Weak Injective and Weak Flat Complexes[*][†]

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Abstract

Let R be an arbitrary ring. We introduce and study a generalization of injective and flat complexes of modules, called weak injective and weak flat complexes of modules respectively. We show that a complex C is weak injective (resp. weak flat) if and only if C is exact and all cycles of C are weak injective (resp. weak flat) as R-modules. In addition, we discuss the weak injective and weak flat dimensions of complexes of modules. Finally, we show that the category of weak injective (resp. weak flat) complexes is closed under direct pure subcomplexes, pure epimorphic images and direct limits. As a result, we then determine the existence of weak injective (resp. weak flat) covers and preenvelopes of complexes.

1. Introduction

Throughout this paper, R denotes an associative ring with unity, $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{op}$) denotes the category of left (resp. right) R-modules and $\mathscr{C}$ (resp. $\mathscr{C}^{op}$) denotes the abelian category of complexes of left (resp. right) R-modules. A complex

$$\cdots \longrightarrow C_2 \xrightarrow{\delta_2^C} C_1 \xrightarrow{\delta_1^C} C_0 \xrightarrow{\delta_0^C} C_{-1} \xrightarrow{\delta_{-1}^C} \cdots$$

in $\mathscr{C}$ (or $\mathscr{C}^{op}$) is denoted by (C, δ) or C. The nth *cycle* and *boundary* of C are denoted by $Z_n(C) = \operatorname{Ker} \delta_n^C$ and $B_n(C) = \operatorname{Im} \delta_{n+1}^C$ respectively; and C is *exact* if $Z_n(C) = B_n(C)$ for any $n \in \mathbb{Z}$. General background materials are referred to [17, 13, 11, 24].

As one of important abelian categories, the category of complexes of modules has been studied by many authors (see, for example [1, 4, 13, 10, 11, 17, 25]), and many results of the category of modules which have been generalized to the category of complexes of modules. As we know, injective and flat complexes play important roles in the study of the category of complexes of modules, and a complex C is injective (resp. flat) if and only if C is exact and $Z_m(C)$ is injective (resp. flat) as R-modules for any $m \in \mathbb{Z}$; In [25, 23], Liu et al. introduced the notion of FP-injective complexes, they obtained many nice characterizations of them over coherent rings, and they showed that some properties of injective complexes have counterparts for FP-injective complexes. More recently, we introduced and investigated in [16, 14] weak injective and weak flat modules, and generalized many results from coherent rings to arbitrary rings. In this process finitely presented modules were replaced by super finitely presented

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modules. Following the above philosophy, it is natural to extend the notions of weak injective and weak flat modules to that of complexes, and then establish the relationship between the weak injectivity (resp. weak flatness) of a complex and and its cycles.

In this paper, we introduce the notions of weak injective and weak flat complexes and show that some properties of injective and flat complexes have counterparts for weak injective and weak flat complexes respectively, and there exists a close link between the weak injective dimension and weak flat dimension of complexes. We also study the existence of weak injective and weak flat covers and preenvelopes of complexes. This paper is organized as follows.

In Section 2, we collect some notations and preliminary results.

In Section 3, we introduce the notions of weak injective and weak flat complexes. We show that a complex C is weak flat (resp. weak injective) if and only if C^+ is weak injective (resp. weak flat), where C^+ stands for the character complex of C. Then we get that a complex C in $\mathscr{C}$ is weak injective if and only if C is exact and $Z_m(C)$ is weak injective in $\operatorname{Mod} R$ for any $m \in \mathbb{Z}$; a complex C in $\mathscr{C}^{op}$ is weak flat if and only if C is exact and $Z_m(C)$ is weak flat in $\operatorname{Mod} R^{op}$ for any $m \in \mathbb{Z}$.

In Section 4, we introduce and study the weak injective dimension $\operatorname{wid} C$ and the weak flat dimension $\operatorname{wfd} C$ of a complex C. For a complex C in $\mathscr{C}$, we prove that $\operatorname{wid} C \leq n$ if and only if C is exact and $\operatorname{wid}_R Z_m(C)$ (the weak injective dimension of $Z_m(C)$ in $\operatorname{Mod} R$) $\leq n$ for any $m \in \mathbb{Z}$. Dually, for a complex C in $\mathscr{C}^{op}$, we have that $\operatorname{wfd} C \leq n$ if and only if C is exact and $\operatorname{wfd}_{R^{op}} Z_m(C)$ (the weak flat dimension of $Z_m(C)$ in $\operatorname{Mod} R^{op}$) $\leq n$ for any $m \in \mathbb{Z}$. As a consequence, we get that if C is an exact complex in $\mathscr{C}$ (resp. $\mathscr{C}^{op}$), then $\operatorname{wid} C = \sup\{\operatorname{wid}_R Z_m(C) \mid m \in \mathbb{Z}\}$ (resp. $\operatorname{wfd} C = \sup\{\operatorname{wfd}_{R^{op}} Z_m(C) \mid m \in \mathbb{Z}\}$). Moreover, for a complex C, we prove that $\operatorname{wid} C = \operatorname{wfd} C^+$ and $\operatorname{wfd} C = \operatorname{wid} C^+$.

In Section 5, we show that the category of weak injective complexes and the category of weak flat complexes are closed under pure subcomplexes, pure epimorphic images and direct limits. As a consequence, we get that any complex has a weak injective (resp. weak flat) cover and a weak injective (resp. weak flat) preenvelope.

2. Preliminaries

In this paper, we use the superscripts to distinguish complexes and the subscripts for a complex. For example, if $\{C^i\}_{i\in I}$ is a family of complexes in $\mathscr{C}$, then C^i_n denotes the degree-n term of the complex C^i. Given an R-module M, we use $\overline{M}$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with the M in the 1st and 0th positions; and we denote by $S^n(M)$ the complex with M in the nth place and 0 in the other places. Given a complex C in $\mathscr{C}$ and an integer m, $C[m]$ denotes the complex such that $C[m]_n = C_{-m+n}$ and whose boundary operators are $(-1)^m\delta^C_{-m+n}$.

For complexes C and D in $\mathscr{C}$, $\operatorname{Hom}(C, D)$ is the abelian group of morphisms from C to D in the category of complexes, and $\operatorname{Ext}^i(C, D)$ for $i \geq 1$ will denote the groups we get from the right derived functor of Hom. We let $\mathscr{H}\mathrm{om}(C, D)$ be the complex of abelian groups

$$\cdots \xrightarrow{\delta_{n+1}} \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, D_{n+i}) \xrightarrow{\delta_n} \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, D_{n-1+i}) \xrightarrow{\delta_{n-1}} \cdots$$

(where $\mathbb{Z}$ is the additive group of integers) such that if $f \in \mathscr{H}\mathrm{om}(C, D)_n$, then

$$(\delta_n f)_m = \delta^D_{n+m} f_m - (-1)^n f_{m-1} \delta^C_m.$$

Let $\underline{\mathrm{Hom}}(C, D) = Z(\mathscr{H}\mathrm{om}(C, D))$. Then $\underline{\mathrm{Hom}}(C, D)$ can be made into a complex with $\underline{\mathrm{Hom}}(C, D)_n$ the abelian group of morphisms from C to $D[n]$ and with a boundary operator given by $\delta_n(f) : C \to D[n-1]$, where $f \in \underline{\mathrm{Hom}}(C, D)_n$ and $(\delta_n f)_m = (-1)^n \delta^D f_m$ for any $m \in \mathbb{Z}$. Note that the new functor $\underline{\mathrm{Hom}}(C, D)$ will have right derived functors whose values will be complexes. These values are denoted by $\underline{\mathrm{Ext}}^i(C, D)$. One easily sees that $\underline{\mathrm{Ext}}^i(C, D)$ is the complex

$$\cdots \to \mathrm{Ext}^i(C, D[n+1]) \to \mathrm{Ext}^i(C, D[n]) \to \mathrm{Ext}^i(C, D[n-1]) \to \cdots$$

with boundary operator induced by the boundary operator of D. For any complex C, the character complex $C^+ = \underline{\mathrm{Hom}}(C, \overline{\mathbb{Q}/\mathbb{Z}})$, where $\mathbb{Q}$ is the additive group of rational numbers.

For any $D \in \mathscr{C}^{op}$ and $C \in \mathscr{C}$, let $D \otimes^{\cdot} C$ be the usual tensor product of the complexes. We define $D \otimes C$ to be $\frac{D\otimes^{\cdot} C}{B(D\otimes^{\cdot} C)}$ with the maps

$$\frac{(D \otimes^{\cdot} C)_n}{B_n(D \otimes^{\cdot} C)} \to \frac{(D \otimes^{\cdot} C)_{n-1}}{B_{n-1}(D \otimes^{\cdot} C)}, \quad x \otimes y \mapsto \delta^D(x) \otimes y$$

where $x \otimes y$ is used to denote the coset in $\frac{(D\otimes^{\cdot} C)_n}{B_n(D\otimes^{\cdot} C)}$, we get a complex of abelian groups. It is obvious that the new functor $- \otimes C$ is a right exact functor, so we can construct the corresponding left derived functor $\mathrm{Tor}_i(-, C)$.

Recall from [11] that a complex C is called *finitely generated* if, in case $C = \sum_{i\in I} D_i$ with $D_i \in \mathscr{C}$ subcomplexes of C, there exists a finite subset $J \subseteq I$ such that $C = \sum_{i\in J} D_i$; and a complex C is called *finitely presented* if C is finitely generated and for any exact sequence of complexes

$$0 \to K \to L \to C \to 0$$

with L finitely generated, K is also finitely generated. A complex C is called *bounded above* (respectively, *bounded below*, *bounded*) [4] if there exists an $n \in \mathbb{Z}$ such that $C_i = 0$ for $i < n$ (respectively, $i > n$, $|i| \geq n$). By [11, Lemma 2.2], a complex C in $\mathscr{C}$ is finitely generated (resp. finitely presented) if and only if C is bounded and C_n is finitely generated (resp. finitely presented) in $\mathrm{Mod}\, R$ for any $n \in \mathbb{Z}$.

A complex P is called *projective* [13] if for any morphism $P \to D$ and any epimorphism $C \to D$, the diagram

$$\begin{tikzcd}
 & P \arrow[d] \arrow[dl, dashed] \\
C \arrow[r] & D
\end{tikzcd}$$

can be completed to a commutative diagram by a morphism $P \to C$. Dually, the notion of injective complexes is defined. Also a complex C in $\mathscr{C}$ is projective (resp. injective) if and only if C is exact and $Z_m(C)$ is projective (resp. injective) in $\operatorname{Mod} R$ for any $m \in \mathbb{Z}$.

Following [9], for any subcategory $\mathscr{F}$ of an abelian category $\mathscr{A}$, the morphism $f : F \to M$ in $\mathscr{A}$ with $F \in \mathscr{F}$ is called an *$\mathscr{F}$-precover* of M if for any morphism $g : F_0 \to M$ in $\mathscr{A}$ with $F_0 \in \mathscr{F}$, there exists a morphism $h : F_0 \to F$ such that the following diagram commutes:

$$\begin{tikzcd}
 & F_0 \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\
F \arrow[r, "f"] & M.
\end{tikzcd}$$

The morphism $f : F \to M$ is called *right minimal* if an endomorphism $h : F \to F$ is an automorphism whenever $f = fh$. An $\mathscr{F}$-precover $f : F \to M$ is called an *$\mathscr{F}$-cover* if f is right minimal. $\mathscr{F}$ is called *(pre)covering* in $\mathscr{A}$ if every object in $\mathscr{A}$ has an $\mathscr{F}$-(pre)cover. Dually, the notions of *$\mathscr{F}$-(pre)envelopes*, *left minimal morphisms* and *(pre)enveloping subcategories* are defined.

Recall from [15] that a left R-module M is called *super finitely presented* if there exists an exact sequence:

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\operatorname{Mod} R$ with each P_i finitely generated projective. Note that the super finitely presented modules are also called *strongly finitely presented* in [19], or FP_∞ in [6, 3, 20]. A left R-module M (resp. right R-module N) is called *weak injective* (resp. *weak flat*) if $\operatorname{Ext}^1_R(F, M) = 0$ (resp. $\operatorname{Tor}^R_1(N, F) = 0$) for any super finitely presented left R-module F. The *weak injective dimension* of M, denoted by $\operatorname{wid}_R M$, is defined as $\inf\{n \mid \operatorname{Ext}^{n+1}_R(F, M) = 0$ for any super finitely presented left R-module $F\}$. If no such n exists, set $\operatorname{wid}_R M = \infty$. The weak flat dimension $\operatorname{wfd}_{R^{op}} N$ of N is defined dually.

3. Weak Injective and Weak Flat Complexes

In this section, we give a treatment of weak injective and weak flat complexes. It is showed that some properties of injective and flat complexes have counterparts for weak injective and weak flat complexes respectively.

Definition 3.1. A complex C is called *super finitely presented* if there exists an exact sequence of complexes of R-modules

$$\cdots \to P^n \to \cdots \to P^1 \to P^0 \to C \to 0$$

with each P^i finitely generated projective.

From the definition, it follows that every super finitely presented complex is finitely presented.

Proposition 3.2. *The following statements are equivalent for a complex C in $\mathscr{C}$.*

(1) C is super finitely presented.

(2) C is bounded and C_m is super finitely presented in $\operatorname{Mod} R$ *for any $m \in \mathbb{Z}$.*

(3) There exists an exact sequence

$$0 \to K \to P \to C \to 0$$

in $\mathscr{C}$ with P finitely generated projective and K super finitely presented.

(4) For any exact sequence

$$0 \to K \to P \to C \to 0$$

in $\mathscr{C}$ with P finitely generated projective, K is super finitely presented.

Proof. (1) ⇒ (2) Let C be a super finitely presented complex in $\mathscr{C}$. Then there exists an exact sequence

$$\cdots \to P^n \to \cdots \to P^1 \to P^0 \to C \to 0$$

in $\mathscr{C}$ with each P^i finitely generated projective. Then, for any $m \in \mathbb{Z}$, we have the exactness of

$$\cdots \to P^n_m \to \cdots \to P^1_m \to P^0_m \to C_m \to 0$$

in $\operatorname{Mod} R$ with each P^i_m finitely generated projective. So C_m is super finitely presented for any $m \in \mathbb{Z}$. Because P^0 is bounded and $P^0 \to C$ is an epimorphism, it follows that C is bounded.

(2) ⇒ (1) Let C be the complex

$$C := \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_l \to 0 \to \cdots$$

in $\mathscr{C}$ with each C_i a super finitely presented left R-module. For each m, there exists an exact sequence

$$P^0_m \overset{\partial^0_m}{\to} C_m \to 0$$

in $\operatorname{Mod} R$ with P^0_m finitely generated projective. Then we have the following commutative diagram:

$$\begin{tikzcd}
P^0 : & 0 \arrow[r] & P^0_n \arrow[r] \arrow[d, "\partial^0_n"] & P^0_n \oplus P^0_{n-1} \arrow[r] \arrow[d, "(d^C_n \partial^0_n, \partial^0_{n-1})"] & \cdots \arrow[r] & P^0_{l+1} \oplus P^0_l \arrow[r] \arrow[d, "(d^C_{l+1} \partial^0_{l+1}, \partial^0_l)"] & P^0_l \arrow[r] \arrow[d, "0"] & 0 \\
C : & 0 \arrow[r] & C_n \arrow[r, "d^C_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] & \cdots \arrow[r, "d^C_{l+1}"] & C_l \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

in $\mathscr{C}$, where P^0 is a finitely generated projective complex. Set $K^1 = \operatorname{Ker}(P^0 \to C)$. Then K^1 is bounded and K^1_m is super finitely presented in $\operatorname{Mod} R$ for any $m \in \mathbb{Z}$ by [20, Lemma 2.3]. By repeating this process, we obtain an exact sequence

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to C \to 0$$

in $\mathscr{C}$ with each P^i finitely generated projective and C is super finitely presented.

(2) ⇒ (4) Let

$$0 \to K \to P \to C \to 0$$

be an exact sequence in $\mathscr{C}$ with P finitely generated projective. Then K is bounded because it is a subcomplex of a bounded complex P. Since C_m is a super finitely presented module for any $m \in \mathbb{Z}$, there exists an exact sequence

$$0 \to K_m \to P_m \to C_m \to 0$$

in $\operatorname{Mod} R$ with P_m finitely generated projective and K_m super finitely presented by [20, Lemma 2.3]. So K is super finitely presented by the equivalence between (1) and (2).

(4) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Suppose that there exists an exact sequence

$$0 \to K \to P \to C \to 0$$

in $\mathscr{C}$ with P finitely generated projective and K super finitely presented. Since K is super finitely presented, there exists an exact sequence

$$\cdots \to P_2^{'} \to P_1^{'} \to P_0^{'} \to K \to 0$$

in $\mathscr{C}$ with each $P_i^{'}$ finitely generated projective. Assembling the above two exact sequences, we obtain the exactness of

$$\cdots \to P_2^{'} \to P_1^{'} \to P_0^{'} \to P \to C \to 0$$

and C is super finitely presented. $\square$

We now introduce the notions of weak injective and weak flat complexes as follows.

Definition 3.3. A complex C in $\mathscr{C}$ is called *weak injective* if $\underline{\operatorname{Ext}}^1(F, C) = 0$ for any super finitely presented complex F in $\mathscr{C}$. A complex D in $\mathscr{C}^{op}$ is called *weak flat* if $\operatorname{Tor}_1(D, F) = 0$ for any super finitely presented complex F in $\mathscr{C}$.

Remark 3.4. (1) Because every super finitely presented complex is finitely presented, every FP-injective (resp. flat) complex is weak injective (resp. weak flat). When R is left coherent, the category of super finitely presented complexes coincides with that of finitely presented complexes by Proposition 3.2, and so a complex is weak injective (resp. weak flat) if and only if it is FP-injective (resp. flat).

(2) By definition, one easily checks that the category of weak injective complexes is closed under extensions, direct products and direct summands; and the category of weak flat complexes is closed under extensions, direct sums and direct summands.

Proposition 3.5. *The category of weak injective complexes is closed under direct sums.*

Proof. Let $\{C_i\}_{i \in I}$ be a family of weak injective complexes and F a super finitely presented complex in $\mathscr{C}$. Then there exists an exact sequence

$$0 \to K \to P \to F \to 0$$

in $\mathscr{C}$ with P finitely generated projective and K super finitely presented by Proposition 3.2. By [25, Lemma 2.8], we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\operatorname{Hom}}(F, \bigoplus_{i \in i} C_i) \arrow[r] \arrow[d, "\cong"] & \underline{\operatorname{Hom}}(P, \bigoplus_{i \in i} C_i) \arrow[r] \arrow[d, "\cong"] & \underline{\operatorname{Hom}}(K, \bigoplus_{i \in i} C_i) \arrow[d, "\cong"] & \\
0 \arrow[r] & \bigoplus_{i \in i} \underline{\operatorname{Hom}}_R(F, C_i) \arrow[r] & \bigoplus_{i \in i} \underline{\operatorname{Hom}}_R(P, C_i) \arrow[r] & \bigoplus_{i \in i} \underline{\operatorname{Hom}}_R(K, C_i) \arrow[r] & 0.
\end{tikzcd}$$

Because $\mathrm{Ext}^1(P, \bigoplus_{i\in i} C_i) = 0$, we have that $\mathrm{Ext}^1(F, \bigoplus_{i\in i} C_i) = 0$ and $\bigoplus_{i\in i} C_i$ is weak injective. $\square$

The following result shows that there exists a dual between weak injective complexes in $\mathscr{C}$ and weak flat complexes in $\mathscr{C}^{op}$.

Proposition 3.6.

(1) A complex C in $\mathscr{C}$ is weak flat if and only if C^+ is weak injective in $\mathscr{C}^{op}$

(2) A complex C in $\mathscr{C}$ is weak injective if and only if C^+ is weak flat in $\mathscr{C}^{op}$.

Proof. (1) By [17, Lemma 5.4.2], we have that $\underline{\mathrm{Ext}}^1(G, C^+) \cong \mathrm{Tor}_1(G, C)^+$ for any complex C in $\mathscr{C}$ and any complex G in $\mathscr{C}^{op}$. So the assertion follows.

(2) Let F be a super finitely presented complex in $\mathscr{C}$. Then there exists an exact sequence

$$0 \to K \to P \to F \to 0$$

in $\mathscr{C}$ with P finitely generated projective and K super finitely presented. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1(C^+, F) \arrow[r] \arrow[d] & C^+ \otimes K \arrow[r] \arrow[d, "\theta_K"] & C^+ \otimes P \arrow[d, "\theta_P"] \\
0 \arrow[r] & \underline{\mathrm{Ext}}^1(F, C)^+ \arrow[r] & \underline{\mathrm{Hom}}(K, C)^+ \arrow[r] & \underline{\mathrm{Hom}}(P, C)^+.
\end{tikzcd}$$

Since θ_K and θ_P are isomorphisms by [10, Lemma 2.3], we have $\underline{\mathrm{Ext}}^1(F, C)^+ \cong \mathrm{Tor}_1(C^+, F)$. Thus the desired result follows. $\square$

Proposition 3.7.

(1) If C is a weak injective left R-module, then $\overline{C}[n]$ is a weak injective complex.

(2) If D is a weak flat right R-module, then $\overline{D}[n]$ is a weak flat complex.

Proof. (1) We will show that $\underline{\mathrm{Ext}}^1(F, \overline{C}[n]) = 0$ for any super finitely presented complex F in $\mathscr{C}$. Let

$$0 \longrightarrow C \longrightarrow X \xrightarrow{\beta} F_n \longrightarrow 0$$

be an exact sequence in $\mathrm{Mod}\, R$ with F_n super finitely presented. By the factor theorem ([2, Theorem 3.6(2)]), we have the following commutative diagram:

$$\begin{tikzcd}
 & & F_{n+2} \arrow[dl, dashed, "\theta"'] \arrow[d, "\delta^F_{n+2}"] \\
0 \arrow[r] & \mathrm{Ker}\,\delta^F_{n+1} \arrow[r, "\lambda"'] & F_{n+1},
\end{tikzcd}$$

where λ is the inclusion. Consider the pullback of $X \xrightarrow{\beta} F_n$ and $F_{n+1} \xrightarrow{\delta^F_{n+1}} F_n$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \operatorname{Ker}\delta^F_{n+1} \arrow[r, equal] \arrow[d, "\gamma"] & \operatorname{Ker}\delta^F_{n+1} \arrow[d, "\lambda"] & \\
0 \arrow[r] & C \arrow[r, "\alpha_{n+1}"] \arrow[d, equal] & D \arrow[r, "u"] \arrow[d, "v"] & F_{n+1} \arrow[r] \arrow[d, "\delta^F_{n+1}"] & 0 \\
0 \arrow[r] & C \arrow[r, "\alpha_n"] & X \arrow[r, "\beta"] & F_n \arrow[r] & 0.
\end{tikzcd}$$

Then we get the following commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_{n+3} \arrow[r, "\mathrm{id}"] \arrow[d, "\delta^F_{n+3}"] & F_{n+3} \arrow[r] \arrow[d, "\delta^F_{n+3}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_{n+2} \arrow[r, "\mathrm{id}"] \arrow[d, "\gamma\theta"] & F_{n+2} \arrow[r] \arrow[d, "\delta^F_{n+2}"] & 0 \\
0 \arrow[r] & C \arrow[r, "\alpha_{n+1}"] \arrow[d, equal] & D \arrow[r, "u"] \arrow[d, "v"] & F_{n+1} \arrow[r] \arrow[d, "\delta^F_{n+1}"] & 0 \\
0 \arrow[r] & C \arrow[r, "\alpha_n"] \arrow[d] & X \arrow[r, "\beta"] \arrow[d, "\delta^F_n\beta"] & F_n \arrow[r] \arrow[d, "\delta^F_n"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_{n-1} \arrow[r, "\mathrm{id}"] \arrow[d, "\delta^F_{n-1}"] & F_{n-1} \arrow[r] \arrow[d, "\delta^F_{n-1}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_{n-2} \arrow[r, "\mathrm{id}"] \arrow[d] & F_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and a complex

$$H := \quad \cdots \to F_{n+3} \to F_{n+2} \to D \to X \to F_{n-1} \to \cdots .$$

Thus we obtain an exact sequence

$$0 \longrightarrow \overline{C}[n] \xrightarrow{\alpha} H \longrightarrow F \longrightarrow 0 \tag{3.1}$$

in $\mathscr{C}$. By Proposition 3.2, F_n is a super finitely presented left R-module. Since C is weak injective, we have $\operatorname{Ext}^1_R(F_n, C) = 0$. So the exact sequence

$$0 \longrightarrow C \xrightarrow{\alpha_n} X \longrightarrow F_n \longrightarrow 0$$

in $\operatorname{Mod} R$ splits, and there exists an R-homomorphism $f_n : X \to C$ such that $f_n\alpha_n = 1_C$. Now define $f_{n+1} : D \to C$ by $f_{n+1} = f_n v$ and $f_i = 0$ for $i \neq n, n+1$. Then we get a morphism of complexes $f : H \to \overline{C}[n]$ such that $f\alpha = 1_{\overline{C}[n]}$, and so the sequence (3.1) splits. It follows that $\underline{\operatorname{Ext}}^1(F, \overline{C}[n]) = 0$ for any super finitely presented complex F in $\mathscr{C}$, as desired.

(2) Let D be a weak flat right R-module. Then D^+ is weak injective in $\operatorname{Mod} R$ by [16, Remark 2.2(2)], and so $\overline{D^+}[n]$ is a weak injective complex in $\mathscr{C}$ by (1). One easily sees that $\overline{D^+}[n] \cong \overline{D}[n]^+$, it follows that $\overline{D}[n]$ is weak flat in $\mathscr{C}^{op}$ by Proposition 3.6(1). The desired assertion follows. ☐

Lemma 3.8. *The following statements are equivalent for a complex C in $\mathscr{C}$.*

(1) C is a weak injective complex.

(2) C_n is weak injective in $\operatorname{Mod} R$ for any $n \in \mathbb{Z}$ and $\mathscr{H}\mathrm{om}(F, C)$ is exact for any super finitely presented complex F in $\mathscr{C}$.

(3) For any exact sequence

$$0 \to Q \to X \to F \to 0$$

in $\mathscr{C}$ with F super finitely presented, the functor $\underline{\operatorname{Hom}}(-, C)$ preserves the exactness.

Proof. (1) ⇒ (2) Let G be a super finitely presented left R-module. Then there exists an exact sequence

$$0 \to N \to P_0 \to G \to 0$$

in $\operatorname{Mod} R$ with P_0 finitely generated projective and N super finitely presented. So

$$0 \to \overline{N} \to \overline{P_0} \to \overline{G} \to 0$$

is exact in $\mathscr{C}$, where $\overline{G}$ is a super finitely presented complex. Let C be a weak injective complex in $\mathscr{C}$. Then, by [10, Proposition 2.1], we have the following commutative diagram with the upper row exact:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\operatorname{Hom}}(\overline{G}, C) \arrow[r] \arrow[d, "\cong"] & \underline{\operatorname{Hom}}(\overline{P_0}, C) \arrow[r] \arrow[d, "\cong"] & \underline{\operatorname{Hom}}(\overline{N}, C) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(G, C)[1] \arrow[r] & \operatorname{Hom}_R(P_0, C)[1] \arrow[r] & \operatorname{Hom}_R(N, C)[1].
\end{tikzcd}$$

So

$$0 \to \operatorname{Hom}_R(G, C) \to \operatorname{Hom}_R(P_0, C) \to \operatorname{Hom}_R(N, C) \to 0$$

is exact, which gives the exactness of

$$0 \to \operatorname{Hom}_R(G, C_n) \to \operatorname{Hom}_R(P_0, C_n) \to \operatorname{Hom}_R(N, C_n) \to 0$$

for any $n \in \mathbb{Z}$. Since $\operatorname{Ext}^1_R(P_0, C_n) = 0$, we have that $\operatorname{Ext}^1_R(G, C_n) = 0$ and C_n is weak injective.

Now let F be a super finitely presented complex and $f : F \to C[i]$ any morphism in $\mathscr{C}$. Then, for any $i \in \mathbb{Z}$, there exists a split exact sequence

$$0 \to C[i] \to M(f) \to F[-1] \to 0$$

in $\mathscr{C}$, where $M(f)$ is the mapping cone of f. Thus f is homotopic to 0 by [17, Lemma 2.3.2]. It follows that $\mathscr{H}\mathrm{om}(F, C)$ is exact, as desired.

(2) ⇒ (1) Let

$$0 \to C \to H \to F \to 0$$

be an exact sequence in $\mathscr{C}$ with F super finitely presented. Since each C_i is weak injective by (2), this exact sequence splits at the module level and it is isomorphic to

$$0 \to C \to M(f) \to F \to 0,$$

where $f : F[1] \to C$ is a map of complexes. Since $\mathscr{H}\mathrm{om}(F[1], C)$ is exact by (2), f is homotopic to 0. It follows that

$$0 \to C \to M(f) \to F \to 0$$

is a split exact sequence in $\mathscr{C}$ by [17, Lemma 2.3.2]. Therefore $\underline{\mathrm{Ext}}^1(F, C) = 0$ and C is weak injective.

(1) ⇒ (3) is trivial.

(3) ⇒ (1) Let F be any super finitely presented complex in $\mathscr{C}$. Then there exists an exact sequence

$$0 \to Q \to P \to F \to 0$$

in $\mathscr{C}$ with P finitely generated projective. Applying $\underline{\mathrm{Hom}}(-, C)$ to it we get the exactness of

$$\underline{\mathrm{Hom}}(P, C) \to \underline{\mathrm{Hom}}(Q, C) \to \underline{\mathrm{Ext}}^1(F, C) \to 0.$$

But the sequence

$$\underline{\mathrm{Hom}}(P, C) \to \underline{\mathrm{Hom}}(Q, C) \to 0$$

is exact by (3). Consequently $\underline{\mathrm{Ext}}^1(F, C) = 0$ and C is weak injective. □

We are now in the position to give our main result.

Theorem 3.9. *The following statements are equivalent for a complex C in $\mathscr{C}$.*

(1) C is weak injective.

(2) C is exact and $Z_m(C)$ is weak injective in $\mathrm{Mod}\, R$ *for any $m \in \mathbb{Z}$.*

Proof. (1) ⇒ (2) Let C be a weak injective complex in $\mathscr{C}$. Then $\mathrm{Ext}^1(S^n(R), C) = 0$ since $S^n(R)$ is super finitely presented for any $n \in \mathbb{Z}$. Because $H_{-n+1}(C) = \mathrm{Ext}^1(S^n(R), C)$ for any $n \in \mathbb{Z}$ (see [17, p.33]), it follows that C is exact. Next we will show that $\underline{\mathrm{Ext}}^1(G, Z_m(C)) = 0$ for any super finitely presented left R-module G and $m \in \mathbb{Z}$.

Let G be a super finitely presented left R-module and

$$0 \to Q \to P \to G \to 0 \tag{3.2}$$

an exact sequence in $\mathrm{Mod}\, R$ with P finitely generated projective and Q super finitely presented. It induces an exact sequence

$$0 \to S^n(Q) \to S^n(P) \to S^n(G) \to 0$$

in $\mathscr{C}$. Then $\underline{\mathrm{Ext}}^1(S^n(G),C)=0$ by assumption. So we have the exactness of

$$\underline{\mathrm{Hom}}(S^n(P),C)\to\underline{\mathrm{Hom}}(S^n(Q),C)\to 0. \tag{3.3}$$

Now suppose $f:Q\to Z_n(C)$ be an R-homomorphism. Since C is exact, we have the following diagram with exact row:

$$\begin{tikzcd}
 & Q \arrow[d, "f"'] \arrow[dr, dashed, "\alpha_n"] & & & \\
0 \arrow[r] & Z_n(C) \arrow[r, "i"] & C_n \arrow[r] & Z_{n-1}(C) \arrow[r] & 0
\end{tikzcd}$$

in $\mathrm{Mod}\,R$. Define $\alpha_n:Q\to C_n$ by $\alpha_n=if$ and $\alpha_j=0$ for $j\neq n$. Then we obtain a morphism $\alpha:S^n(Q)\to C$ in $\mathscr{C}$. Because the sequence (3.3) is exact, there exists $\beta:S^n(P)\to C$ such that the following diagram commutes:

$$\begin{tikzcd}
S^n(Q) \arrow[r] \arrow[d, "\alpha"'] & S^n(P) \arrow[dl, dashed, "\beta"] \\
C. &
\end{tikzcd}$$

Therefore, we have a commutative diagram

$$\begin{tikzcd}
 & & Q \arrow[r] \arrow[d, "\alpha_n"'] & P \arrow[dl, dashed, "\beta_n"'] \\
0 \arrow[r] & Z_n(C) \arrow[r] & C_n \arrow[r, "\delta_n^C"] & C_{n-1}
\end{tikzcd}$$

in $\mathrm{Mod}\,R$. It is clear that $\delta_n^C\beta_n=0$, which implies that $\mathrm{Im}\,\beta_n\subseteq\mathrm{Ker}\,\delta_n^C=Z_n(C)$. So we can define a morphism $g:P\to Z_n(C)$ by $g=\beta_n$. Consequently, the sequence

$$\mathrm{Hom}_R(P,Z_n(C))\to\mathrm{Hom}_R(Q,Z_n(C))\to 0$$

is exact.

On the other hand, applying $\mathrm{Hom}_R(-,Z_n(C))$ to the exact sequence (3.2), we get the exactness of

$$\mathrm{Hom}_R(P,Z_n(C))\to\mathrm{Hom}_R(Q,Z_n(C))\to\mathrm{Ext}_R^1(G,Z_n(C))\to 0.$$

It follows that $\mathrm{Ext}_R^1(G,Z_n(C))=0$ and $Z_n(C)$ is weak injective.

(2) $\Rightarrow$ (1) Because C is exact by (2), for any $n\in\mathbb{Z}$ we have an exact sequence

$$0\to Z_n(C)\to C_n\to Z_{n-1}(C)\to 0$$

in $\mathrm{Mod}\,R$. Since both $Z_n(C)$ and $Z_{n-1}(C)$ are weak injective, C_n is weak injective. Now, by Lemma 3.8, it suffices to prove that $\mathscr{H}\mathrm{om}(G,C)$ is exact for any super finitely presented complex G in $\mathscr{C}$.

Let G be a super finitely presented complex in $\mathscr{C}$. Then G is bounded by Proposition 3.2. Thus we may suppose that

$$G:=\cdots\longrightarrow 0\longrightarrow G_n\xrightarrow{\delta_n^G}G_{n-1}\xrightarrow{\delta_{n-1}^G}\cdots\xrightarrow{\delta_2^G}G_1\xrightarrow{\delta_1^G}G_0\longrightarrow 0\longrightarrow\cdots.$$

Since $\mathscr{H}\mathrm{om}(G,C)$ is a complex of abelian groups with

$$\mathscr{H}\mathrm{om}(G,C) := \quad \cdots \xrightarrow{\delta_{n+1}} \prod_{t\in\mathbb{Z}} \mathrm{Hom}_R(G_t, C_{n+t}) \xrightarrow{\delta_n} \prod_{t\in\mathbb{Z}} \mathrm{Hom}_R(G_t, C_{n-1+t}) \xrightarrow{\delta_{n-1}} \cdots .$$

It follows that $\delta_{n-1}\delta_n = 0$, which implies that $\mathrm{Im}\,\delta_n \subseteq \mathrm{Ker}\,\delta_{n-1}$ for any $n \in \mathbb{Z}$. So we only need to show that $\mathrm{Ker}\,\delta_{n-1} \subseteq \mathrm{Im}\,\delta_n$.

Let $f \in \mathrm{Ker}\,\delta_{n-1}$. Then $\delta_{n-1}(f) = (\delta^C_{n-1+t} f_t - (-1)^{n-1} f_{t-1}\delta^G_t)_{t\in\mathbb{Z}} = 0$. Next we will construct a morphism

$$g \in \mathscr{H}\mathrm{om}_{\mathscr{C}}(G,C)_n = \prod_{t\in\mathbb{Z}} \mathrm{Hom}_R(G_t, C_{n+t}),$$

such that $\delta_n(g) = (\delta^C_{n+t} g_t - (-1)^n g_{t-1}\delta^G_t)_{t\in\mathbb{Z}} = (f_t)_{t\in\mathbb{Z}} = f$.

Notice that $f_t = 0$ for $t \leq -1$, so we take $g_t = 0$ if $t \leq -1$.

If $t = 0$, then $\delta^C_{n-1} f_0 = 0$. It follows that $\mathrm{Im}\, f_0 \subseteq \mathrm{Ker}\,\delta^C_{n-1} = \mathrm{Im}\,\delta^C_n$. Now consider the following diagram:

$$\begin{tikzcd}
 & & & G_0 \arrow[dl, dashed, "g_0"'] \arrow[d, "f_0"] & \\
0 \arrow[r] & Z_n(C) \arrow[r] & C_n \arrow[r, "\delta^C_n"'] & Z_{n-1}(C) \arrow[r] & 0.
\end{tikzcd}$$

Since $Z_n(C)$ is weak injective and G_0 is super finitely presented in $\mathrm{Mod}\, R$, there exists a homomorphism $g_0 : G_0 \to C_n$ in $\mathrm{Mod}\, R$ such that $f_0 = \delta^C_n g_0$.

If $t = 1$, then we have

$$\delta^C_n(f_1 - (-1)^{n-1} g_0\delta^G_1) = \delta^C_n f_1 - (-1)^{n-1}\delta^C_n g_0 \delta^G_1 = \delta^C_n f_1 - (-1)^{n-1} f_0\delta^G_1 = 0,$$

and so

$$\mathrm{Im}(f_1 - (-1)^{n-1} g_0\delta^G_1) \subseteq \mathrm{Ker}\,\delta^C_n = \mathrm{Im}\,\delta^C_{n+1}.$$

Putting $h_1 = f_1 - (-1)^{n-1} g_0\delta^G_1$, we have the following diagram:

$$\begin{tikzcd}
 & & & G_1 \arrow[dl, dashed, "g_1"'] \arrow[d, "h_1"] & \\
0 \arrow[r] & Z_{n+1}(C) \arrow[r] & C_{n+1} \arrow[r, "\delta^C_{n+1}"'] & Z_n(C) \arrow[r] & 0.
\end{tikzcd}$$

Since $Z_{n+1}(C)$ is weak injective and G_1 is super finitely presented in $\mathrm{Mod}\, R$, there exists a homomorphism $g_1 : G_1 \to C_{n+1}$ in $\mathrm{Mod}\, R$ such that $h_1 = \delta^C_{n+1} g_1$. Thus

$$f_1 = h_1 - (-1)^n g_0\delta^G_1 = \delta^C_{n+1} g_1 - (-1)^n g_0 \delta^G_1.$$

If $t = 2$, then

$$\begin{aligned}
\delta^C_{n+1}(f_2 - (-1)^{n-1} g_1\delta^G_2) &= \delta^C_{n+1} f_2 - (-1)^{n-1}\delta^C_{n+1} g_1\delta^G_2 \\
&= \delta^C_{n+1} f_2 - (-1)^{n-1} h_1\delta^G_2 \\
&= \delta^C_{n+1} f_2 - (-1)^{n-1}(f_1 - (-1)^{n-1} g_0\delta^G_1)\delta^G_2 \\
&= \delta^C_{n+1} f_2 - (-1)^{n-1} f_1\delta^G_2 = 0.
\end{aligned}$$

Thus $\operatorname{Im}(f_2 - (-1)^{n-1} g_1 \delta_2^G) \subseteq \operatorname{Ker} \delta_{n+1}^C = \operatorname{Im} \delta_{n+2}^C$. Set $h_2 = f_2 - (-1)^{n-1} g_1 \delta_2^G$. Consider the following diagram:

$$\begin{tikzcd}
 & & & G_2 \arrow[dl, dashed, "g_2"'] \arrow[d, "h_2"] & \\
0 \arrow[r] & Z_{n+2}(C) \arrow[r] & C_{n+2} \arrow[r, "\delta_{n+2}^C"'] & Z_{n+1}(C) \arrow[r] & 0.
\end{tikzcd}$$

Then there exists a homomorphism $g_2 : G_2 \to C_{n+2}$ in $\operatorname{Mod} R$ such that $h_2 = \delta_{n+2}^C g_2$. Also,

$$f_2 = h_2 - (-1)^n g_1 \delta_2^G = \delta_{n+2}^C g_2 - (-1)^n g_1 \delta_2^G.$$

Continuing this process, we get that $f = (f_t)_{t \in \mathbb{Z}} = \delta_n g \in \operatorname{Im} \delta_n$. Consequently $\operatorname{Ker} \delta_{n-1} \subseteq \operatorname{Im} \delta_n$. The proof is finished. $\square$

Similar to the proof of [11, Theorem 2.4], we have the following

Theorem 3.10. *The following statements are equivalent for a complex D in $\mathscr{C}^{op}$.*

(1) D is a weak flat complex.

(2) D is exact and $Z_i(D)$ is weak flat in $\operatorname{Mod} R^{op}$ *for any $i \in \mathbb{Z}$.*

(3) D^+ is a weak inective complex in $\mathscr{C}$, where

$$D^+ := \quad \cdots \to (D_{i-2})^+ \to (D_{i-1})^+ \to (D_i)^+ \to \cdots .$$

4. Weak Injective and Weak Flat Dimensions of Complexes

In this section, we introduce and investigate weak injective and weak flat dimensions of complexes. Some known results in [17] are generalized. We also show that there exists a close link between the weak injective dimension and the weak flat dimension of complexes.

Definition 4.1. (1) The *weak injective dimension* of a complex C in $\mathscr{C}$, written $\operatorname{wid} C$, is defined as $\inf\{n \mid$ there exists an exact sequence

$$0 \to C \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

in $\mathscr{C}$ with each E^i weak injective$\}$. If no such n exists, set $\operatorname{wid} C = \infty$.

(2) The *weak flat dimension* of a complex D in $\mathscr{C}^{op}$, written $\operatorname{wfd} D$, is defined as $\inf\{n \mid$ there exists an exact sequence

$$0 \to F^n \to \cdots \to F^1 \to F^0 \to D \to 0$$

in $\mathscr{C}^{op}$ with each F^i weak flat$\}$. If no such n exists, set $\operatorname{wfd} C = \infty$.

García Rozas proved in [17, Theorem 3.1.3] that for any complex C in $\mathscr{C}$, the injective dimension of C in $\mathscr{C}$ is at most n if and only if C is exact and the injective dimension of $Z_m(C)$ in $\operatorname{Mod} R$ is at most n for any $m \in \mathbb{Z}$. The following theorem generalizes this result.

Theorem 4.2. *Let C be a complex in $\mathscr{C}$. Then the following statements are equivalent.*

(1) $\operatorname{wid} C \leq n$.

(2) C is exact and $\operatorname{wid}_R Z_m(C) \leq n$ *for any $m \in \mathbb{Z}$.*

Proof. (1) ⇒ (2) Assume that $\operatorname{wid} C \leq n$ and

$$0 \to C \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

is a weak injective resolution of C in $\mathscr{C}$. By Theorem 3.9, each E^i is an exact complex. Thus we easily deduce that C is exact by [21, Theorem 6.3]. On the other hand, for any $m \in \mathbb{Z}$, we have the following exact sequence

$$0 \to Z_m(C) \to Z_m(E^0) \to Z_m(E^1) \to \cdots \to Z_m(E^n) \to 0$$

in $\operatorname{Mod} R$. By Theorem 3.9, each $Z_m(E^i)$ is weak injective. Therefore $\operatorname{wid}_R Z_m(C) \leq n$ for any $m \in \mathbb{Z}$.

(2) ⇒ (1) Let

$$0 \to C \to E^0 \to E^1 \to \cdots \to E^{n-1} \to L^n \to 0$$

be an exact sequence in $\mathscr{C}$ with each E^i weak injective. We only need to show that L^n is weak injective. Consider the following exact sequence:

$$0 \to Z_m(C) \to Z_m(E^0) \to \cdots \to Z_m(E^{n-1}) \to Z_m(L^n) \to 0$$

in $\operatorname{Mod} R$. Because $\operatorname{wid}_R Z_m(C) \leq n$ and each $Z_m(E^i)$ is weak injective by Theorem 3.9, we have $Z_m(L^n)$ is weak injective. Because C and all E^i are exact, one easily gets that L^n is exact by [21, Theorem 6.3]. Consequently, L^n is a weak injective complex by Theorem 3.9 again, and the assertion follows. □

For any complex D in $\mathscr{C}^{op}$, it is known that the flat dimension of D in $\mathscr{C}^{op}$ is at most n if and only if D is exact and the flat dimension of $Z_m(D)$ in $\operatorname{Mod} R^{op}$ is at most n for any $m \in \mathbb{Z}$ (see [17, Lemma 5.4.1]). By a dual argument to that in Theorem 4.2, we get the following

Theorem 4.3. *Let D be a complex in $\mathscr{C}^{op}$. Then the following statements are equivalent.*

(1) $\operatorname{wfd} D \leq n$.

(2) D is exact and $\operatorname{wfd}_{R^{op}} Z_m(D) \leq n$ *for any $m \in \mathbb{Z}$.*

As an application of Theorems 4.2 and 4.3, we have the following

Corollary 4.4. *Let C (resp. D) be an exact complex in $\mathscr{C}$ (resp. $\mathscr{C}^{op}$). Then we have*

(1) $\operatorname{wid} C = \sup\{\operatorname{wid}_R Z_m(C) \mid m \in \mathbb{Z}\}$.

(2) $\operatorname{wfd} D = \sup\{\operatorname{wfd}_{R^{op}} Z_m(D) \mid m \in \mathbb{Z}\}$.

Proof. The assertions follows from Theorems 4.2 and 4.3 respectively with standard arguments. □

Similar to the proofs of [16, Propositions 3.1, 3.3 and 3.4], we get the following two results.

Proposition 4.5. *For a complex C in $\mathscr{C}$, the following conditions are equivalent.*

(1) $\operatorname{wid} C \leq n$

(2) $\underline{\operatorname{Ext}}^{n+1}(F, C) = 0$ *for any super finitely presented complex F in $\mathscr{C}$.*

(3) $\underline{\mathrm{Ext}}^{n+i}(F,C)=0$ *for any super finitely presented complex* F *in* $\mathscr{C}$ *and* $i\geq 1$.

Proposition 4.6. *For a complex* D *in* $\mathscr{C}^{op}$*, the following conditions are equivalent.*
(1) $\mathrm{wfd}\, D\leq n$
(2) $\mathrm{Tor}_{n+1}(D,F)=0$ *for any super finitely presented complex* F *in* $\mathscr{C}$.
(3) $\mathrm{Tor}_{n+i}(D,F)=0$ *for any super finitely presented complex* F *in* $\mathscr{C}$ *and* $i\geq 1$.

We finish this section with the following theorem, which illustrates that there exists a close link between the weak injective and the weak flat dimension of complexes.

Theorem 4.7. *For a complex* C *in* $\mathscr{C}$ *(or* $\mathscr{C}^{op}$*), we have*
(1) $\mathrm{wid}\, C=\mathrm{wfd}\, C^{+}$.
(2) $\mathrm{wfd}\, C=\mathrm{wid}\, C^{+}$.

Proof. (1) Let F a super finitely presented complex in $\mathscr{C}$. There exists an exact sequence

$$0\to K\to P^0\to F\to 0$$

in $\mathscr{C}$ with P^0 finitely generated projective and K super finitely presented by Proposition 3.2. For any $i\geq 1$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Ext}}^{i+1}(F,C)^{+} \arrow[r] \arrow[d] & \underline{\mathrm{Ext}}^{i}(K,C)^{+} \arrow[r] \arrow[d, "\theta_K"] & \underline{\mathrm{Ext}}^{i}(P^0,C)^{+}=0 \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Tor}_{i+1}(C^{+},F) \arrow[r] & \mathrm{Tor}_{i}(C^{+},K) \arrow[r] & \mathrm{Tor}_{i}(C^{+},P^0)=0
\end{tikzcd}$$

By Proposition 3.6(2), θ_K is an isomorphism for $i=1$. Thus $\underline{\mathrm{Ext}}^{2}(F,C)^{+}\cong \mathrm{Tor}_2(C^{+},F)$ by the five lemma. By using induction, we get that $\underline{\mathrm{Ext}}^{i+1}(F,C)^{+}\cong \mathrm{Tor}_{i+1}(C^{+},F)$ for any super finitely presented complex F in $\mathscr{C}$, and so (1) holds true.

(2) It is dual to (1). □

5. Weak Injective Covers and Preenvelopes of Complexes

In this section, we show that any complex has a weak injective (resp. weak flat) cover and preenvelope.

Recall from [17] that an exact sequence

$$0\to S\to C\to C/S\to 0$$

in $\mathscr{C}$ is called *pure* if

$$\underline{\mathrm{Hom}}(P,C)\to \underline{\mathrm{Hom}}(P,C/S)\to 0$$

is exact for any finitely presented complex P in $\mathscr{C}$, or equivalently, if

$$0\to D\otimes S\to D\otimes C$$

is exact for any (finitely presented) complex D in $\mathscr{C}$. In this case, S and C/S are called a *pure subcomplex* and a *pure epimorphic image* of C respectively.

Proposition 5.1. *The category of weak injective complexes and the category of weak flat complexes are closed under pure subcomplexes, pure epimorphic images and direct limits.*

Proof. Let B be a pure subcomplex of a weak injective complex C and

$$0 \to B \to C \to C/B \to 0$$

a pure exact sequence in $\mathscr{C}$. Then for any super finitely presented complex F in $\mathscr{C}$, we get the exactness of

$$0 \to \underline{\mathrm{Hom}}(F, B) \to \underline{\mathrm{Hom}}(F, C) \to \underline{\mathrm{Hom}}(F, C/B) \to 0.$$

It follows that $\underline{\mathrm{Ext}}^1(F, B) = 0$ since $\underline{\mathrm{Ext}}^1(F, C) = 0$. Therefore, B is weak injective. On the other hand, one can easily conclude that C/B is also weak injective by Proposition 4.5, and hence the category of weak injective complexes is closed under pure epimorphic images.

Let $\{C_i\}_{i\in I}$ be a direct system of weak injective complexes and F a super finitely presented complex in $\mathscr{C}$. Then there exists an exact sequence

$$0 \to K \to P \to F \to 0$$

in $\mathscr{C}$ with P finitely generated projective and K super finitely presented. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(P, \varinjlim C_i) \arrow[r] \arrow[d, "\cong"] & \underline{\mathrm{Hom}}(K, \varinjlim C_i) \arrow[r] \arrow[d, "\cong"] & \underline{\mathrm{Ext}}^1(F, \varinjlim C_i) \arrow[r] \arrow[d] & 0 \\
\varinjlim \underline{\mathrm{Hom}}(P, C_i) \arrow[r] & \varinjlim \underline{\mathrm{Hom}}(K, C_i) \arrow[r] & \varinjlim \underline{\mathrm{Ext}}^1(F, C_i) \arrow[r] & 0.
\end{tikzcd}$$

Because $\mathscr{C}$ is locally finitely generated in the sense of [22], we have that $\underline{\mathrm{Hom}}(P, \varinjlim C_i) \cong \varinjlim \underline{\mathrm{Hom}}(P, C_i)$ and $\underline{\mathrm{Hom}}(K, \varinjlim C_i) \cong \varinjlim \underline{\mathrm{Hom}}(K, C_i)$ by [22, Chapter V, Proposition 3.4]. Consequently we have that $\underline{\mathrm{Ext}}^1(F, \varinjlim C_i) \cong \varinjlim \underline{\mathrm{Ext}}^1(F, C_i) = 0$ and $\varinjlim C_i$ is weak injective.

Now suppose that A is a pure subcomplex of a weak flat complex C in $\mathscr{C}^{op}$. Then there exists a pure exact sequence

$$0 \to A \to C \to C/A \to 0$$

in $\mathscr{C}^{op}$, which induces a split exact sequence

$$0 \to (C/A)^+ \to C^+ \to A^+ \to 0$$

in $\mathscr{C}$. By Proposition 3.6(1), C^+ is weak injective. Since A^+ is isomorphic to a direct summand of C^+, A^+ is also weak injective by Remark 3.4(2). Therefore A is weak flat by Proposition 3.6(1) again. On the other hand, let F be a super finitely presented complex in $\mathscr{C}$. Then we have the following exact sequence

$$0 \to A \otimes F \to C \otimes F \to C/A \otimes F \to 0.$$

It follows that $\mathrm{Tor}_1(C/A, F) = 0$ since $\mathrm{Tor}_1(C, F) = 0$. Thus C/A is weak flat and the category of weak flat complexes is closed under pure epimorphic images.

Let $\{D_i\}_{i\in I}$ be a direct system of weak flat complexes in $\mathscr{C}^{op}$ and F a super finitely presented complex in $\mathscr{C}$. Then there exists an exact sequence

$$0 \to L \to P \to F \to 0$$

in $\mathscr{C}$ with P finitely generated projective and L super finitely presented. By [17, Proposition 4.2.1], we obtain the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1(\varinjlim D_i, F) \arrow[r] \arrow[d] & (\varinjlim D_i) \otimes L \arrow[r] \arrow[d, "\cong"] & (\varinjlim D_i) \otimes P \arrow[d, "\cong"] \\
0 \arrow[r] & \varinjlim \mathrm{Tor}_1(D_i, F) \arrow[r] & \varinjlim (D_i \otimes L) \arrow[r] & \varinjlim (D_i \otimes P)
\end{tikzcd}$$

It follows that $\mathrm{Tor}_1(\varinjlim D_i, F) \cong \varinjlim \mathrm{Tor}_1(D_i, F) = 0$, and so $\varinjlim D_i$ is weak flat. $\square$

Recall from [8] that a category $\mathscr{D}$ is called *finitely accessible* (or *locally finitely presented* in [7]) if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects. It was showed in [8] that if $\mathscr{D}$ is a finitely accessible category and $\mathscr{B}$ is a class of objects of $\mathscr{D}$ closed under direct limits and pure epimorphic images, then $\mathscr{B}$ is covering; if $\mathscr{D}$ is a finitely accessible additive category with products and $\mathscr{B}$ is a class of objects of $\mathscr{D}$ closed under products and pure subobjects, then $\mathscr{B}$ is a preenveloping class.

We now are in a position to prove the following

Theorem 5.2.

(1) Any complex in $\mathscr{C}$ has a weak injective cover.

(2) Any complex in $\mathscr{C}^{op}$ has a weak flat cover.

Proof. (1) By [10, Lemma 2.2], any complex is a direct limit of finitely presented complexes. It is easy to see that $\mathscr{C}$ is finitely accessible. Because the category of weak injective complexes is closed under direct limits and pure epimorphic images by Proposition 5.1, it follows from [8, Theorem 2.6] that any complex in $\mathscr{C}$ has a weak injective cover.

(2) It is dual to (1). $\square$

For a complex C, its *cardinality* is defined to be $|\coprod_{n\in\mathbb{Z}} C_n|$ in [18].

Theorem 5.3.

(1) Any complex in $\mathscr{C}^{op}$ has a weak flat preenvelope.

(2) Any complex in $\mathscr{C}$ has a weak injective preenvelope.

Proof. (1) The proof is modelled on that of [17, Theorem 5.2.2].

Because any direct product of weak flat modules is weak flat by [16, Theorem 2.13], it follows that a direct product of weak flat complexes is also a weak flat complex since it is exact and the kernels of the boundary operators are weak flat.

Let C be a complex in $\mathscr{C}^{op}$ and $\mathcal{N}_\beta$ an infinite cardinal number such that $\mathrm{Card}(C) \cdot \mathrm{Card}(R) \leq \mathcal{N}_\beta$. Set $Y = \{D \mid D \text{ is a weak flat complex in } \mathscr{C}^{op} \text{ and } \mathrm{Card}(D) \leq \mathcal{N}_\beta\}$. Let $\{D_i\}_{i\in I}$ be a family of representatives of this class with the index set I. Let $H_i = \mathrm{Hom}(C, D_i)$ for any $i \in I$, and let $F = \prod D_i^{H_i}$. Then F is a weak flat complex in $\mathscr{C}^{op}$. Define $\varphi : C \to F$ such that the composition of φ with the projective map $F \to D_i^{H_i}$ maps $x \in F^k$ to $(h^k(x))_{h\in H_i}$. Then it is easy to see that $\varphi : C \to F$ is a map of complexes. We claim that $\varphi : C \to F$ is a weak flat preenvelope. Now let $\varphi^{'} : C \to G$ with G a weak flat complex. By [17, Lemma 5.2.1], the subcomplex $\varphi^{'}(C)$ can be enlarged to a pure subcomplex $G^{'} \subseteq G$ with $\mathrm{Card}(G^{'}) \leq \mathcal{N}_\beta$. Note that $G^{'}$ is weak flat by Proposition 5.1. So $G^{'}$ is isomorphic to one of the D_i. By the construction of the map φ, one easily sees that $\varphi^{'}$ can be factored through φ. Consequently, the first assertion follows.

(2) It is dual to (1). $\square$

Remark 5.4. From the proof of Theorem 5.3, it follows that the category of weak flat complexes is closed under direct products. Note that the category of weak injective complexes is closed under direct products by Remark 3.4(2). We can also obtain Theorem 5.3 directly from [8, Theorem 4.1] because the category of weak injective complexes and the category of weak flat complexes are closed under pure subcomplexes by Proposition 5.1.

Proposition 5.5. *Let C be a complex in $\mathscr{C}$.*

(1) If $f : G \to C$ is a weak injective precover in $\mathscr{C}$, then $f_n : G_n \to C_n$ is a weak injective precover in $\mathrm{Mod}\, R$ *for any $n \in \mathbb{Z}$.*

(2) If $g : C \to D$ is a weak injective preenvelope in $\mathscr{C}$, then $g_n : C_n \to D_n$ is a weak injective preenvelope in $\mathrm{Mod}\, R$ *for any $n \in \mathbb{Z}$.*

Proof. (1) Let E be a weak injective left R-module and $h : E \to C_n$ an R-homomorphism. Define a morphism $\overline{h} : \overline{E}[n-1] \to C$ in $\mathscr{C}$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & E \arrow[r, "\mathrm{id}"] \arrow[d, "h"] & E \arrow[r] \arrow[d, "\delta_n^C h"] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & C_{n-2} \arrow[r] & \cdots .
\end{tikzcd}$$

Since $\overline{E}[n-1]$ is a weak injective complex by Proposition 3.7, and since $f : G \to C$ is a weak injective precover of C in $\mathscr{C}$ by assumption, there exists a morphism $\alpha : \overline{E}[n-1] \to G$ in $\mathscr{C}$ such that $f\alpha = \overline{h}$. So we have a commutative diagram:

$$\begin{tikzcd}
 & E \arrow[dl, dashed, "\alpha_n"'] \arrow[d, "h"] \\
G_n \arrow[r, "f_n"'] & C_n
\end{tikzcd}$$

in $\mathrm{Mod}\, R$. This means that $f_n : G_n \to C_n$ is a weak injective precover of C_n in $\mathrm{Mod}\, R$.

(2) Let F be a weak injective left R-module and $\beta : C_n \to F$ an R-homomorphism. Define a morphism $\overline{\beta} : C \to \overline{F}[n]$ in $\mathscr{C}$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & C_{n+2} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d, "\beta\delta_{n+1}^C"] & C_n \arrow[r] \arrow[d, "\beta"] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & F \arrow[r, "\mathrm{id}"] & F \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Because $\overline{F}[n]$ is a weak injective complex by Proposition 3.7, and since $g : C \to D$ is a weak injective preenvelope of C in $\mathscr{C}$ by assumption, there exists a morphism $\gamma : D \to \overline{F}[n]$ in $\mathscr{C}$ such that $\gamma g = \overline{\beta}$. So we have a commutative diagram:

$$\begin{tikzcd}
C_n \arrow[r, "g_n"] \arrow[d, "\beta"'] & D_n \arrow[dl, dashed, "\gamma_n"] \\
F &
\end{tikzcd}$$

in $\operatorname{Mod} R$. This shows that $g_n : C_n \to D_n$ is a weak injective preenvelope of $C_n$in $\operatorname{Mod} R$. □

Dually, we have the following

Proposition 5.6. *Let C be a complex in $\mathscr{C}^{op}$.*

(1) If $f : G \to C$ is a weak flat precover in $\mathscr{C}^{op}$, then $f_n : G_n \to C_n$ is a weak flat precover in $\operatorname{Mod}R^{op}$ *for any $n \in \mathbb{Z}$.*

(2) If $g : C \to D$ is a weak flat preenvelope in $\mathscr{C}^{op}$, then $g_n : C_n \to D_n$ is a weak flat preenvelope in $\operatorname{Mod}R^{op}$ *for any $n \in \mathbb{Z}$.*

In the following result, we give some equivalent characterizations for ${}_RR$ being weak injective in terms of the properties of weak injective and weak flat complexes.

Theorem 5.7. *The following statements are equivalent.*

(1) ${}_RR$ is weak injective.

(2) Every injective complex in $\mathscr{C}^{op}$ is weak flat.

(3) Every flat complex in $\mathscr{C}$ is weak injective.

(4) Every complex in $\mathscr{C}^{op}$ has a monic weak flat preenvelope.

(5) Every complex in $\mathscr{C}$ has an epic weak injective cover.

Proof. (1) ⇒ (2) Let C be an injective complex in $\mathscr{C}^{op}$. Then C is exact and $Z_m(C)$ is an injective right R-module for any $m \in \mathbb{Z}$. Since ${}_RR$ is weak injective, $Z_m(C)$ is a weak flat right R-module by [16, Proposition 2.17]. Thus C is weak flat by Theorem 3.10.

(2) ⇒ (1) Let M be an injective right R-module. Then $\overline{M}$ is an injective complex in $\mathscr{C}^{op}$, and hence $\overline{M}$ is a weak flat complex by (2). It follows that M is a weak flat right R-module. Then ${}_RR$ is weak injective by [16, Proposition 2.17].

(1) ⇔ (3) It is dual to (1) ⇔ (2).

(1) ⇒ (4) Since ${}_RR$ is weak injective, every injective right R-module is weak flat by [16, Proposition 2.17]. Thus every injective complex in $\mathscr{C}^{op}$ is weak flat, and so (4) follows.

(4) ⇒ (2) Let I be an injective complex in $\mathscr{C}^{op}$. By (4), there exists an exact sequence

$$0 \to I \to F \to N \to 0$$

in $\mathscr{C}^{op}$ with $I \to F$ a weak flat preenvelope of I. The sequence is split since I is injective. Thus I is weak flat as a direct summand of F by Remark 3.4(2).

(1) ⇒ (5) Let C be a complex in $\mathscr{C}$. Then, by Theorem 5.2, C has a weak injective cover $f : E \to C$ in $\mathscr{C}$. On the other hand, there exists an exact sequence

$$F \to C \to 0$$

in $\mathscr{C}$ with F free. Then $F \cong \bigoplus_{n\in\mathbb{Z}} \overline{R}^{(X_n)}[n]$. Since ${}_RR$ is weak injective by (1), we have that $\overline{R}^{(X_n)}[n]$ is weak injective, and so f is an epimorphism.

(5) ⇒ (1) Let $E \to \overline{R}$ be an epic weak injective cover of $\overline{R}$ in $\mathscr{C}$. Then ${}_RR$ is isomorphic to a direct summand of a weak injective left R-module E_0, and so ${}_RR$ is weak injective by [16, Proposition 2.3]. □

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