy equivalence of two marked ribbon surfaces. Thus, $A \cong B$ if and only if $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A) \sim \mathbf{S}_{\bullet}^{\mathcal{E}_B}(B)$.

2.2. **Permissive curves.** The definition of permissive curves is given by Baur and Coelho Simões [7, Definition 3.1], which is used to describe indecomposable modules over gentle algebras. Now we recall some related notions.

For simplicity, we define that a *curve* on the marked ribbon surface $\mathbf{S}_{\bullet}^{\mathcal{E}} = (\mathcal{S}, \mathcal{M}, \Delta_{\bullet})^{\mathcal{E}}$ is a function $c : [0,1] \to \mathcal{S}$ such that $c(t) \in \mathcal{S}\backslash\partial\mathcal{S}$ for any $0 < t < 1$, one of $c(0), c(1) \in \mathcal{M}\cup\mathcal{E}$ and $c(0) = c(1) \in \mathcal{S}\backslash\partial\mathcal{S}$ holds; and say that c is *consecutively crossing* $u, v \in \Delta_{\bullet}$ if the segment of c between the points $p_1 = c \cap u$ and $p_2 = c \cap v$ does not cross any other arc in $\Delta_{\bullet}$. Let B be a boundary component with no marked point of $\mathbf{S}_{\bullet}^{\mathcal{E}}$, called a *unmarked boundary component* of $\mathcal{S}$. In particular, for any curve c, we always assume that c has minimal intersection number with $\Delta_{\bullet}$.

Definition 2.8. [7, Definition 3.1] Let $\mathbf{S}_{\bullet}^{\mathcal{E}} = (\mathcal{S}, \mathcal{M}, \Delta_{\bullet})^{\mathcal{E}}$ be a marked ribbon surface and let $c : [0,1] \to \mathcal{S}$ be a curve.

(1) The curve c is called *permissible* if it the following conditions are satisfied.
  (a) The winding number of c arounding any unmarked boundary component of $\mathcal{S}$ is either 0 or 1;
  (b) If c consecutively crosses two (possibly not distinct) arcs u and v in Γ, then u and v have a common endpoint $p \in \mathcal{M}$, and locally we have a triangle with p a vertex.
(2) A *permissible closed curve* is a closed curve c satisfying Condition (1)(b).

Definition 2.9. [7, Definition 3.5] Two permissible curves c_1 and c_2 in $\mathbf{S}_{\bullet}^{\mathcal{E}}$ are called *equivalent*, if one of the following conditions holds:

(1) There is a sequence of consecutive edges $\delta_1, \cdots, \delta_k$ of an elementary $\bullet$-polygon $\mathcal{P}$ such that is not of Case I shown in Figure 2.2 such that:
  – c_1 is homotopic to the concatenation of c_2 and $\delta_1, \cdots, \delta_k$; and
  – c_1 starts at an endpoint of δ_1 (respectively, δ_k), c_2 starts at an endpoint of δ_k (respectively, δ_1), and their first crossing with $\Delta_{\bullet}$ has the same edge of $\mathcal{P}$.
(2) The starting points of c_1 and c_2 are marked points of an elementary $\bullet$-polygon $\mathcal{P}$ of Case I or II shown in Figure 2.2; their first crossing with $\Delta_{\bullet}$, say δ, is with the same edge of $\mathcal{P}$ and the segments of c_1 and c_2 between δ and their ending points are isotopic.

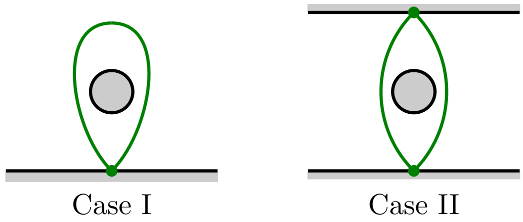


FIGURE 2.2. The elementary $\bullet$-polygon $\mathcal{P}$ is of the form Case I (respectively, Case II); in this case, $\mathcal{S}$ has an unmarked boundary which is an edge of $\mathcal{P}$, thus the number of edges equals 2 (respectively, 3).

We use $\mathrm{PC}(\mathbf{S}_\bullet^{\mathcal{E}})$ to denote the set of all equivalent classes of permissible curves with endpoints lying in $\mathcal{M} \cup \mathcal{E}$, and use $\mathrm{CC}(\mathbf{S}_\bullet^{\mathcal{E}})$ to denote the set of all homotopy classes of permissible curves without endpoints. Then, for any two curves $c_1, c_2 \in [c]$, it is easy to see that

$$\{a \in \Delta_\bullet \mid a \cap c_1 \cap (\mathcal{S} \backslash \partial\mathcal{S}) \neq \varnothing\} = \{a \in \Delta_\bullet \mid a \cap c_2 \cap (\mathcal{S} \backslash \partial\mathcal{S}) \neq \varnothing\}. \tag{2.1}$$

For each curve c, we always suppose that it consecutively crossing $a_1^c, a_2^c, \cdots, a_{n(c)}^c$ ($n(c) \in \mathbb{N}$), and denote:

- $c_{i,j}$ ($1 \le i < j \le n(c) - 1$) the segment of c obtained by a_i^c and a_j^c cutting c; and
- $c_{0,i}$ (respectively, $c_{j,n(c)+1}$) the segment of c obtained by $c(0)$ and a_i^c (respectively, a_j^c and $c(1)$) cutting c.

Furthermore, by (2.1), we say that $c_{i,j}$ is *elementary* if $[c] \in \mathrm{PC}(\mathbf{S}_\bullet^{\mathcal{E}})$ and $j = i + 1$, and say that c is *trivial* (or *zero*) if $n(c) = 1$, that is, $c = c_{0,1}$ is an elementary segment.

Theorem 2.10.

(1) (Butler-Ringel [23, Proposition 2.3] and Wald-Waschbüsch [9, Section 3]) *Let A be a gentle algebra and $\mathscr{J}$ the set of all Jordan blocks with non-zero eigenvalue. Then any indecomposable right A-module is either a string module or a band module.*

(2) (Baur-Coelho-Simoes [7, Theorem 3.8, Theorem 3.9]) *Furthermore, there exists a bijection*

$$\mathfrak{M} : \mathrm{PC}(\mathbf{S}_\bullet^{\mathcal{E}_A}(A)) \cup (\mathrm{CC}(\mathbf{S}_\bullet^{\mathcal{E}_A}(A)) \times \mathscr{J}) \to \mathsf{ind}(\mathsf{mod}A)$$

Remark 2.11. The indecomposable module isomorphic to $\mathfrak{M}([c])$ is string if $c \in \mathrm{PC}(\mathbf{S}_\bullet^{\mathcal{E}_A}(A))$; and the indecomposable module isomorphic to $\mathfrak{M}([b], \boldsymbol{J})$ is band if $(b, \boldsymbol{J}) \in \mathrm{CC}(\mathbf{S}_\bullet^{\mathcal{E}_A}(A)) \times \mathscr{J}$. The original definitions of string modules and band modules are given by Butler and Ringel in [9, Section 3].

3. Representation-finite gentle algebras

From now on, assume that $A = \Bbbk\mathcal{Q}/\mathcal{I}$ is a gentle algebra.

3.1. Subsurfaces and subquivers.

Definition 3.1 (Marked ribbon subsurfaces)**.** Given a marked ribbon surface $\mathbf{S}_\bullet^{\mathcal{E}} = (\mathcal{S}, \mathcal{M}, \Delta_\bullet)^{\mathcal{E}}$. A *marked ribbon subsurface* (or simply, *subsurface*) of $\mathbf{S}_\bullet^{\mathcal{E}}$ is a quadruple $(\widehat{\mathcal{S}}, \widehat{\mathcal{M}}, \widehat{\Delta}_\bullet, \widehat{\mathcal{E}})$, say $\widehat{\mathbf{S}}_\bullet^{\widehat{\mathcal{E}}} = (\widehat{\mathcal{S}}, \widehat{\mathcal{M}}, \widehat{\Delta}_\bullet)^{\widehat{\mathcal{E}}} \precsim \mathbf{S}_\bullet^{\mathcal{E}}$, such that

$$\widehat{\mathcal{S}} \subseteq \mathcal{S}, \ \widehat{\mathcal{M}} \subseteq \mathcal{M}, \ \widehat{\Delta}_\bullet \subseteq \Delta_\bullet \text{ and } \widehat{\mathcal{E}} \text{ is induced by } \widehat{\Delta}_\bullet.$$

Notation 3.2. We use $\mathcal{M}|_{\widehat{\mathcal{S}}}$ and $\Delta_\bullet|_{\widehat{\mathcal{S}}}$ to denote the restrictions of $\mathcal{M}$ and $\Delta_\bullet$ on $\widehat{\mathcal{S}}$ respectively.

Lemma 3.3. *Let $\widehat{A} = \Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}}$, where $\widehat{\mathcal{Q}}$ is a subquiver of $\mathcal{Q}$ and $\widehat{\mathcal{I}} = \{\alpha\beta \in \mathcal{I} \mid \alpha \in \widehat{\mathcal{Q}}, \beta \in \widehat{\mathcal{Q}}\}$. Then there is a unique subsurface $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}$ of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{Q}}_0 & \xrightarrow{\subseteq} & \mathcal{Q}_0 \\ {\scriptstyle\mathfrak{v}}\downarrow & & \downarrow{\scriptstyle\mathfrak{v}} \\ \widehat{\Delta}_{\bullet} & \xrightarrow{\subseteq} & \Delta_{\bullet} \end{array} \tag{3.1}$$

commutates.

Proof. Consider the $\circ$-FFAS $\Delta_{\circ} = \Delta^{\star}_{\bullet}$, which divides $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ to some elementary $\circ$-polygons $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$ and any elementary $\circ$-polygons $\mathcal{P}_i$ is homotopic to the marked surface of $\overrightarrow{\mathbb{A}}_{m_i}$, which is of the form as shown in Figure 3.1. The $\bullet$-marked point p

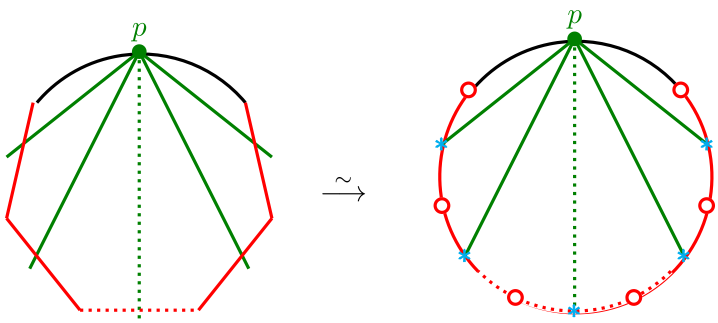


Figure 3.1. The marked ribbon surface of a gentle algebra which is type $\overrightarrow{\mathbb{A}}$.

corresponds to the path $\wp = \alpha_1 \cdots \alpha_{m_i}$ over $\mathcal{Q}$ such that the following conditions hold:

- $\alpha_j\alpha_{j+1} \notin \mathcal{I}$ for any $1 \le j \le m_i - 1$;
- $\beta\alpha_1 \in \mathcal{I}$ holds for any arrow β satisfying $t(\beta) = s(\alpha_1)$; and
- $\alpha_{m_i}\gamma \in \mathcal{I}$ holds for any arrow γ satisfying $t(\alpha_{m_i}) = s(\gamma)$.

Indeed, $\wp$ is the path $\alpha^i_1 \cdots \alpha^i_{t-1}$ on the subquiver

$$\overrightarrow{\mathbb{A}}_{m_i} = v_{(i,1)} \xrightarrow{\alpha^i_1} v_{(i,2)} \xrightarrow{\alpha^i_2} \cdots \xrightarrow{\alpha^i_{t-1}} v_{(i,m_i)},$$

and $\mathbf{S}_{\bullet}(A)$ is the union of all elementary $\circ$-polygons. Here $\wp$ is called a *non-trivial permitted thread* in [6] or a *maximal path* in [20].

On the other hand, it is easy to see that the quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$ of A satisfies the following unions

$$\mathcal{Q}_0 = \bigcup_{i=1}^{n} (\overrightarrow{\mathbb{A}}_{m_i})_0, \ \mathcal{Q}_1 = \bigcup_{i=1}^{n} (\overrightarrow{\mathbb{A}}_{m_i})_1,$$

and for any arrows $\alpha : v_{(i,j-1)} \to v_{(i,j)}$ and $\beta : v_{(i',j')} \to v_{(i',j'+1)}$ satisfying $v_{(i,j)} = v_{(i',j')}$, we have that $\alpha\beta \in \mathcal{I}$ if and only if

$$v_{(i,j-1)} \xrightarrow{\alpha^i_{j-1}} (v_{(i,j)} = v_{(i',j')}) \xrightarrow{\alpha^{i'}_{j'}} v_{(i',j'+1)}$$

is not a subquiver of any $\overrightarrow{\mathbb{A}}_{m_i}$. Thus for any subquiver $\widehat{\mathcal{Q}}$ of $\mathcal{Q}$, we have a union $\widehat{\mathcal{Q}} = \bigcup\limits_{i=1}^{n} \widehat{\mathcal{Q}}^i$ such that $\widehat{\mathcal{Q}}^i \subseteq \overrightarrow{\mathbb{A}}_{m_i}$, where $\widehat{\mathcal{Q}}^i$ is a disjoint union of some connected subquivers of $\overrightarrow{\mathbb{A}}_{m_i}$. Without loss of generality, suppose $\widehat{\mathcal{Q}}^i = \bigcup_{k=1}^{\hbar_i} \overrightarrow{\mathbb{A}}(i, \imath_k, \jmath_k)$, where $\overrightarrow{\mathbb{A}}(i, \imath_k, \jmath_k)$ is some connected subquiver of $\overrightarrow{\mathbb{A}}_{m_i}$ with $1 \le \imath_k \le \jmath_k \le m_i$ and $\jmath_k < \imath_{k+1}$ for all k. Then $\overrightarrow{\mathbb{A}}(i, \imath_k, \jmath_k)$ is the subquiver

$$v_{(i,\imath_k)} \xrightarrow{\alpha_{(i,\imath_k)}} v_{(i,\imath_k+1)} \xrightarrow{\alpha_{(i,\imath_k+1)}} \cdots \xrightarrow{\alpha_{(i,\jmath_k-2)}} v_{(i,\jmath_k-1)} \xrightarrow{\alpha_{(i,\jmath_k-1)}} v_{(i,\jmath_k)}$$

corresponding to the subsurface which is of the form as shown in FIGURE 3.2.

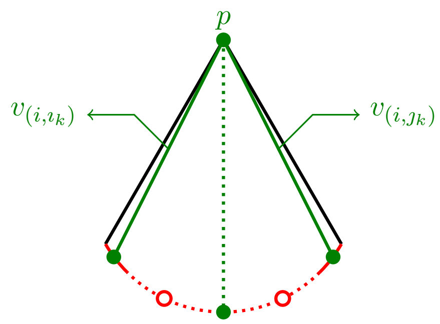


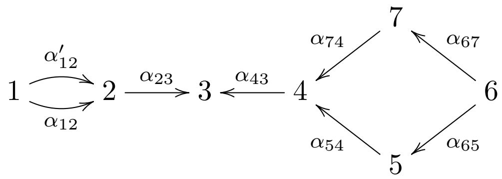
FIGURE 3.2. The subsurface corresponding to $A(i,\imath_k,\jmath_k) = \Bbbk\overrightarrow{\mathbb{A}}(i,\imath_k,\jmath_k)$.

Thus, the marked ribbon surface of $\widehat{A}$ is a union of some subsurfaces of $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$:

$$\widehat{\mathbf{S}}_{\bullet}^{\widehat{\mathcal{E}}} = \bigcup_{i=1}^{n} \bigcup_{k=1}^{\hbar_i} \widehat{\mathbf{S}}_{\bullet}^{\widehat{\mathcal{E}}(A(i,\imath_k,\jmath_k))}(A(i,\imath_k,\jmath_k)) \ \sim \ \mathbf{S}_{\bullet}^{\mathcal{E}_{\widehat{A}}}(\widehat{A}),$$

where the second union is a disjoint union. Which is as desired. □

Example 3.4. Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$, where $\mathcal{Q} =$



and $\mathcal{I} = \langle \alpha_{12}\alpha_{23}, \alpha_{74}\alpha_{43}\rangle$. Then $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A) = (\mathcal{S}(A), \mathcal{M}(A), \Delta_{\bullet}(A))^{\mathcal{E}_A}$ and $\Delta_{\circ}(A) = \Delta_{\bullet}^{\star}(A)$ is shown as in FIGURE 3.3.

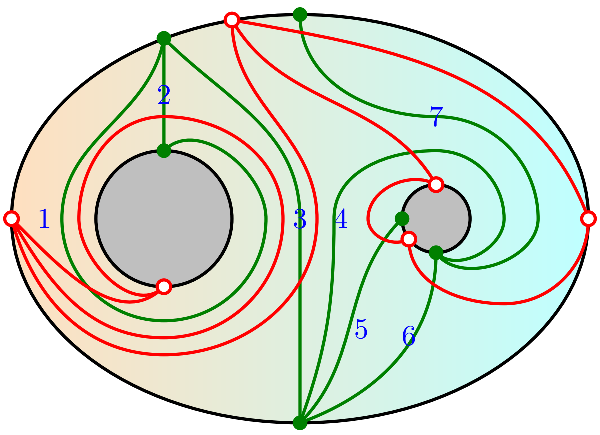
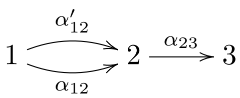


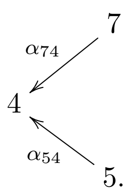
FIGURE 3.3. $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$

Consider the subquiver $\widehat{\mathcal{Q}} = \widehat{\mathcal{Q}}^1 \times \widehat{\mathcal{Q}}^2$ which is obtained by removing arrows α_{43}, α_{65}, α_{67} and the vertex 6, where

$\widehat{\mathcal{Q}}^1 =$



$\widehat{\mathcal{Q}}^2 =$



Then $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}(\Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}})$ is of the form as shown in FIGURE 3.4, and $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}(\Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}})$ contains two parts: the marked ribbon surface of $\Bbbk\widehat{\mathcal{Q}}^1/\langle\alpha_{12}\alpha_{23}\rangle$ and that of $\Bbbk\widehat{\mathcal{Q}}^2$.

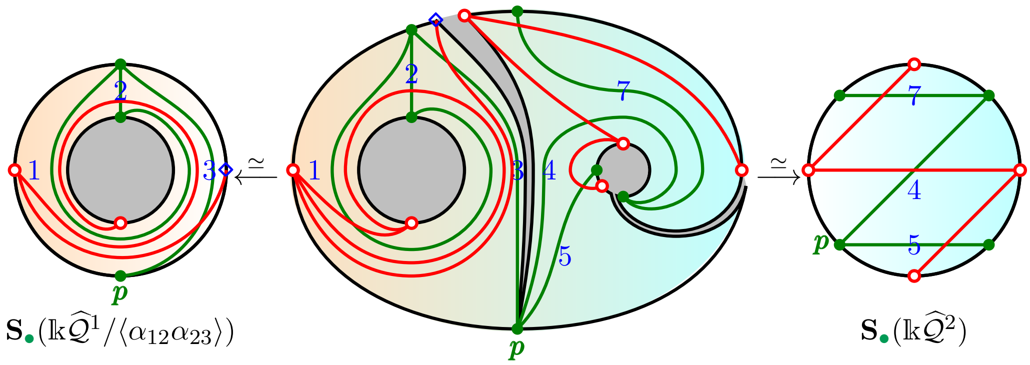


FIGURE 3.4. $\mathbf{S}_\bullet(\Bbbk\widehat{\mathcal{Q}}^1/\langle\alpha_{12}\alpha_{23}\rangle) \cup \mathbf{S}_\bullet(\Bbbk\widehat{\mathcal{Q}}^2) \sim \mathbf{S}_\bullet(\widehat{A}) \subseteq \mathbf{S}_\bullet(A)$.

3.2. **Representation-finite gentle algebras.** Recall that a finite dimensional $\Bbbk$-algebra Λ is *representation-finite* if $\mathsf{ind}(\mathsf{mod}\Lambda)$, the set of all isoclasses of indecomposable Λ-modules, is a finite set; otherwise Λ is *representation-infinite*.

Notation 3.5.

(1) We say that the quiver $(\mathcal{Q},\mathcal{I})$ of a gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ *has a cycle* $\mathscr{C}$ *(of length* n*)*, if there is a path $\wp$ of length n on the underlying graph of $\mathcal{Q}$ such that its starting point equals to its ending point. Moreover, the cycle $\mathscr{C}$ is called *basic* if it is type of $\widetilde{\mathbb{A}}_n$; and $\mathscr{C}$ is called a *cycle without relation* if any path on $\mathscr{C}$ does not belong to $\mathcal{I}$. Furthermore, a cycle $\mathscr{C}$ without relation is called a *band* if it is not a power $\mathscr{C}^t$ $(t \ge 2)$ of some cycle $\mathscr{C}$.

(2) We define that $\Lambda(\mathscr{C}) = \Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}}$ induced by a cycle $\mathscr{C}$ is the algebra whose quiver $\widehat{\mathcal{Q}}$ is the subquiver of $\mathcal{Q}$ such that all vertices and arrows are given by $\mathscr{C}$ and the admissible ideal $\widehat{\mathcal{I}}$ is generated by all paths of length two in $\mathcal{I}$.

The following proposition is probably known, we give here a simple proof of it by the first Brauer-Thrall Theorem [3, IV.5] and the classification of indecomposable modules over gentle algebras.

Proposition 3.6. *A gentle algebra* $A = \Bbbk\mathcal{Q}/\mathcal{I}$ *is representation-finite if and only if* $(\mathcal{Q},\mathcal{I})$ *has no band.*

Proof. If $(\mathcal{Q},\mathcal{I})$ has a band $\mathscr{C}$, then we can obtain all band modules $B(\mathscr{C},\boldsymbol{J}_n(\lambda))$ corresponding to the pair $(\mathscr{C},\boldsymbol{J}_n(\lambda))$ by Theorem 2.10(1), where $\boldsymbol{J}_n(\lambda)$ is an $n\times n$ Jordan block with non-zero eigenvalue $\lambda\in\Bbbk$. Thus, A is representation-infinite because $B(\mathscr{C},\boldsymbol{J}_n(\lambda)) \cong B(\mathscr{C},\boldsymbol{J}_m(\mu))$ if and only if $m=n$ and $\lambda=\mu$.

If $(\mathcal{Q},\mathcal{I})$ has no band, then all indecomposable modules are strings by Theorem 2.10(1). Notice that $(\mathcal{Q},\mathcal{I})$ is a bounded quiver, thus there exists a string s such that $l(s)\ge l(s')$ holds for every string s', where $l(s)$ and $l(s')$ are the lengths of s and s' respectively. Then the $\Bbbk$-dimension $\dim_\Bbbk M$ of any string module M is at most $l(s)$. By the first Brauer-Thrall Theorem, A is representation-finite. □

By Proposition 3.6, it is easy to see that any gentle algebra A is representation-infinite if and only if $\mathrm{CC}(\mathbf{S}^{\mathcal{E}_A}_\bullet(A)) = \varnothing$. Note that any band on the quiver of A corresponds to some permissible curve without endpoints in $\mathrm{CC}(\mathbf{S}^{\mathcal{E}_A}_\bullet(A))$ by Theorem 2.10(2). We may give another proof of Proposition 3.6 by using the marked surfaces $\mathbf{S}^{\mathcal{E}_A}_\bullet(A)$, which is independent of the first Brauer-Thrall Theorem as follows.

Since $\mathrm{CC}(\mathbf{S}^{\mathcal{E}_A}_\bullet(A)) \ne \varnothing$ and $\mathscr{J}$ is an infinite set, A is representation-infinite. Thus, we only need to prove the following assertion:

If $(\mathcal{Q},\mathcal{I})$ has no band, then A is representation-finite.

For this, consider the following two cases:

(1) A is gentle-tree, that is, $\mathcal{Q}$ contains no cycle.
(2) For any subquiver $\widehat{\mathcal{Q}}$ of $\mathcal{Q}$ induced by a cycle $\mathscr{C}$, there exists a path $\wp=\alpha\beta$ on $\mathcal{Q}$ which is a relation.

In the case (1), $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ is a disc and $\sharp\mathcal{M}=\sharp\mathcal{Q}_0+2$ (see [20, Corollary 1.23]). Thus $\mathrm{CC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))=\varnothing$ and

$$\sharp\mathsf{ind}(\mathsf{mod}A)=\sharp\mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))\le\sum_{r=0}^{\sharp\mathcal{Q}_0+2}\binom{\sharp\mathcal{Q}_0+2}{r}=2^{\sharp\mathcal{Q}_0+2}<\infty.$$

In the case (2), we have that $\widehat{A}=\Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}}$ such that $\widehat{\mathcal{I}}\ne 0$, and $\mathbf{S}_{\bullet}(\Bbbk\widehat{\mathcal{Q}})\sim\mathbf{S}_{\bullet}(\Lambda(\mathscr{C}))$ (the length of $\mathscr{C}$ is $n\ge 2$), where $\Lambda(\mathscr{C})$ is the algebra whose quiver is the subquiver of $(\mathcal{Q},\mathcal{I})$ induced by $\mathscr{C}$. Then every curve b without endpoint encircles some boundary component(s) of $\mathbf{S}_{\bullet}(\Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}})$. Moreover, there is no permissible curve $c\in\mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))$ such that c contains a segment consecutively crossing a^c_i, a^c_{i+1}, $\cdots$, $a^c_j=a^c_i$ $(j>i)$. Otherwise, without loss of generality, assume $a^c_\imath\ne a^c_\jmath$ (for any $i\le\imath\ne\jmath<j$), and consider the curve $\hat{c}$ without endpoints such that the following conditions hold:

- $a^{\hat{c}}_1=a^{\hat{c}}_{n(\hat{c})+1}=a^c_1$; and
- for any $1\le k\le n(\hat{c})$, $\hat{c}_{k,k+1}$ and $c_{i+k-1,i+k}$ are segments corresponding to the same arrow (notice that $i+n(\hat{c})=j$) (for example, see the cyan curve as shown in FIGURE 3.5).

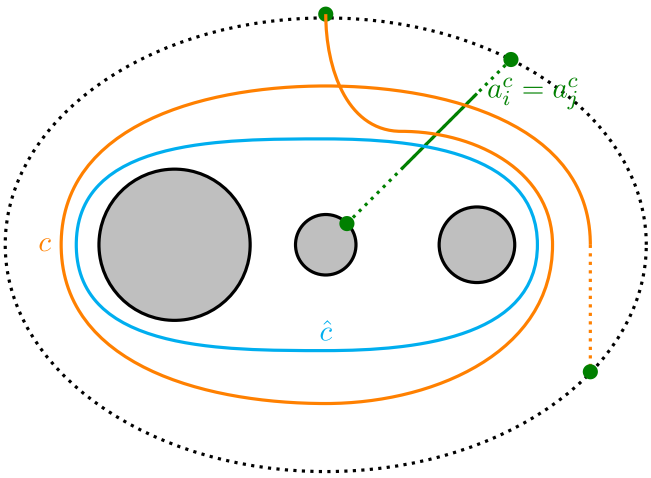


FIGURE 3.5. $\hat{c}$ is a permissible curve without endpoint induced by c.

Then $\hat{c}$ is a permissible curve in $\mathrm{CC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))$, and so $\mathsf{mod}A$ contains a band module corresponding to $\mathfrak{M}(\hat{c},\boldsymbol{J}_n(\lambda))$ $(\lambda\ne 0)$. By Construction 2.6, $\mathcal{Q}$ has a band which is of the form

$$\mathfrak{v}(a^c_i)\,\frac{\quad}{}\,\mathfrak{v}(a^c_{i+1})\cdots\cdots\frac{\quad}{}\,\mathfrak{v}(a^c_{j-1}),$$

which contradicts assumption.

Immediately, $\mathrm{CC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))=\varnothing$, and so $\sup\limits_{c\in\mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))}n(c)<\infty$. Then $\sharp\mathsf{ind}(\mathsf{mod}A)=\sharp\mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))<\infty$ because the number of marked points of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ is finite.

4. Gorenstein projective support τ-tilting modules

4.1. **Generalized dissections.** He, Zhou and Zhu [16] provided a description of support τ-tilting modules over gentle algebras by marked ribbon surfaces.

Definition 4.1. Let $\mathbf{S}_{\bullet}^{\mathcal{E}} = (\mathcal{S}, \mathcal{M}, \Delta_{\bullet})^{\mathcal{E}}$ be a marked ribbon surface, and let $D_{\bullet}$ be a set of some permissible curves.

- We say that $D_{\bullet}$ is a *partial generalized dissection* (=PGD) if the following conditions are satisfied.
  - every curve in $D_{\bullet}$ can not be homotopic to any boundary segment;
  - $(c_1 \cap c_2) \cap (\mathcal{S} \backslash \partial \mathcal{S}) = \varnothing$ for any curves $c_1, c_2 \in D_{\bullet}$; and
  - $D_{\bullet}$ contains no closed curve.
- We say that $D_{\bullet}$ is a *generalized dissection* (=GD) if $D_{\bullet}$ is a maximal PGD, that is, for any permissible curve $c' \notin D_{\bullet}$, we have $(c \cap c') \cap (\mathcal{S} \backslash \partial \mathcal{S}) \notin \varnothing$ for some $c \in D_{\bullet}$.

We use $\mathrm{PGD}(\mathbf{S}_{\bullet}^{\mathcal{E}})$ (respectively, $\mathrm{GD}(\mathbf{S}_{\bullet}^{\mathcal{E}})$) to denote the set of all partial dissections (respectively, maximal partial dissections) of the marked ribbon surface $\mathbf{S}_{\bullet}^{\mathcal{E}}$.

Theorem 4.2. [16, Theorem B] *There is a bijection*

$$\begin{array}{rccc} \mathfrak{M}: & \mathrm{GD}(\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)) & \to & \text{s-}\tau\text{tilt}(A) \\ & D_{\bullet} & \mapsto & \mathfrak{M}(D_{\bullet}) := \bigoplus\limits_{c \in D_{\bullet}} \mathfrak{M}([c]) \end{array}$$

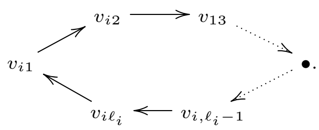
from the collection of all generalized dissections of $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$ *to that of all isomorphism classes of support* τ*-tilting modules.*

4.2. **Gorenstein projective τ-tilting modules.** A module $G \in \mathsf{mod}A$ is called *Gorenstein projective* (=*G-projective*) if there is an exact sequence of projective modules

$$\cdots \longrightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} P^{1} \xrightarrow{d^{1}} P^{2} \longrightarrow \cdots$$

in $\mathsf{mod}A$ which remains still exact after applying the functor $\mathrm{Hom}_A(-, A)$, such that $G \cong \mathrm{Im} d^{-1}$ [5, 13]. Obviously, every projective module is G-projective. We use $\mathsf{G\text{-}proj}A$ to denote the subcategory of $\mathsf{mod}A$ consisting of all G-projective modules.

The quiver $(\mathcal{Q}, \mathcal{I})$ of some gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ has at least one cycle $C_i = \alpha_{i1} \cdots \alpha_{i\ell_i}$ $(1 \le i \le r)$ with $s(\alpha_{i\overline{j}}) = v_{i\overline{j}} = t(\alpha_{i\overline{j-1}})$, where $\overline{j} = (j \bmod \ell_j) + 1$. For example,



We call $\mathscr{C}_i$ an *oriented cycle* of A. Furthermore, it is called *full relational* if any path of length two on it lies in $\mathcal{I}$. We use $\mathrm{froc}(A)$ to denote the set of all full relational oriented cycles of A.

Let $M \in \mathsf{mod}A$. We use $|M|$ to denote the number of non-isomorphic indecomposable direct summands of M. Recall from [1] that M is called τ*-tilting* if $\mathrm{Hom}_A(M, \tau M) = 0$ and $|M| = |A|$, where τ is the Auslander-Reiten translation functor. Moreover, M is called *support* τ*-tilting* if M is τ-tilting over $A/(e)$ with e an idempotent of A.

Definition 4.3. [19] An A-module T is called *G-projective* (*support*) τ*-tilting* if T is G-projective and (support) τ-tilting; in particular, we say it is *non-trivial* if $T \notin \mathsf{proj}A$.

We use $\mathrm{GPs}\tau\text{-tilt}(A)$ to denote the subcategory of $\mathsf{mod}A$ consisting of all G-projective support τ-tilting A-modules.

Definition 4.4. Let $\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A)$ be a marked ribbon surface. A *G-projective curve* c is a permissible curve in $\mathrm{PC}(\mathbf{S}_{\bullet}^{\mathcal{E}_A}(A))$ such that one of following conditions is satisfied.

(a) c is a *projective curve* corresponding to a_i^c, say $\wp(a_i^c)$, that is, there is a unique integer $1 \le i \le n(c)$ such that

(i) $a_1^c, a_2^c, \cdots, a_i^c$ have a common endpoint p which is right to c, and there is no arc $a \in \Delta_\bullet$ with endpoint p such that a is left to a_1^c at the point p;

(ii) $a_i^c, a_{i+1}^c, \cdots, a_{n(c)}^c$ have common endpoint q which is left to c, and there is no arc $\hat{a} \in \Delta_\bullet$ with endpoint q such that $\hat{a}$ is left to a_1^c at the point q.

(See Figure 4.1, the permissible curve c is projective, the point $\circ$ is an element in $\mathcal{M} \cup \mathcal{E}$.)

(b) c is the permissible curve such that: $a_1^c, a_2^c, \cdots, a_{n(c)}^c$ satisfies condition (a)(i); a_1^c is an edge of an ∞-elementary $\bullet$-polygon $\mathcal{P}$; and $c_{0,1}$ lies in the inner of $\mathcal{P}$.

(We provide an example in Figure 4.2, the permissible curve set $\mathrm{PC}(\mathbf{S}_\bullet^{\mathcal{E}})$ of the marked ribbon surface $\mathbf{S}_\bullet^{\mathcal{E}}$ contains three G-projective curves).

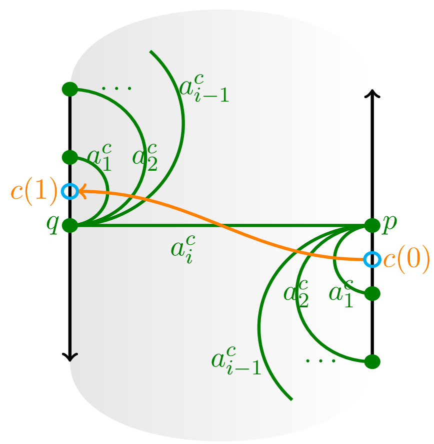


Figure 4.1. Projective.

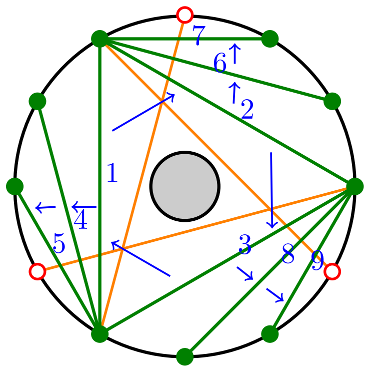


Figure 4.2. G-Projective.

Kalck proved in [17, Theorem 2.5] that any indecomposable module G is G-projective if and only if G is isomorphic to either eA (e is a primitive idempotent of A) or αA (α is an arbitrary arrow on any full relational oriented cycle). Thus, we immediately obtain the following proposition to describe all indecomposable G-projective modules over gentle algebras by marked ribbon surfaces by Construction 2.6.

Proposition 4.5. *For any* $c \in \mathrm{PC}(\mathbf{S}_\bullet^{\mathcal{E}_A}(A))$, *the module* $\mathfrak{M}([c])$ *is indecomposable projective (respectively, G-projective) if and only if* c *is a projective (respectively, G-projective) curve.*

Let $\mathrm{froc}(A) = \{C_i \mid 1 \le i \le r\}$. We use $\mathcal{P}_i$ to denote the ∞-elementary $\bullet$-polygon corresponding to C_i, and use $\mathrm{EP}_\bullet^\infty(\mathbf{S}_\bullet^{\mathcal{E}_A}(A))$ to denote the set of all ∞-elementary $\bullet$-polygons. As shown in Figure 4.3, in clockwise direction, we denote by $p_{i1}, \cdots, p_{i\ell_i}$ the vertices of $\mathcal{P}_i$ (ℓ_i is the length of C_i). The arc with endpoints $p_{i\bar{j}}$ and $p_{i\overline{j+1}}$ is denoted by $a_{i\bar{j}}$, where $1 \le j \le \ell_i$ and $\bar{j} = j \bmod \ell_i + 1$. Moreover, the non-projective G-projective curve with endpoint $p_{i\bar{j}}$ is denoted by $g_{i\bar{j}}$.

Notation 4.6. We say that a GD is a *Gorenstein-projective generalized dissection* (=GPGD) if all elements of GD are G-projective curves.

By Theorem 4.2, we have the following corollary.

Corollary 4.7. *Let* $D_\bullet$ *be a set of some permissible curves in* $\mathbf{S}_\bullet^{\mathcal{E}_A}(A)$. *Then* $\mathfrak{M}(D_\bullet) \in \mathrm{GPs}\tau\text{-tilt}(A)$ *if and only if* $D_\bullet$ *is a GPGD.*

4.3. **Endomorphism algebras of G-projective support τ-tilting modules.** Set $\mathfrak{S}(A) := \bigoplus_{M \in \mathsf{G\text{-}proj}(A)} M$. The *Cohen-Macaulay-Auslander* (= *CM-Auslander*) *algebra*

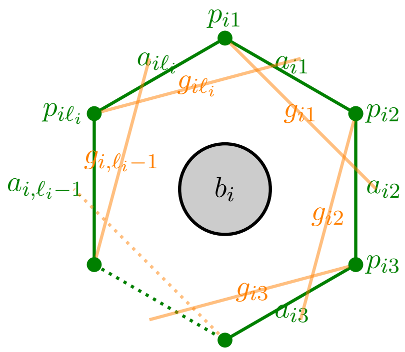


FIGURE 4.3. Non-projective G-projective curves and ∞-elementary $\bullet$-polygon.

A^{CMA} of A is defined by $A^{\text{CMA}} := \text{End}_A \mathfrak{G}(A)$. Now we recall the descriptions of CM-Auslander algebras of gentle algebras introduced by Chen and Lu [10]. We use the following notations:

- $\mathcal{Q}_1^{\text{froc}}$: the set of all arrows on full relational oriented cycles of A;
- $\mathcal{Q}^{\text{CMA}}$: the quiver $(\mathcal{Q}_0^{\text{CMA}}, \mathcal{Q}_1^{\text{CMA}}, s, t)$, where
  - $\mathcal{Q}_0^{\text{CMA}} = \mathcal{Q}_0 \cup \widetilde{\mathcal{Q}}_0$ where $\widetilde{\mathcal{Q}}_0$ is the set with bijection $\omega : \mathcal{Q}_1^{\text{froc}} \to \widetilde{\mathcal{Q}}_0$;
  - $\mathcal{Q}_1^{\text{CMA}} = (\mathcal{Q}_1 \backslash \mathcal{Q}_1^{\text{froc}}) \cup \widetilde{\mathcal{Q}}_1$ where $\widetilde{\mathcal{Q}}_1 := \mathcal{Q}_1^{\text{froc},-} \cup \mathcal{Q}_1^{\text{froc},+}$ is the disjoint union of $\mathcal{Q}_1^{\text{froc},-} := \{\alpha^- \mid \alpha \in \mathcal{Q}_1^{\text{froc}}\}$ and $\mathcal{Q}_1^{\text{froc},+} := \{\alpha^+ \mid \alpha \in \mathcal{Q}_1^{\text{froc}}\}$;
  - $s(\alpha^-) = s(\alpha)$, $t(\alpha^+) = t(\alpha)$ and $t(\alpha^-) = \omega(\alpha) = s(\alpha^+)$ hold for any $\alpha \in \mathcal{Q}_1^{\text{froc}}$.
- $\mathcal{I}^{\text{CMA}}$: the ideal of $\Bbbk\mathcal{Q}^{\text{CMA}}$ generated by the following paths of length two:
  - $\alpha\beta$ with $\alpha, \beta \in \mathcal{Q}_1 \backslash \mathcal{Q}_1^{\text{froc}}$;
  - $\alpha^+\beta^-$ with $t(\alpha) = s(\beta)$.

The following result was proved by Chen and Lu [10, Theorem 3.5]. It should be pointed out that it can be easily induced by the descriptions of G-projective modules (Proposition 4.5) and irreducible morphisms (see [7, Proposition 3.21]).

Theorem 4.8. [10, Theorem 3.5] *For the algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$, we have*

$$A^{\text{CMA}} = \Bbbk\mathcal{Q}^{\text{CMA}}/\mathcal{I}^{\text{CMA}}.$$

Now we give a description of the CM-Auslander algebra of A by the marked ribbon surfaces $\mathbf{S}_\bullet^{\mathcal{E}_A}(A) = (\mathcal{S}(A), \mathcal{M}(A), \Delta_\bullet(A))^{\mathcal{E}_A}$. Assume that $\mathbf{S}_\bullet^{\mathcal{E}_A}(A)$ has r ∞-elementary polygons $\mathcal{P}_1, \cdots, \mathcal{P}_r$. For any $\mathcal{P}_i$, let $\alpha_{i\overline{j}}$ be the arrow corresponding to the angle of $\mathcal{P}_i$ whose edges are $p_{\overline{ij-1}}$ and $p_{i\overline{j}}$ (see FIGURE 4.4 I). Let

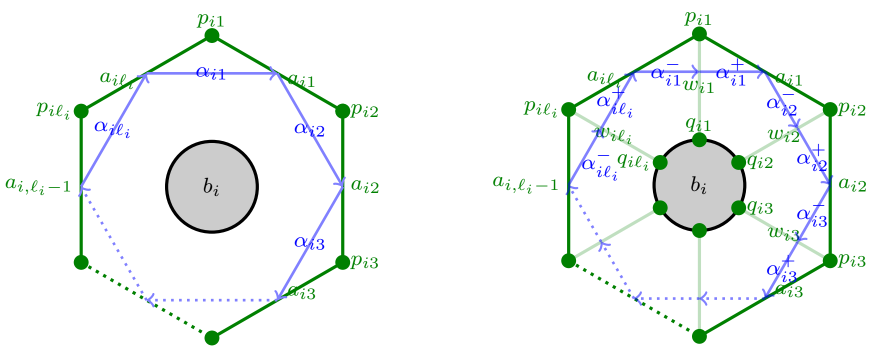
- $\mathcal{S}^{\text{CMA}}(A) = \mathcal{S}(A)$;
- $\mathcal{M}^{\text{CMA}}(A) = \mathcal{M}(A) \bigcup \bigcup_{i=1}^{t} \widetilde{\mathcal{M}}_i$, where $\widetilde{\mathcal{M}}_1, \cdots, \widetilde{\mathcal{M}}_t$ are sets of new marked points such that
  - there is a bijection between $\bigcup_{i=1}^{t} \widetilde{\mathcal{M}}_i$ and $\widetilde{\mathcal{Q}}_0$; and
  - all marked points in $\widetilde{\mathcal{M}}_i = \{q_{ij} \mid 1 \le j \le \ell_i\}$ clockwise arrange on b_i, where ℓ_i is the number of edges of $\mathcal{P}_i$ which belong to $\Delta_\bullet(A)$;
- $\Delta_\bullet^{\text{CMA}}(A) = \Delta_\bullet(A) \cup \widetilde{\Delta_\bullet}$, where $\widetilde{\Delta_\bullet}$ is the set of all new curves with endpoints p_{ij} and q_{ij};
- $\mathcal{E}_A^{\text{CMA}}$ is induced by $\Delta_\bullet^{\text{CMA}}(A)$.

Theorem 4.9. *There is a homotopy equivalence*

$$\mathbf{S}_\bullet^{\mathcal{E}_{A^{\text{CMA}}}}(A^{\text{CMA}}) \sim \left(\mathcal{S}^{\text{CMA}}(A), \mathcal{M}^{\text{CMA}}(A), \Delta_\bullet^{\text{CMA}}(A)\right)^{\mathcal{E}_A^{\text{CMA}}}.$$

Proof. Assume $\text{froc}(A) = \{C_i \mid 1 \le i \le r\}$. Let $(\mathcal{S}^{\text{CMA}}(A), \mathcal{M}^{\text{CMA}}(A), \Delta_\bullet^{\text{CMA}}(A))^{\mathcal{E}_A^{\text{CMA}}} = \mathbf{S}_\bullet^{\mathcal{E}}$ and $A(\mathbf{S}_\bullet^{\mathcal{E}}) = \Bbbk\mathcal{Q}'/\mathcal{I}'$. It suffices to prove $A(\mathbf{S}_\bullet^{\mathcal{E}}) = A^{\text{CMA}}$.

Note that there is a bijection between $\text{froc}(A)$ and $\text{EP}_\bullet^{\infty}(\mathbf{S}_\bullet^{\mathcal{E}})$. As shown in the FIGURE 4.4 I, any angle (whose vertex is p_{ij} where $1 \le j \le \ell_i$) of the ∞-elementary $\bullet$-polygon $\mathcal{P}_i$



I: the ∞-elementary $\bullet$-polygon $\mathcal{P}_i$ II: $\mathcal{M}^{\mathrm{CMA}}(A)$ and $\Delta_\bullet^{\mathrm{CMA}}(A)$

FIGURE 4.4. The marked ribbon surface of the CM-Auslander algebra of a gentle algebra: the change of ∞-elementary $\bullet$-polygon $\mathcal{P}_i$.

corresponds to an arrow $\alpha_{ij} \in \mathcal{Q}_1$, it is changed to two arrows α_{ij}^- and α_{ij}^+ in $\mathcal{Q}_1^{\mathrm{CMA}}$ by adding the arc w_{ij} (whose endpoints are p_{ij} and q_{ij}) in $\bullet$-FFAS $\Delta_\bullet(A)$. Then it is easy to see that $\mathcal{Q}'_0 = \mathcal{Q}_0^{\mathrm{CMA}}$ and $\mathcal{Q}'_1 = \mathcal{Q}_1^{\mathrm{CMA}}$ by Construction 2.6 and Theorem 4.8.

On the other hand, we have $\alpha_{ij}^+\alpha_{i\ j+1}^- \in \mathcal{I}'$ and $\alpha_{ij}^-\alpha_{ij}^+ \notin \mathcal{I}'$ by Construction 2.6 and Theorem 4.8, because $s(\alpha_{ij}^-)$, $t(\alpha_{ij}^-) = \omega(\alpha_{ij}) = w_{ij} = s(\alpha_{ij}^+)$ and $t(\alpha_{ij}^+)$ are sides of same elementary $\bullet$-polygon in $\mathrm{EP}_\bullet(\mathbf{S}_\bullet^{\mathcal{E}})$ (see FIGURE 4.4 II). Thus $\mathcal{I}^{\mathrm{CMA}} = \mathcal{I}'$. □

The proof of Theorem 4.9 induces immediately the following corollary.

Corollary 4.10. [10, Theorem 4.4] *A gentle algebra A is representation-finite if and only if A^{CMA} is representation-finite.*

Proof. Indeed, there is a bijection between $\mathrm{CC}(\mathbf{S}_\bullet^{\mathcal{E}_A}(A))$ and $\mathrm{CC}(\mathbf{S}_\bullet^{\mathcal{E}_{A^{\mathrm{CMA}}}}(A^{\mathrm{CMA}}))$. □

Remark 4.11. The vertices of the quiver $\mathcal{Q}^{\mathrm{CMA}}$ correspond to the indecomposable G-projective A-modules as follows:

$$\mathcal{Q}_0 \overset{1-1}{\to} \mathsf{proj}A,\ v \mapsto P(v) \text{ and } \widetilde{\mathcal{Q}}_0 \overset{1-1}{\to} \mathsf{proj}A\backslash\mathsf{G}\text{-}\mathsf{proj}A,\ \omega(\alpha_{ij}) \mapsto \mathfrak{M}([g_{ij}]),$$

where α_{ij} is the arrow on the cycle $C_i \in \mathrm{froc}(A)$ such that $s(\alpha_{ij}) = v_{ij}$ and g_{ij} the non-projective G-projective curve with endpoint p_{ij} ($1 \le j \le \ell_i$). Thus, $\omega(\alpha_{ij})$ corresponds to the arc w_{ij} in the $\bullet$-FFAS $\Delta_\bullet(A^{\mathrm{CMA}})$ of $\mathbf{S}_\bullet^{\mathcal{E}_{A^{\mathrm{CMA}}}}(A^{\mathrm{CMA}})$.

The following result provide a description of the endomorphism algebra of a G-projective support τ-tilting module by marked ribbon surfaces.

Proposition 4.12. *Let $T \in \mathrm{GPs}\tau\text{-tilt}(A)$ and $B = \mathrm{End}_A T$. Then $\mathbf{S}_\bullet^{\mathcal{E}_B}(B)$ is homotopic to a subsurface of $\mathbf{S}_\bullet^{\mathcal{E}_{A^{\mathrm{CMA}}}}(A^{\mathrm{CMA}})$ which is induced by removing some arc in $\Delta_\bullet(A^{\mathrm{CMA}})$.*

Proof. Let $\mathrm{End}_A T = \Bbbk\mathcal{Q}'/\mathcal{I}'$. Since T is a direct summand of $\mathfrak{G}(A)$, we have $\mathcal{Q}'_0 \subseteq \mathcal{Q}_0^{\mathrm{CMA}}$. Then $\Delta_\bullet(B) \subseteq \Delta_\bullet(A^{\mathrm{CMA}})$ by the bijection between the $\bullet$-FFAS of the marked ribbon surfaces and the vertices of the quiver of A. Now, removing all arcs in $\Delta_\bullet(A^{\mathrm{CMA}})\backslash\Delta_\bullet(B)$ and then removing all marked points in $\mathcal{M}(A^{\mathrm{CMA}})$ which are not endpoints of arcs in $\Delta_\bullet(B)$, we obtain a subsurface of $\mathbf{S}_\bullet^{\mathcal{E}_{A^{\mathrm{CMA}}}}(A^{\mathrm{CMA}})$, which is homotopic to $\mathbf{S}_\bullet^{\mathcal{E}_B}(B)$. □

5. Main results

5.1. **The existence of non-trivial G-projective τ-tilting modules.** Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ a (*trivial*) *Gorenstein torsion pair* if $\mathcal{T}$ is a (*trivial*) *G-projective torsion class*, that is, the basic Ext-projective generator in $\mathcal{T}$ is (trivial) G-projective. In [24], Xie and Zhang used G-projective support τ-tilting modules to describe all Gorenstein torsion pairs over finite dimensional algebras. On the other hand, Li and Zhang provided a construction of non-trivial G-projective τ-tilting modules over tensor algebras and lower triangular matrix algebras [19, Corollary 3.11 and Proposition 3.14].

In this subsection, we provide another construction of non-trivial G-projective τ-tilting modules over gentle algebras.

Lemma 5.1. *Let A be a gentle algebra with at least one full relational oriented cycle C_i and $T \in \mathrm{GPs}\tau\text{-tilt}(A)$. Then at most one of the indecomposable G-projective module $G(v_{i\bar{j}}) := \alpha_{i\bar{j}}A$ and the indecomposable projective module $P(v_{i\,\overline{j-1}})$ is a direct summand of T.*

Proof. First of all, we have $\mathrm{EP}^{\infty}_{\bullet}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)) \neq \varnothing$ by assumption. Let $\mathcal{P}_i \in \mathrm{EP}^{\infty}_{\bullet}(\mathbf{S}^{\mathcal{E}}_{\bullet})$ be the elementary $\bullet$-polygon corresponding to C_i. Then

$$G(v_{i\bar{j}}) \cong \mathfrak{M}([g_{i\bar{j}}]) \text{ and } P(v_{i\overline{j-1}}) \cong \mathfrak{M}([\wp(a_{i\overline{j-1}})]).$$

It is easy to see that $g_{i\bar{j}} \cap \wp(a_{i\overline{j-1}}) \cap (\mathcal{S}\backslash\partial\mathcal{S}) \neq \varnothing$ for any $1 \le j \le \ell_i$. Thus, if both $G(v_{i\bar{j}})$ and $P(v_{i\overline{j-1}})$ are direct summands of T, then

$$\mathfrak{M}^{-1}(T) := \{c \in \mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)) \mid \mathfrak{M}([c]) \text{ is a direct summand of } T\}$$

is not a GPGD, which contradicts Corollary 4.7. □

Now we are in a position to state the first main result.

Theorem 5.2. *Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be a gentle algebra. Then A has a non-trivial G-projective τ-tilting module if and only if there is a cycle $C_i \in \mathrm{froc}(A)$ such that C_i has a vertex $v_{i\bar{j}}$ ($\bar{j} = (j \bmod \ell_i) + 1$) which is not a target of any arrow except $\alpha_{i\overline{j-1}}$.*

Proof. If every vertex v_{ij} of any full relational oriented cycle C_i is a target of some arrow α with $\alpha \neq \alpha_{i\overline{j-1}}$, then there exists an arc a in the $\bullet$-FFAS $\Delta_{\bullet}(A)$ of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ such that $s(\alpha) = \mathfrak{v}(a)$ (in this case, we have $t(\alpha) = s(\alpha_{i\overline{j-1}})$ ($= \mathfrak{v}(a_{i\bar{j}}) = v_{i\bar{j}}$)). Thus,

$$g_{i\bar{j}} \cap \wp(a) \cap (\mathcal{S}\backslash\partial\mathcal{S}) \neq \varnothing \text{ and } g_{i\bar{j}} \cap \wp(a_{i\overline{j-1}}) \cap (\mathcal{S}\backslash\partial\mathcal{S}) \neq \varnothing \text{ (see Figure 5.1)} \tag{5.1}$$

by Theorem 4.2. Assume that $T \in \mathrm{GPs}\tau\text{-tilt}(A)$ is non-trivial G-projective τ-tilting, $d_1(T)$ is the number of projective direct summands of T and $d_2(T)$ is the number of non-projective G-projective direct summands of T. Then $|T| = d_1(T) + d_2(T)$ and $d_2(T) > 0$.

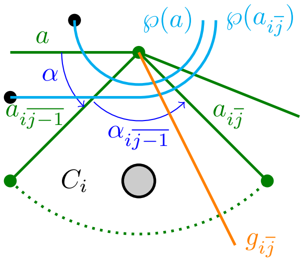


Figure 5.1. In this figure, the point "$\bullet$" is an element in $\mathcal{E}_A$.

Thus, if $G(v_{i\bar{j}}) \cong \mathfrak{M}([g_{i\bar{j}}]) \le_{\oplus} T$, then, by (5.1), we have that $P(\mathfrak{v}(a)) \cong \mathfrak{M}(\wp(a))$ and $P(\mathfrak{v}(a_{i\bar{j}})) \cong \mathfrak{M}(\wp(a_{i\overline{j-1}}))$ and neither of them is a direct summand of T; that is, if $d_2(T) = 1$, then $d_1(T) \le \sharp\mathcal{Q}_0 - 2$. Furthermore, by using induction, we have $d_1(T) \le \sharp\mathcal{Q}_0 - 2d_2(T)$. It follows that

$$\sharp\mathcal{Q}_0 = |T| \le d_2(T) + \sharp\mathcal{Q}_0 - 2d_2(T) = \sharp\mathcal{Q}_0 - d_2(T) < \sharp\mathcal{Q}_0,$$

which is a contradiction.

If there is a vertex $v_{i\bar{j}}$ of $C_i \in \mathrm{froc}(A)$ such that $v_{i\bar{j}}$ is not a target of any arrow except $\alpha_{i\overline{j-1}}$, then there is a unique $a_{i\overline{j-1}}$ such that $\wp(a_{i\overline{j-1}}) \cap g_{i\bar{j}} \cap (\mathcal{S}\backslash\partial\mathcal{S}) \neq \varnothing$. Thus,

$$\mathfrak{M}(g_{i\overline{j-1}}) \bigoplus \bigoplus_{v_{i\overline{j-1}} \neq v \in \mathcal{Q}_0} e_v A \in \mathrm{GPs}\tau\text{-tilt}(A)$$

is non-trivial. □

Corollary 5.3. *Assume that* $\mathrm{froc}(A)=\{C_i \mid 1\le i\le t\}$. *Then for any* $T\in \mathrm{GPs}\tau\text{-tilt}(A)$, *we have*

$$T=\bigoplus_{i\in I} G_i \bigoplus \bigoplus_{j\in J} P_j \ \textit{and}\ \sharp I \le \sum_{1\le i\le t} \lfloor \ell_i/2 \rfloor,$$

where $G_i\in \mathsf{ind}(\mathsf{G}\text{-}\mathsf{proj}A)\backslash \mathsf{proj}A$ *and* $P_i\in \mathsf{proj}A$.

Proof. It is easy to see that

$$g_{i\bar{j}}\cap a_{i\bar{j}}=p_{i\bar{j}},\quad g_{i\bar{j}}\cap a_{i\overline{j+1}}\cap(\mathcal{S}\backslash\partial\mathcal{S})\neq\varnothing \ \text{ and }\ g_{i\bar{j}}\cap g_{i\overline{j+2}}\cap(\mathcal{S}\backslash\partial\mathcal{S})=\varnothing.$$

Thus any GPGD contains at most $\lfloor \ell_i/2\rfloor$ G-projective curves which belong to $\{g_{i\bar{j}} \mid 1\le \bar{j}\le \ell_i\}$. Therefore, $\sharp I\le \sum_{i=1}^{t}\lfloor \ell_i/2\rfloor$ by Corollary 4.7. □

5.2. The representation-types of gentle algebras and endomorphism algebras of G-projective support τ-tilting modules.

Lemma 5.4. *Let* $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet} \precsim \mathbf{S}^{\mathcal{E}}_{\bullet}$. *If* $A(\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet})$ *is representation-infinite, then so is* $A(\mathbf{S}^{\mathcal{E}}_{\bullet})$.

Proof. Since $A(\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet})$ is representation-infinite, the quiver of $A(\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet})$ has a band $\mathscr{C}$ (of length n) by Proposition 3.6. Then, up to homotopy, we have $\mathbf{S}^{\mathcal{E}_{\Lambda(\mathscr{C})}}_{\bullet}(\Lambda(\mathscr{C}))\precsim \widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}\precsim \mathbf{S}^{\mathcal{E}}_{\bullet}$, Thus

$$\varnothing\neq \mathrm{CC}(\mathbf{S}^{\mathcal{E}_{\Lambda(\mathscr{C})}}_{\bullet}(\Lambda(\mathscr{C})))\subseteq \mathrm{CC}(\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet})\subseteq \mathrm{CC}(\mathbf{S}^{\mathcal{E}}_{\bullet}).$$

Then $A(\mathbf{S}^{\mathcal{E}}_{\bullet})$ is representation-infinite by Theorem 2.10. □

The following theorem is the second main result.

Theorem 5.5. *Let* $A=\Bbbk\mathcal{Q}/\mathcal{I}$ *be a gentle algebra. Then* A *is representation-finite if and only if* $B=\mathrm{End}_A T$ *is representation-finite for any* $T\in\mathrm{GPs}\tau\text{-tilt}(A)$.

Proof. If B is representation-finite for any G-projective support τ-tilting module T, then $A\cong \mathrm{End}_A A$ is representation-finite because A is a trivial G-projective support τ-tilting.

In the following, we will prove the necessity. If $\mathrm{gl.dim}A<\infty$, then all G-projective modules are projective. Thus any $T\in\mathrm{GPs}\tau\text{-tilt}(A)$ is a direct summand of A. We have $B:=\mathrm{End}_A T\le_{\oplus}\mathrm{End}_A\widehat{T}\cong A$. Therefore, B is representation-finite.

If $\mathrm{gl.dim}A=\infty$, assume that there is a module $T\in\mathrm{GPs}\tau\text{-tilt}(A)$ such that $B=\mathrm{End}_A T=\Bbbk\mathcal{Q}^B/\mathcal{I}^B$ is representation-finite, then, up to homotopy, we have

$$\mathbf{S}^{\mathcal{E}_{\Lambda(\mathscr{C})}}_{\bullet}(\Lambda(\mathscr{C}))\precsim \mathbf{S}^{\mathcal{E}_B}_{\bullet}(B)\overset{\text{Theorem 4.9}}{\precsim}\mathbf{S}^{\mathcal{E}_{A^{\mathrm{CMA}}}}_{\bullet}(A^{\mathrm{CMA}}) \tag{5.2}$$

by Proposition 4.12, where $\mathscr{C}$ is the subquiver induced by some band of $\mathcal{Q}^B$ (the existence of $\mathscr{C}$ is given by Proposition 3.6). Thus, A^{CMA} is representation-infinite by (5.2) and Lemma 5.4. It follows from Corollary 4.10 that A is representation-infinite. □

6. Examples

In this section, we provide some examples for G-projective support τ-tilting modules. First of all, we give the following example shows that the G-projective condition in Theorem 5.5 is necessary.

Example 6.1. Let $A=\Bbbk\mathcal{Q}/\mathcal{I}$ be a gentle algebra where $\mathcal{Q}$ is as shown in the first figure in Figure 6.1 and $\mathcal{I}=\langle\alpha\beta,\gamma\delta,\zeta\eta,\eta\zeta\rangle$.

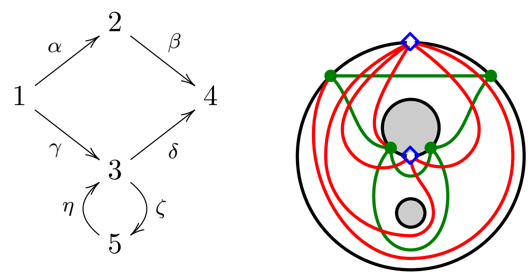


FIGURE 6.1. The gentle algebra in Example 6.1 and its marked ribbon surface.

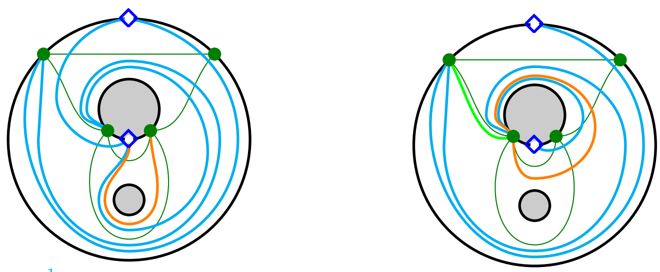
We get a GD of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ as shown in the second figure in FIGURE 6.1, which corresponds to the support τ-tilting module

$$T = 2 \oplus \substack{2\\4} \oplus {}_{2}{}^{1}{}_{\substack{3\\5}} \oplus {}_{2}{}_{4}{}^{\substack{5\\3}} \oplus {}_{2}{}^{1}{}_{3}{}^{5},$$

which is not G-projective. Then, by the Auslander-Reiten quiver of A, we have $B := \mathrm{End}_A T = \Bbbk\mathcal{Q}^B/\mathcal{I}^B$, where

$$\mathcal{Q}^B = \begin{array}{ccccc} 2 & \xrightarrow{\alpha_{23}} & 3 & & \\ {\scriptstyle\alpha_{12}}\big\uparrow\big\downarrow{\scriptstyle\alpha_{21}} & & \big\downarrow{\scriptstyle\alpha_{34}} & & \\ 1 & \xrightarrow[\alpha_{14}]{} & 4 & \xrightarrow[\alpha_{45}]{} & 5 \end{array} \qquad \text{and} \qquad \mathcal{I}^B = \langle \alpha_{12}\alpha_{21}, \alpha_{21}\alpha_{12}, \alpha_{14}\alpha_{45} \rangle.$$

By Proposition 3.6, we have that B is a representation-infinite, but A is representation-finite.



$(T_1 = {}_{\substack{3\\5}}{}^{1}{}_{2} \oplus \substack{2\\4} \oplus {}_{5}{}^{3}{}_{4} \oplus 4 \oplus 5)$ $\qquad (T_2 = 0 \oplus \substack{2\\4} \oplus {}_{5}{}^{3}{}_{4} \oplus 4 \oplus \substack{3\\4})$

FIGURE 6.2. The GPGDs corresponding to T_1 and T_2, respectively.

Moreover, we can check that any non-trivial G-projective support τ-tilting A-module is a direct summand of T_1 or T_2, where T_1 and T_2 correspond to GPGD as shown in FIGURE 6.2. We have $\mathrm{End}_A T_1 = \Bbbk\mathcal{Q}^1/\mathcal{I}^1$ and $\mathrm{End}_A T_2 = \Bbbk\mathcal{Q}^2/\mathcal{I}^2$, where

$$\mathcal{Q}^1 = \begin{array}{ccccc} & \overset{\alpha^1_{12}}{\nearrow} & 2 & \overset{\alpha^1_{24}}{\searrow} & \\ 1 & & & & 4 \xrightarrow{\alpha^1_{45}} 5, \\ & \underset{\alpha^1_{13}}{\searrow} & 3 & \underset{\alpha^1_{34}}{\nearrow} & \end{array} \qquad \mathcal{I}^1 = \langle \alpha^1_{12}\alpha^1_{24}, \alpha^1_{13}\alpha^1_{24} \rangle;$$

and

$$\mathcal{Q}^2 = \quad 1 \xrightarrow{\alpha^2_{12}} 2 \xleftarrow{\alpha^2_{32}} 3 \xleftarrow{\alpha^2_{43}} 4, \qquad \mathcal{I}^2 = 0.$$

It is easy to see that $\mathrm{End}_A T_1$ and $\mathrm{End}_A T_2$ are representation-finite. It follows that $\mathrm{End}_A T$ is representation-finite for any non-trivial G-projective support τ-tilting module T.

In the following, we provide a general method to construct a non-trivial G-projective support τ-tilting module T over a representation-infinite gentle algebra A with infinite global dimension such that $\mathrm{End}_A T$ is representation-infinite. We need the following lemma to show that this construction is always feasible.

Lemma 6.2. *Assume that $A = \Bbbk\mathcal{Q}/\mathcal{I}$ is a gentle algebra with at least one full relational oriented cycle. Let $\widehat{\mathcal{Q}}$ be a subquiver of $\mathcal{Q}$ and $\widehat{\mathcal{I}} = \langle \alpha\beta \in \mathcal{I} \mid \alpha \in \widehat{\mathcal{Q}}_1, \beta \in \widehat{\mathcal{Q}}_1 \rangle$ such that $\widehat{A} := \Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}}$ is a gentle algebra with a unique full relational oriented cycle. Then for any $\widehat{A}$-module $\widehat{T} \in \mathrm{GPs}\tau\text{-tilt}(\widehat{A})$, there is an A-module $T \in \mathrm{GPs}\tau\text{-tilt}(A)$ such that if $\mathrm{End}_{\widehat{A}}\widehat{T}$ is representation-infinite, then so is $\mathrm{End}_A T$.*

Proof. Let $g = g_{ij} \in \mathrm{PC}(\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet})$ be a non-projective G-projective curve consecutively crossing arcs $\widehat{a}^g_1, \cdots, \widehat{a}^g_{n(g)}$ in $\Delta_{\bullet}(\widehat{A})$ (see Figure 6.3 (1)), where

$$\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet} := \mathbf{S}^{\mathcal{E}_{\widehat{A}}}_{\bullet}(\widehat{A}) = (\mathcal{S}(\widehat{A}), \mathcal{M}(\widehat{A}), \Delta_{\bullet}(\widehat{A}))^{\mathcal{E}_{\widehat{A}}}$$

is the marked ribbon surface of $\widehat{A}$. Since $\widehat{\mathcal{Q}}$ is a subquiver of $\mathcal{Q}$, we have, up to homotopy,

$$\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet} \precsim \mathbf{S}^{\mathcal{E}_A}_{\bullet}(A) \tag{6.1}$$

by Lemma 3.3. Furthermore, consider the set

$$\begin{aligned} X &= \{a \in \Delta_{\bullet}(A) \mid a \text{ is left to the } a_{i,j-1} \text{ at the marked point } p_{ij}\} \\ &\xlongequal{\text{denoted by}} \{a_k \in \Delta_{\bullet}(A) \mid 1 \le k \le N\}. \end{aligned}$$

We have $\{\widehat{a}^g_j \mid 1 \le j \le n(g)\} \subseteq X$. Let $g' \in \mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))$ be the curve consecutively crossing $a_1, \cdots, a_N$. Then g' is a non-projective G-projective curve by Proposition 4.5 (see Figure 6.3 (2)).

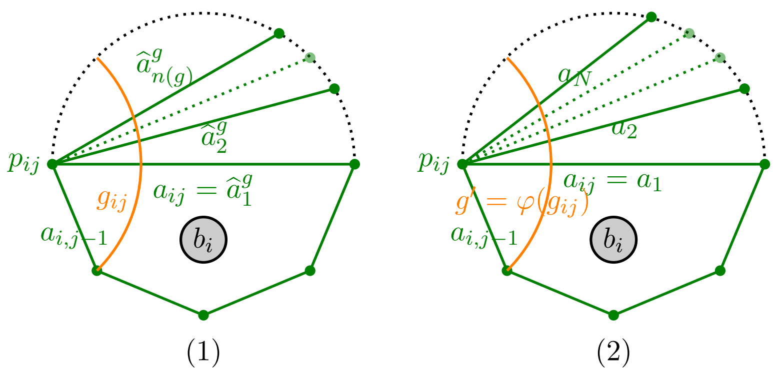


Figure 6.3.

Similarly, for any projective curve $\wp$ in $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}$, we can get a projective curve $\wp'$ in $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$. It induces an injection φ between the set of all G-projective curves in $\mathrm{PC}(\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet})$ and that of all G-projective curves in $\mathrm{PC}(\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A))$ by $g \mapsto g'$.

Suppose $\widehat{T} = \bigoplus_{j=1}^{r} \mathfrak{M}([c_j])$. Then there is a GPGD D of $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}$ such that $c_j \in D$ is G-projective for any $1 \le j \le r$, and $\bigoplus_{j=1}^{r} \mathfrak{M}([\varphi(c_j)])$ is a support τ-tilting A-module by the definition of support τ-tilting modules. Thus, for any $c \neq c'$, $\varphi(D) = \{\varphi(c) \mid c \in D\}$ satisfies $\varphi(c) \cap \varphi(c') \cap (\mathcal{S} \backslash \partial\mathcal{S}) = \varnothing$, that is, $\varphi(D)$ is a PGD of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$. Then, by [7, Theorem 3.15], we have

$$\mathrm{End}_{\widehat{A}}\widehat{T} \cong \mathrm{End}_A\Big(\bigoplus_{c \in D} \mathfrak{M}([\varphi(c)])\Big) \ (= \mathrm{End}_A \mathfrak{M}([\varphi(D)])).$$

(Indeed, all irreducible morphisms can be corresponded by the pivot elementary moves of permissible curves. Thus, all arrows of the quivers of $\mathrm{End}_{\widehat{A}}\widehat{T}$ and $\mathrm{End}_A \mathfrak{M}([\varphi(D)])$ can be given by the angles of elementary $\bullet$-polygons obtained by GPGD cutting $\widehat{\mathbf{S}}^{\widehat{\mathcal{E}}}_{\bullet}$ and $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$, respectively).

For any arc $a \in \Delta_{\bullet}(A)\backslash\Delta_{\bullet}(\widehat{A})$, define $\mathfrak{c}(a)$ to be:

- a, if $a \cap \eta \cap (\mathcal{S}\backslash\partial\mathcal{S}) = \varnothing$ for all $\eta \in \varphi(D)$;
- $\wp(a)$, otherwise.

Then $D' = \varphi(D) \cup \{\mathfrak{c}(a) \mid a \in \Delta_{\bullet}(A)\backslash\Delta_{\bullet}(\widehat{A})\}$ is a GPGD of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$, and so $T := \mathfrak{M}(D') \in \mathrm{GPs}\tau\text{-tilt}(A)$.

Since $\mathfrak{M}([\varphi(D)])$ is a direct summand of T, we have that $\mathrm{End}_A\mathfrak{M}([\varphi(D)])$ is a direct summand of $\mathrm{End}_A T$. Furthermore, if $\mathrm{End}_{\widehat{A}}\widehat{T}$ is representation-infinite, then so is $\mathrm{End}_A T$. □

All gentle algebras with infinite global dimension to the following three classes:

(I) The quiver of A contains the subquiver $\mathcal{Q}^{\mathbf{I}}$ which is an oriented cycle C_i connected to a band $\mathscr{C}$ such that C_i and $\mathscr{C}$ have no common arrow (cf. Figure 6.4). In this case, write $e^{\mathbf{I}} := \sum_{v\in\mathcal{Q}^{\mathbf{I}}_0} \varepsilon_v$, where ε_v is the path of length zero corresponding to the vertex $v \in \mathcal{Q}^{\mathbf{I}}_0$ (naturally, $\mathcal{I}^{\mathbf{I}} = \langle \alpha\beta \in \mathcal{I} \mid \alpha, \beta \in \mathcal{Q}^{\mathbf{I}}_1\rangle$).

(II) The quiver of A contains the subquiver $\mathcal{Q}^{\mathbf{II}}$ which is an oriented cycle C_i with a band $\mathscr{C}$ such that C_i and $\mathscr{C}$ have at least one common arrow (cf. Figure 6.5). In this case, write $e^{\mathbf{II}} := \sum_{v\in\mathcal{Q}^{\mathbf{II}}_0} \varepsilon_v$ (naturally, $\mathcal{I}^{\mathbf{II}} = \langle \alpha\beta \in \mathcal{I} \mid \alpha, \beta \in \mathcal{Q}^{\mathbf{II}}_1\rangle$).

(III) The quiver of A has no band. In this case, A is representation-finite, and its quiver has at least one full relational oriented cycle.

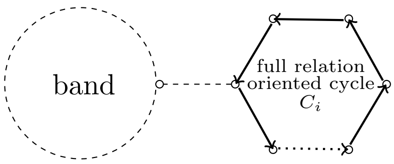


Figure 6.4. (**I**)

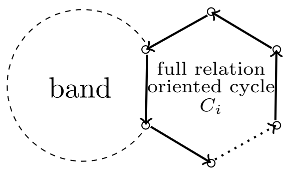


Figure 6.5. (**II**)

Set $A_{\mathbf{I}} := \Bbbk\mathcal{Q}^{\mathbf{I}}/\mathcal{I}^{\mathbf{I}}$ and $A_{\mathbf{II}} := \Bbbk\mathcal{Q}^{\mathbf{II}}/\mathcal{I}^{\mathbf{II}}$. Assume that A is a representation-infinite. Then the quiver $\mathcal{Q}$ has at least one band $\mathscr{C}_n$ and has at least one full relational oriented cycle C_i.

Now we provide a method to construct a non-trivial G-projective support τ-tilting A-module T such that $B = \mathrm{End}_A T$ is representation-infinite. Note that there are the following two cases:

Case **I**. $\mathcal{Q}$ has a subquiver which is of the form $\mathcal{Q}^{\mathbf{I}}$;
Case **II**. $\mathcal{Q}$ has a subquiver which is of the form $\mathcal{Q}^{\mathbf{II}}$.

We only give the construction in Case **I**, the construction in Case **II** is similar.

Construction 6.3.

Step 1. Select the subsurface $\mathbf{S}^{\mathcal{E}_{\mathbf{I}}(A_{\mathbf{I}})}_{\bullet}$ of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ (cf. the left picture in Figure 6.6). The existence of $\mathbf{S}^{\mathcal{E}_{\mathbf{I}}(A_{\mathbf{I}})}_{\bullet}$ is given by Lemma 3.3, and the number of projective curves crossed by g_{i1} is at least one.

Step 2. Set

$$X := \{g_{ij} \in \mathrm{PC}(\mathbf{S}^{\mathcal{E}_{\mathbf{I}}}_{\bullet}(A_{\mathbf{I}})) \mid 2 \le j \le \ell_i \text{ is even}\};$$
$$Y := \{\wp \in \mathrm{PC}(\mathbf{S}^{\mathcal{E}_{\mathbf{I}}}_{\bullet}(A_{\mathbf{I}})) \mid \wp \text{ is projective}\}.$$

Then $\sharp Y = \sharp\mathcal{Q}^{\mathbf{I}}_0$. By Lemma 5.1, we have $\wp(a_{i,j-1}) \cap g_{ij} \cap (\mathcal{S}\backslash\partial\mathcal{S}) \neq \varnothing$.

Step 3. Removing all the above projective curves $\wp(a_{i,j-1})$ from Y, we obtain a subset Y' of Y such that $\sharp Y' = \sharp\mathcal{Q}^{\mathbf{I}}_0 - \sharp X$. Then

$$g_{i\overline{j}} \cap \wp(a) \cap (\mathcal{S}\backslash\partial\mathcal{S}) = \varnothing \text{ for any } a \in Y', \text{ and } g_{i\overline{j}} \cap g_{i\overline{j+2}} \cap (\mathcal{S}\backslash\partial\mathcal{S}) = \varnothing,$$

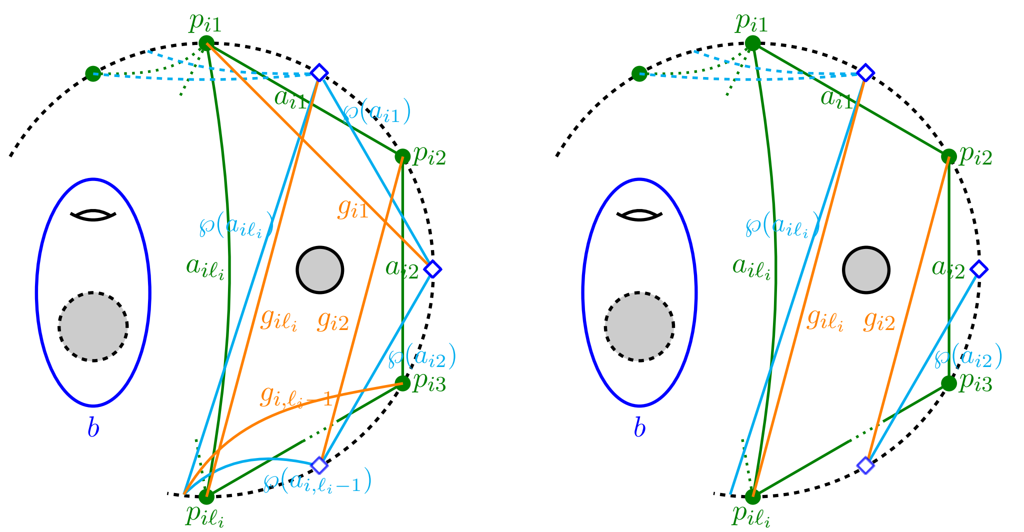


FIGURE 6.6. There exists a permissible curve b without endpoint in Case **I** or **II**.

Step 4. Let $D_\bullet := X \cup Y'$ be a GPGD. Then $\mathfrak{M}(D_\bullet) \in \mathrm{GPs}\tau\text{-tilt}(A)$ is non-trivial and its endomorphism algebra $\mathrm{End}_A \mathfrak{M}(D_\bullet)$ is representation-infinite.

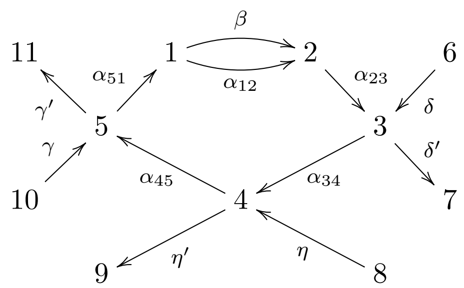
Indeed, $D_\bullet$ is a GPD satisfying

$$\sharp D_\bullet = \sharp X + \sharp Y' = \sharp X + (\sharp \mathcal{Q}_0^{\mathbf{I}} - \sharp X) = \sharp \mathcal{Q}_0^{\mathbf{I}}.$$

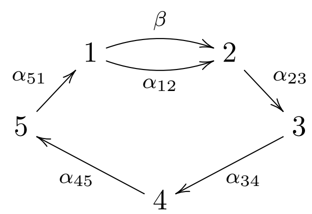
It follows from [16, Corollary 5.8], Proposition 4.5 and Corollary 4.7 that $\mathfrak{M}(D_\bullet) \in \mathrm{GPs}\tau\text{-tilt}(A_{\mathbf{I}})$. Note that for each vertex v in band, the projective module $P(v)_{A_{\mathbf{I}}}$ corresponding to $\wp(\mathfrak{v}^{-1}(v)) \in \mathbf{S}_\bullet^{\mathcal{E}_{\mathbf{I}}}(A_{\mathbf{I}})$ is a direct summand of $\widehat{T}$. Thus, $\mathrm{End}_{\widehat{A}}\widehat{T}$ is representation-infinite. By Lemma 6.2, $\mathrm{End}_A T$ is representation-infinite, where $T = \mathfrak{M}(D_\bullet) \in \mathrm{GPs}\tau\text{-tilt}(A)$ is non-trivial.

We also give another example to illustrate our results.

Example 6.4. Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be a gentle algebra where $\mathcal{Q} =$



and $\mathcal{I} = \langle \alpha_{12}\alpha_{23}, \alpha_{23}\alpha_{34}, \alpha_{34}\alpha_{45}, \alpha_{45}\alpha_{51}, \alpha_{51}\alpha_{12}, \gamma\gamma', \delta\delta', \eta\eta' \rangle$. Consider the gentle algebra $\widehat{A} = \Bbbk\widehat{\mathcal{Q}}/\widehat{\mathcal{I}}$ given by $\widehat{\mathcal{Q}} =$



and $\widehat{\mathcal{I}} = \langle \alpha_{12}\alpha_{23}, \alpha_{23}\alpha_{34}, \alpha_{34}\alpha_{45}, \alpha_{45}\alpha_{51}, \alpha_{51}\alpha_{12} \rangle$. Then $\mathbf{S}_\bullet^{\mathcal{E}_{\widehat{A}}}(\widehat{A}) \precsim \mathbf{S}_\bullet^{\mathcal{E}_A}(A)$ and the GPDG $\widehat{D} = \{g_{i2}, g_{i4}, \wp(a_{i2}), \wp(a_{i4}), \wp(a_{i5})\}$ of $\mathbf{S}_\bullet^{\mathcal{E}_{\widehat{A}}}(\widehat{A})$ can be seen as a PGD D of $\mathbf{S}_\bullet^{\mathcal{E}_A}(A)$, see FIGURE 6.7. In this case, $\widehat{D}$ corresponds to the G-projective τ-tilting $\widehat{A}$-module

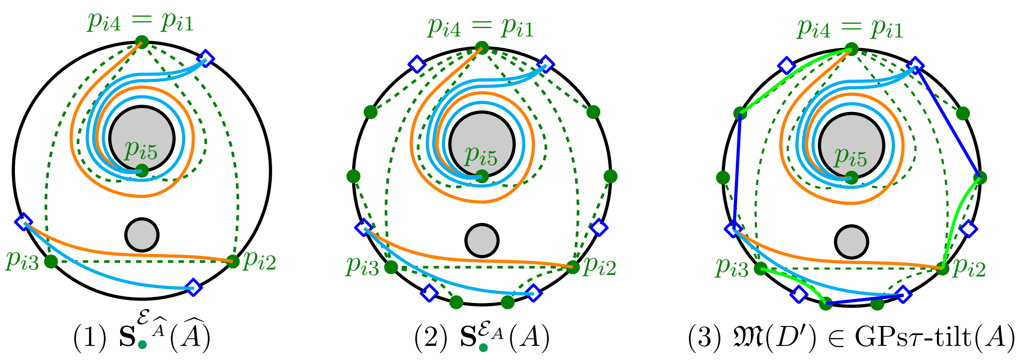


FIGURE 6.7. Constructing $T = \mathfrak{M}(D')$ such that $\mathrm{End}_A T$ is representation-infinite.

$$\widehat{T} = 2 \oplus 5 \oplus {}_{2}{}^{1}{}_{\substack{2\\3}} \oplus \substack{2\\3} \oplus \substack{4\\5},$$

and $\varphi(\widehat{D})$ corresponds to the support τ-tilting A-module

$$2 \oplus \substack{5\\10} \oplus {}_{2}{}^{1}{}_{\substack{2\\3\\7}} \oplus \substack{2\\3\\7} \oplus {}_{9}{}^{4}{}_{\substack{5\\11}},$$

where φ is the injection defined in the proof of Lemma 6.2. Thus, $\varphi(\widehat{D})$ can be embedded into a GPGD $D' = \varphi(\widehat{D}) \cup \{\mathfrak{c}(a) \mid a \in \Delta_{\bullet}(A) \backslash \Delta_{\bullet}(\widehat{A})\}$ of $\mathbf{S}^{\mathcal{E}_A}_{\bullet}(A)$ which corresponds to the G-projective τ-tilting A-module

$$T := \mathfrak{M}(D') = 2 \oplus \substack{5\\10} \oplus {}_{2}{}^{1}{}_{\substack{2\\3\\7}} \oplus \substack{2\\3\\7} \oplus {}_{9}{}^{4}{}_{\substack{5\\11}} \oplus 7 \oplus 9 \oplus 11. \text{ (see FIGURE 6.7 (3)).}$$

Then $\mathrm{End}_A T$ is representation-infinite by Proposition 3.6.

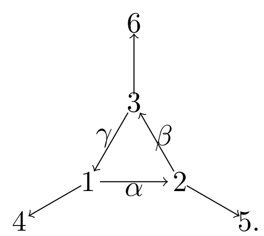
It is known that any tilting right A-module T over an arbitrary finite dimensional $\Bbbk$-algebra A is a tilting left B-module ${}_BT$ with $B = \mathrm{End}(T_A)$ and $\mathrm{End}({}_BT) \cong A$. However, we will give some examples to show that the following statements might be false when T_A is a G-projective τ-tilting right A-module over a gentle algebra A. That is, the following properties (a) and (b) do not hold in general.

(a) ${}_BT$ is G-projective τ-tilting, where $B = \mathrm{End}(T_A)$.
(b) $\mathrm{End}({}_BT) \cong A$.

See Examples 6.5 and 6.6.

The following example shows that there is a G-projective τ-tilting right A-module T over a gentle algebra A such that neither (a) nor (b) holds.

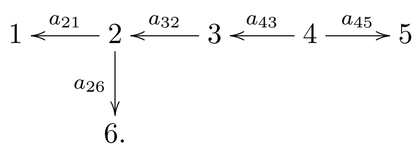
Example 6.5. Let $A = \Bbbk\mathcal{Q}/\langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle$, where $\mathcal{Q}$ is



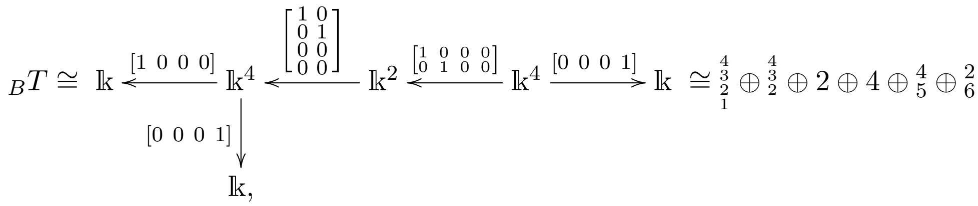
We write

$$T_1 = 4, \qquad T_2 = {}_{4}{}^{1}{}_{\substack{2\\5}}, \qquad T_3 = \substack{1\\4}, \qquad T_4 = {}_{6}{}^{3}{}_{\substack{1\\4}}, \qquad T_5 = 6, \qquad T_6 = 5.$$

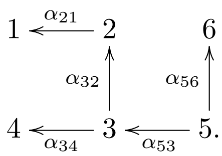
Then we may verify that $T_A = \bigoplus_{i=1}^{6} T_i$ is a G-projective support τ-tilting module, and $B = \mathrm{End}(T_A)$ is isomorphic to $\Bbbk\mathcal{Q}'/\langle a_{32}a_{21} \rangle$, where $\mathcal{Q}'$ is



We have

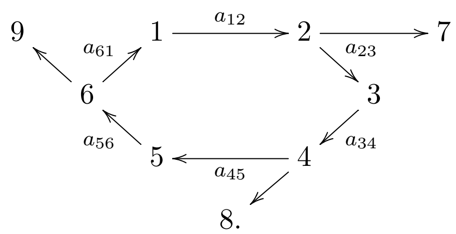


is neither G-projective nor G-injective because the simple module 2 is neither projective nor injective. Furthermore, we have $\mathrm{End}({}_BT) = \Bbbk\mathcal{Q}''/\langle \alpha_{53}\alpha_{32}\rangle$ which is not isomorphic to A, where $\mathcal{Q}''$ is



The following example shows that there is a G-projective τ-tilting right A-module T over a gentle algebra A such that (a) holds but (b) does not.

Example 6.6. Let $A = \Bbbk\mathcal{Q}/\langle a_{12}a_{23}, a_{23}a_{34}, a_{34}a_{45}, a_{45}a_{56}, a_{56}a_{61}\rangle$, where $\mathcal{Q}$ is



Then

$$T = \begin{smallmatrix}1\\2\\7\end{smallmatrix} \oplus \begin{smallmatrix}2\\7\end{smallmatrix} \oplus 7 \oplus \begin{smallmatrix}3\\4\\8\end{smallmatrix} \oplus \begin{smallmatrix}4\\8\end{smallmatrix} \oplus 8 \oplus \begin{smallmatrix}5\\6\\9\end{smallmatrix} \oplus \begin{smallmatrix}6\\9\end{smallmatrix} \oplus 9$$

is a G-projective τ-tilting right A-module. We have $B := \mathrm{End}(T_A) \cong (\Bbbk A_3^1) \oplus (\Bbbk A_3^2) \oplus (\Bbbk A_3^3)$, where

$$A_3^i = \ (i,1) \longrightarrow (i,2) \longrightarrow (i,3)$$

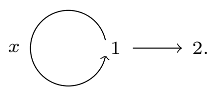
for any $1 \le i \le 3$. Furthermore, we have that ${}_BT$ is isomorphic to

$$\bigoplus_{i=1}^{3} \left(\begin{smallmatrix}(i,1)\\(i,2)\\(i,3)\end{smallmatrix} \oplus \begin{smallmatrix}(i,1)\\(i,2)\end{smallmatrix} \oplus (i,1) \right),$$

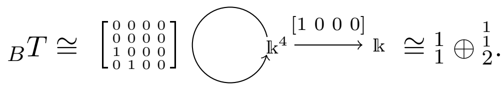
which is a G-projective τ-tilting left B-module (Note that the indecomposable projective left B-module corresponding to the vertex i in the quiver of B is induced by paths with the ending point i, this is the opposite of the case of the indecomposable projective right A-module).

The following example shows that there is a G-projective τ-tilting right A-module T over a gentle algebra A with infinite global dimension such that both (a) and (b) hold.

Example 6.7. Let $A = \Bbbk\mathcal{Q}/\langle x^2\rangle$, where $\mathcal{Q}$ is



Then $T_A = {}_{2}{}^{1}{}_{\begin{smallmatrix}1\\2\end{smallmatrix}} \oplus 2$ is a G-projective τ-tilting right A-module and $B := \mathrm{End}(T_A) \cong A$. Furthermore,



It is easy to check that T is a G-projective τ-tilting left B-module.

We do not find any example of a non-trivial G-projective τ-tilting right A-module T over a gentle algebra A such that (b) holds but (a) does not.

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School of Mathematics and Statistics, Guizhou University, Guiyang 550025, P. R. China
E-mail address: yzliu3@163.com

Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China
E-mail address: 470985396@163.com

Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China
E-mail address: huangzy@nju.edu.cn