On the number of $\tau$-tilting modules over the Auslander algebras of radical square zero Nakayama algebras

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Abstract

Let $\Lambda_n$ be a radical square zero Nakayama algebra with $n$ simple modules and $\Gamma_n$ the Auslander algebra of $\Lambda_n$. We calculate the number $|\tau$-tilt $\Gamma_n|$ of $\tau$-tilting modules and the number $|s\tau$-tilt $\Gamma_n|$ of support $\tau$-tilting modules over $\Gamma_n$. In particular, we prove that there are recurrence relations:

$$|\tau$-tilt $\Gamma_n| = 3|\tau$-tilt $\Gamma_{n-1}| + |\tau$-tilt $\Gamma_{n-2}|,$
$$|s\tau$-tilt $\Gamma_n| = 6|s\tau$-tilt $\Gamma_{n-1}| + 3|s\tau$-tilt $\Gamma_{n-2}|,$

from which the exact values of $|\tau$-tilt $\Gamma_n|$ and $|s\tau$-tilt $\Gamma_n|$ are derived.

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1 Introduction

The starting point of tilting theory was the introduction of tilting modules over a hereditary algebra by Happel and Ringel [11]. Ever since, the study of tilting modules has been an important branch in the representation theory of finite dimensional algebras.

In 2014, Adachi, Iyama and Reiten [1] introduced $\tau$-tilting theory replacing the rigidity condition $\text{Ext}^1_{\Lambda}(M,M) = 0$ for a tilting module by the weaker condition $\text{Hom}_{\Lambda}(M,\tau M) = 0$ for a $\tau$-tilting module, where $\Lambda$ is a finite dimensional algebra and $\tau$ is the Auslander-Reiten translation. The support $\tau$-tilting modules are in bijection with some important objects in representation theory including functorially finite torsion classes introduced in [6], 2-term silting complexes introduced in [14], cluster-tilting objects in the cluster category and left finite semibricks introduced in [3]. Therefore, it is important to calculate the number of support $\tau$-tilting modules over a given algebra.

For hereditary algebras, the (support) $\tau$-tilting modules are exactly the (support) tilting modules. For algebras Dynkin type, the numbers of these modules were first calculated via
cluster algebras \[8\], and later via representation theory \[15\]. In particular, over a hereditary algebra of type \(A_n\), the number of tilting modules is \(C_n\) and the number of support tilting modules is \(C_{n+1}\), where \(C_i\) is the \(i\)-th Catalan number \(\frac{1}{i+1} \binom{2i}{i}\).

Recall from \[4, V.3.2\] that a finite dimensional algebra is Nakayama if its quiver is one of the following:

\[
A_n : 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n,
\]

\[
\overline{A}_n : 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n.
\]

Adachi \[2\] gave a recurrence relation for the number of \(\tau\)-tilting modules over Nakayama algebras of type \(A_n\). Asai \[3\] also gave a recurrence relation for the number of support \(\tau\)-tilting modules over Nakayama algebras \(KA_n/\text{rad}^r\) and \(K\overline{A}_n/\text{rad}^r\). More recently, Gao and Schiffler \[10\] extended the recurrence relation of Adachi to \(\tau\)-tilting modules over \(K\overline{A}_n/\text{rad}^r\).

It was showed in \[7\] that the number of tilting modules over the Auslander algebra of \(K[x]/(x^n)\) is \(n!\). Kajita \[13\] calculated the number of tilting modules over the Auslander algebra of hereditary algebra of Dynkin type. Iyama and Zhang \[12\] classified the support \(\tau\)-tilting modules over the Auslander algebra of \(K[x]/(x^n)\), and they also showed that there is a bijection between the set of support \(\tau\)-tilting modules over the Auslander algebra of \(K[x]/(x^n)\) and the symmetric group of degree \(n\). More recently, Zhang \[17\] calculated the number of tilting modules over the Auslander algebra \(\Gamma_n\) of a radical square zero Nakayama algebra \(\Lambda_n\). In particular, Zhang proved that the number of tilting modules over \(\Gamma_n\) is \(2^n-1\) if \(\Lambda_n\) is of type \(A_n\); and is \(2^n\) if \(\Lambda_n\) is of type \(\overline{A}_n\).

In this paper, we focus on the number \(|\tau\text{-tilt } \Gamma_n|\) of \(\tau\)-tilting modules and the number \(|\text{st}-\text{tilt } \Gamma_n|\) of support \(\tau\)-tilting modules over the Auslander algebra \(\Gamma_n\) of a radical square zero Nakayama algebra \(\Lambda_n\). Our result is as follows.

**Theorem 1.1.** (Theorems 3.1, 3.5, 4.2 and 4.3) Let \(\Gamma_n\) be the Auslander algebra of a radical square zero Nakayama algebra \(\Lambda_n\).

1. If \(\Lambda_n\) is of type \(A_n\), then

   \[
   |\tau\text{-tilt } \Gamma_n| = \frac{(3 + \sqrt{13})^n - (3 - \sqrt{13})^n}{\sqrt{13} \cdot 2^n}
   \]

   and

   \[
   |\text{st}-\text{tilt } \Gamma_n| = \frac{(3 + 2\sqrt{3})^n - (3 - 2\sqrt{3})^n}{2\sqrt{3}}.
   \]

2. If \(\Lambda_n\) is of type \(\overline{A}_n\), then

   \[
   |\tau\text{-tilt } \Gamma_n| = \frac{(3 + \sqrt{13})^n + (3 - \sqrt{13})^n}{2^n}
   \]

   and

   \[
   |\text{st}-\text{tilt } \Gamma_n| = (3 + 2\sqrt{3})^n + (3 - 2\sqrt{3})^n.
   \]

The paper is organized as follows. In Section 2, we fix some notations and recall several results about \(\tau\)-tilting modules and Auslander algebras of radical square zero Nakayama algebras which are useful in the sequel. In Section 3, we show that if \(\Lambda_n\) is of type \(A_n\), then there are recurrence relations

\[
|\tau\text{-tilt } \Gamma_n| = 3|\tau\text{-tilt } \Gamma_{n-1}| + |\tau\text{-tilt } \Gamma_{n-2}|
\]
and 

$$|s\tau\text{-tilt}\Gamma_n| = 6|s\tau\text{-tilt}\Gamma_{n-1}| + 3|s\tau\text{-tilt}\Gamma_{n-2}|.$$ 

In Section 4, we show that if $\Lambda_n$ is of type $A_n$, then there are the same recurrence relations as above. From these recurrence relations the exact values of $|\tau\text{-tilt}\Gamma_n|$ and $|s\tau\text{-tilt}\Gamma_n|$ are derived. Finally, we list the table of $|\tau\text{-tilt}\Gamma_n|$ and $|s\tau\text{-tilt}\Gamma_n|$ for $1 \leq n \leq 8$ in Section 5.

### 2 Preliminaries

Throughout this paper, all algebras are basic, connected, finite dimensional $K$-algebras over an algebraically closed field $K$. For an algebra $\Lambda$, we denote by $\text{mod}\Lambda$ the category of finitely generated right $\Lambda$-modules and by $\tau$ the Auslander-Reiten translation of $\Lambda$. We use $P_i$, $I_i$ and $S_i$ to denote the indecomposable projective, injective, and simple modules of an algebra corresponding to the vertex $i$ respectively. For any $i, j \in \{1, 2, \cdots, n\}$, we denote by $[i, j] = \{i, i + 1, \cdots, j\}$ if $i \leq j$; otherwise, $[i, j] = \emptyset$. Let $e_i$ be the primitive idempotent element of an algebra corresponding to the vertex $i$. We write $e_{[i, j]} := e_i + e_{i+1} + \cdots + e_j$.

For a module $M \in\text{mod}\Lambda$, we use $|M|$ to denote the number of pairwise non-isomorphic indecomposable summands of $M$, and use $l(M)$ and $\text{pd}_\Lambda M$ to denote the Loewy length and projective dimension of $M$ respectively. For a finite set $X$, we use $|X|$ to denote the cardinality of $X$. For two sets $X_1$ and $X_2$, $X_1 \coprod X_2$ stands for their disjoint union.

**Definition 2.1.** ([1, Definition 0.1]) Let $\Lambda$ be an algebra and $M \in\text{mod}\Lambda$.

1. $M$ is called $\tau$-rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$.
2. $M$ is called $\tau$-tilting if it is $\tau$-rigid and $|M| = |\Lambda|$.
3. $M$ is called support $\tau$-tilting if it is a $\tau$-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent $e$ of $\Lambda$.
4. $M$ is called proper support $\tau$-tilting if it is a support $\tau$-tilting but not a $\tau$-tilting $\Lambda$-module.

Recall that $M \in\text{mod}\Lambda$ is called sincere if every simple $\Lambda$-module appears as a composition factor in $M$. It is well-known that the $\tau$-tilting modules are exactly the sincere support $\tau$-tilting modules ([1, Proposition 2.2(a)]).

We denote by $\tau\text{-tilt}\Lambda$ (respectively, $s\tau\text{-tilt}\Lambda$, $ps\tau\text{-tilt}\Lambda$) the set of isomorphism classes of basic $\tau$-tilting (respectively, support $\tau$-tilting, proper support $\tau$-tilting) $\Lambda$-modules.

Set

$$\text{ps}\tau\text{-tilt}_{np}\Lambda := \{M \in \text{ps}\tau\text{-tilt}\Lambda \mid M \text{ has no projective direct summands}\}.$$ 

**Theorem 2.2.** ([2, Theorem 2.6]) Let $\Lambda$ be a Nakayama algebra. Then there is a bijection between $\tau\text{-tilt}\Lambda$ and $\text{ps}\tau\text{-tilt}_{np}\Lambda$.

The following result is very useful.

**Proposition 2.3.** ([2, Proposition 2.32]) Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then each $\tau$-tilting $\Lambda$-module has $P_1$ as a direct summand.

As a consequence, we get the following

**Lemma 2.4.** Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then each support $\tau$-tilting $\Lambda$-module which has $S_1, S_2, \cdots, S_{l(P_1)}$ as composition factors has $P_1$ as a direct summand.
Proof. Let $M$ be a support $\tau$-tilting $\Lambda$-module which has $S_1, S_2, \cdots, S_{l(P)}$ as composition factors. If $M$ is $\tau$-tilting, then it has $P_1$ as a direct summand by Proposition 2.3. Now, assume that $M$ has $S_1, S_2, \cdots, S_{l(P)}, \cdots, S_j$ as composition factors but not $S_{j+1}$. Let $N$ be the maximal direct summand of $M$ which only has $S_1, S_2, \cdots, S_{l(P)}, \cdots, S_j$ as composition factors. Then $N$ is a $\tau$-tilting $\Lambda/\langle e_{j+1,n}\rangle$-module. By Proposition 2.3, $N$ has $P_1$ as a direct summand. \hfill \Box

**Theorem 2.5.** ([2, Theorem 2.33 and Corollary 2.34]) Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then there are mutually inverse bijections

$$\tau\text{-}\text{tilt} \; \Lambda \leftrightarrow \prod_{i=1}^{l(P)} \tau\text{-}\text{tilt}(\Lambda/\langle e_i \rangle)$$

given by $\tau\text{-}\text{tilt} \; \Lambda \ni M \mapsto M/P_1$ and $N \mapsto N \oplus P_1 \in \tau\text{-}\text{tilt} \; \Lambda$. In particular,

$$|\tau\text{-}\text{tilt} \; \Lambda| = \sum_{i=1}^{l(P)} C_{i-1} \cdot |\tau\text{-}\text{tilt}(\Lambda/\langle e_{1,i} \rangle)|.$$

**Remark 2.6.** Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then every $\tau$-tilting $\Lambda$-module can be decomposed $M$ as $M = P_1 \oplus N_1 \oplus N_2$ where $N_1$ is a maximal direct summand of $M$ without $S_1$ as composition factors. Moreover, $N_1 \oplus N_2$ is a $\tau$-tilting $\Lambda/\langle e_{j+1} \rangle$-module where $j := l(N_2)$ (see the proof of [2, Theorem 2.33]).

An algebra $\Lambda$ is of *finite representation type* if there are only finite many indecomposable $\Lambda$-modules $X_1, X_2, \cdots, X_m$ up to isomorphism. In this case, the endomorphism algebra $\text{End}_\Lambda(\oplus_{i=1}^m X_i)$ is called the *Auslander algebra* of $\Lambda$.

By using a straight calculation, we get the quiver of the Auslander algebra of radical square zero Nakayama algebras as follows.

**Proposition 2.7.**

(1) The Auslander algebra $\Gamma_n$ of $\Lambda_n := KA_n/\text{rad}^2$ is given by the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{2n - 2} 2n - 1$$

with the relations: $a_{2k-1}a_{2k} = 0$ for $1 \leq k \leq n - 1$.

(2) The Auslander algebra $\Gamma_n'$ of $\Lambda_n := K\tilde{A}_n/\text{rad}^2$ is given by the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{2n - 1} 2n$$

with the relations: $a_{2k-1}a_{2k} = 0$ for $1 \leq k \leq n$.

3 **The case for $\Gamma_n$**

In this section, we will give the formulas for calculating $|\tau\text{-}\text{tilt} \; \Gamma_n|$ and $|s\tau\text{-}\text{tilt} \; \Gamma_n|$. Let $\Delta_n$ be the algebra is given by the following quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{2n - 2} 2n - 1$$

with the relations: $a_{2k-1}a_{2k} = 0$ for $1 \leq k \leq n - 1$.

The following result gives the formula for calculating $|\tau\text{-}\text{tilt} \; \Gamma_n|$.  


**Theorem 3.1.** We have
\[
| \tau\text{-}\text{tilt } \Gamma_n | = 3 | \tau\text{-}\text{tilt } \Gamma_{n-1} | + | \tau\text{-}\text{tilt } \Gamma_{n-2} | \\
with | \tau\text{-}\text{tilt } \Gamma_1 | = 1 \text{ and } | \tau\text{-}\text{tilt } \Gamma_2 | = 3. \text{ Hence}
\]
\[
| \tau\text{-}\text{tilt } \Gamma_n | = \frac{(3 + \sqrt{13})^n - (3 - \sqrt{13})^n}{\sqrt{13} \cdot 2^n}.
\]

*Proof.* Applying Theorem 2.5 to $\Gamma_n$ and $\Delta_n$ respectively, we have
\[
| \tau\text{-}\text{tilt } \Gamma_n | = C_0 \cdot | \tau\text{-}\text{tilt}(\Gamma_n/\langle e_1 \rangle) | + C_1 \cdot | \tau\text{-}\text{tilt}(\Gamma_n/\langle e_1 + e_2 \rangle) | = | \tau\text{-}\text{tilt } \Delta_{n-1} | + | \tau\text{-}\text{tilt } \Gamma_{n-1} | 
\]
(3.1)

and
\[
| \tau\text{-}\text{tilt } \Delta_n | = C_0 \cdot | \tau\text{-}\text{tilt}(\Delta_n/\langle e_0 \rangle) | + C_1 \cdot | \tau\text{-}\text{tilt}(\Delta_n/\langle e_0 + e_1 \rangle) | \\
+ C_2 \cdot | \tau\text{-}\text{tilt}(\Delta_n/\langle e_0 + e_1 + e_2 \rangle) | = | \tau\text{-}\text{tilt } \Gamma_n | + | \tau\text{-}\text{tilt } \Delta_{n-1} | + 2 | \tau\text{-}\text{tilt } \Gamma_{n-1} |. 
\]
(3.2)

The equation (3.1) implies
\[
| \tau\text{-}\text{tilt } \Delta_{n-1} | = | \tau\text{-}\text{tilt } \Gamma_n | - | \tau\text{-}\text{tilt } \Gamma_{n-1} |.
\]

Applying it to the equation (3.2), we have
\[
| \tau\text{-}\text{tilt } \Gamma_n | = 3 | \tau\text{-}\text{tilt } \Gamma_{n-1} | + | \tau\text{-}\text{tilt } \Gamma_{n-2} | 
\]
(3.3)

The equation (3.3) is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 3x - 1 = 0$. The proof is finished.

Let $\Lambda$ be an algebra. Recall that a module $M \in \text{mod } \Lambda$ is called *tilting* if the following conditions are satisfied.

1. $\text{pd}_\Lambda M \leq 1$;
2. $\text{Ext}_\Lambda^1(M, M) = 0$;
3. $|M| = |\Lambda|$.

Thus a module $M \in \text{mod } \Lambda$ is tilting if and only if it is a $\tau$-tilting and $\text{pd}_\Lambda M \leq 1$ by the Auslander-Reiten formula.

The set of all tilting $\Lambda$-modules is denoted by $\text{tilt } \Lambda$. The following result is part of [17, Theorem 2.8]. Here we give it a new proof.

**Proposition 3.2.** $| \text{tilt } \Gamma_n | = 2^{n-1}$.

*Proof.* Since $P_1$ is the unique $\Gamma_n$-module which has $S_1$ as a composition factor and its projective dimension is at most one. By Remark 2.6 and the above argument, we have that $P_1 \oplus N_1$ is a tilting $\Gamma_n$-module if and only if $N_1$ is a tilting $\Gamma_n/\langle e_1 \rangle$-module, since $\text{pd}_{\Gamma_n} N_1 = \text{pd}_{\Gamma_n/\langle e_1 \rangle} N_1$. Thus
\[
| \text{tilt } \Gamma_n | = | \text{tilt}(\Gamma_n/\langle e_1 \rangle) | = | \text{tilt } \Delta_{n-1} |.
\]

Note that $P_0$ and $S_0$ are the only two $\Delta_n$-modules which have $S_0$ as a composition factor and their projective dimension are at most one. Similarly, we may get
\[
| \text{tilt } \Delta_n | = | \text{tilt}(\Delta_n/\langle e_0 \rangle) | + | \text{tilt}(\Delta/\langle e_0 + e_1 \rangle) | = | \text{tilt } \Gamma_n | + | \text{tilt } \Delta_{n-1} |.
\]

Thus $| \text{tilt } \Gamma_n | = 2 | \text{tilt } \Gamma_{n-1} |$ with $| \text{tilt } \Gamma_1 | = 1$, and therefore $| \text{tilt } \Gamma_n | = 2^{n-1}$. \qed
As generalizations of simple modules and semisimple modules, bricks and semibricks were introduced and studied in \cite{9,16}. Let $\Lambda$ be an algebra. A $\Lambda$-module $M$ is called a brick if $\text{Hom}_\Lambda(M, M)$ is a $K$-division algebra, and a semibrick is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. Recall from \cite{3} that a semibrick $S$ is called left finite if the smallest torsion class $T(S)$ containing $S$ is functorially finite. There exists a bijection between $s\tau$-tilt $\Lambda$ and the set of left finite semibricks of $\Lambda$ (\cite[Theorem 2.3]). Note that every torsion class is functorially finite for a representation-finite algebra. So, for a Nakayama algebra $\Lambda$, there exists a bijection between $s\tau$-tilt $\Lambda$ and the set of left finite semibricks of $\Lambda$, and hence $|s\tau$-tilt $\Lambda| = |s\text{brick} \Lambda|$. Asai gave a method to calculate the number of semibricks over $KA_n/\text{rad}'$. In fact, we have the following more general result.

**Proposition 3.3.** Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then we have

\begin{enumerate}[(1)]
    \item $|s\tau$-tilt $\Lambda| = 2|s\tau$-tilt($\Lambda/\langle e_n \rangle$)$| + \sum_{i=2}^{l(I_n)} C_{i-1} \cdot |s\tau$-tilt($\Lambda/\langle e_{[n-i+1,n]} \rangle$)$|$. \\
    \item $|s\tau$-tilt $\Lambda| = 2|s\tau$-tilt($\Lambda/\langle e_1 \rangle$)$| + \sum_{i=2}^{l(I_n)} C_{i-1} \cdot |s\tau$-tilt($\Lambda/\langle e_{[1,n]} \rangle$)$|$. \\
\end{enumerate}

**Proof.** (1) For a given brick $X$ of $\Lambda$ with top $X = S_i$ and soc $X = S_j$, we will denote $S_{i,j} := X$. We define $W_0$ as the subset of $s\text{brick} \Lambda$ consisting of the semibricks without $S_n$ as a composition factor. It is clear that $|W_0| = |s\text{brick} \Lambda/\langle e_n \rangle|$.

Let $W_i$ ($i = 1, 2, \cdots, l(I_n)$) be the subset of $s\text{brick} \Lambda$ consisting of the semibricks which obtain the brick $S_{n-i+1,n}$.

First, there is a bijection

$$W_1 \mapsto s\text{brick} \Lambda/\langle e_n \rangle$$

defined by $S \mapsto S \setminus \{S_{n,n}\}$. So $|W_0| = |s\text{brick} \Lambda/\langle e_n \rangle|$.

Secondly, for $i = 2, 3, \cdots, l(I_n)$, there exists a bijection

$$W_1 \mapsto s\text{brick} \Lambda/\langle e_{[n-i+1,n]} \rangle \times s\text{brick} \Lambda/\langle 1 - e_{[n-i+2,n-1]} \rangle$$

defined by $S \mapsto (\{S \in S \mid \text{Supp} S \cap [n - i + 1, n] = \emptyset\}, \{S \in S \mid \text{Supp} S \subset [n - i + 2, n - 1]\}$),

where $\text{Supp} S$ stands for the support of $S$. Note that $s\text{brick} \Lambda = \bigcup_{i=0}^{l(I_n)} W_i$. Thus we obtain

$$|s\tau$-tilt $\Lambda| = |s\text{brick} \Lambda| = \sum_{i=0}^{l(I_n)} |W_i|$$

$$= 2|s\text{brick} \Lambda/\langle e_n \rangle| + \sum_{i=2}^{l(I_n)} |s\text{brick} \Lambda/\langle e_{[n-i+1,n]} \rangle| \cdot |s\text{brick} \Lambda/\langle 1 - e_{[n-i+2,n-1]} \rangle|$$

$$= 2|s\text{brick} \Lambda/\langle e_n \rangle| + \sum_{i=2}^{l(I_n)} |s\text{brick} \Lambda/\langle e_{[n-i+1,n]} \rangle| \cdot |s\text{brick} KA_{i-2}|$$

$$= 2|s\tau$-tilt($\Lambda/\langle e_n \rangle$)$| + \sum_{i=2}^{l(I_n)} |s\tau$-tilt($\Lambda/\langle e_{[n-i+1,n]} \rangle$)$| \cdot |s\tau$-tilt($KA_{i-2}$)$|$$

$$= 2|s\tau$-tilt($\Lambda/\langle e_n \rangle$)$| + \sum_{i=2}^{l(I_n)} C_{i-1} \cdot |s\tau$-tilt($\Lambda/\langle e_{[n-i+1,n]} \rangle$)$|$. \\

(2) Note that there is a bijection between $s\tau$-tilt $\Lambda$ and $s\tau$-tilt $\Lambda^{op}$ (\cite[Theorem 2.14]). Now the assertion follows from (1). \hfill \square
We give the following example to illustrate Proposition 3.3.

**Example 3.4.** Let $\Lambda$ be an algebra given by the quiver

$$
1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{} 4
$$

with the relation $\alpha \beta = 0$. By Proposition 3.3(1), we have

$$
|\text{st-tilt} \Lambda| = 2|\text{st-tilt}(\Lambda/\langle e_4 \rangle)| + |\text{st-tilt}(\Lambda/\langle e_3 + e_4 \rangle)| + 2|\text{st-tilt}(\Lambda/\langle e_2 + e_3 + e_4 \rangle)|
$$

$$
= 2 \times 12 + 5 + 2 \times 2 = 33.
$$

On the other hand, by Proposition 3.2(2) we also have

$$
|\text{st-tilt} \Lambda| = 2|\text{st-tilt}(\Lambda/\langle e_1 \rangle)| + |\text{st-tilt}(\Lambda/\langle e_1 + e_2 \rangle)| = 2 \times 14 + 5 = 33.
$$

The following result gives the formula for calculating $|\text{st-tilt} \Gamma_n|$.

**Theorem 3.5.** We have

$$
|\text{st-tilt} \Gamma_n| = 6|\text{st-tilt} \Gamma_{n-1}| + 3|\text{st-tilt} \Gamma_{n-2}|
$$

with $|\text{st-tilt} \Gamma_1| = 2$ and $|\text{st-tilt} \Gamma_2| = 12$. Hence

$$
|\text{st-tilt} \Gamma_n| = \frac{(3 + 2\sqrt{3})^n - (3 - 2\sqrt{3})^n}{2\sqrt{3}}.
$$

**Proof.** Applying Proposition 3.3(2) to $\Gamma_n$ and $\Delta_n$ respectively, we have

$$
|\text{st-tilt} \Gamma_n| = 2|\text{st-tilt}(\Gamma_n/\langle e_1 \rangle)| + C_1 \cdot |\text{st-tilt}(\Gamma_n/\langle e_1 + e_2 \rangle)|
$$

$$
= 2|\text{st-tilt} \Delta_{n-1}| + |\text{st-tilt} \Gamma_{n-1}|
$$

and

$$
|\text{st-tilt} \Delta_n| = 2|\text{st-tilt}(\Delta_n/\langle e_0 \rangle)| + C_1 \cdot |\text{st-tilt}(\Delta_n/\langle e_0 + e_1 \rangle)|
$$

$$
+ C_2 \cdot |\text{st-tilt}(\Delta_n/\langle e_0 + e_1 + e_2 \rangle)|
$$

$$
= 2|\text{st-tilt} \Gamma_n| + |\text{st-tilt} \Delta_{n-1}| + 2|\text{st-tilt} \Gamma_{n-1}|.
$$

This implies

$$
|\text{st-tilt} \Gamma_n| = 6|\text{st-tilt} \Gamma_{n-1}| + 3|\text{st-tilt} \Gamma_{n-2}|
$$

(3.5)

The equation (3.5) is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 6x - 3 = 0$. The proof is finished.

Let $\overline{\Gamma}_n$ be the algebra given by the quiver

$$
1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{} 2n - 2 \xrightarrow{a_{2n-2}} 2n - 1 \xrightarrow{a_{2n-1}} 2n
$$

with the relations: $a_{2k-1}a_{2k} = 0$ for $1 \leq k \leq n - 1$, and let $\overline{\Delta}_n$ be the algebra given by the following quiver

$$
0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{} 2n - 2 \xrightarrow{a_{2n-2}} 2n - 1 \xrightarrow{a_{2n-1}} 2n
$$

with the relations: $a_{2k-1}a_{2k} = 0$ for $1 \leq k \leq n - 1$. By using the same argument as that in Theorem 3.5, we can obtain

$$
|\text{st-tilt} \overline{\Delta}_n| = 6|\text{st-tilt} \overline{\Delta}_{n-1}| + 3|\text{st-tilt} \overline{\Delta}_{n-2}|.
$$
4 The case for $\Gamma'_n$

In this section, we will give the formulas for calculating $|\tau\text{-tilt } \Gamma'_n|$ and $|s\tau\text{-tilt } \Gamma'_n|$.

Let $X_n$ be the set of all support $\tau$-tilting $\Gamma_n$-modules which have no $P_1, P_2, \cdots, P_{2n-3}$ as direct summands, and let $Y_n$ be the set of all support $\tau$-tilting $\Delta_n$-modules which have no $P_0, P_1, P_2, \cdots, P_{2n-3}$ as direct summands. Let $X'_n$ be the set of all support $\tau$-tilting $\Gamma_n$-modules which have no $P_1, P_2, \cdots, P_{2n-2}$ as direct summands, and let $Y'_n$ be the set of all support $\tau$-tilting $\Delta_n$-modules which have no $P_0, P_1, P_2, \cdots, P_{2n-2}$ as direct summands.

We need the following lemma.

Lemma 4.1.

1. $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ and $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.
2. $|X'_n| = 3|X'_{n-1}| + |X'_{n-2}|$ and $|Y'_n| = 3|Y'_{n-1}| + |Y'_{n-2}|$.

Proof. (1) By Lemma 2.4, we have that all support $\tau$-tilting $\Gamma_n$-modules which have $S_1, S_2$ as composition factors must have $P_1$ as a direct summand. Hence $X_n$ consists of two parts: the first part comes from all support $\tau$-tilting $\Gamma_n$-modules which has no $P_1, P_2, \cdots, P_{2n-3}$ as direct summands and has no $S_1$ as its composition factor (the number is exactly $|Y_{n-1}|$), the second part comes from all support $\tau$-tilting $\Gamma_n$-modules which has no $P_1, P_2, \cdots, P_{2n-3}$ as direct summands and has $S_1$ as its composition factor but not $S_2$ (the number is exactly $|X_{n-1}|$). Hence, $|X_n| = |Y_{n-1}| + |X_{n-1}|$. Similarly, we have $|Y_n| = |X_n| + |Y_{n-1}| + 2|X_{n-1}|$. These two equations imply $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ and $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.

(2) It is similar to (1).

The following result gives the formula for calculating $|\tau\text{-tilt } \Gamma'_n|$.

Theorem 4.2. We have

$$|\tau\text{-tilt } \Gamma'_n| = 3|\tau\text{-tilt } \Gamma'_n-1| + |\tau\text{-tilt } \Gamma'_n-2|$$

with $|\tau\text{-tilt } \Gamma'_1| = 3$ and $|\tau\text{-tilt } \Gamma_2| = 11$. Hence

$$|\tau\text{-tilt } \Gamma'_n| = \frac{(3 + \sqrt{3})^n + (3 - \sqrt{3})^n}{2^n}$$

Proof. We claim that every proper support $\tau$-tilting $\Gamma'_n$-module $M$ which has $S_1, S_2$ as its composition factors must have a projective $\Gamma'_n$-module as direct summand. Indeed, if $M$ has no $S_{2n}$ as a composition factor, then it has $P_1$ as its direct summand by Lemma 2.4. Now, assume that $M$ has $S_1, S_{i+1}, \cdots, S_{2n}, S_1, S_2$ as composition factors but not $S_{i-1}$. Then $M$ has $P_i$ as its direct summand by Lemma 2.4.

Now, $\text{ps}\tau\text{-tilt}_{np} \Gamma'_n$ consists of the following two parts:

(i) $U_1$: the subset of $\text{ps}\tau\text{-tilt}_{np} \Gamma'_n$ in which every module has no $S_2$ as its composition factor.

(ii) $U_2$: the subset of $\text{ps}\tau\text{-tilt}_{np} \Gamma'_n$ in which every module has $S_2$ as its composition factor but not $S_1$.

Since $\mathcal{X} := \Gamma'_n/\langle e_2 \rangle$ is the following quiver

$$3 \xrightarrow{a_3} \cdots \xrightarrow{a_{2n-3}} 2n - 2 \xrightarrow{a_{2n-2}} 2n - 1 \xrightarrow{a_{2n-1}} 2n - \xrightarrow{a_{2n}} 1$$
with the relations: $a_{2k-1}a_{2k} = 0$ for $2 \leq k \leq n$, $U_1$ is exactly the set of support $\tau$-tilting $\Gamma$-modules which has no $P_3, P_4, \ldots, P_{2n-1}$ as direct summands, and so $|U_1| = |X_n|$. Note that $\Gamma := \Gamma / \langle e_1 \rangle$ is the following quiver
\[
\begin{array}{cccccccc}
2 & \overset{a_2}{\rightarrow} & 3 & \overset{a_3}{\rightarrow} & \cdots & \overset{a_{2n-3}}{\rightarrow} & 2n & -2^{a_{2n-2}}2n - 1 \end{array}
\]
with the relations: $a_{2k-1}a_{2k} = 0$ for $2 \leq k \leq n - 1$. Thus, the number of support $\tau$-tilting $\Gamma$-modules which has no $P_2, P_4, \ldots, P_{2n-2}$ as direct summands is exactly $|Y'_{n-1}|$. Moreover, the number of support $\tau$-tilting $\Gamma$-modules which has no $P_2, P_4, \ldots, P_{2n-2}$ as direct summands and has no $S_2$ as a composition factor is exactly $|X'_{n-1}|$. Therefore, $|U_2| = |Y'_{n-1}| - |X'_{n-1}|$. By Theorem 2.2, we obtain
\[
|\tau\text{-tilt } \Gamma_n'| = |ps\tau\text{-tilt}_{np} \Gamma_n'| = |U_1| + |U_2| = |X_n| + |Y'_{n-1}| - |X'_{n-1}|.
\]
Now, the recurrence relation of $|\tau\text{-tilt } \Gamma_n'|$ follows from Lemma 4.1.

The following result gives the formula for calculating $|s\tau\text{-tilt } \Gamma_n'|$.

**Theorem 4.3.** We have
\[
|s\tau\text{-tilt } \Gamma_n'| = 6|s\tau\text{-tilt } \Gamma_{n-1}'| + 3|s\tau\text{-tilt } \Gamma_{n-2}'|
\]
with $|s\tau\text{-tilt } \Gamma_1'| = 6$ and $|s\tau\text{-tilt } \Gamma_2'| = 42$. Hence
\[
|s\tau\text{-tilt } \Gamma_n'| = (3 + 2\sqrt{3})^n + (3 - 2\sqrt{3})^n.
\]

**Proof.** The set $\text{sbrick } \Gamma_n$ of semibricks of $\Gamma_n$ consists of the following five parts:

(i) $V_0$: the subset of $\text{sbrick } \Gamma_n$ consisting of the semibricks without $S_1$ as a composition factor.

(ii) $V_1$: the subset of $\text{sbrick } \Gamma_n$ which obtain the brick $S_1$ but not the brick $I_2$.

(iii) $V_2$: the subset of $\text{sbrick } \Gamma_n$ which obtain the brick $I_1$.

(iv) $V_3$: the subset of $\text{sbrick } \Gamma_n$ which obtain the brick $P_1$.

(v) $V_4$: the subset of $\text{sbrick } \Gamma_n$ which obtain the brick $I_2$.

Obviously, $|V_0| = |\text{sbrick}(\Gamma_n / \langle e_1 \rangle)| = |\text{sbrick } \Delta_{n-1}|$.

There is a bijection $V_1 \mapsto \text{sbrick}(\Lambda / \langle e_1 \rangle)$ defined by $S \mapsto S \setminus \{S_1\}$, so we have
\[
|V_1| = |\text{sbrick}(\Gamma_n / \langle e_1 \rangle)| = |\text{sbrick } \Delta_{n-1}|.
\]

Similarly, there are bijections
\[
V_2 \mapsto \text{sbrick}(\Lambda / \langle e_1 + e_{2n} \rangle)
\]
and
\[
V_3 \mapsto \text{sbrick}(\Lambda / \langle e_1 + e_2 \rangle),
\]
so we have
\[
|V_2| = |\text{sbrick}(\Gamma_n / \langle e_1 + e_{2n} \rangle)| = |\text{sbrick } \Delta_{n-1}|
\]
and
\[
|V_3| = |\text{sbrick}(\Gamma_n / \langle e_1 + e_2 \rangle)| = |\text{sbrick } \Delta_{n-1}^{op}|.
\]
Finally, we can define a bijection
\[ V_4 \mapsto \text{sbrick}(\Lambda/(e_1 + e_2 + e_{2n})) \times \text{sbrick}(\Lambda/(1 - e_1)) \]

given by \( V_4 \ni S \mapsto (S \setminus \{S_1, I_2\}, \overline{S_1} \cap \overline{S}) \). Thus
\[ |V_4| = \text{cbrik}(\Lambda/(e_1 + e_2 + e_{2n})) \cdot \text{cbrik}(\Lambda/(1 - e_1)) = 2|\text{cbrik} \Gamma_{n-1}|. \]

Therefore
\[ |\text{s-tilt} \Gamma'_n| = |\text{cbrik} \Gamma'_n| = \sum_{i=0}^{4} |V_i| = 2|\text{cbrik} \overline{\Delta}_{n-1}| + |\text{cbrik} \Delta_{n-1}| + |\text{cbrik} \Delta_{n-1}^{op}| + 2|\text{cbrik} \Gamma_{n-1}| = 2|\text{s-tilt} \overline{\Delta}_{n-1}| + |\text{s-tilt} \Delta_{n-1}| + |\text{s-tilt} \Delta_{n-1}^{op}| + 2|\text{s-tilt} \Gamma_{n-1}|. \]

Note that \( |\text{s-tilt} \Delta_{n-1}| \) is a linear combination of \( |\text{s-tilt} \Gamma_n| \) and \( |\text{s-tilt} \Gamma_{n-1}| \) by the equation (3.4), so \( |\text{s-tilt} \Delta_n| \) has the same recurrence relation as \( |\text{s-tilt} \Gamma_n| \). In particular, \( |\text{s-tilt} \overline{\Delta}_n|, |\text{s-tilt} \Delta_n|, |\text{s-tilt} \Gamma_n| \) have the same recurrence relations, and so \( |\text{s-tilt} \Gamma'_n| \) also has the same recurrence relation as them.

5 Examples

In this section, we give the list the numbers of (support) \( \tau \)-tilting modules over \( \Gamma_n \) and \( \Gamma'_n \) in the following table. The sequence \( |\tau\text{-tilt} \Gamma_n| \) is listed on the On-line Encyclopedia of Integer Sequences (OEIS) as the sequence A006190 and \( |\tau\text{-tilt} \Gamma'_n| \) as the sequence A006497.

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References


