

Pure semisimple conjecture and compactly generated triangulated categories

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Abstract: For a ring R , the famous pure semisimple conjecture states that R is left pure semisimple if and only if it is right pure semisimple, which remains still open. By a work of Beligiannis we know that the derived pure semisimple conjecture implies the pure semisimple conjecture. In this paper, we prove that the derived pure semisimple conjecture is a derived equivalent invariant, which provides new examples of rings that satisfy the pure semisimple conjecture. We then continue Beligiannis' effort for another familiar compactly generated triangulated categories and use their triangle equivalences to construct derived equivalences. Finally, we prove that the pure semisimplicity of locally finitely presented Grothendieck categories with enough projectives can be obtained from the pure semisimplicity of some certain compactly generated triangulated categories.

Keywords: (derived) pure semisimple conjecture; locally finitely presented Grothendieck categories; compactly generated triangulated categories; (pure) derived categories; homotopy categories; derived equivalence.

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1. Introduction

Throughout this article, all rings R are assumed to be associative rings with identity and all modules are unitary. Let R be a ring. In what follows, by an “ R -module” we always mean a left R -module; for a right R -module, we view it as an R^{op} -module, where R^{op} is the opposite ring of R .

Recall that a ring R is *left pure semisimple* if any R -module is pure projective, or equivalently, any R -module is pure injective, in which an R -module is called *pure projective* (resp. *pure injective*) if it is projective (resp. injective) with respect to any pure exact short exact sequence of R -modules. A ring R is called *right pure semisimple* if R^{op} is left pure semisimple. It is well known that a ring R is semisimple if and only if any R -modules is projective (or injective), and this happens if and only if any R^{op} -module is projective (or injective). While the question whether a ring is left pure semisimple if and only if it is right pure semisimple remains still open, which is known as the *pure semisimple conjecture* (PSC for short) in the literature. This conjecture has been verified for several classes of rings, such

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as, von Neumann regular rings, Artin algebras, QF rings, rings having a self-duality and rings with polynomial identity, (see Lemma 5.6 and the remarks between Example 5.8 and Remark 5.9). For more details of **PSC**, we refer to [8, 9, 17, 18, 16].

For any ring R , the category $R\text{-Mod}$ of R -modules is an important example of locally finitely presented Grothendieck categories. They were extensively studied; see [6, 7, 2, 14, 21]. In particular, the pure semisimplicity of them was proved to be closely related to **PSC**; see [7] for details. On the other hand, for a compactly generated triangulated category $\mathcal{T}$, in order to establish homological algebra as in Abelian categories, Beligiannis [2] introduced the proper class for $\mathcal{T}$ and investigated the pure semisimplicity of $\mathcal{T}$. It follows that pure semisimple compactly generated triangulated categories are an analogue of pure semisimple locally finitely presented Grothendieck categories. Our first motivation is to obtain some result which further supports such a viewpoint.

Theorem A. (Theorem 3.3). Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives with $\text{proj}(\mathcal{A})$ its subcategory consisting of finitely generated projective objects, and let $\mathcal{T}_{\mathcal{A}}$ be a compactly generated category consisting of complexes of objects in $\mathcal{A}$ with $\mathcal{T}_{\mathcal{A}}^c$ its triangulated subcategory consisting of compact objects. Suppose that the following condition is satisfied:

($\mathbf{T}_{\mathcal{A}}$) there are fully faithful functors $\mathcal{A} \rightarrow \mathcal{T}_{\mathcal{A}}$, and $\text{proj}(\mathcal{A}) \rightarrow \mathcal{T}_{\mathcal{A}}^c$ and the objects in $\mathcal{T}_{\mathcal{A}}^c$ are complexes of finitely presented objects in $\mathcal{A}$.

If $\mathcal{T}_{\mathcal{A}}$ is pure semisimple, then so is $\mathcal{A}$.

Note from Examples 3.4 and 3.7 that the condition ($\mathbf{T}_{\mathcal{A}}$) is satisfied under various circumstances.

It follows from [2] that **PSC** is closely related to the pure semisimplicity of the usual derived category $\mathbf{D}(R\text{-Mod})$ of R -modules. The *derived pure semisimple conjecture* (**DPSC** for short) [2, 12.17] states that $\mathbf{D}(R\text{-Mod})$ is pure semisimple if and only if so is $\mathbf{D}(R^{\text{op}}\text{-Mod})$. It was shown in [2, Remark 12.18] that **DPSC** implies **PSC**, more precisely, **PSC** for a ring R is valid provided that $\mathbf{D}(R\text{-Mod})$ is pure semisimple if and only if so is $\mathbf{D}(R^{\text{op}}\text{-Mod})$.

The second motivation of this paper is to continue Beligiannis' effort for another compactly generated triangulated categories, such as the pure derived category $\mathbf{D}_{\text{pur}}(R\text{-Mod})$, the homotopy category $\mathbf{K}(R\text{-Proj})$ (resp. $\mathbf{K}(R\text{-Inj})$) of projective (resp. injective) modules and the derived category $\mathbf{D}(R\text{-FP-I})$ of FP-injective modules, which are closely related to $\mathbf{D}(R\text{-Mod})$. For more applications of such triangulated categories, we refer to [11, 12, 21].

Our second main result is as follows, which establishes a link between **PSC** and the pure semisimplicity of $\mathbf{D}_{\text{pur}}(R\text{-Mod})$, $\mathbf{K}(R\text{-Proj})$, $\mathbf{K}(R\text{-Inj})$ and $\mathbf{D}(R\text{-FP-I})$, respectively.

Theorem B. (Corollaries 4.6 and 4.7). Let R be a ring.

- (1) **PSC** for R is valid if one of the following conditions is satisfied.
 - (i) $\mathbf{D}_{\text{pur}}(R\text{-Mod})$ is pure semisimple if and only if $\mathbf{D}_{\text{pur}}(R^{\text{op}}\text{-Mod})$ is pure semisimple;
 - (ii) $\mathbf{D}(R\text{-FP-I})$ is pure semisimple if and only if $\mathbf{D}(R^{\text{op}}\text{-FP-I})$ is pure semisimple.
- (2) **PSC** for R is valid if R is a left and right coherent ring satisfying one of the following conditions.
 - (iii) $\mathbf{K}(R\text{-Proj})$ is pure semisimple if and only if $\mathbf{K}(R^{\text{op}}\text{-Proj})$ is pure semisimple;

- (iv) $K(R\text{-Inj})$ is pure semisimple if and only if $K(R^{\text{op}}\text{-Inj})$ is pure semisimple.
- (3) If R is a hereditary ring, then all of the above conditions (i)–(iv) and the following two conditions are equivalent.
 - (v) $\mathbf{D}(R\text{-Mod})$ is pure semisimple if and only if $\mathbf{D}(R^{\text{op}}\text{-Mod})$ is pure semisimple;
 - (vi) **PSC** for R is valid.

Motivated by the classical Morita theory of categories of modules, Rickard developed a derived Morita theory in terms of one-sided tilting complexes, which plays an important role not only in modern mathematics but also in some neighbouring fields in mathematics; see [10, Introduction] for details. Recall that two rings are *derived equivalent* if their derived categories are triangle equivalent. As a generalization of Morita equivalences of rings, derived equivalences of rings share many important invariants; see [10, Introduction] for example. One topic on the derived Morita theory, as a comparison with the classical Morita theory, is to investigate new invariants which are (not) preserved under derived equivalences of rings.

Note that “**DPSC**” is an invariant under derived equivalences of rings (see Remark 5.4). Combining this with Theorem B, we obtain the following result.

Theorem C. (Theorem 5.5). Let R be a ring derived equivalent to a hereditary ring S . If S satisfies **PSC**, then so does R .

Although Theorem C does not recover the known result on rings which satisfy **PSC**, it broadens the classes of rings satisfy this conjecture. For these classes, see Example 5.8, which shows that “having a self-duality” is not a property of rings preserved under derived equivalences of rings. However, rings having a self-duality satisfy **PSC** as mentioned above. On the other hand, it has been known that all rings satisfy **PSC** whenever they possess the properties “being von Neumann regular”, “being an Artin algebra” and “admitting a polynomial identity”, but such properties are invariants under derived equivalences of rings (Remark 5.9).

In addition, we show that derived equivalences of rings can be obtained from triangle equivalences of another compactly generated triangulated categories (Proposition 5.3). Combining this result with Theorem C, we obtain several sufficient conditions for **PSC** to be valid (Corollaries 5.10 and 5.11).

2. Preliminaries

Let R be a ring. We denote by $R\text{-Mod}$ the category of all left R -modules (R -modules for short) and by $R\text{-Proj}$ (resp. $R\text{-Inj}$) the subcategory of $R\text{-Mod}$ consisting of all projective (resp. injective) R -modules, and denote by $\text{Ch}(R\text{-Mod})$ (resp. $K(R\text{-Mod})$, $\mathbf{D}(R\text{-Mod})$ and $\mathbf{D}_{\text{pur}}(R\text{-Mod})$) the complex category (resp. the unbounded homotopy category, the unbounded derived category and the unbounded pure derived category) of $R\text{-Mod}$. Besides, we denote by $K(R\text{-Proj})$ (resp. $K(R\text{-Inj})$) the homotopy category of $R\text{-Proj}$ (resp. $R\text{-Inj}$). On the other hand, we denote by $R\text{-mod}$ the category of all finitely presented R -modules and by $R\text{-proj}$ the subcategory of $R\text{-mod}$ consisting of all projective modules, denote by $\text{Ch}(R\text{-mod})$ (resp. $K^b(R\text{-mod})$ and $\mathbf{D}^b(R\text{-mod})$) the complex category (resp. the bounded homotopy

category and the bounded derived category) of $R\text{-mod}$, and denote by $K^b(R\text{-proj})$ the bounded homotopy category of $R\text{-proj}$.

We refer to the readers that when we consider the corresponding categories involving R^{op} -modules (i.e., right R -modules), we just replace R by R^{op} , where R^{op} stands for the opposite ring of R .

2.1. Functor categories. Let $\mathcal{A}$ be a skeletally small additive category. Its functor category is denoted by $\text{Mod}(\mathcal{A})$, which consists of additive contravariant functors from $\mathcal{A}$ to the category Ab of Abelian groups. A functor $F \in \text{Mod}(\mathcal{A})$ is called a *representable functor* (or a *projective functor*) if $F = \text{Hom}_{\mathcal{A}}(-, A)$ for some $A \in \mathcal{A}$. A functor $F \in \text{Mod}(\mathcal{A})$ is called a *flat functor* if F is a direct limit of representable functors. By the Yoneda's lemma, there is a fully faithful inclusion

$$\mathbf{Y} : \mathcal{A} \rightarrow \text{Mod}(\mathcal{A}) \text{ via } A \mapsto \text{Hom}_{\mathcal{A}}(-, A),$$

which is known as the *Yoneda embedding* in the literature.

2.2. Locally finitely presented Grothendieck categories and their homotopy categories and (pure) derived categories. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category, that is, $\mathcal{A}$ is a Grothendieck category that has (exact) direct limits and a small set $\mathcal{S}$ of finitely presented objects, and any object in $\mathcal{A}$ can be expressed as a direct limits of objects from $\mathcal{S}$. Here, an object F is said to be *finitely presented* if the functor $\text{Hom}_{\mathcal{A}}(F, -)$ commutes with direct limits. On the other hand, an object F is said to be *finitely generated* if the functor $\text{Hom}_{\mathcal{A}}(F, -)$ commutes with direct sums. The subcategory of $\mathcal{A}$ consisting of finitely presented (resp. finitely generated) objects is denoted by $\text{fp}(\mathcal{A})$ (resp. $\text{fg}(\mathcal{A})$). Trivially $\text{fp}(\mathcal{A}) \subseteq \text{fg}(\mathcal{A})$. Note that $R\text{-Mod}$ is a locally finitely presented Grothendieck category (with enough projectives and injectives) for any ring R .

A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{A}$ is called *pure exact* if for any $F \in \text{fp}(\mathcal{A})$, there is a short exact sequence of Abelian groups

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(F, A) \rightarrow \text{Hom}_{\mathcal{A}}(F, B) \rightarrow \text{Hom}_{\mathcal{A}}(F, C) \rightarrow 0.$$

An object P (resp. E) in $\mathcal{A}$ is called *pure projective* (resp. *pure injective*) if P (resp. E) is projective (resp. injective) with respect to any pure exact short exact sequence in $\mathcal{A}$. The category $\mathcal{A}$ is called *pure semisimple* if any object in $\mathcal{A}$ is pure projective (or equivalently, pure injective).

We denote by $\text{Ch}(\mathcal{A})$ the category of cochain complexes of objects from $\mathcal{A}$. Here a cochain complex (simply complex) $X^\bullet$ is a sequence

$$X^\bullet = \cdots \rightarrow X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \rightarrow \cdots$$

in $\mathcal{A}$ with $d^k d^{k-1} = 0$ for any $k \in \mathbb{Z}$; and a *morphism* $f^\bullet : X^\bullet \rightarrow Y^\bullet$ of complexes is a family of morphisms $f^\bullet = (f^k : X^k \rightarrow Y^k)_{k \in \mathbb{Z}}$ in $\mathcal{A}$ satisfying $d_Y^k \cdot f^k = f^{k+1} \cdot d_X^k$ for each k . A morphism $f^\bullet : X^\bullet \rightarrow Y^\bullet$ of complexes is called *null-homotopic* if there is a family $(h^k : X^k \rightarrow Y^{k-1})_{k \in \mathbb{Z}}$ of morphisms in $\mathcal{A}$ such that

$$f^k = d_Y^{k-1} h^k + h^{k+1} d_X^k.$$

for any $k \in \mathbb{Z}$. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is the category $\text{Ch}(\mathcal{A})$ with morphisms of the cochain maps up to null-homotopy.

Given a complex $X^\bullet$, the k th *cycle* (resp., *homology*) of $X^\bullet$ is defined as $\text{Ker} d^k$ (resp., $\text{Ker} d^k / \text{Im} d^{k-1}$) and denoted by $Z^k(X^\bullet)$ (resp., $H^k(X^\bullet)$). A complex $X^\bullet$ is called *bounded* (resp., *homology bounded*) if $X^k = 0$ (resp., $H^k(X^\bullet) = 0$) for $|k| \gg 0$, and $X^\bullet$ is called *exact* if $H^k(X^\bullet) = 0$ for any $k \in \mathbb{Z}$, or equivalently, there is a short exact sequence $0 \rightarrow Z^k(X^\bullet) \rightarrow X^k \rightarrow Z^{k+1}(X^\bullet) \rightarrow 0$ in $\mathcal{A}$ for any $k \in \mathbb{Z}$. A complex $X^\bullet$ is called *pure exact* if $X^\bullet$ is exact and any short exact sequence $0 \rightarrow Z^k(X^\bullet) \rightarrow X^k \rightarrow Z^{k+1}(X^\bullet) \rightarrow 0$ is pure exact for any $k \in \mathbb{Z}$. We denote by $K_{\text{ac}}(\mathcal{A})$ (resp., $K_{\text{pac}}(\mathcal{A})$) the homotopy subcategory of $K(\mathcal{A})$ consisting of all exact (resp., pure exact) complexes. Now we can define the derived category $\mathbf{D}(\mathcal{A})$ (resp., the pure derived category $\mathbf{D}_{\text{pur}}(\mathcal{A})$) of $\mathcal{A}$ as the Verdier quotients

$$\mathbf{D}(\mathcal{A}) := K(\mathcal{A})/K_{\text{ac}}(\mathcal{A}) \quad \text{and} \quad \mathbf{D}_{\text{pur}}(\mathcal{A}) := K(\mathcal{A})/K_{\text{pac}}(\mathcal{A}).$$

Note that $K(\mathcal{A})$, $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}_{\text{pur}}(\mathcal{A})$ are triangulated categories, their autofunctors are denoted by [1].

2.3. Compactly generated triangulated categories. Let $\mathcal{T}$ be a triangulated category with small coproducts. Recall that an object $C \in \mathcal{T}$ is *compact* if for any collection $\{T_j \mid j \in J\}$ of objects in $\mathcal{T}$, the canonical morphism

$$\coprod_{j \in J} \text{Hom}_{\mathcal{T}}(C, T_j) \rightarrow \text{Hom}_{\mathcal{T}}(C, \coprod_{j \in J} T_j)$$

is an isomorphism. The category $\mathcal{T}$ is *compactly generated* if there exists a set $\mathcal{U} \subseteq \mathcal{T}$ of compact objects such that for each $0 \neq T \in \mathcal{T}$ there is a morphism $0 \neq f : \Sigma^m U \rightarrow T$ for some $U \in \mathcal{U}$ and $m \in \mathbb{Z}$, where Σ denotes the autofunctor of $\mathcal{T}$. We denote by $\mathcal{T}^c$ the triangulated subcategory of $\mathcal{T}$ consisting of compact objects in $\mathcal{T}$.

A triangle $T_1 \rightarrow T \rightarrow T_2$ in $\mathcal{T}$ is called *pure exact* if for any $P \in \mathcal{T}^c$, there is a short exact sequence of Abelian groups

$$0 \rightarrow \text{Hom}_{\mathcal{T}}(P, T_1) \rightarrow \text{Hom}_{\mathcal{T}}(P, T) \rightarrow \text{Hom}_{\mathcal{T}}(P, T_2) \rightarrow 0.$$

It follows that all pure exact triangles form a *proper class* (in the sense of [2]) in $\mathcal{T}$ that can establish homological algebra in $\mathcal{T}$: the compactly generated triangulated category $\mathcal{T}$ equipped with such a proper class has enough projectives and injectives by [2, Lemma 8.1 and Theorem 8.6]. The “projective” and “injective” objects in the category $\mathcal{T}$ equipped with pure triangles are called *pure projective* and *pure injective* respectively.

$\mathcal{T}$ is called *pure semisimple* if any object in $\mathcal{T}$ is pure projective (or equivalently, pure injective) (see [2, Terminology 11.1]). It follows from [2, Proposition 11.2] that $\mathcal{T}$ is pure semisimple if and only all flat functors in $\text{Mod}(\mathcal{T}^c)$ are projective. For more characterizations of pure semisimple compactly generated triangulated categories, we refer to [2, Theorem 9.3].

3. Pure semisimplicity of locally finitely presented Grothendieck category with enough projectives

We start with the following observation, which is essentially made by [13, Chapter 3, Exercise 3.6.4].

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be skeletally small additive categories with a fully faithful functor $inc : \mathcal{A} \hookrightarrow \mathcal{B}$. Then the restriction functor*

$$inc_* : \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B})$$

has a fully faithful right adjoint.

Proof. Let $\mathbf{Y}_{\mathcal{A}} : \mathcal{A} \hookrightarrow \text{Mod}(\mathcal{A})$ and $\mathbf{Y}_{\mathcal{B}} : \mathcal{B} \hookrightarrow \text{Mod}(\mathcal{B})$ be the Yoneda embedding functors, which are given by

$$A \mapsto \text{Hom}_{\mathcal{A}}(-, A) \quad (\forall A \in \mathcal{A}) \quad \text{and} \quad B \mapsto \text{Hom}_{\mathcal{B}}(-, B) \quad (\forall B \in \mathcal{B})$$

respectively. Then by [13, Chapter 3, Exercise 3.6.4], the restriction functor inc_* has a right adjoint ${}^*inc : \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B})$ with a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{inc} & \mathcal{B} \\ \mathbf{Y}_{\mathcal{A}} \downarrow & & \downarrow \mathbf{Y}_{\mathcal{B}} \\ \text{Mod}(\mathcal{A}) & \xrightarrow{{}^*inc} & \text{Mod}(\mathcal{B}). \end{array}$$

Now ${}^*inc$ is fully faithful since so are inc , $\mathbf{Y}_{\mathcal{A}}$ and $\mathbf{Y}_{\mathcal{B}}$. □

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives. We denote by $\text{proj}(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting of all finitely generated projective objects (or equivalently, consisting of all finitely presented projective objects).

Lemma 3.2. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives. Then there is an equivalence of Grothendieck categories*

$$\text{Mod}(\text{proj}(\mathcal{A})) \rightarrow \mathcal{A},$$

which can be viewed as an identity.

Proof. The result is well-known, one can see [19, remarks in p.301]: the assumption that $\mathcal{A}$ is a locally finitely presented Grothendieck category with enough projectives, implies that it is a locally finitely presented Grothendieck category admitting a family of finitely generated projective generators. □

Given an Abelian category $\mathcal{A}$, we call that $\mathcal{A}$ is a *Frobenius category* if $\mathcal{A}$ has enough projectives and injectives such that the projective objects and the injective objects coincide. For an additive subcategory $\mathcal{B}$ of $\mathcal{A}$, we denote by $\text{Add}(\mathcal{B})$ the subcategory of $\mathcal{A}$ consisting of all objects which can be expressed as a direct summand of $\bigoplus_{j \in J} X_j$ for a family $\{X_j \mid i \in J\}$ of objects in $\mathcal{B}$.

Given a Grothendieck category $\mathcal{A}$, we say that $\mathcal{A}$ is *locally Noetherian* if $\mathcal{A}$ has set of generators consisting of Noetherian objects. It is known that a locally finitely presented Grothendieck category $\mathcal{A}$ is locally Noetherian if $\text{fp}(\mathcal{A}) = \text{fg}(\mathcal{A})$, equivalently, if $\text{fg}(\mathcal{A})$ forms an Abelian subcategory of $\mathcal{A}$. For a locally finitely presented Grothendieck category $\mathcal{A}$, we denote by $\text{Ch}(\text{fp}(\mathcal{A}))$ the subcategory of $\text{Ch}(\mathcal{A})$ consisting of complexes of finitely presented objects in $\mathcal{A}$.

We are now in a position to prove a result, as Theorem A advertised in the introduction, for locally finitely presented Grothendieck categories with enough projectives, which illustrates that their pure simplicity can be obtained from the pure simplicity of certain triangulated categories.

Theorem 3.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives, and let $\mathcal{T}_{\mathcal{A}}$ be a compactly generated triangulated category consisting of complexes in $\text{Ch}(\mathcal{A})$ satisfying the following condition:*

($\mathbf{T}_{\mathcal{A}}$) *there are fully faithful functors $\mathcal{A} \hookrightarrow \mathcal{T}_{\mathcal{A}}$ and $\text{proj}(\mathcal{A}) \hookrightarrow \mathcal{T}_{\mathcal{A}}^{\circ}$, and the objects in $\mathcal{T}_{\mathcal{A}}^{\circ}$ are complexes of finitely presented objects in $\mathcal{A}$.*

If $\mathcal{T}_{\mathcal{A}}$ is pure semisimple, then so is $\mathcal{A}$.

Proof. Suppose that $\mathcal{A}$ is a locally finitely presented Grothendieck category with enough projectives, and $\mathcal{T}_{\mathcal{A}}$ is a pure semisimple compactly generated triangulated category consisting of complexes in $\text{Ch}(\mathcal{A})$ satisfying the condition **($\mathbf{T}_{\mathcal{A}}$)**.

Let $\text{inc} : \text{proj}(\mathcal{A}) \hookrightarrow \mathcal{T}_{\mathcal{A}}^{\circ}$ be the inclusion functor, and consider its restriction functor

$$\text{inc}_* : \text{Mod}(\mathcal{T}_{\mathcal{A}}^{\circ}) \rightarrow \mathcal{A} \stackrel{\text{Lemma 3.2}}{=} \text{Mod}(\text{proj}(\mathcal{A})).$$

By Lemma 3.1, we have that inc_* admits a fully faithful right adjoint. Hence, $\mathcal{A}$ is a localized quotient of $\text{Mod}(\mathcal{T}_{\mathcal{A}}^{\circ})$. Since $\mathcal{T}_{\mathcal{A}}$ is pure semisimple by assumption, it follows from [2, Theorem 9.3] that $\text{Mod}(\mathcal{T}_{\mathcal{A}}^{\circ})$ is Frobenius, and hence it is locally Noetherian by [2, Proposition 9.2]. It in turn induces that $\mathcal{A}$ is locally Noetherian.

Since $\mathcal{T}_{\mathcal{A}}$ is pure semisimple, we have

$$\mathcal{T}_{\mathcal{A}} = \text{Add}(\mathcal{T}_{\mathcal{A}}^{\circ})$$

by [2, Theorem 9.3]. For any $M \in \mathcal{A}$, let $S(M)$ be the stalk complex in $\text{Ch}(\mathcal{A})$ with M in degree 0. It is in $\mathcal{T}_{\mathcal{A}}$ by assumption. Thus, there is a family $\{X^{\bullet}_j \mid j \in J\} \subseteq \mathcal{T}_{\mathcal{A}}^{\circ} \subseteq \text{Ch}(\text{fp}(\mathcal{A}))$ and a complex $Y^{\bullet} \in \mathcal{T}_{\mathcal{A}}$ such that

$$S(M) \oplus Y^{\bullet} = \bigoplus_{j \in J} X^{\bullet}_j,$$

which yields

$$M \oplus H^0(Y^{\bullet}) = \bigoplus_{j \in J} H^0(X^{\bullet}_j).$$

Since $\mathcal{A}$ is locally Noetherian, it is easy to see that each $H_0(X^{\bullet}_j)$ is finitely presented. Now, M is pure projective, as it is a summand of a coproduct of some objects in $\text{fp}(\mathcal{A})$. Consequently, $\mathcal{A}$ is pure semisimple. $\square$

We denote by $K^b(\text{proj}(\mathcal{A}))$ (resp. $K^b(\text{fp}(\mathcal{A}))$) the subcategory of $K(\mathcal{A})$ consisting of bounded complexes of finitely generated projective (resp. finitely presented) objects in $\mathcal{A}$. We give the first application of Theorem 3.3.

Example 3.4. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives.

- (1) Consider the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$. Since $\mathcal{A}$ is a locally finitely presented Grothendieck category with enough projectives, it is a routine to obtain that $\mathcal{A}$ has a set $\mathcal{S}$ of finitely presented (or equivalently, finitely generated) projective objects, and that $\mathcal{S}$ can serve as the set of compact generators for $\mathbf{D}(\mathcal{A})$. It follows that $\mathbf{D}(\mathcal{A})$ is compactly generated. Furthermore, applying [20, Chapter 1, Proposition 1.3(2)] one has a triangle equivalence

$$\mathbf{D}(\mathcal{A})^c \simeq \mathbf{K}^b(\text{proj}(\mathcal{A})).$$

It is trivial that all the objects of $\mathbf{K}^b(\text{proj}(\mathcal{A}))$ are in $\text{Ch}(\text{fp}(\mathcal{A}))$. Furthermore, it is well-known that there are fully faithful functors $\mathcal{A} \rightarrow \mathbf{D}(\mathcal{A})$ and $\text{proj}(\mathcal{A}) \rightarrow \mathbf{K}^b(\text{proj}(\mathcal{A}))$. Now $\mathbf{D}(\mathcal{A})$ satisfies the condition $(\mathbf{T}_{\mathcal{A}})$. It follows from Theorem 3.3 that $\mathcal{A}$ is pure semisimple whenever $\mathbf{D}(\mathcal{A})$ is pure semisimple.

- (2) Consider the pure derived category $\mathbf{D}_{\text{pur}}(\mathcal{A})$ of $\mathcal{A}$. It follows from [11, Theorem 8] that $\mathbf{D}_{\text{pur}}(\mathcal{A})$ is a compactly generated triangulated category with a triangle equivalence

$$\mathbf{D}_{\text{pur}}(\mathcal{A})^c \simeq \mathbf{K}^b(\text{fp}(\mathcal{A})).$$

It is trivial that all the objects of $\mathbf{K}^b(\text{fp}(\mathcal{A}))$ are in $\text{Ch}(\text{fp}(\mathcal{A}))$. Furthermore, it is well-known that there are fully faithful functors $\mathcal{A} \rightarrow \mathbf{D}_{\text{pur}}(\mathcal{A})$ and $\text{proj}(\mathcal{A}) \rightarrow \mathbf{K}^b(\text{fp}(\mathcal{A}))$. Now $\mathbf{D}_{\text{pur}}(\mathcal{A})$ satisfies the condition $(\mathbf{T}_{\mathcal{A}})$. It follows from Theorem 3.3 that $\mathcal{A}$ is pure semisimple whenever $\mathbf{D}_{\text{pur}}(\mathcal{A})$ is pure semisimple.

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. We denote by $\mathbf{D}^b(\text{fp}(\mathcal{A}))$ the subcategory of $\mathbf{D}(\mathcal{A})$ consisting of homology bounded complexes of finitely presented objects in $\mathcal{A}$. We denote by $\text{FP-I}(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting of all FP-injective objects in $\mathcal{A}$. Here an object $A \in \mathcal{A}$ is said to be *FP-injective* if $\text{Ext}_{\mathcal{A}}^1(F, A) = 0$ for any $F \in \text{fp}(\mathcal{A})$. In order to better understand $\mathbf{K}(\text{Inj}(\mathcal{A}))$, the homotopy subcategory of $\mathbf{K}(\mathcal{A})$ consisting of complexes of injective objects in $\mathcal{A}$, Šťovíček in [21] studied $\mathbf{D}(\text{FP-I}(\mathcal{A}))$, the derived category of FP-injective objects of $\mathcal{A}$, which is defined as the Verdier quotient

$$\mathbf{D}(\text{FP-I}(\mathcal{A})) =: \mathbf{K}(\text{FP-I}(\mathcal{A}))/\mathbf{K}_{\text{pac}}(\text{FP-I}(\mathcal{A})),$$

where $\mathbf{K}(\text{FP-I}(\mathcal{A}))$ (resp. $\mathbf{K}_{\text{pac}}(\text{FP-I}(\mathcal{A}))$) is the homotopy subcategory of $\mathbf{K}(\mathcal{A})$ (resp. $\mathbf{K}_{\text{pac}}(\mathcal{A})$) consisting of complexes of FP-injective objects in $\mathcal{A}$.

Recall that $\mathcal{A}$ is called *locally coherent* if $\text{fp}(\mathcal{A})$ forms an Abelian subcategory of $\mathcal{A}$. It follows from [21, Proposition B.3] that a locally finitely presented Grothendieck category $\mathcal{A}$ is locally coherent if and only if $\text{FP-I}(\mathcal{A})$ is closed under cokernels of monomorphisms. Krause in [11] and Šťovíček in [21] studied the derived category $\mathbf{D}(R\text{-FP-I})$ of FP-injective modules. It is defined as the Verdier quotient

$$\mathbf{D}(R\text{-FP-I}) := \mathbf{K}(R\text{-FP-I})/\mathbf{K}_{\text{pac}}(R\text{-FP-I}).$$

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and $X^\bullet \in \text{Ch}(\mathcal{A})$. Recall that $X^\bullet$ is *DG-injective* if $X^\bullet \in \text{Ch}(\text{Inj}(\mathcal{A}))$ and any morphism $f^\bullet : E^\bullet \rightarrow X^\bullet$ with $E^\bullet$ exact is null-homotopic. We denote by $\mathbf{K}(\text{DG-Inj-Ch}(\mathcal{A}))$ the homotopy subcategory of $\mathbf{K}(\mathcal{A})$ consisting of DG-injective complexes. It is easy to see that the inclusion functor $\mathbf{K}(\text{DG-Inj-Ch}(\mathcal{A})) \hookrightarrow \mathbf{K}(\text{Inj}(\mathcal{A}))$ is a fully faithful.

We have the following standard observations.

Proposition 3.5. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then it holds that*

(1) *there is a fully faithful triangulated functor*

$$\iota : \mathbf{K}(\text{Inj}(\mathcal{A})) \rightarrow \mathbf{D}(\text{FP-I}(\mathcal{A})).$$

If $\mathcal{A}$ is locally coherent, then ι is a triangle equivalence; in particular, for any ring R , there is a fully faithful triangulated functor

$$\iota : \mathbf{K}(R\text{-Inj}) \rightarrow \mathbf{D}(R\text{-FP-I}),$$

and it is a triangle equivalence if R is left coherent;

(2) *there is a fully faithful additive functor*

$$\tau : \mathcal{A} \rightarrow \mathbf{K}(\text{Inj}(\mathcal{A}));$$

in particular, for a left coherent ring R , there is a fully faithful additive functor

$$\tau : R\text{-Mod} \rightarrow \mathbf{K}(R\text{-Inj}).$$

Proof. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category.

(1) By the proof of [21, Lemma 6.14], we know that there is a triangle equivalence

$$\mathbf{K}(\text{Inj}(\mathcal{A})) \simeq \mathbf{K}(\text{FP-I}(\mathcal{A})) / {}^\perp \mathbf{K}(\text{Inj}(\mathcal{A})) \cap \mathbf{K}(\text{FP-I}(\mathcal{A})),$$

where ${}^\perp \mathbf{K}(\text{Inj}(\mathcal{A})) = \{X^\bullet \in \mathbf{K}(\mathcal{A}) \mid \text{Hom}_{\mathbf{K}(\mathcal{A})}(X^\bullet, I^\bullet) = 0 \text{ for any } I^\bullet \in \mathbf{K}(\text{Inj}(\mathcal{A}))\}$. Note from [21, Lemma 6.10] that $\mathbf{K}_{\text{pac}}(\text{FP-I}(\mathcal{A})) \subseteq {}^\perp \mathbf{K}(\text{Inj}(\mathcal{A})) \cap \mathbf{K}(\text{FP-I}(\mathcal{A}))$. Now comparing the definition of $\mathbf{D}(R\text{-FP-I})$, one has a fully faithful triangulated functor

$$\iota : \mathbf{K}(\text{Inj}(\mathcal{A})) \rightarrow \mathbf{D}(\text{FP-I}(\mathcal{A})).$$

The last assertion follows from [21, Theorem 6.12].

(2) By [14, Corollary 8.5], there is a triangle equivalence

$$\lambda : \mathbf{D}(\mathcal{A}) \simeq \mathbf{K}(\text{DG-Inj-Ch}(\mathcal{A}))$$

Note that there is a fully faithful functor $\alpha : \mathcal{A} \rightarrow \mathbf{D}(\mathcal{A})$. Now let τ be the composition of functors

$$\mathcal{A} \xrightarrow{\alpha} \mathbf{D}(\mathcal{A}) \xrightarrow{\lambda} \mathbf{K}(\text{DG-Inj-Ch}(\mathcal{A})) \hookrightarrow \mathbf{K}(\text{Inj}(\mathcal{A}))$$

It follows that τ is a fully faithful additive functor. □

Remark 3.6. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives. We denote by $\mathbf{K}(\text{Proj}(\mathcal{A}))$ the homotopy subcategory of $\mathbf{K}(\mathcal{A})$ consisting of complexes of projective objects in $\mathcal{A}$. According to [14, Corollary 6.7], there is a triangle equivalence

$$\mathbf{D}(\mathcal{A}) \simeq \mathbf{K}(\text{DG-Proj-Ch}(\mathcal{A}))$$

in which $\mathbf{K}(\text{DG-Proj-Ch}(\mathcal{A}))$ is the homotopy subcategory of $\mathbf{K}(\mathcal{A})$ consisting of DG-projective complexes. Here a complex $X^\bullet$ is called *DG-projective* if $X^\bullet \in \text{Ch}(\text{Proj}(\mathcal{A}))$ and any morphism $f^\bullet : X^\bullet \rightarrow$

$E^\bullet$ with $E^\bullet$ exact is null-homotopic. Now by using an argument similar to that in Proposition 3.5(2), we obtain a fully faithful additive functor $\mathcal{A} \rightarrow \mathbf{K}(\text{Proj}(\mathcal{A}))$.

Now we give another application of Theorem 3.3.

Example 3.7. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projectives.

- (1) Consider $\mathbf{D}(\text{FP-I}(\mathcal{A}))$, the derived category of FP-injective objects of $\mathcal{A}$. Due to [21, Proposition 6.4 and Corollary 6.13], it is a compactly generated triangulated category with a triangle equivalence

$$\mathbf{D}(\text{FP-I}(\mathcal{A}))^c \simeq \mathbf{D}^b(\text{fp}(\mathcal{A})).$$

Trivially all the objects of $\mathbf{D}^b(\text{fp}(\mathcal{A}))$ are in $\text{Ch}(\text{fp}(\mathcal{A}))$. Furthermore, there is a fully faithful functor $\text{proj}(\mathcal{A}) \rightarrow \mathbf{D}^b(\text{fp}(\mathcal{A}))$. On the other hand, by Proposition 3.5 there is a fully faithful functor $\mathcal{A} \rightarrow \mathbf{D}(\text{FP-I}(\mathcal{A}))$. Now $\mathbf{D}(\text{FP-I}(\mathcal{A}))$ satisfies the condition $(\mathbf{T}_{\mathcal{A}})$. It follows from Theorem 3.3 that $\mathcal{A}$ is pure semisimple whenever $\mathbf{D}(\text{FP-I}(\mathcal{A}))$ is pure semisimple.

- (2) Suppose that $\mathcal{A}$ is also locally coherent. Consider, $\mathbf{K}(\text{Inj}(\mathcal{A}))$, the homotopy subcategory of $\mathbf{K}(\mathcal{A})$ consisting of complexes of injective objects of $\mathcal{A}$. Due to [21, Corollary 6.13], it is a compactly generated triangulated category with a triangle equivalence

$$\mathbf{K}(\text{Inj}(\mathcal{A}))^c \simeq \mathbf{D}^b(\text{fp}(\mathcal{A})).$$

Trivially all the objects of $\mathbf{D}^b(\text{fp}(\mathcal{A}))$ are in $\text{Ch}(\text{fp}(\mathcal{A}))$. Furthermore, there is a fully faithful functor $\text{proj}(\mathcal{A}) \rightarrow \mathbf{D}^b(\text{fp}(\mathcal{A}))$. On the other hand, by Proposition 3.5 there is a fully faithful functor $\mathcal{A} \rightarrow \mathbf{K}(\text{Inj}(\mathcal{A}))$. Now $\mathbf{K}(\text{Inj}(\mathcal{A}))$ satisfies the condition $(\mathbf{T}_{\mathcal{A}})$. It follows from Theorem 3.3 that $\mathcal{A}$ is pure semisimple whenever $\mathbf{K}(\text{Inj}(\mathcal{A}))$ is pure semisimple.

We now turn out the case for rings. It is well known that a ring R is left pure semisimple if and only if $R\text{-Mod}$ is pure semisimple.

Corollary 3.8. *Let R be a ring, and let $\mathcal{T}_R$ be a compactly generated triangulated category consisting of complexes in $\text{Ch}(R\text{-Mod})$ satisfying the following condition*

$(\mathbf{T}_R)$ there is an embedding $R\text{-Mod} \hookrightarrow \mathcal{T}_R$ which restricts an embedding $R\text{-proj} \hookrightarrow \mathcal{T}_R^c$, and the objects in $\mathcal{T}_R^c$ are complexes of modules in $R\text{-mod}$.

If $\mathcal{T}_R$ is pure semisimple, then so is R .

Proof. It is well-known that $R\text{-Mod}$ is a locally finitely presented Grothendieck category with enough projectives. Then the assertion follows from Theorem 3.3. $\square$

Remark 3.9. Let R be a ring. It follows from Examples 3.4 and 3.7(1) that R is left pure semisimple provided that $\mathbf{D}(R\text{-Mod})$ (resp. $\mathbf{D}_{\text{pur}}(R\text{-Mod})$ and $\mathbf{D}(R\text{-FP-I})$) is pure semisimple. On the other hand, if R is a left coherent ring such that $\mathbf{K}(R\text{-Inj})$ is pure semisimple, then R is left pure semisimple by Example 3.7(2).

Remark 3.10. From Examples 3.4 and 3.7, we get the following assertions.

- (1) Over any ring R , all of $\mathbf{D}(R\text{-Mod})$, $\mathbf{D}_{\text{pur}}(R\text{-Mod})$ and $\mathbf{D}(R\text{-FP-I})$ satisfy the condition $(\mathbf{T}_R)$.
- (2) If R is a left coherent ring, then $\mathbf{K}(R\text{-Inj})$ satisfies the condition $(\mathbf{T}_R)$ by Proposition 3.5.

4. Pure semisimple conjecture

Let R be a ring and let $\mathcal{T}_R$ be a triangulated category consisting of complexes in $\text{Ch}(R\text{-Mod})$. In what follows, we denote by $\mathcal{T}_{R^{\text{op}}}$ the triangulated category consisting of complexes in $\text{Ch}(R^{\text{op}}\text{-Mod})$ which shares the same properties as $\mathcal{T}_R$. It can be viewed as the opposite category of $\mathcal{T}_R$. For example, if we take $\mathcal{T}_R$ as $\mathbf{D}(R\text{-Mod})$ (resp. $\mathbf{D}_{\text{pur}}(R\text{-Mod})$), then $\mathcal{T}_{R^{\text{op}}}$ means $\mathbf{D}(R^{\text{op}}\text{-Mod})$ (resp. $\mathbf{D}_{\text{pur}}(R^{\text{op}}\text{-Mod})$).

Definition 4.1. Let R be a ring and let $\mathcal{T}_R$ (resp. $\mathcal{T}_{R^{\text{op}}}$) be a compactly generated triangulated category consisting of complexes in $\text{Ch}(R\text{-Mod})$ (resp. $\text{Ch}(R^{\text{op}}\text{-Mod})$). The ring R is said to satisfy *the triangulated pure semisimple conjecture* with respect to $\mathcal{T}$ if $\mathcal{T}_R$ is pure semisimple if and only if so $\mathcal{T}_{R^{\text{op}}}$. In particular, we say that R satisfies the (*resp. pure-*)*derived pure semisimple conjecture* if $\mathbf{D}(R\text{-Mod})$ (resp. $\mathbf{D}_{\text{pur}}(R\text{-Mod})$) is pure semisimple if and only if so is $\mathbf{D}(R^{\text{op}}\text{-Mod})$ (resp. $\mathbf{D}_{\text{pur}}(R^{\text{op}}\text{-Mod})$).

By investigating the pure global dimension of the usual derived categories, Beligiannis in [2, Remark 12.18] proved that **DPSC** implies **PSC**, that is, a ring R satisfies **PSC** provided that it satisfies **DPSC**. In this section, we investigate the behavior of pure derived categories and the other compactly generated triangulated categories. To achieve it, we need the following lemma.

Lemma 4.2. *Let R be a ring, and let $\mathcal{T}_R$ be a compactly generated triangulated category consisting of complexes in $\text{Ch}(R\text{-Mod})$ satisfying the following condition:*

$(\mathbf{ST}_R)$ there are fully faithful triangulated functors $\mathbf{D}(R\text{-Mod}) \hookrightarrow \mathcal{T}_R$ and $\mathbf{K}^b(R\text{-proj}) \hookrightarrow \mathcal{T}_R^c$ such that $\mathcal{T}_R^c$ consists of complexes in $\text{Ch}(R\text{-mod})$, and any triangle in $\mathcal{T}_R$ is a triangle in $\mathbf{D}(R\text{-Mod})$.

Then, $\mathcal{T}_R$ is pure semisimple whenever $\mathbf{D}(R\text{-Mod})$ is pure semisimple.

Proof. Suppose that $\mathcal{T}_R$ is a compactly generated triangulated category consisting of complexes in $\text{Ch}(R\text{-Mod})$ satisfying the condition $(\mathbf{ST}_R)$ and suppose that $\mathbf{D}(R\text{-Mod})$ is pure semisimple. We claim that any pure triangle in $\mathcal{T}_R$ is split. For this, let

$$T_1^\bullet \rightarrow T^\bullet \rightarrow T_2^\bullet \rightarrow T_1^\bullet[1]$$

be any pure triangle in $\mathcal{T}_R$. That is, for any $X^\bullet \in \mathcal{T}_R^c$, there is a short exact sequence of Abelian groups

$$0 \rightarrow \text{Hom}_{\mathcal{T}_R}(X^\bullet, T_1^\bullet) \rightarrow \text{Hom}_{\mathcal{T}_R}(X^\bullet, T^\bullet) \rightarrow \text{Hom}_{\mathcal{T}_R}(X^\bullet, T_2^\bullet) \rightarrow 0.$$

Now let $P^\bullet \in \mathbf{K}^b(R\text{-proj})$. Then $P^\bullet \in \mathcal{T}_R^c$ as there is an embedding $\mathbf{K}^b(R\text{-proj}) \hookrightarrow \mathcal{T}_R^c$. It yields a short exact sequence of Abelian groups

$$0 \rightarrow \text{Hom}_{\mathcal{T}_R}(P^\bullet, T_1^\bullet) \rightarrow \text{Hom}_{\mathcal{T}_R}(P^\bullet, T^\bullet) \rightarrow \text{Hom}_{\mathcal{T}_R}(P^\bullet, T_2^\bullet) \rightarrow 0.$$

By assumption on $\mathcal{T}_R$, the triangle $T_1^\bullet \rightarrow T^\bullet \rightarrow T_2^\bullet \rightarrow T_1^\bullet[1]$ is also in $\mathbf{D}(R\text{-Mod})$. Using the embedding $\mathbf{D}(R\text{-Mod}) \hookrightarrow \mathcal{T}_R$, we have the following isomorphisms

$$\text{Hom}_{\mathbf{D}(R\text{-Mod})}(P^\bullet, T_i^\bullet) \cong \text{Hom}_{\mathcal{T}_R}(P^\bullet, T_i^\bullet) \text{ for } i = 1, 2, \quad \text{and} \quad \text{Hom}_{\mathbf{D}(R\text{-Mod})}(P^\bullet, T^\bullet) \cong \text{Hom}_{\mathcal{T}_R}(P^\bullet, T^\bullet).$$

Now there is a short exact sequence of Abelian groups

$$0 \rightarrow \mathrm{Hom}_{\mathbf{D}(R\text{-Mod})}(P^\bullet, T_1^\bullet) \rightarrow \mathrm{Hom}_{\mathbf{D}(R\text{-Mod})}(P^\bullet, T^\bullet) \rightarrow \mathrm{Hom}_{\mathbf{D}(R\text{-Mod})}(P^\bullet, T_2^\bullet) \rightarrow 0,$$

in other words, $T_1^\bullet \rightarrow T^\bullet \rightarrow T_2^\bullet \rightarrow T_1^\bullet[1]$ is also a pure triangle in $\mathbf{D}(R\text{-Mod})$. It is split as $\mathbf{D}(R\text{-Mod})$ is pure semisimple (see the remarks in [2, p.331, line (26)–(28)]). The claim is proved.

To prove that $\mathcal{T}_R$ is pure semisimple, it suffices to prove

$$\mathcal{T}_R = \mathrm{Add}(\mathcal{T}_R^c)$$

by [2, Theorem 9.3]. For any $T^\bullet \in \mathcal{T}_R$, using the assumption that $\mathcal{T}_R$ is compactly generated, one obtains a morphism $f^\bullet : X^\bullet \rightarrow T^\bullet$ in $\mathcal{T}_R$ with $X^\bullet = \coprod_{j \in J} C_j^\bullet$ for a set $\{C_j^\bullet \mid j \in J\} \subseteq \mathcal{T}_R^c$. Now by [2, Lemma 8.1] one can fit $f^\bullet$ into a pure triangle

$$S^\bullet \rightarrow X^\bullet \xrightarrow{f^\bullet} T^\bullet \rightarrow S^\bullet[1]$$

in $\mathcal{T}_R$. By what we have proved, such a pure triangle splits. It yields

$$T^\bullet \oplus S^\bullet = X^\bullet = \bigoplus_{j \in J} C_j^\bullet.$$

Thus $\mathcal{T}_R \subseteq \mathrm{Add}(\mathcal{T}_R^c)$, and therefore $\mathcal{T}_R = \mathrm{Add}(\mathcal{T}_R^c)$ since the converse containment is trivial. $\square$

We are now in a position to give our main result in this section.

Theorem 4.3. *Let R be a ring, and let $\mathcal{T}_R$ (resp. $\mathcal{T}_{R^{\mathrm{op}}}$) be a compactly generated triangulated category consisting of complexes in $\mathrm{Ch}(R\text{-Mod})$ (resp. $\mathrm{Ch}(R^{\mathrm{op}}\text{-Mod})$). Suppose that $\mathcal{T}_R$ satisfies the condition as in Lemma 4.2, and that $\mathcal{T}_{R^{\mathrm{op}}}$ satisfies the following condition:*

(ST_{R^{op}}) *there are fully faithful triangulated functors $\mathbf{D}(R^{\mathrm{op}}\text{-Mod}) \hookrightarrow \mathcal{T}_{R^{\mathrm{op}}}$ and $\mathbf{K}^b(R^{\mathrm{op}}\text{-proj}) \hookrightarrow \mathcal{T}_{R^{\mathrm{op}}}^c$ such that $\mathcal{T}_{R^{\mathrm{op}}}^c$ consists of complexes in $\mathrm{Ch}(R^{\mathrm{op}}\text{-mod})$, and any triangle in $\mathcal{T}_{R^{\mathrm{op}}}$ is a triangle in $\mathbf{D}(R^{\mathrm{op}}\text{-Mod})$.*

*Then, R satisfies **PSC** whenever it satisfies the the triangulated pure semisimple conjecture with respect to $\mathcal{T}$.*

Proof. Suppose that R satisfies the triangulated pure semisimple conjecture with respect to $\mathcal{T}$ and both conditions **(ST_R)** (resp. **(ST_{R^{op}})**) are satisfied.

To see that R satisfies **PSC**, according to [8], we may suppose that R be a left pure semisimple ring which is hereditary and prove that R is right pure semisimple. Indeed, since R is a left pure semisimple ring which is left hereditary, it follows from [2, Proposition 12.8] that $\mathbf{D}(R\text{-Mod})$ is pure semisimple. Applying Lemma 4.2 it in turn implies that $\mathcal{T}_R$ is pure semisimple. Thus, the assumption yields that $\mathcal{T}_{R^{\mathrm{op}}}$ is pure semisimple as well. It follows from Corollary 3.8 that R is right pure semisimple. $\square$

Remark 4.4. Let R be a ring. As mentioned above, [2, Remark 12.18] shows that R satisfies **PSC** whenever R satisfies **DPSC**. Note that this result can be reobtained by Theorem 4.3 as $\mathbf{D}(R\text{-Mod})$ (resp. $\mathbf{D}(R^{\mathrm{op}}\text{-Mod})$) satisfies the condition **(ST_R)** (resp. **(ST_{R^{op}})**).

The next result shows that the condition **(ST_R)** is satisfied under various circumstances.

Lemma 4.5. *For any ring R , it holds that*

- (1) $\mathbf{D}_{\text{pur}}(R\text{-Mod})$ (resp. $\mathbf{D}(R\text{-FP-I})$) satisfies the condition $(\mathbf{ST}_R)$;
- (2) if R is left coherent, then $\mathbf{K}(R\text{-Inj})$ satisfies the condition $(\mathbf{ST}_R)$;
- (3) if R is right coherent, then $\mathbf{K}(R\text{-Proj})$ satisfies the condition $(\mathbf{ST}_R)$.

Proof. It is known that any triangle in $\mathbf{D}_{\text{pur}}(R\text{-Mod})$ (resp. $\mathbf{D}(R\text{-FP-I})$)

$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$$

is equivalent to the one induced from a pure short exact sequence

$$0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$$

of R -complexes (resp. with all terms of $A^\bullet$, $B^\bullet$ and $C^\bullet$ FP-injective); and that any triangle

$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$$

in $\mathbf{K}(R\text{-Inj})$ (resp. $\mathbf{K}(R\text{-Proj})$) is equivalent to the one induced from a (degreewise split) short exact sequence of R -complexes

$$0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$$

with all terms of $A^\bullet$, $B^\bullet$ and $C^\bullet$ injective, which is a triangle in $\mathbf{D}(R\text{-Mod})$ (as we know, a triangle in $\mathbf{D}(R\text{-Mod})$ is equivalent to the one induced from only a short exact sequence of R -complexes). In addition, it follows from Remark 3.10 that $\mathbf{D}_{\text{pur}}(R\text{-Mod})$ (resp. and $\mathbf{D}(R\text{-FP-I})$) satisfies the condition $(\mathbf{T}_R)$ over any ring R and that $\mathbf{K}(R\text{-Inj})$ satisfies the condition $(\mathbf{T}_R)$ over a left coherent ring R . Now (1) and (2) follows.

To see (3), assume that R is a right coherent ring. Then $\mathbf{K}(R\text{-Proj})$ is compactly generated by [12, Proposition 7.14]. Furthermore, by [20, Chapter 1, Proposition 2.6(1)] there is a triangle equivalence

$$\mathbf{K}(R\text{-Proj})^c \simeq \mathbf{D}^b(R^{\text{op}}\text{-mod})^{\text{op}},$$

where the category on the right side is the opposite category of $\mathbf{D}^b(R^{\text{op}}\text{-mod})$. By [1, Corollary 2.2.3], there is a recollement of compactly generated triangulated categories

$$\begin{array}{ccccc} & \longleftarrow & & \longleftarrow & \\ \mathbf{K}_{\text{ac}}(R\text{-Proj}) & \longrightarrow & \mathbf{K}(R\text{-Proj}) & \longrightarrow & \mathbf{D}(R\text{-Mod}), \\ & \longleftarrow & & \longleftarrow & \end{array}$$

which yields an embedding $\mathbf{D}(R\text{-Mod}) \hookrightarrow \mathbf{K}(R\text{-Proj})$ that restricts an embedding

$$\mathbf{D}(R\text{-Mod})^c \hookrightarrow \mathbf{K}(R\text{-Proj})^c.$$

It is well known that there are triangle equivalences (see [22, Lemma 6.5 and its proof]):

$$\mathbf{D}^b(R\text{-mod}) \simeq \mathbf{D}(R\text{-Mod})^c \simeq \mathbf{D}^b(R^{\text{op}}\text{-mod})^{\text{op}}.$$

Thus there is an embedding $\mathbf{D}(R\text{-Mod}) \hookrightarrow \mathbf{K}(R\text{-Proj})$ that restricts an embedding

$$\mathbf{K}^b(R\text{-proj}) \hookrightarrow \mathbf{D}^b(R^{\text{op}}\text{-mod})^{\text{op}} = \mathbf{K}(R\text{-Proj})^c.$$

It follows that $\mathbf{K}(R\text{-Proj})$ satisfies the condition $(\mathbf{T}_R)$, as required. □

The following corollary is an immediate consequence of Theorem 4.3 and Lemma 4.5, which is a part of Theorem B advertised in the introduction.

Corollary 4.6. *For any ring R , it holds that*

- (1) *if R satisfies the pure-derived pure semisimple conjecture, then it also satisfies **PSC**;*
- (2) *if the pure semisimplicity of $\mathbf{D}(R\text{-FP-I})$ is equivalent to that of $\mathbf{D}(R^{\text{op}}\text{-FP-I})$ (that is, $\mathbf{D}(R\text{-FP-I})$ is pure semisimple if and only if so is $\mathbf{D}(R^{\text{op}}\text{-FP-I})$), then R satisfies **PSC**;*
- (3) *if R is left and right coherent and the pure semisimplicity of $\mathbf{K}(R\text{-Inj})$ is equivalent to that of $\mathbf{K}(R^{\text{op}}\text{-Inj})$, then R satisfies **PSC**;*
- (4) *if R is left and right coherent and the pure semisimplicity of $\mathbf{K}(R\text{-Proj})$ is equivalent to that of $\mathbf{K}(R^{\text{op}}\text{-Proj})$, then R satisfies **PSC**.*

We end the section by the following result, which, as a part of Theorem B advertised in the introduction, gives some equivalent characterizations of **PSC**.

Corollary 4.7. *Let R be a hereditary ring. Then the following statements are equivalent.*

- (1) *R satisfies **PSC**.*
- (2) *R satisfies **DPSC**.*
- (3) *R satisfies the pure-derived pure semisimple conjecture.*
- (4) *$\mathbf{K}(R\text{-Inj})$ is pure semisimple if and only if $\mathbf{K}(R^{\text{op}}\text{-Inj})$ is pure semisimple.*
- (5) *$\mathbf{K}(R\text{-Proj})$ is pure semisimple if and only if $\mathbf{K}(R^{\text{op}}\text{-Proj})$ is pure semisimple.*
- (6) *$\mathbf{D}(R\text{-FP-I})$ is pure semisimple if and only if $\mathbf{D}(R^{\text{op}}\text{-FP-I})$ is pure semisimple.*

Proof. (1) $\iff$ (2) It follows [2, Remark 12.18 and Proposition 12.8] as R is hereditary.

By Corollary 4.6(1) and (2), we have (3) $\implies$ (1) and (6) $\implies$ (1).

Since any hereditary ring is left and right coherent, we have (4) $\implies$ (1) and (5) $\implies$ (1) by Corollary 4.6(3) and (4).

(2) $\implies$ (3) + (4) + (5) + (6) Suppose that R is hereditary (of course left and right coherent) which satisfies **DPSC**. Then all of triangulated categories $\mathbf{D}_{\text{pur}}(R\text{-Mod})$, $\mathbf{D}_{\text{pur}}(R^{\text{op}}\text{-Mod})$, $\mathbf{K}(R\text{-Inj})$, $\mathbf{K}(R^{\text{op}}\text{-Inj})$, $\mathbf{K}(R\text{-Proj})$, $\mathbf{K}(R^{\text{op}}\text{-Proj})$, $\mathbf{D}(R\text{-FP-I})$ and $\mathbf{D}(R^{\text{op}}\text{-FP-I})$ are compactly generated. Assume that

$$\mathbf{D}_{\text{pur}}(R\text{-Mod}) \text{ (resp. } \mathbf{K}(R\text{-Inj}), \mathbf{K}(R\text{-Proj}) \text{ and } \mathbf{D}(R\text{-FP-I}))$$

is pure semisimple. Note that such a compactly generated triangulated category satisfies the condition $(\mathbf{T}_R)$ (see Lemma 4.5). Then Corollary 3.8 applies to show that R is left pure semisimple, and hence $\mathbf{D}(R\text{-Mod})$ is pure semisimple by [2, Proposition 12.8]. Thus $\mathbf{D}(R^{\text{op}}\text{-Mod})$ is pure semisimple by (2). Lemma 4.5 tells us that the compactly generated triangulated category

$$\mathbf{D}_{\text{pur}}(R^{\text{op}}\text{-Mod}), \mathbf{K}(R^{\text{op}}\text{-Inj}), \mathbf{K}(R^{\text{op}}\text{-Proj}) \text{ and } \mathbf{D}(R^{\text{op}}\text{-FP-I})$$

satisfy the condition $(\mathbf{ST}_{R^{\text{op}}})$, it follows from Lemma 4.2 that

$$\mathbf{D}_{\text{pur}}(R^{\text{op}}\text{-Mod}) \text{ (resp. } \mathbf{K}(R^{\text{op}}\text{-Inj}), \mathbf{K}(R^{\text{op}}\text{-Proj}) \text{ and } \mathbf{D}(R^{\text{op}}\text{-FP-I}))$$

is pure semisimple. □

5. Derived equivalences

Recall that two rings R and S are *derived equivalent* if there is a triangle equivalence

$$\mathbf{D}(R\text{-Mod}) \simeq \mathbf{D}(S\text{-Mod}).$$

In this section, we will investigate the behavior of **PSC** under derived equivalences of rings, as applications of Corollary 4.7. Recall from [10] that a complex $T^\bullet$ in $\text{Ch}(R\text{-Mod})$ is called *one-sided tilting* if the following conditions are satisfied.

- (TC1) $T^\bullet$ is a bounded complex with each T^m in $R\text{-proj}$.
- (TC2) $\text{Hom}_{\mathbf{D}(R\text{-Mod})}(T^\bullet, T^\bullet[n]) = 0$ for any $n \geq 1$.
- (TC3) $T^\bullet$ generates $\mathbf{K}^b(R\text{-proj})$ as an idempotent complete triangulated category; equivalently, the smallest triangulated subcategory closed under summands that contains $T^\bullet$ equals $\mathbf{K}^b(R\text{-proj})$.

We need the following result, which can be found in [10, Theorems 2.10 and 2.22] and [15, Theorem 9.1].

Lemma 5.1. *Let R and S be rings. Then the following statements are equivalent.*

- (1) R and S are derived equivalent.
- (2) There is a triangle equivalence $\mathbf{D}^b(R\text{-Mod}) \simeq \mathbf{D}^b(S\text{-Mod})$.
- (3) There is a triangle equivalence $\mathbf{K}^b(R\text{-proj}) \simeq \mathbf{K}^b(S\text{-proj})$.
- (4) There exists a one-sided tilting S -complex $T^\bullet$ such that $R \cong \text{End}_{\mathbf{D}(S\text{-Mod})}(T^\bullet)$.
- (5) R^{op} and S^{op} are derived equivalent.

If R and S are left coherent, then all the above and below statements are equivalent.

- (6) There is a triangle equivalence $\mathbf{D}^b(R\text{-mod}) \simeq \mathbf{D}^b(S\text{-mod})$.

Let $n \geq 0$. Cline, Parshall and Scott [3] initially introduced and studied an R -module T satisfying the following conditions and called it a *generalized tilting module*.

(cTM1) There exists an exact sequence of R -modules

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow T \rightarrow 0$$

with all P_i in $R\text{-proj}$.

(cTM2) $\text{Ext}_R^{\geq 1}(T, T) = 0$.

(cTM3) There exists an exact sequence of R -modules

$$0 \rightarrow R \rightarrow T^0 \rightarrow T^1 \rightarrow \cdots \rightarrow T^n \rightarrow 0$$

with all T^i being direct summands of finite direct sums of copies of T .

To avoid confusion with some notions in the literature, we call T as above a *classical n -tilting module*.

Example 5.2. Let R be a ring and T a classical n -tilting R -module for some $n \geq 0$. Consider the bounded complex

$$T^\bullet =: 0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

in $\text{Ch}(R\text{-proj})$, which is from (cTM1). Then, as shown in [15, Introduction], it is a routine to check that $T^\bullet$ is a one-sided tilting complex with an isomorphism

$$\text{End}_{\mathbf{D}(R\text{-Mod})}(T^\bullet) \cong \text{End}_R(T).$$

Now by Lemma 5.1, the rings R and $\text{End}_R(T)$ are derived equivalent. Note that such a derived equivalence was initially proved in [3, Theorem 2.1] by using a different method.

As applications of the equivalences $(1) \iff (5) \iff (6)$ in Lemma 5.1, the first two assertions in the following result show that derived equivalences can be obtained by triangle equivalences between certain compactly generated categories, and the third one focuses on **PSC**.

Proposition 5.3. *Let R and S be rings.*

(1) *If R and S are left coherent rings which admit one of the following triangle equivalences*

$$\mathbf{D}_{\text{pur}}(R\text{-Mod}) \simeq \mathbf{D}_{\text{pur}}(S\text{-Mod}), \quad \mathbf{D}(R\text{-FP-I}) \simeq \mathbf{D}(S\text{-FP-I}), \quad \text{and} \quad \mathbf{K}(R\text{-Inj}) \simeq \mathbf{K}(S\text{-Inj}),$$

then R and S are derived equivalent.

(2) *If R and S are right coherent rings which admit a triangle equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(S\text{-Proj})$, then R and S are derived equivalent.*

(3) *Assume that R and S are derived equivalent. If one of them satisfies **DPSC**, then the other satisfies **PSC**.*

Proof. Note that any triangle equivalence between compactly generated triangulated categories can be restricted to their triangulated subcategories of compact objects. It will be used freely in the sequel.

(1) Suppose that R and S are left coherent rings which admit one of the triangle equivalences listed as above. By Examples 3.4 and 3.7, all of triangulated categories

$$\mathbf{D}_{\text{pur}}(R\text{-Mod}), \quad \mathbf{D}(R\text{-FP-I}), \quad \text{and} \quad \mathbf{K}(R\text{-Inj})$$

are compactly generated triangulated categories with triangle equivalences

$$\mathbf{D}_{\text{pur}}(R\text{-Mod})^c = \mathbf{K}^b(R\text{-mod}) \quad \text{and} \quad \mathbf{K}(R\text{-Inj})^c = \mathbf{D}(R\text{-FP-I})^c = \mathbf{D}^b(R\text{-mod}).$$

The same behavior holds for

$$\mathbf{D}_{\text{pur}}(S^{\text{op}}\text{-Mod}), \quad \mathbf{D}(S^{\text{op}}\text{-FP-I}), \quad \text{and} \quad \mathbf{K}(S^{\text{op}}\text{-Inj}).$$

Case 1. If there is one of the following triangle equivalences

$$\mathbf{D}(R\text{-FP-I}) \simeq \mathbf{D}(S\text{-FP-I}) \quad \text{and} \quad \mathbf{K}(R\text{-Inj}) \simeq \mathbf{K}(S\text{-Inj}),$$

then there is a restricted triangle equivalence

$$\mathbf{D}^b(R\text{-mod}) \simeq \mathbf{D}^b(S\text{-mod}).$$

It follows from Lemma 5.1 that R and S are derived equivalent.

Case 2. If there is a triangle equivalence $\mathbf{D}_{\text{pur}}(R\text{-Mod}) \simeq \mathbf{D}_{\text{pur}}(S\text{-Mod})$, then there is a restricted triangle equivalence $\mathbf{K}^b(R\text{-mod}) \simeq \mathbf{K}^b(S\text{-mod})$. It induces the following triangle equivalence

$$\mathbf{D}^b(R\text{-mod}) \simeq \mathbf{D}^b(S\text{-mod}).$$

It also follows from Lemma 5.1 that R and S are derived equivalent.

(2) Suppose that R and S are right coherent rings which admit a triangle equivalence

$$\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(S\text{-Proj}).$$

By the proof of Lemma 4.5(3), we know that $\mathbf{K}(R\text{-Proj})$ and $\mathbf{K}(S\text{-Proj})$ are compactly generated triangulated categories such that

$$\mathbf{K}(R\text{-Proj})^c = \mathbf{D}^b(R^{\text{op}}\text{-mod})^{\text{op}} \quad \text{and} \quad \mathbf{K}(S\text{-Proj})^c = \mathbf{D}^b(S^{\text{op}}\text{-mod})^{\text{op}}.$$

Thus there is a restricted triangle equivalence

$$\mathbf{D}^b(R^{\text{op}}\text{-mod})^{\text{op}} \simeq \mathbf{D}^b(S^{\text{op}}\text{-mod})^{\text{op}}.$$

It induces the following triangle equivalence

$$\mathbf{D}^b(R^{\text{op}}\text{-mod}) \simeq \mathbf{D}^b(S^{\text{op}}\text{-mod}).$$

It again follows from Lemma 5.1 that R and S are derived equivalent.

(3) Suppose that R and S are derived equivalent in which S satisfies **DPSC**. Due to Remark 4.4, we are done once we see that R satisfies **DPSC**. For this, let $\mathbf{D}(R\text{-Mod})$ be pure semisimple. By [2, Remark 11.5], the assumption that the triangle equivalence $\mathbf{D}(R\text{-Mod}) \simeq \mathbf{D}(S\text{-Mod})$ yields that $\mathbf{D}(S\text{-Mod})$ is pure semisimple, and hence $\mathbf{D}(S^{\text{op}}\text{-Mod})$ is pure semisimple as S satisfies **DPSC**. Again by [2, Remark 11.5], it in turn induces that $\mathbf{D}(R^{\text{op}}\text{-Mod})$ is pure semisimple as Lemma 5.1 means that the triangle equivalence $\mathbf{D}(R^{\text{op}}\text{-Mod}) \simeq \mathbf{D}(S^{\text{op}}\text{-Mod})$ holds true as well. $\square$

Remark 5.4. By the proof of Proposition 5.3(3), we have that **DPSC** is an invariant under derived equivalences of rings.

According to [2, Proposition 12.3], we know that if a ring R is derived equivalent to a left pure semisimple and left hereditary ring, then $\mathbf{D}(R\text{-Mod})$ is pure semisimple. As an immediate consequence of Proposition 5.3(3) and Corollary 4.7, we get a more nuanced result, as Theorem C advertised in the introduction.

Theorem 5.5. *Assume that a ring R is derived equivalent to a hereditary ring S . If S satisfies **PSC**, then so does R .*

Recall from [8] that a ring R has a *self-duality* if there is a Morita duality $F : R\text{-mod} \rightarrow R^{\text{op}}\text{-mod}$. It is well-known that any Artin algebra has a self-duality. Based on this, the next lemma includes the fact that any Artin algebra satisfies **PSC**, which was originally obtained by Auslander.

Lemma 5.6. *If R is a ring having a self-duality, then R satisfies **PSC**.*

Proof. It follows from [8, Corollary 5.3]. $\square$

As an immediate consequence of Theorem 5.5 and Lemma 5.6, we get the following result.

Corollary 5.7. *Let R and S be rings which are derived equivalent. If S is a hereditary ring which has a self-duality, then R satisfies **PSC**.*

As mentioned above, rings being Artin algebras implies that they have a self-duality. Although as a special case, “being a Artin algebra” is an invariant under derived equivalences of rings (see Remark 5.9), the next example shows that “having a self-duality” is not an invariant under derived equivalences of rings in general. It also provides a new example of rings which satisfies **PSC**.

Example 5.8. Consider the ring

$$S := \begin{pmatrix} k[[x]] & k[[x]] \\ xk[[x]] & k[[x]] \end{pmatrix},$$

where k is a field and $k[[x]]$ is the formal power series ring in a variable x over k . Then $k[[x]]$ is a complete discrete valuation ring with maximal ideal $xk[[x]]$, and S is the two-by-two Iwahori order over $k[[x]]$. It follows that S is a two-sided Noetherian hereditary serial ring. In addition, S is linearly compact as $k[[x]]$ is complete. It follows from [4, Propositions 3.5 and 3.6] that S has a self-duality.

Let

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and put

$$P_1 := Se_1, \quad P_2 := Se_2 \quad \text{and} \quad L_2 := P_2/J P_2,$$

where $J := J(S)$ is the Jacobson radical of S . Consider the finitely presented S -module

$$T = P_2 \oplus L_2.$$

By [5, Example 5.4.6], we have that T is a 1-tilting S -module with an endomorphism ring

$$R := \text{End}_S(T) \cong \begin{pmatrix} k[[x]] & k[[x]]/xk[[x]] \\ 0 & k[[x]]/xk[[x]] \end{pmatrix} \cong \begin{pmatrix} k[[x]] & k \\ 0 & k \end{pmatrix}.$$

Now Example 5.2 yields that R and S are derived equivalent. Thus R satisfies **PSC** by Corollary 5.7.

In what follows, all of R (resp R^{op})-modules will be expressed as a form $\begin{pmatrix} A \\ B \end{pmatrix}$ in which A, B are $k[[x]]$ -module and k -module (resp. $k[[x]]^{\text{op}}$ -module and k^{op} -module) respectively.

Next, we show that R does not have a self-duality. Assume, for a contradiction, that R has a self-duality

$$F : R\text{-mod} \rightarrow R^{\text{op}}\text{-mod}.$$

Then the image $F(R)$ is an injective cogenerator for $R^{\text{op}}\text{-mod}$, of course it is finitely generated. Note that R is two-sided Noetherian as k is finitely generated as a two-sided $k[[x]]$ (resp. k)-module. So every right ideal of R is finitely presented, and hence, the Baer’s criterion shows that an object injective

in $R^{\text{op}}\text{-mod}$ is injective in the whole category $R^{\text{op}}\text{-Mod}$. Thus $F(R)$ is a finitely generated injective cogenerator of $R^{\text{op}}\text{-Mod}$.

Let $k((x))$ be the field of fractions of $k[[x]]$ and let

$$I := k((x))/k[[x]].$$

It is well-known that I is an injective $k[[x]]^{\text{op}}\text{-module}$. Furthermore, I is the injective envelope of the $k[[x]]^{\text{op}}\text{-module}$ $k[[x]]/xk[[x]] \cong k$. It in turn induces that the $R^{\text{op}}\text{-module}$ $E := \begin{pmatrix} I \\ 0 \end{pmatrix}$ is an injective envelope of the simple $R^{\text{op}}\text{-module}$ $\begin{pmatrix} k \\ 0 \end{pmatrix}$.

Note that I contains elements of arbitrarily large x -power order, whereas every finite subset is annihilated by some power of x , which implies that E is not a finitely generated $R^{\text{op}}\text{-module}$. But E embeds into $F(R)$. It forces that E is a direct summand of $F(R)$, which is finitely generated as well, a contradiction. Therefore we conclude that R has no self-duality.

Let R be a ring. If R is a von Neumann regular ring, then all R -modules are flat, and hence, all short exact sequences of R -modules are pure exact. Hence pure projective R -modules are exactly projective R -modules. It follows that R satisfies **PSC**. If R is an Artin algebras or a QF ring, then R has a self-duality, and hence R satisfies **PSC** by Lemma 5.6. If R is a ring with a polynomial identity, then it satisfies **PSC** by [8, Theorem 6.7]. Thus the following classes of rings satisfy **PSC**.

- (R1) von Neumann regular rings;
- (R2) Artin algebras or QF rings;
- (R3) rings with a polynomial identity.

The following remark is a supplement to this result, which illustrates that the properties of rings “being von Neumann”, “being an Artin algebra” and “admitting a polynomial identity” are preserved under derived equivalences. For any finitely generated projective R -module P , there exist a projective R -module Q and $n \geq 0$ such that $R^{(n)} = P \oplus Q$. Then we have

$$\text{End}_R(P) = e\mathbb{M}_n(R)e,$$

where $\mathbb{M}_n(R)$ is the full matrix ring over R of order n , and e is the canonical projection $R^{(n)} \rightarrow P$, viewed as an idempotent matrix in $\mathbb{M}_n(R)$. This fact will be used freely.

Remark 5.9. (1) Note that hereditary QF rings are exactly semisimple rings, while “rings being semisimple” is an invariant under the derived equivalence of rings ([2, Proposition 12.3(3)]).

(2) According to [23, Corollary 5.3], we know that, if R and S are Artin algebras over an arbitrary field such that they are derived equivalent, then they are QF whenever so is one of them. However, in general, we do not know whether “rings being QF” is an invariant under the derived equivalence of rings.

(3) “Rings being von Neumann regular” is an invariant under the derived equivalence of rings. Indeed, suppose that R and S are derived equivalent and S is a von Neumann regular ring. Then there

is a one-sided tilting complex $T^\bullet$ in $\text{Ch}(S\text{-Mod})$ such that

$$R \cong \text{End}_{\mathbf{D}(S\text{-Mod})}(T^\bullet).$$

Note that each finitely generated submodule of a finitely generated projective S -module is its a summand as S is von Neumann regular. Now we have an equality

$$T^\bullet = \bigoplus_{i \in \mathbb{Z}} H^i(T^\bullet)[-i]$$

with each $H^i(T^\bullet)$ a finitely generated projective S -module in $\mathbf{D}(S\text{-Mod})$. It follows that

$$\text{End}_{\mathbf{D}(S\text{-Mod})}(T^\bullet) \cong \prod_i \text{End}_S(H^i(T^\bullet)).$$

But $\text{End}_S(H^i(T^\bullet)) \cong e_i \mathbb{M}_n(S) e_i$ for some $n \geq 0$ and some idempotent matrix $e_i \in \mathbb{M}_n(S)$ as each $H^i(T^\bullet)$ is a finitely generated projective S -module. Thus,

$$R \cong \text{End}_{\mathbf{D}(S\text{-Mod})}(T^\bullet) \cong \prod_i e_i \mathbb{M}_n(S) e_i$$

is a von Neumann regular ring.

(4) “Rings being an Artin algebra” is an invariant under the derived equivalence of rings. Indeed, suppose that R and S are derived equivalent and S is an Artin k -algebra over a commutative Artin ring k . Then there is a one-sided tilting complex $T^\bullet$ in $\text{Ch}(S\text{-Mod})$ such that

$$R \cong \text{End}_{\mathbf{D}(S\text{-Mod})}(T^\bullet).$$

So R is isomorphic to a quotient k -algebra of the form

$$\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)/N,$$

where N is a subalgebra of $\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)$ consisting of null-homotopic endomorphisms of $T^\bullet$. Since each $T^i \in S\text{-proj}$ and $T^\bullet$ is bounded, we have that each $\text{Hom}_S(T^l, T^m)$ is a finitely generated S -module, and hence, as a k -module, $\text{End}_S(\bigoplus_l T^l)$ is finitely generated. Notice further that $\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)$ is a submodule of $\text{End}_S(\bigoplus_l T^l)$, it is again a finitely generated k -module. Thus

$$R \cong \text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)/N$$

is a finitely generated k -module as well. It follows that R is an Artin k -algebra.

(5) “Rings admitting a polynomial identity” is an invariant under the derived equivalence of rings. Indeed, suppose that R and S are derived equivalent and S admits a polynomial identity. Then there is a one-sided tilting complex $T^\bullet$ in $\text{Ch}(S\text{-Mod})$ such that

$$R \cong \text{End}_{\mathbf{D}(S\text{-Mod})}(T^\bullet).$$

So R is isomorphic to a quotient ring of the form

$$\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)/N,$$

where N is a subring of $\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)$ consisting of null-homotopic endomorphisms of $T^\bullet$. Note that $T^\bullet$ is a bounded complex in $\text{Ch}(S\text{-Mod})$ with each T^l a finitely generated projective S -module. So $\bigoplus_l T^l$ is also a finitely generated projective S -module, which yields an isomorphism

$$\text{End}_S(\bigoplus_l T^l) \cong e_i \mathbb{M}_n(S) e_i$$

for some $n \geq 0$ and some idempotent matrix $e_i \in \mathbb{M}_n(S)$. It follows that $\text{End}_S(\bigoplus_l T^l)$ admits a polynomial identity as so does $\mathbb{M}_n(S)$. Now, as a subring, $\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)$ admits a polynomial identity, which in turn induces that the quotient ring

$$\text{End}_{\text{Ch}(S\text{-Mod})}(T^\bullet)/N \cong R$$

admits a polynomial identity.

Finally, we list some results which are consequences of Proposition 5.3, and Theorem 5.5 and Corollary 5.7.

Corollary 5.10. *Let S be a hereditary ring which satisfies **PSC**.*

(1) *If R is a left coherent ring which admits one of the following triangle equivalences*

$$\mathbf{D}_{\text{pur}}(R\text{-Mod}) \simeq \mathbf{D}_{\text{pur}}(S\text{-Mod}), \quad \mathbf{D}(R\text{-FP-I}) \simeq \mathbf{D}(S\text{-FP-I}), \quad \text{and} \quad \mathbf{K}(R\text{-Inj}) \simeq \mathbf{K}(S\text{-Inj}),$$

*then R satisfies **PSC**.*

(2) *If R is a right coherent ring which admit a triangle equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(S\text{-Proj})$, then R satisfies **PSC**.*

Corollary 5.11. *Let S be a hereditary ring which has a self-duality.*

(1) *If R is a left coherent ring which admits one of the following triangle equivalences*

$$\mathbf{D}_{\text{pur}}(R\text{-Mod}) \simeq \mathbf{D}_{\text{pur}}(S\text{-Mod}), \quad \mathbf{D}(R\text{-FP-I}) \simeq \mathbf{D}(S\text{-FP-I}), \quad \text{and} \quad \mathbf{K}(R\text{-Inj}) \simeq \mathbf{K}(S\text{-Inj}),$$

*then R satisfies **PSC**.*

(2) *If R is a right coherent ring which admit a triangle equivalence $\mathbf{K}(R\text{-Proj}) \simeq \mathbf{K}(S\text{-Proj})$, then R satisfies **PSC**.*

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