Selforthogonal Modules of Finite Projective Dimension*†

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Abstract

Let R be a ring and ${}_R\omega$ a selforthogonal module. We introduce the notion of the right orthogonal dimension (relative to ${}_R\omega$) of modules. We give a criterion for computing this relative right orthogonal dimension of modules. For a left coherent and semi-local ring R and a finitely presented selforthogonal module ${}_R\omega$, we show that the projective dimension of ${}_R\omega$ and the right orthogonal dimension (relative to ${}_R\omega$) of R/J are identical, where J is the Jacobson radical of R. As a consequence, we get that ${}_R\omega$ has finite projective dimension if and only if every left (finitely presented) R-module has finite right orthogonal dimension (relative to ${}_R\omega$). If ω is a tilting module, we then prove that a left R-module has finite right orthogonal dimension (relative to ${}_R\omega$) if and only if it has a special $\omega^{\perp}$-preenvelope.

Tilting modules and cotilting modules are very important research objects in representation theory of artin algebras, which are some special kinds of orthogonal modules. Let ${}_R\omega$ be a finitely generated selforthogonal module over an Artinian algebra R. Huang in [H] introduced the notion of left orthogonal dimension (relative to ${}_R\omega$) of modules, and proved that the injective dimension of ${}_R\omega$ is finite if and only if every finitely generated left R-module has finite left orthogonal dimension (relative to ${}_R\omega$). In [CT] Colpi and Trlifaj investigated the properties of tilting torsion theories. Futhermore, Angeleri Hügel and Coelho established in [AnC] the relationship between the tilting theory and relative homological theory.

Motivated by the papers mentioned above, in this paper we introduce in Section 1 the notion of right orthogonal dimension (relative to a given selfothogonal module) of modules. In Section 2, we first give a criterion for computing this relative right orthogonal dimension of modules and prove that for a left R-module M and a non-negative integer n, the right orthogonal dimension (relative to a selfothogonal module ${}_R\omega$) of M is at most n if and only if the n-th cosyzygy of M is right orthogonal with ${}_R\omega$. Let R be a left coherent and semi-local ring and ${}_R\omega$ a finitely presented selforthogonal module. We show that the projective

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dimension of ${}_R\omega$ and the right orthogonal dimension (relative to ${}_R\omega$) of R/J are identical, where J is the Jacobson radical of R. As a consequence of the obtained results, we get that ${}_R\omega$ has finite projective dimension if and only if every left (finitely presented) R-module has finite right orthogonal dimension (relative to ${}_R\omega$). In Section 3 we study the properties of right orthogonal dimension (relative to ${}_R\omega$) of modules when ω is a tilting module. We prove that a left R-module has finite right orthogonal dimension (relative to ${}_R\omega$) if and only if it has a special $\omega^{\perp}$-preenvelope if and only if it has a special $\mathcal{X}$-precover, where $\mathcal{X}$ denotes the subcategory of left R-modules consisting of all X with $\text{Add}_R\omega\text{-coresol.dim}_R(X)$ (see Section 3 for the defintion) at most n.

1. Definitions and Notations

For a ring R, We use Mod R to denote the category of left R-modules. For a left R-module M, we use $\text{l.pd}_R(M)$ to denote the left projective dimension of M.

Definition 1.1 Let $\omega \in$ Mod R. We call ω a *selforthogonal module* if $\text{Ext}^i_R(\omega, \omega) = 0$ for any $i \geq 1$.

Definition 1.2 Let $\omega \in$ Mod R be a selforthogonal module and $X \in$ Mod R. X is said to be *right orthogonal* with ω if $\text{Ext}^i_R(\omega, X)=0$ for any $i \geq 1$. We use $\omega^{\perp}$ to denote the subcategory of Mod R consisting of the modules which are right orthogonal with ω. An exact sequence $0 \to X \to X_0 \to \cdots \to X_n \to \cdots$ in Mod R is called a $\omega^{\perp}$-*coresolution* of X if all $X_i \in \omega^{\perp}$.

We now introduce the notion of the right orthogonal dimension (relative to a given module) of modules as follows.

Definition 1.3 Let ω and M be in Mod R. If M has a $\omega^{\perp}$-coresolution $0 \to M \to X_0 \to \cdots \to X_n \to \cdots$, then set $\omega^{\perp}\text{-dim}_R(M) = \inf\{n | 0 \to M \to X_0 \to \cdots \to X_n \to 0$ is a right orthogonal coresolution of $M\}$. If no such an integer exists set $\omega^{\perp}\text{-dim}_R(M) = \infty$. We call $\omega^{\perp}\text{-dim}_R(M)$ the *right orthogonal dimension* of M.

For any $M \in$Mod R, it is clear that $\omega^{\perp}\text{-dim}_R(M)$ is at most the injective dimension of M, and $\omega^{\perp}\text{-dim}_R(M) = 0$ if and only if $M \in \omega^{\perp}$. So the notion of this relative right orthogonal dimension can be regarded as a generalization of that of the injective dimension. On the other hand, there is a close relation between the right orthogonal dimension of modules (relative to a given selforthogonal module ${}_R\omega$) and the projective dimension of ${}_R\omega$. In fact, in next section we will show that if ${}_R\omega$ is selforthogonal, then for any $M \in$Mod R, $\omega^{\perp}\text{-dim}_R(M)$ is at most the projective dimension of ${}_R\omega$.

2. The Right Orthogonal Dimension of Modules

In this section, R is a ring and ${}_R\omega \in$ Mod R is a selforthogonal module. The right orthogonal dimension relative to ${}_R\omega$ of a module M is called the *right orthogonal dimension* of M for short.

Definition 2.1 ([R]) Let $B \in$Mod R and $0 \to B \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} E_i \xrightarrow{d_i} \cdots$ be an injective coresolution; denote Imε by co$\Omega^0(B)$ and, for $n \geq$1, denote Imd_{n-1} by co$\Omega^n(B)$. For any $n \geq$0, co$\Omega^n(B)$ is called the n-th *cosyzygy* of B.

The following theorem gives a criterion for computing the right orthogonal dimension of modules.

Theorem 2.2 *Let $M \in$Mod R. Then $\omega^{\perp}$-$\dim_R(M) \leq n$ if and only if* co$\Omega^n(M) \in \omega^{\perp}$.

Proof. Assume that

$$0 \to M \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} E_i \xrightarrow{d_i} \cdots$$

is a minimal injective coresolution of ${}_RM$. The sufficiency is trivial. We next prove the necessity.

The case n=0 is trivial. Now suppose that $n \geq$1 and M has the following right orthogonal coresolution: $0 \to M \to Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots Y_{n-1} \xrightarrow{f_{n-1}} Y_n \longrightarrow 0$.

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & E_0 & \longrightarrow & \mathrm{co}\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_0 & \longrightarrow & E_0 \bigoplus \mathrm{Im} f_0 & \longrightarrow & \mathrm{co}\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Im} f_0 & = & \mathrm{Im} f_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the exactness of the middle row in the above diagram we get that $\mathrm{Ext}_R^i(\omega, \mathrm{Im} f_0) \cong \mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^1(M))$ for any $i \geq$1. On the other hand, it is easy to see that $\mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^n(M)) \cong \mathrm{Ext}_R^{i+n-1}(\omega, \mathrm{co}\Omega^1(M))$ for any $i \geq$1 and $\mathrm{Ext}_R^{t+n-1}(\omega, \mathrm{Im} f_0) \cong \mathrm{Ext}_R^t(\omega, Y_n) = 0$ for any $t \geq 1$. So $\mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^n(M)) = 0$ for any $i \geq$1 and co$\Omega^n(M) \in \omega^{\perp}$. This completes the proof. $\square$

Lemma 2.3 *Let* $M \in$Mod R *and* $\omega^{\perp}$-$\dim_R(M) < \infty$. *Then* $\omega^{\perp}$-$\dim_R(M)$=$\sup\{t \mid \mathrm{Ext}_R^t(\omega, M) \neq 0\}$.

Proof. Suppose $\omega^{\perp}$-$\dim_R(M)$=$n < \infty$. By Theorem 2.2, $\mathrm{Ext}_R^k(\omega, M) \cong \mathrm{Ext}_R^{k-n}(\omega, \mathrm{co}\Omega^n(M))$=0 for any $k \geq n+1$. So $\sup\{t \mid \mathrm{Ext}_R^t(\omega, M) \neq 0\} \leq n$. Suppose that

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

is a minimal injective coresolution of M. For any $n \geq 1$, from the exact sequence $0 \to \mathrm{co}\Omega^{n-1}(M) \to E_{n-1} \to \mathrm{co}\Omega^n(M) \to 0$ we get a long exact sequence:

$$\cdots \to \mathrm{Ext}_R^{i-1}(\omega, \mathrm{co}\Omega^n(M)) \to \mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^{n-1}(M)) \to \mathrm{Ext}_R^i(\omega, E_{n-1})$$

$$\to \mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^n(M)) \to \mathrm{Ext}_R^{i+1}(\omega, \mathrm{co}\Omega^{n-1}(M)) \to \cdots .$$

So $\mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^{n-1}(M))$=0 for any $i \geq 2$. We claim that $\mathrm{Ext}_R^1(\omega, \mathrm{co}\Omega^{n-1}(M)) \neq 0$. Otherwise, if $\mathrm{Ext}_R^1(\omega, \mathrm{co}\Omega^{n-1}(M))$=0, then $\mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^{n-1}(M))$=0 for any $i \geq 1$ and $\mathrm{co}\Omega^{n-1}(M) \in \omega^{\perp}$. It follows from Theorem 2.2 that $\omega^{\perp}$-$\dim_R(M) \leq n-1$, which is a contradiction. In addition $\mathrm{Ext}_R^n(\omega, M) \cong \mathrm{Ext}_R^1(\omega, \mathrm{co}\Omega^{n-1}(M))$, so $\mathrm{Ext}_R^n(\omega, M) \neq 0$, which implies $\sup\{t \mid \mathrm{Ext}_R^t(\omega, M) \neq 0\} \geq n$. This finishes the proof. $\square$

Lemma 2.4 $\omega^{\perp}$-$\dim_R(M) \leq \mathrm{l.pd}_R(\omega)$ *for any* $M \in$Mod R.

Proof. Without loss of generalization, suppose $\mathrm{l.pd}_R(\omega) = n < \infty$. Then for any $M \in$Mod R and $i \geq n+1$, we have that $\mathrm{Ext}_R^i(\omega, \mathrm{co}\Omega^n(M)) \cong \mathrm{Ext}_R^{n+i}(\omega, M)$=0 for any $i \geq 1$ and $\mathrm{co}\Omega^n(M) \in \omega^{\perp}$. It follows from Theorem 2.2 that $\omega^{\perp}$-$\dim_R(M) \leq n$. We are done. $\square$

Recall from [S] that R is called a *semi-local ring* if R/J is an Artinian semi-simple ring. It is well known that a semi-perfect ring (more specially, an Artinian ring or a semi-primary ring) is semi-local (see [AF]). Also recall from [S] that a left R-module M is called *finitely presented* if there is a finitely generated projective left R-module P and a finitely generated submodule N of P such that $P/N \cong M$. A left R-module M is said to *admit a finitely generated projective resolution* if there is an exact sequence: $\cdots \longrightarrow P_i \xrightarrow{f_i} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \longrightarrow M \longrightarrow 0$, where P_i is a finitely generated projective left R-module for any $i \geq 0$ (see [G]). It is clear that if M admits a finitely generated projective resolution as above, then M and $\mathrm{Im} f_i$ are finitely presented for any $i \geq 1$.

Theorem 2.5 *Let* R *be a semi-local ring. If* ${}_R\omega$ *admits a finitely generated projective resolution, then* $\mathrm{l.pd}_R(\omega) = \omega^{\perp}$-$\dim_R(R/J)$.

Proof. By Lemma 2.4, we have that $\omega^{\perp}$-$\dim_R(R/J) \leq \mathrm{l.pd}_R(\omega)$. Then we need to prove that $\mathrm{l.pd}_R(\omega) \leq \omega^{\perp}$-$\dim_R(R/J)$. Without loss of generalization, suppose $\omega^{\perp}$-$\dim_R(R/J) = n < \infty$. Then by Lemma 2.3, we have that $\mathrm{Ext}_R^i(\omega, R/J)$=0 for any $i \geq n+1$.

Now suppose that

$$\cdots \to P_i \xrightarrow{f_i} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \to {}_R\omega \to 0$$

is a finitely generated projective resolution of ${}_R\omega$. Then $\mathrm{Ext}_R^i(\mathrm{Im} f_n, R/J) \cong \mathrm{Ext}_R^{n+i}(\omega, R/J)$=0 for any $i \geq 1$. So by [XC, Lemma 3], we have that $\mathrm{Im} f_n$ is projective and hence $\mathrm{l.pd}_R(\omega) \leq n$. We are done. □

Recall from [S] that R is called a *left coherent ring* if every finitely generated submodule of a finitely presented left R-module is also finitely presented. It is clear that R is left coherent if R is a left Noetherian ring. On the other hand, it is not difficult to see that if R is a left coherent ring, then every presented left R-module admits a finitely generated projective resolution. So, the following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6 *Let R be a left coherent and semi-local ring. If ${}_R\omega$ is finitely presented, then* $\mathrm{l.pd}_R(\omega) = \omega^{\perp}$-$\dim_R(R/J)$.

The following result is a dual version of [H, Theorem 2].

Corollary 2.7 *Let R be a left coherent and semi-local ring and ${}_R\omega$ finitely presented. Then the following statements are equivalent.*

(1) $\mathrm{l.pd}_R(\omega) < \infty$.

(2) *Every (finitely presented) left R-module has finite right orthogonal dimension.*

(3) *Every cyclic left R-module has finite right orthogonal dimension.*

(4) $\omega^{\perp}$-$\dim_R(R/J) < \infty$.

Proof. By Lemma 2.4 and Corollary 2.6. □

3. Tilting Case

Recall from [AnC] that a module $M \in$Mod R is called a *tilting module* provided the following conditions are satisfied:

(1) $\mathrm{l.pd}_R(M) < \infty$.

(2) $\mathrm{Ext}_R^i(M, M^{(I)})$=0 for any $i \geq 1$ and index set I.

(3) There exists an exact sequence $0 \to {}_RR \to M_0 \to M_1 \to \cdots \to M_r \to 0$ with $M_i \in \mathrm{Add}_R M$ for any $0 \leq i \leq r$, where $\mathrm{Add}_R M$ denotes the full subcategory of Mod R consisting of all modules isomorphic to direct summands of direct sums of copies of ${}_RM$.

In this section, $\omega\in$Mod R is a tilting module.

Let $\mathscr{A}$ be a subcategory of Mod R. We use ${}^{\perp}\mathscr{A}$ the subcategory of Mod R consisting of the modules X with $\mathrm{Ext}_R^i(A,X)=0$ for any $A\in\mathscr{A}$ and $i\geq 1$.

Lemma 3.1 ([AnC, Lemma 2.4]) (1) *For any* $X\in\omega^{\perp}$, *there exists an exact sequence* $0\to K\to M\to X\to 0$ *with* $M\in\mathrm{Add}_R\omega$ *and* $K\in\omega^{\perp}$.

(2) *Every homomorphism* $A\to X$ *with* $A\in{}^{\perp}(\omega^{\perp})$ *and* $X\in\omega^{\perp}$ *factors through* $\mathrm{Add}_R\omega$. *In particular,* $\mathrm{Add}_R\omega=\omega^{\perp}\bigcap{}^{\perp}(\omega^{\perp})$.

Definition 3.2 ([EJ]) Assume that $\mathscr{C}\supset\mathscr{D}$ are subcategories of Mod R and $C\in\mathscr{C}$, $D\in\mathscr{D}$. The morphism $C\to D$ is said to be a *preenvelope* of C if $\mathrm{Hom}_R(D,X)\to\mathrm{Hom}_R(C,X)\to 0$ is exact for all $X\in\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be *preenveloping* in $\mathscr{C}$ if every C in $\mathscr{C}$ has a preenvelope. An $\mathscr{D}$*-preenvelope* f of C is called *special* if it is a monomorphism and $\mathrm{Ext}_R^1(\mathrm{Coker}f,X)$=0 for any $X\in\mathscr{D}$. Dually, the morphism $D\to C$ is said to be a *precover* of C if $\mathrm{Hom}_R(X,D)\to\mathrm{Hom}_R(X,C)\to 0$ is exact for all $X\in\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be *precovering* in $\mathscr{C}$ if every C in $\mathscr{C}$ has a precover. An $\mathscr{D}$*-precover* f of C is called *special* if it is an epimorphism and $\mathrm{Ext}_R^1(X,\mathrm{Ker}f)$=0 for any $X\in\mathscr{D}$.

Lemma 3.3 $\omega^{\perp}$-$\mathrm{dim}_R(M)\leq 1$ *if and only if there exists an exact sequence* $0\to M\to V\to M'\to 0$ *with* $V\in\omega^{\perp}$ *and* $M'\in\mathrm{Add}_R\omega$.

Proof. Since ${}_R\omega$ is a tilting module, $\mathrm{Add}_R\omega\in\omega^{\perp}$. So the sufficiency is trivial. Hence it suffices to prove the necessity. Suppose $\omega^{\perp}$-$\mathrm{dim}_R(M)\leq 1$, then there exists an exact sequence $0\to M\to V_0\to V_1\to 0$ with $V_0,V_1\in\omega^{\perp}$. By Lemma 3.1(1), there exists an exact sequence $0\to K\to M'\to V_1\to 0$ with $M'\in\mathrm{Add}_R\omega$ and $K\in\omega^{\perp}$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & V & \to & M' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & V_0 & \to & V_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle column in the above diagram, We know that $V\in\omega^{\perp}$. So the middle row is desired. $\square$

Let $\mathcal{X}$ be a full subcategory of Mod R and A a module in Mod R. If there is an exact sequence $0 \to A \to X_0 \to X_1 \to \cdots \to X_n \to \cdots$ in Mod R with $X_i \in \mathcal{X}$ for any $i \geq 0$, then we define the $\mathcal{X}$-*coresolution dimension* of A, denoted by $\mathcal{X}$-coresol.dim$_R(A)$, as $\inf\{n |$there is an exact sequence $0 \to A \to X_0 \to X_1 \to \cdots \to X_n \to 0$ in Mod R with $X_i \in \mathcal{X}$ for any $0 \leq i \leq n\}$. We set $\mathcal{X}$-coresol.dim$_R(A)$ infinity if no such an integer exists (see [AuB]).

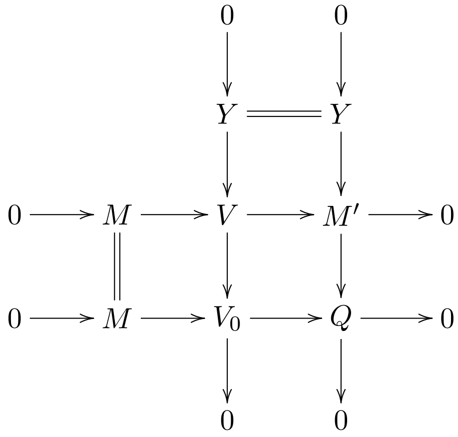
Proposition 3.4 *Let n be a non-negative integer. For any $M \in$ Mod$R$, $\omega^{\perp}$-dim$_R(M) \leq n$ if and only if there exists an exact sequence $0 \to M \to V \to M' \to 0$ with* Add$_R\omega$-coresol.dim$_R(M') \leq n-1$ *and $V \in \omega^{\perp}$.*

Proof. The sufficiency is trivial. In the following we will prove the necessity by using induction on n. The case for $n \leq 1$ follows from Lemma 3.3. Now suppose $n \geq 2$. We have an exact sequence $0 \to M \to V_0 \to Q \to 0$ in Mod R with $V_0 \in \omega^{\perp}$ and $\omega^{\perp}$-dim$_R(Q) \leq n-1$. By induction assumption, we have an exact sequence $0 \to Q \to V_1 \to K \to 0$ with Add$_R\omega$-coresol.dim$_R(K) \leq n-2$ and $V_1 \in \omega^{\perp}$. Since $V_1 \in \omega^{\perp}$, by Lemma 3.1(1) there exists an exact sequence $0 \to Y \to X \to V_1 \to 0$ with $X \in$Add$_R\omega$ and $Y \in \omega^{\perp}$. First, consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & M' & \to & Q & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & X & \to & V_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle column in the above diagram, we have that Add$_R\omega$-coresol.dim$_R(M') \leq n-1$.

Next, consider the following pull-back diagram:



From the middle column in the above diagram, we know that $E \in \omega^{\perp}$. So the middle row is desired. □

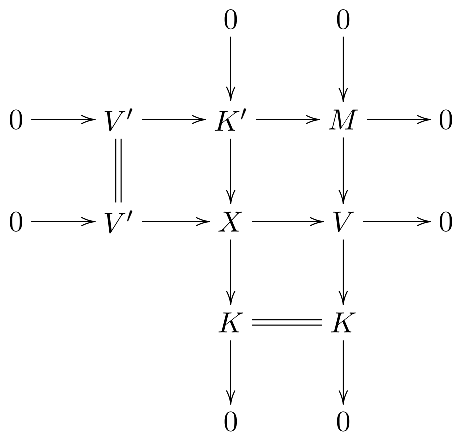
Proposition 3.5 *The following statements are equivalent for a non-negative integer* n.

(1) *For any* M $\in$Mod R, *there exists an exact sequence* $0 \to M \to V \to K \to 0$ *with* $\mathrm{Add}_R\omega$-coresol.$\dim_R(K) \leq n-1$ *and* $V \in \omega^{\perp}$.

(2) *For any* M $\in$Mod R, *there exists an exact sequence* $0 \to V' \to K' \to M \to 0$ *with* $\mathrm{Add}_R\omega$-coresol.$\dim_R(K') \leq n$ *and* $V' \in \omega^{\perp}$.

Proof. (1)⇒(2) Let M $\in$Mod R. Then by (1), there exists an exact sequence $0 \to M \to V \to K \to 0$ with $\mathrm{Add}_R\omega$-coresol.$\dim_R(K) \leq n-1$ and $V \in \omega^{\perp}$. By Lemma 3.1(1) there exists an exact sequence $0 \to V' \to X \to V \to 0$ with X $\in$$\mathrm{Add}_R\omega$ and $V' \in \omega^{\perp}$.

Consider the following pull-back diagram:



From the middle column in the above diagram, we have $\mathrm{Add}_R\omega$-coresol.$\dim_R(K') \leq n$. So the first row is desired.

(2)⇒(1) Let $M\in$Mod R. Then by (2), there exists an exact sequence $0\to V'\to K'\to M\to 0$ with $\mathrm{Add}_R\omega$-coresol.$\dim_R(K')\leq n$ and $V'\in\omega^{\perp}$. So there exists an exact sequence $0\to K'\to X\to K\to 0$ with $X\in\mathrm{Add}_R\omega$ and $\mathrm{Add}_R\omega$-coresol.$\dim_R(K)\leq n-1$.

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & V' & = & V' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K' & \to & X & \to & K & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & V & \to & K & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the middle column in the above diagram, we have $V\in\omega^{\perp}$. So the last row is desired. □

It is easy to verify that the exact sequence in Proposition 3.5(1) is a special $\omega^{\perp}$-preenvelope of M, and that of Proposition 3.5(2) is a special $\mathcal{X}$-precover of M, where $\mathcal{X}$ denotes the subcategory of Mod R consisting of all X with $\mathrm{Add}_R\omega$-coresol.$\dim_R(X)\leq n$. Thus we get the main result in this section as follows.

Theorem 3.6 *For any $M\in$Mod R, the following are equivalent*:

(1) $\omega^{\perp}$-$\dim_R(M)\leq n$.

(2) $\mathrm{co}\Omega^n(M)\in\omega^{\perp}$.

(3) *M has a special $\omega^{\perp}$-preenvelope $f: M\to V$ with* $\mathrm{Add}_R\omega$-coresol.$\dim_R(\mathrm{Coker}f)\leq n-1$.

(4) *M has a special $\mathcal{X}$-precover $g: K'\to M$ with* $\mathrm{Ker}g\in\omega^{\perp}$, *where $\mathcal{X}$ denotes the subcategory of* Mod R *consisting of all X with* $\mathrm{Add}_R\omega$-coresol.$\dim_R(X)\leq n$.

Proof. By Theorem 2.2 we have (1)⇔(2). By Propositions 3.4 and 3.5 we have (1)⇔(3)⇔(4). □

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