Special Precovered Categories of Gorenstein Categories*†

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Abstract

Let $\mathscr{A}$ be an abelian category and $\mathscr{P}(\mathscr{A})$ the subcategory of $\mathscr{A}$ consisting of projective objects. Let $\mathscr{C}$ be a full, additive and self-orthogonal subcategory of $\mathscr{A}$ with $\mathscr{P}(\mathscr{A})$ a generator, and let $\mathcal{G}(\mathscr{C})$ be the Gorenstein subcategory of $\mathscr{A}$. Then the right 1-orthogonal category $\mathcal{G}(\mathscr{C})^{\perp_1}$ of $\mathcal{G}(\mathscr{C})$ is both projectively resolving and injectively coresolving in $\mathscr{A}$. We also get that the subcategory $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ of $\mathscr{A}$ consisting of objects admitting special $\mathcal{G}(\mathscr{C})$-precovers is closed under extensions and $\mathscr{C}$-stable direct summands (*). Furthermore, if $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$, then we have that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is the minimal subcategory of $\mathscr{A}$ containing $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C})$ with respect to the property (*), and that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving in $\mathscr{A}$ with a $\mathscr{C}$-proper generator $\mathscr{C}$.

1 Introduction

As a generalization of finitely generated projective modules, Auslander and Bridger introduced in [3] the notion of finitely generated modules of Gorenstein dimension zero over commutative Noetherian rings. Then Enochs and Jenda generalized it in [7] to arbitrary modules over a general ring and introduced the notion of Gorenstein projective modules and its dual (that is, the notion of Gorenstein injective modules). Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ an additive and full subcategory of $\mathscr{A}$. Recently Sather-Wagstaff, Sharif and White introduced in [14] the notion of the Gorenstein subcategory $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$, which is a common generalization of the notions of modules of Gorenstein dimension zero [3], Gorenstein projective modules, Gorenstein injective modules [7], V-Gorenstein projective modules and V-Gorenstein injective modules [9], and so on.

Let R be an associative ring with identity, and let $\mathrm{Mod}\, R$ be the category of left R-modules and $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)$ the subcategory of $\mathrm{Mod}\, R$ consisting of Gorenstein projective modules. Let $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ and $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ be the subcategories of $\mathrm{Mod}\, R$ consisting of modules admitting a $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$-precover and admitting a special $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$-precover respectively. The following question in relative homological algebra remains still open: does $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)) = \mathrm{Mod}\, R$ always hold true? Several authors have gave some partially positive answers to this question, see [2, 4, 5, 16]. Note that in these references, $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R)) = \mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$, see Example 4.8 below for details. In particular, any module in $\mathrm{Mod}\, R$ with finite Gorenstein projective dimension admits a $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$-precover which is

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also special ([10]). In fact, it is unknown whether $\mathrm{PC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R)) = \mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R))$ always holds true. Based on the above, it is necessary to study the properties of these two subcategories.

Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ an additive and full subcategory of $\mathscr{A}$. We use $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ to denote the subcategory of $\mathscr{A}$ consisting of objects admitting special $\mathcal{G}(\mathscr{C})$-precovers. The aim of this paper is to investigate the structure of $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ in terms of the properties of the right 1-orthogonal category $\mathcal{G}(\mathscr{C})^{\perp_1}$ of $\mathcal{G}(\mathscr{C})$. This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Assume that $\mathscr{C}$ is self-orthogonal and the subcategory of $\mathscr{A}$ consisting of projective objects is a generator for $\mathscr{C}$. In Section 3, we prove that $\mathcal{G}(\mathscr{C})^{\perp_1}$ is both projectively resolving and injectively coresolving in $\mathscr{A}$. We also characterize when all objects in $\mathscr{A}$ are in $\mathcal{G}(\mathscr{C})^{\perp_1}$.

In Section 4, we prove that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under extensions and $\mathscr{C}$-stable direct summands (*). Furthermore, if $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$, then we get the following two results: (1) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is the minimal subcategory of $\mathscr{A}$ containing $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C})$ with respect to the property (*); and (2) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving in $\mathscr{A}$ with a $\mathscr{C}$-proper generator $\mathscr{C}$.

2 Preliminaries

Throughout this paper, $\mathscr{A}$ is an abelian category and all subcategories of $\mathscr{A}$ are full, additive and closed under isomorphisms. We use $\mathscr{P}(\mathscr{A})$ (resp. $\mathscr{I}(\mathscr{A})$) to denote the subcategory of $\mathscr{A}$ consisting of projective (resp. injective) objects. For a subcategory $\mathscr{C}$ of $\mathscr{A}$ and an object A in $\mathscr{A}$, the *$\mathscr{C}$-dimension* $\mathscr{C}\text{-}\dim A$ of A is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to A \to 0$$

in $\mathscr{A}$ with all C_i in $\mathscr{C}\}$. Set $\mathscr{C}\text{-}\dim A = \infty$ if no such integer exists (cf. [12]). For a non-negative integer n, we use $\mathscr{C}^{\leq n}$ (resp. $\mathscr{C}^{<\infty}$) to denote the subcategory of $\mathscr{A}$ consisting of objects with $\mathscr{C}$-dimension at most n (resp. finite $\mathscr{C}$-dimension).

Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$. Recall that a sequence in $\mathscr{A}$ is called $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$*-exact* if it is exact after applying the functor $\mathrm{Hom}_{\mathscr{A}}(X,-)$ for any $X \in \mathscr{X}$. Dually, the notion of a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$*-exact sequence* is defined. Set

$$\mathscr{X}^{\perp} := \{M \mid \mathrm{Ext}^{\geq 1}_{\mathscr{A}}(X, M) = 0 \text{ for any } X \in \mathscr{X}\},$$

$$^{\perp}\mathscr{X} := \{M \mid \mathrm{Ext}^{\geq 1}_{\mathscr{A}}(M, X) = 0 \text{ for any } X \in \mathscr{X}\},$$

and

$$\mathscr{X}^{\perp_1} := \{M \mid \mathrm{Ext}^{1}_{\mathscr{A}}(X, M) = 0 \text{ for any } X \in \mathscr{X}\},$$

$$^{\perp_1}\mathscr{X} := \{M \mid \mathrm{Ext}^{1}_{\mathscr{A}}(M, X) = 0 \text{ for any } X \in \mathscr{X}\}.$$

We call $\mathscr{X}^{\perp_1}$ (resp. $^{\perp_1}\mathscr{X}$) the *right (resp. left) 1-orthogonal category of* $\mathscr{X}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\mathscr{A}$. We write $\mathscr{X} \perp \mathscr{Y}$ if $\mathrm{Ext}^{\geq 1}_{\mathscr{A}}(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Definition 2.1. (cf. [6]) Let $\mathscr{X} \subseteq \mathscr{Y}$ be subcategories of $\mathscr{A}$. The morphism $f : X \to Y$ in $\mathscr{A}$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$ is called an *$\mathscr{X}$-precover* of Y if $\mathrm{Hom}_{\mathscr{A}}(X', f)$ is epic for any $X' \in \mathscr{X}$. An $\mathscr{X}$-precover $f : X \to Y$ is called *special* if f is epic and $\mathrm{Ker}\, f \in \mathscr{X}^{\perp_1}$. $\mathscr{X}$ is called *special precovering* in $\mathscr{Y}$

if any object in $\mathscr{Y}$ admits a special $\mathscr{X}$-precover. Dually, the notions of a *(special) $\mathscr{X}$-(pre)envelope* and a *special preenveloping subcategory* are defined.

Definition 2.2. (cf. [10]) A subcategory of $\mathscr{A}$ is called *projectively resolving* if it contains $\mathscr{P}(\mathscr{A})$ and is closed under extensions and under kernels of epimorphisms. Dually, the notion of *injectively coresolving subcategories* is defined.

From now on, assume that $\mathscr{C}$ is a given subcategory of $\mathscr{A}$.

Definition 2.3. (cf. [14]) The *Gorenstein subcategory* $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$ is defined as $\mathcal{G}(\mathscr{C}) = \{M$ is an object in $\mathscr{A} \mid$ there exists an exact sequence:

$$\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots \tag{2.1}$$

in $\mathscr{C}$, which is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact, such that $M \cong \mathrm{Im}(C_0 \to C^0)\}$; in this case, (2.1) is called a *complete $\mathscr{C}$-resolution* of M.

In what follows, R is an associative ring with identity, $\mathrm{Mod}\, R$ is the category of left R-modules and $\mathrm{mod}\, R$ is the category of finitely generated left R-modules.

Remark 2.4.

1. Let R be a left and right Noetherian ring. Then $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))$ coincides with the subcategory of $\mathrm{mod}\, R$ consisting of modules with Gorenstein dimension zero ([3]).
2. $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ (resp. $\mathcal{G}(\mathscr{I}(\mathrm{Mod}\, R))$) coincides with the subcategory of $\mathrm{Mod}\, R$ consisting of Gorenstein projective (resp. injective) modules ([7]).
3. Let R be a left Noetherian ring, S a right Noetherian ring and ${}_RV_S$ a dualizing bimodule. Put $\mathscr{W} = \{V \bigotimes_S P \mid P \in \mathscr{P}(\mathrm{Mod}\, S)\}$ and $\mathscr{U} = \{\mathrm{Hom}_S(V, E) \mid E \in \mathscr{I}(\mathrm{Mod}\, S^{op})\}$. Then $\mathcal{G}(\mathscr{W})$ (resp. $\mathcal{G}(\mathscr{U})$) coincides with the subcategory of $\mathrm{Mod}\, R$ consisting of V-Gorenstein projective (resp. injective) modules ([9]).

Definition 2.5. (cf. [14]) Let $\mathscr{X} \subseteq \mathscr{T}$ be subcategories of $\mathscr{A}$. Then $\mathscr{X}$ is called a *generator* (resp. *cogenerator*) for $\mathscr{T}$ if for any $T \in \mathscr{T}$, there exists an exact sequence $0 \to T' \to X \to T \to 0$ (resp. $0 \to T \to X \to T' \to 0$) in $\mathscr{T}$ with $X \in \mathscr{X}$; and $\mathscr{X}$ is called a *projective generator* (resp. *an injective cogenerator*) for $\mathscr{T}$ if $\mathscr{X}$ is a generator (resp. cogenerator) for $\mathscr{T}$ and $\mathscr{X} \perp \mathscr{T}$ (resp. $\mathscr{T} \perp \mathscr{X}$).

We have the following easy observation.

Lemma 2.6. *Assume that $\mathscr{C} \perp \mathscr{C}$ and $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$. Then for any $G \in \mathcal{G}(\mathscr{C})$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact exact sequence*

$$0 \to G' \to P \to G \to 0$$

in $\mathscr{A}$ with $P \in \mathscr{P}(\mathscr{A})$ and $G' \in \mathcal{G}(\mathscr{C})$.

Proof. Let $G \in \mathcal{G}(\mathscr{C})$. Then there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact exact sequence

$$0 \to G_1 \to C_0 \to G \to 0$$

in $\mathscr{A}$ with $C_0 \in \mathscr{C}$ and $G_1 \in \mathcal{G}(\mathscr{C})$. Because $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$ by assumption, there exists an exact sequence

$$0 \to C' \to P \to C_0 \to 0$$

in $\mathscr{A}$ with $P \in \mathscr{P}(\mathscr{A})$ and $C' \in \mathscr{C}$. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C' & = & C' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & G' & \dashrightarrow & P & \dashrightarrow & G & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G_1 & \longrightarrow & C_0 & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

By [11, Lemma 2.5], the middle row is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact, and hence $G' \in \mathcal{G}(\mathscr{C})$ by [11, Proposition 4.7], that is, the middle row is the desired sequence. $\square$

The following result is useful in the sequel.

Proposition 2.7. *Assume that $\mathscr{C} \perp \mathscr{C}$ and $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$. Then*

(1) $\mathcal{G}(\mathscr{C})^{\perp_1} = \mathcal{G}(\mathscr{C})^{\perp}$.

(2) $\mathcal{G}(\mathscr{C}) \subseteq {}^{\perp}\mathscr{C} \cap \mathscr{C}^{\perp}$.

Proof. (1) It suffices to prove that $\mathcal{G}(\mathscr{C})^{\perp_1} \subseteq \mathcal{G}(\mathscr{C})^{\perp}$. Let $M \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $G \in \mathcal{G}(\mathscr{C})$. By Lemma 2.6, we have an exact sequence

$$0 \to G' \to P \to G \to 0$$

in $\mathscr{A}$ with $P \in \mathscr{P}(\mathscr{A})$ and $G' \in \mathcal{G}(\mathscr{C})$. It induces $\mathrm{Ext}^2_{\mathscr{A}}(G, M) \cong \mathrm{Ext}^1_{\mathscr{A}}(G', M) = 0$, and hence $\mathrm{Ext}^2_{\mathscr{A}}(G', M) = 0$ and $\mathrm{Ext}^3_{\mathscr{A}}(G, M) \cong \mathrm{Ext}^2_{\mathscr{A}}(G', M) = 0$. Repeating this process, we get $\mathrm{Ext}^{\geq 1}_{\mathscr{A}}(G, M) = 0$.

(2) See [11, Lemma 5.7]. $\square$

We remark that if $\mathscr{A}$ has enough projective objects, and if $\mathscr{P}(\mathscr{A}) \subseteq \mathscr{C}$ and $\mathscr{C}$ is closed under kernels of epimorphisms, then $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$.

3 The right 1-orthogonal category of $\mathcal{G}(\mathscr{C})$

In the rest of this paper, assume that the subcategory $\mathscr{C}$ is self-orthogonal (that is, $\mathscr{C} \perp \mathscr{C}$) and $\mathscr{P}(\mathscr{A})$ is a generator for $\mathscr{C}$. In this section, we mainly investigate the homological properties of $\mathcal{G}(\mathscr{C})^{\perp_1}$. We begin with some examples of $\mathcal{G}(\mathscr{C})^{\perp_1}$.

Example 3.1.

(1) By Proposition 2.7 and [11, Theorem 5.8], we have $\mathscr{P}(\mathscr{A}) \subseteq \mathscr{C} \subseteq \mathscr{C}^{<\infty} \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$.

(2) $\mathscr{P}(\mathscr{A})^{<\infty} \cup \mathscr{I}(\mathscr{A})^{<\infty} \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$.

(3) If the global dimension of R is finite, then $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathrm{Mod}\, R$.

(4) By [8, Theorem 11.5.1] and [1, Theorem 31.9], we have that R is quasi-Frobenius if and only if $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathscr{I}(\mathrm{Mod}\, R)$, and if and only if $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathscr{P}(\mathrm{Mod}\, R) = \mathscr{I}(\mathrm{Mod}\, R)$.

For a non-negative integer n, recall that a left and right noetherian ring R is called *n-Gorenstein* if the left and right self-injective dimensions of R are at most n. The following result is a generalization of Example 3.1(4).

Example 3.2. If R is n-Gorenstein, then

$$\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} = \mathscr{P}(\mathrm{Mod}\, R)^{\leq n} = \mathscr{P}(\mathrm{Mod}\, R)^{<\infty} = \mathscr{I}(\mathrm{Mod}\, R)^{\leq n} = \mathscr{I}(\mathrm{Mod}\, R)^{<\infty}.$$

Proof. By [13, Theorem 2] and Example 3.1(2), we have

$$\mathscr{P}(\mathrm{Mod}\, R)^{\leq n} = \mathscr{P}(\mathrm{Mod}\, R)^{<\infty} = \mathscr{I}(\mathrm{Mod}\, R)^{\leq n} = \mathscr{I}(\mathrm{Mod}\, R)^{<\infty} \subseteq \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1}.$$

Now let $M \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1}$ and $N \in \mathrm{Mod}\, R$. Since R is n-Gorenstein, there exists an exact sequence

$$0 \to G_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all P_i in $\mathscr{P}(\mathrm{Mod}\, R)$ and $G_n \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ by [8, Theorem 11.5.1]. Then we have $\mathrm{Ext}_R^{n+1}(N, M) \cong \mathrm{Ext}_R^1(G_n, M) = 0$ and $M \in \mathscr{I}(\mathrm{Mod}\, R)^{\leq n}$, and thus $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1} \subseteq \mathscr{I}(\mathrm{Mod}\, R)^{\leq n}$. $\square$

The following result shows that $\mathcal{G}(\mathscr{C})^{\perp_1}$ behaves well.

Theorem 3.3.

(1) $\mathcal{G}(\mathscr{C})^{\perp_1}$ is closed under direct products, direct summands and extensions.

(2) $\mathcal{G}(\mathscr{C})^{\perp_1}$ is projectively resolving in $\mathscr{A}$.

(3) $\mathcal{G}(\mathscr{C})^{\perp_1}$ is injectively coresolving in $\mathscr{A}$.

Proof. (1) It is trivial.

(2) By Example 3.1(1), $\mathscr{P}(\mathscr{A}) \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$. Let $G \in \mathcal{G}(\mathscr{C})$ and

$$0 \to L \to M \to N \to 0$$

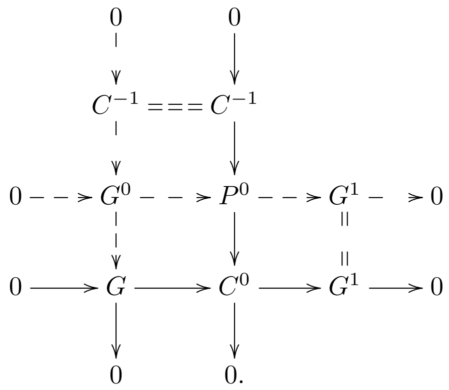
be an exact sequence in $\mathscr{A}$ with $M, N \in \mathcal{G}(\mathscr{C})^{\perp_1}$. By Proposition 2.7(1), we have $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(G, M) = 0 = \mathrm{Ext}_{\mathscr{A}}^{\geq 1}(G, N)$. Then $\mathrm{Ext}_{\mathscr{A}}^{\geq 2}(G, L) = 0$. Because $G \in \mathcal{G}(\mathscr{C})$, we have an exact sequence

$$0 \to G \to C^0 \to G^1 \to 0$$

in $\mathscr{A}$ with $C^0 \in \mathscr{C}$ and $G^1 \in \mathcal{G}(\mathscr{C})$. For C^0, there exists an exact sequence

$$0 \to C^{-1} \to P^0 \to C^0 \to 0$$

in $\mathscr{A}$ with $P^0 \in \mathscr{P}(\mathscr{A})$ and $C^{-1} \in \mathscr{C}$. Consider the following pullback diagram

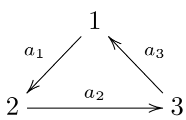


By the above argument, we have $\mathrm{Ext}^1_{\mathscr{A}}(G^0, L) \cong \mathrm{Ext}^2_{\mathscr{A}}(G^1, L) = 0$. Because the leftmost column splits by Proposition 2.7(2), G is isomorphic to a direct summand of G^0 and $\mathrm{Ext}^1_{\mathscr{A}}(G, L) = 0$, which shows that $L \in \mathcal{G}(\mathscr{C})^{\perp_1}$.

(3) It is trivial that $\mathscr{I}(\mathscr{A}) \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$. By Proposition 2.7, we have that $\mathcal{G}(\mathscr{C})^{\perp_1}$ is closed under cokernels of monomorphisms. Thus $\mathcal{G}(\mathscr{C})^{\perp_1}$ is injectively coresolving. □

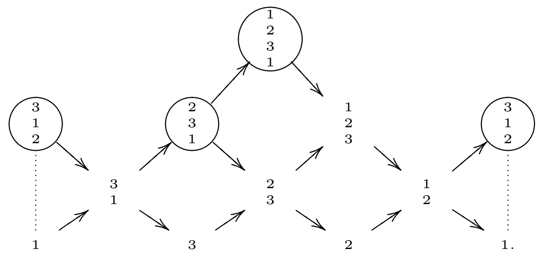
Before giving some applications of Theorem 3.3(2), consider the following example.

Example 3.4. Let Q be a quiver:



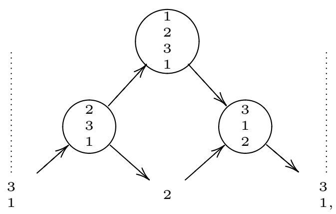
and $I =< a_1a_3a_2, a_2a_1a_3 >$. Let $R = kQ/I$ with k a field. Then the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, R)$ of $\mathrm{mod}\, R$ is as follows.

$\Gamma(\mathrm{mod}\, R)$:

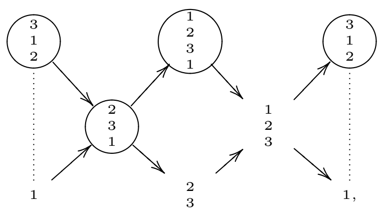


By a direct computation, we have

$\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))$:



$\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))^{\perp_1}$:



where the terms marked by circles are indecomposable projective modules in $\mathrm{mod}\, R$. Then we have $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R)) \cap \mathcal{G}(\mathscr{P}(\mathrm{mod}\, R))^{\perp_1} = \mathscr{P}(\mathrm{mod}\, R)$.

In general, we have the following

Corollary 3.5. *If $\mathscr{C}$ is closed under direct summands, then for any $n \geq 0$, we have*

$$\mathcal{G}(\mathscr{C})^{\leq n} \cap \mathcal{G}(\mathscr{C})^{\perp_1} = \mathscr{C}^{\leq n}.$$

Proof. By Example 3.1(1), we have $\mathscr{C}^{\leq n} \subseteq \mathcal{G}(\mathscr{C})^{\leq n} \cap \mathcal{G}(\mathscr{C})^{\perp_1}$.

Now let $M \in \mathcal{G}(\mathscr{C})^{\leq n} \cap \mathcal{G}(\mathscr{C})^{\perp_1}$. By [11, Theorem 5.8], there exists an exact sequence

$$0 \to K_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$$

in $\mathscr{A}$ with all C_i in $\mathscr{C}$ and $K_n \in \mathcal{G}(\mathscr{C})$. By Theorem 3.3(2), we have $K_n \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Because $\mathscr{C}$ is closed under direct summands by assumption, it follows easily from the definition of $\mathcal{G}(\mathscr{C})$ that $K_n \in \mathscr{C}$ and $M \in \mathscr{C}^{\leq n}$. □

Proposition 3.6. *For any $M \in \mathscr{A}$, the following statements are equivalent.*

(1) $M \in \mathcal{G}(\mathscr{C})^{\perp_1}$.

(2) The functor $\mathrm{Hom}_{\mathscr{A}}(-, M)$ is exact with respect to any short exact sequence in $\mathscr{A}$ ending with an object in $\mathcal{G}(\mathscr{C})$.

(3) Every short exact sequence starting with M is $\mathrm{Hom}_{\mathscr{A}}(\mathcal{G}(\mathscr{C}), -)$-exact.

If, moreover, R is a commutative ring, $\mathscr{A} = \mathrm{Mod}\, R$ and $\mathscr{C} = \mathscr{P}(\mathrm{Mod}\, R)$, then the above conditions are equivalent to the following

(4) $\mathrm{Hom}_R(Q, M) \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1}$ for any $Q \in \mathscr{P}(\mathrm{Mod}\, R)$.

Proof. (1) ⇔ (2) ⇔ (3) It is easy.

Now let R be a commutative ring.

(1) ⇒ (4) For any $G \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$, we have an exact sequence

$$0 \to K \xrightarrow{f} P \to G \to 0 \tag{3.1}$$

in $\mathrm{Mod}\, R$ with $P \in \mathscr{P}(\mathrm{Mod}\, R)$. Let $Q \in \mathscr{P}(\mathrm{Mod}\, R)$. Then

$$0 \to Q \otimes_R K \xrightarrow{1_Q \otimes f} Q \otimes_R P \to Q \otimes_R G \to 0$$

is exact. It is easy to check that $Q \otimes_R G \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$. Then $\mathrm{Ext}^1_R(Q \otimes_R G, M) = 0$ by (1), and so $\mathrm{Hom}_R(1_Q \otimes f, M)$ is epic. By the adjoint isomorphism, we have that $\mathrm{Hom}_R(f, \mathrm{Hom}_R(Q, M))$ is also epic. So applying the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_R(Q, M))$ to (3.1) we get $\mathrm{Ext}^1_R(G, \mathrm{Hom}_R(Q, M)) = 0$, and hence $\mathrm{Hom}_R(Q, M) \in \mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))^{\perp_1}$.

(4) ⇒ (1) It is trivial by setting $Q = R$. □

In the following result, we characterize categories over which all objects are in $\mathcal{G}(\mathscr{C})^{\perp_1}$.

Proposition 3.7. *Assume that $\mathscr{C}$ is closed under direct summands. Consider the following conditions.*

(1) $\mathcal{G}(\mathscr{C})^{\perp_1} = \mathscr{A}$.

(2) $\mathcal{G}(\mathscr{C}) \subseteq \mathcal{G}(\mathscr{C})^{\perp_1}$.

(3) $\mathcal{G}(\mathscr{C}) = \mathscr{C}$.

Then we have (1) ⇒ (2) ⇒ (3). If $\mathscr{C}$ is a projective generator for $\mathscr{A}$, then all of them are equivalent.

Proof. The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let $G \in \mathcal{G}(\mathscr{C})$. Then there exists an exact sequence

$$0 \to G_1 \to C_0 \to G \to 0$$

in $\mathscr{A}$ with $C_0 \in \mathscr{C}$ and $G_1 \in \mathcal{G}(\mathscr{C})$. By (2), we have that $G_1 \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and the above exact sequence splits. Thus as a direct summand of C_0, $G \in \mathscr{C}$ by assumption.

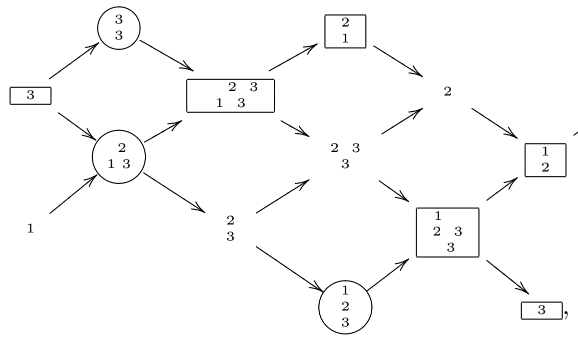
If $\mathscr{C}$ is a projective generator for $\mathscr{A}$, then the implication (3) $\Rightarrow$ (1) follows directly. $\square$

Let $\mathscr{X}$ be a subcategory of $\operatorname{mod} R$ containing $\mathscr{P}(\operatorname{mod} R)$. We use $\underline{\mathscr{X}}$ to denote the stable category of $\mathscr{X}$ modulo $\mathscr{P}(\operatorname{mod} R)$. We end this section by giving two examples about $\mathcal{G}(\mathscr{P}(\operatorname{mod} R))^{\perp_1}$.

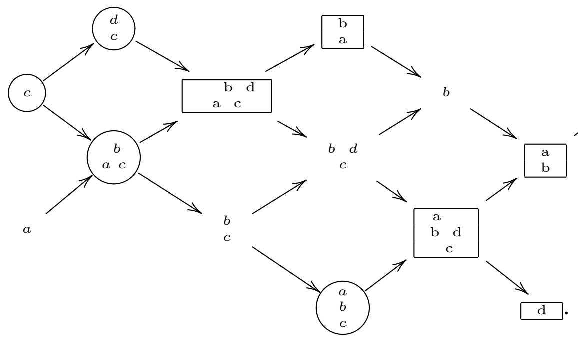
Example 3.8. Let Q_1 and Q_2 be the following two quivers

and let $I_1 =< \alpha_2\alpha_1, \alpha_1\alpha_2, \alpha_4\alpha_3, \alpha_4^2 >$ and $I_2 =< \alpha_b\alpha_a, \alpha_a\alpha_b >$. Let $R_1 = KQ_1/I_1$ and $R_2 = KQ_2/I_2$. Note that R_2 is Gorenstein and R_1 is not Gorenstein. The Auslander-Reiten quivers of $\operatorname{mod} R_1$ and $\operatorname{mod} R_2$ are as follows.

$\Gamma(\operatorname{mod} R_1)$:



$\Gamma(\operatorname{mod} R_2)$:



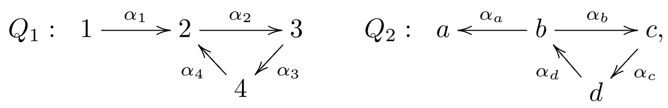
Then we have

(1) The objects marked in a cycle or a box are indecomposable objects in $\mathcal{G}(\mathscr{P}(\operatorname{mod} R_i))^{\perp_1}$ $(i = 1, 2)$; in particular, the objects marked in a cycle are indecomposable objects in $\mathscr{P}(\operatorname{mod} R_i)$ $(i = 1, 2)$.

(2) $\underline{\operatorname{mod}} R_1 \simeq \underline{\operatorname{mod}} R_2$ and $\frac{\operatorname{mod} R_1}{\mathcal{G}(\mathscr{P}(\operatorname{mod} R_1))^{\perp_1}} \simeq \frac{\operatorname{mod} R_2}{\mathcal{G}(\mathscr{P}(\operatorname{mod} R_2))^{\perp_1}}$.

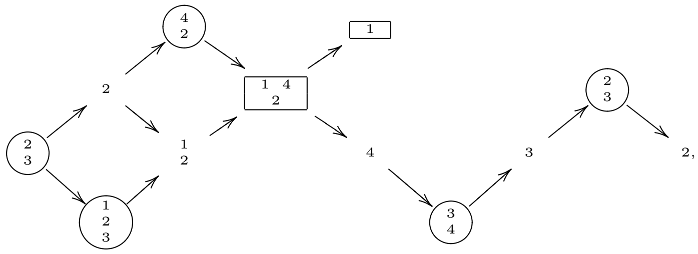
(3) $\mathcal{G}(\mathscr{P}(\operatorname{mod} R_1))^{\perp_1} \not\simeq \mathcal{G}(\mathscr{P}(\operatorname{mod} R_2))^{\perp_1}$ and $\underline{\mathcal{G}(\mathscr{P}(\operatorname{mod} R_1))^{\perp_1}} \simeq \underline{\mathcal{G}(\mathscr{P}(\operatorname{mod} R_2))^{\perp_1}}$.

Example 3.9. Let Q_1 and Q_2 be the following two quivers

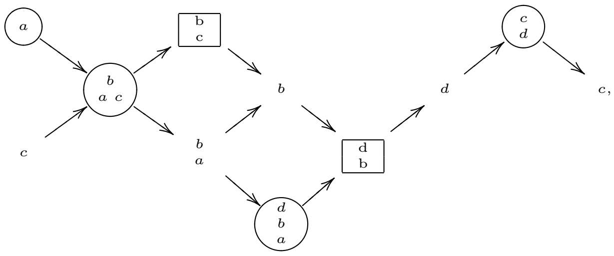


and let $I_1 =< \alpha_3\alpha_2, \alpha_4\alpha_3, \alpha_2\alpha_4 >$ and $I_2 =< \alpha_c\alpha_b, \alpha_d\alpha_c, \alpha_b\alpha_d >$. Let $R_1 = KQ_1/I_1$ and $R_2 = KQ_2/I_2$. Then the Auslander-Reiten quivers of $\mathrm{mod}\, R_1$ and $\mathrm{mod}\, R_2$ are as follows.

$\Gamma(\mathrm{mod}\, R_1)$:



$\Gamma(\mathrm{mod}\, R_2)$:



Then we have

(1) The objects marked in a cycle or a box are indecomposable objects in $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_i))^{\perp_1}$ $(i = 1, 2)$; in particular, the objects marked in a cycle are indecomposable objects in $\mathscr{P}(\mathrm{mod}\, R_i)$ $(i = 1, 2)$.

(2) $\underline{\mathrm{mod}}R_1 \not\simeq \underline{\mathrm{mod}}R_2$ and $\frac{\mathrm{mod}\, R_1}{\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_1))^{\perp_1}} \simeq \frac{\mathrm{mod}\, R_2}{\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_2))^{\perp_1}}$.

(3) $\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_1))^{\perp_1} \not\simeq \mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_2))^{\perp_1}$ and $\underline{\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_1))^{\perp_1}} \simeq \underline{\mathcal{G}(\mathscr{P}(\mathrm{mod}\, R_2))^{\perp_1}}$.

4 The special precovered category of $\mathcal{G}(\mathscr{C})$

In this section, we introduce and investigate the special precovered category of $\mathcal{G}(\mathscr{C})$ in terms of the properties of $\mathcal{G}(\mathscr{C})^{\perp_1}$.

Proposition 4.1.

(1) Let $M \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $f : C \twoheadrightarrow M$ be an epimorphism in $\mathscr{A}$ with $C \in \mathscr{C}$. Then $\mathrm{Ker}\, f \in \mathcal{G}(\mathscr{C})^{\perp_1}$ *and f is a special $\mathcal{G}(\mathscr{C})$-precover of M.*

(2) Consider an exact sequence

$$0 \to M' \to C \to M \to 0. \tag{4.1}$$

If M' admits special $\mathcal{G}(\mathscr{C})$-precover, then so is M. The converse is true if $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$ and (4.1) is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$*-exact.*

Proof. (1) The assertion follows from Example 3.1(1) and Theorem 3.3(2).

(2) Assume that M' admits a special $\mathcal{G}(\mathscr{C})$-precover and

$$0 \to N \to G \to M' \to 0$$

is an exact sequence in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$ and $N \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Combining it with the following $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{C})$-exact exact sequence

$$0 \to G \xrightarrow{i} C^0 \xrightarrow{p} G^1 \to 0$$

in $\mathscr{A}$ with $C^0 \in \mathscr{C}$ and $G^1 \in \mathcal{G}(\mathscr{C})$, we get the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & N & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & G & \xrightarrow{i} & C^0 & \xrightarrow{p} & G^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & M' & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

Adding the exact sequence $0 \longrightarrow 0 \longrightarrow C \xrightarrow{1_C} C \longrightarrow 0$ to the middle row, we obtain the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & N & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & G & \xrightarrow{\binom{i}{0}} & C^0 \oplus C & \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1_C \end{smallmatrix}\right)} & G^1 \oplus C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (g,1_C)} & & \downarrow{\scriptstyle h'} & & \\
0 & \longrightarrow & M' & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}$$

which can be completed to a commutative diagram with exact columns and rows as follows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & N & \dashrightarrow & C' & \dashrightarrow & M'' & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \xrightarrow{\binom{i}{0}} & C^0 \oplus C & \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1_C \end{smallmatrix}\right)} & G^1 \oplus C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (g,1_C)} & & \downarrow{\scriptstyle h'} & & \\
0 & \longrightarrow & M' & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Note that $G^1 \oplus C \in \mathcal{G}(\mathscr{C})$. Moreover, since $N \in \mathcal{G}(\mathscr{C})^{\perp_1}$, we have $M'' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Theorem 3.3(3). Thus the rightmost column in the above diagram is a special $\mathcal{G}(\mathscr{C})$-precover of M.

Now let $\mathscr{C}$ be a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$ and (4.1) be $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. Assume that M admits a special $\mathcal{G}(\mathscr{C})$-precover and

$$0 \to L \to G \to M \to 0,$$

$$0 \to L' \to C' \to L \to 0$$

are exact sequences in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$, $L \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $C' \in \mathscr{C}$. By [11, Lemma 3.1(1)], we get the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & L' & \dashrightarrow & G' & \dashrightarrow & M' & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & C' & \dashrightarrow & C' \oplus C & \dashrightarrow & C & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

By Proposition 2.7(2) and Theorem 3.3(2), we have $L' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and the leftmost column is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. So the middle column is also $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. On the other hand, the middle column is $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{C})$-exact by Proposition 2.7(2). So $G' \in \mathcal{G}(\mathscr{C})$ by [11, Proposition 4.7(5)], and hence the upper row is a special $\mathcal{G}(\mathscr{C})$-precover of M'. □

We introduce the following

Definition 4.2. We call $\mathrm{SPC}(\mathcal{G}(\mathscr{C})) := \{A \in \mathscr{A} \mid A$ admits a special $\mathcal{G}(\mathscr{C})$-precover$\}$ the *special precovered category* of $\mathcal{G}(\mathscr{C})$.

It is trivial that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is the largest subcategory of $\mathscr{A}$ such that $\mathcal{G}(\mathscr{C})$ is special precovering in it. In particular, $\mathrm{SPC}(\mathcal{G}(\mathscr{C})) = \mathscr{A}$ if and only if $\mathcal{G}(\mathscr{C})$ is special precovering in $\mathscr{A}$. For the sake of convenience, we say that a subcategory $\mathscr{X}$ of $\mathscr{A}$ is *closed under $\mathscr{C}$-stable direct summands* provided that the condition $X \oplus C \in \mathscr{X}$ with $C \in \mathscr{C}$ implies $X \in \mathscr{X}$.

Theorem 4.3.

(1) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is closed under extensions.*

(2) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is closed under $\mathscr{C}$-stable direct summands.*

Proof. (1) Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence in $\mathscr{A}$. Assume that L and N admit special $\mathcal{G}(\mathscr{C})$-precovers and

$$0 \to L' \to G_L \xrightarrow{f} L \to 0,$$

$$0 \to N' \to G_N \xrightarrow{g} N \to 0$$

are exact sequences in $\mathscr{A}$ with $G_L, G_N \in \mathcal{G}(\mathscr{C})$ and $L', N' \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & L & \dashrightarrow & Q & \dashrightarrow & G_N & \dashrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

Since $\mathrm{Ext}^2_R(G_N, L') = 0$ by Proposition 2.7(1), we get an epimorphism $\mathrm{Ext}^1_R(G_N, f) : \mathrm{Ext}^1_R(G_N, G_L) \to \mathrm{Ext}^1_R(G_N, L)$. It induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & G_L & \dashrightarrow & G_M & \dashrightarrow & G_N & \dashrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & Q & \longrightarrow & G_N & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

Set $M' := \mathrm{Ker}\, \alpha\beta$. Then we get the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & L' & \dashrightarrow & M' & \dashrightarrow & N' & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_L & \longrightarrow & G_M & \longrightarrow & G_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Note that $G_M \in \mathcal{G}(\mathscr{C})$ (by [14, Corollary 4.5]) and $M' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ (by Theorem 3.3(1)). Thus the middle column in the above diagram is a special $\mathcal{G}(\mathscr{C})$-precover of M. This proves that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under extensions.

(2) Let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and

$$0 \to K \to G \to M \to 0$$

be an exact sequence in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$ and $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$. Assume that $M \cong L \oplus C$ with $C \in \mathscr{C}$, we have an exact and split sequence

$$0 \to C \to M \to L \to 0$$

in $\mathscr{A}$. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K & \dashrightarrow & L' & \dashrightarrow & C & \dashrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since $K, C \in \mathcal{G}(\mathscr{C})^{\perp_1}$, we have $L' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Theorem 3.3(1). Thus the middle column in the above diagram is a special $\mathcal{G}(\mathscr{C})$-precover of L. □

The following question seems to be interesting.

Question 4.4. *Is* $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *closed under direct summands?*

The following result shows that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ possesses certain minimality, which generalizes [15, Theorem 6.8(1)].

Theorem 4.5. *Assume that* $\mathscr{C}$ *is a generator for* $\mathcal{G}(\mathscr{C})^{\perp_1}$. *Then we have*

(1) $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C}) \subseteq \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *and* $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is closed under extensions and* $\mathscr{C}$*-stable direct summands.*

(2) $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ *is the minimal subcategory with respect to the property (1) as above.*

To prove this theorem, we need the following

Lemma 4.6. *Let*

$$0 \to K \to G \to M \to 0$$

be an exact sequence in $\mathscr{A}$ *with* $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$ *and* $G \in \mathcal{G}(\mathscr{C})$. *Then there exists an exact sequence*

$$0 \to G \to M \oplus C \to K' \to 0$$

in $\mathscr{A}$ *with* $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ *and* $C \in \mathscr{C}$.

Proof. Let

$$0 \to K \to G \to M \to 0$$

be an exact sequence in $\mathscr{A}$ with $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$ and $G \in \mathcal{G}(\mathscr{C})$. Since $G \in \mathcal{G}(\mathscr{C})$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to G \to C \to G' \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$ and $G' \in \mathcal{G}(\mathscr{C})$. Consider the following pushout diagram

$$\begin{CD}
@. @. 0 @. 0 @. \\
@. @. @VVV @VVV @. \\
0 @>>> K @>>> G @>>> M @>>> 0 \\
@. @| @VVV @VVV @. \\
0 @>>> K @>>> C @>>> K' @>>> 0 \\
@. @. @VVV @VVV @. \\
@. @. G' @= G' @. \\
@. @. @VVV @VVV @. \\
@. @. 0 @. 0. @.
\end{CD}$$

Since $K, C \in \mathcal{G}(\mathscr{C})^{\perp_1}$, we have $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Theorem 3.3(3).

Consider the following pullback diagram

$$\begin{CD}
@. @. 0 @. 0 @. \\
@. @. @VVV @VVV @. \\
@. @. G @= G @. \\
@. @. @VVV @VVV @. \\
0 @>>> M @>>> Q @>>> C @>>> 0 \\
@. @| @VVV @VVV @. \\
0 @>>> M @>>> K' @>>> G' @>>> 0 \\
@. @. @VVV @VVV @. \\
@. @. 0 @. 0. @.
\end{CD}$$

Since the middle column in the first diagram is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact, so is the rightmost column in this diagram. Then the middle row in the second diagram is also $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact by [11, Lemma 2.4(1)], and in particular, it splits. Thus $Q \cong M \oplus C$ and the middle column in the second diagram is the desired exact sequence. □

Proof of Theorem 4.5. (1) It follows from Proposition 4.1(1) and Theorem 4.3.

(2) Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$ such that $\mathcal{G}(\mathscr{C})^{\perp_1} \cup \mathcal{G}(\mathscr{C}) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and $\mathscr{C}$-stable direct summands. Let $M \in \operatorname{SPC}(\mathcal{G}(\mathscr{C}))$. Then by Lemma 4.6, we have an exact sequence

$$0 \to G \to M \oplus C \to K' \to 0$$

in $\mathscr{A}$ with $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$, $G \in \mathcal{G}(\mathscr{C})$ and $C \in \mathscr{C}$. Because $G, K' \in \mathscr{X}$, we have that $M \oplus C \in \mathscr{X}$ and $M \in \mathscr{X}$. It follows that $\operatorname{SPC}(\mathcal{G}(\mathscr{C})) \subseteq \mathscr{X}$.

As an immediate consequence of Theorem 4.5, we get the following

Corollary 4.7. *Assume that $\mathcal{G}(\mathscr{P}(\operatorname{Mod} R))$ is special precovering in $\operatorname{Mod} R$ and $\mathscr{X}$ is a subcategory of $\operatorname{Mod} R$. If $\mathcal{G}(\mathscr{P}(\operatorname{Mod} R))^{\perp_1} \cup \mathcal{G}(\mathscr{P}(\operatorname{Mod} R)) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and $\mathscr{P}(\operatorname{Mod} R)$-stable direct summands, then $\mathscr{X} = \operatorname{Mod} R$.*

Proof. By assumption, we have $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))) = \mathrm{Mod}\, R$. Now the assertion follows from Theorem 4.5. □

We collect some known classes of rings R satisfying that $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ is special precovering in $\mathrm{Mod}\, R$ as follows.

Example 4.8. For any one of the following rings R, $\mathcal{G}(\mathscr{P}(\mathrm{Mod}\, R))$ is special precovering in $\mathrm{Mod}\, R$.

(1) Commutative Noetherian rings of finite Krull dimension ([5, Remark 5.8]).

(2) Rings in which all projective left R-modules have finite injective dimension ([16, Corollary 4.3]); especially, Gorenstein rings (that is, n-Gorenstein rings for some $n \geq 0$).

(3) Right coherent rings in which all flat R-modules have finite projective dimension ([2, Theorem 3.5] and [4, Proposition 8.10]); especially, right coherent and left perfect rings, and right Artinian rings.

We recall the following definition from [12].

Definition 4.9. Let $\mathscr{C}$, $\mathscr{T}$ and $\mathscr{E}$ be subcategories of $\mathscr{A}$ with $\mathscr{C} \subseteq \mathscr{T}$.

(1) $\mathscr{C}$ is called an *$\mathscr{E}$-proper generator* (resp. *$\mathscr{E}$-coproper cogenerator*) for $\mathscr{T}$ if for any object T in $\mathscr{T}$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)$ (resp. $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{E})$)-exact exact sequence $0 \to T' \to C \to T \to 0$ (resp. $0 \to T \to C \to T' \to 0$) in $\mathscr{A}$ such that C is an object in $\mathscr{C}$ and T' is an object in $\mathscr{T}$.

(2) $\mathscr{T}$ is called *$\mathscr{E}$-preresolving* in $\mathscr{A}$ if the following conditions are satisfied.

(i) $\mathscr{T}$ admits an $\mathscr{E}$-proper generator.

(ii) $\mathscr{T}$ is *closed under $\mathscr{E}$-proper extensions*, that is, for any $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, if both A_1 and A_3 are objects in $\mathscr{T}$, then A_2 is also an object in $\mathscr{T}$.

An $\mathscr{E}$-preresolving subcategory $\mathscr{T}$ of $\mathscr{A}$ is called *$\mathscr{E}$-resolving* if the following condition is satisfied.

(iii) $\mathscr{T}$ is *closed under kernels of $\mathscr{E}$-proper epimorphisms*, that is, for any $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, if both A_2 and A_3 are objects in $\mathscr{T}$, then A_1 is also an object in $\mathscr{T}$.

In the following, we investigate when $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving. We need the following two lemmas.

Lemma 4.10. *For any $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence*

$$0 \to K \to C \to M \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$.

Proof. Let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to K' \to G \to M \to 0$$

in $\mathscr{A}$ with $G \in \mathcal{G}(\mathscr{C})$ and $K' \in \mathcal{G}(\mathscr{C})^{\perp_1}$. For G, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to G' \to C \to G \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$ and $G' \in \mathcal{G}(\mathscr{C})$. Consider the following pullback diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & K & \dashrightarrow & C & \dashrightarrow & M & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K' & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

By [11, Lemma 2.5], the middle row is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact, as desired. □

Lemma 4.11. *Assume that $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$. Given a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence*

$$0 \to L \to M \to N \to 0$$

in $\mathscr{A}$, we have

(1) If $M, N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$, then $L \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$.

(2) If $L, M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to K \to C \to N \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$, then $N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$.

Proof. Let $0 \to L \to M \to N \to 0$ be a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence in $\mathscr{A}$.

(1) Assume that $M, N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. By Lemma 4.10, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to K \to C \to N \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$. Consider the following pullback diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & L & \dashrightarrow & T & \dashrightarrow & C & \dashrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

By Proposition 4.1(2), $K \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then it follows from Theorem 4.3(1) and the exactness of the middle column that $T \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Notice that the middle row is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact by [11, Lemma 2.4(1)], so it splits and $T \cong L \oplus C$. Thus $L \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by Theorem 4.3(2).

(2) Assume $L, M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to K \to C \to N \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$. As in the above diagram, since $L, C \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$, we have $T \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by Theorem 4.3(1). Moreover, the middle column is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact by [11, Lemma 2.4(1)]. So $K \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by (1), and hence $N \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ by Proposition 4.1(2). □

Now we are ready to prove the following

Theorem 4.12. *If $\mathscr{C}$ is a generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$, then $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is $\mathscr{C}$-resolving in $\mathscr{A}$ with a $\mathscr{C}$-proper generator $\mathscr{C}$.*

Proof. Following Theorem 4.3(1) and Lemma 4.11(1), we know that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under $\mathscr{C}$-proper extensions and kernels of $\mathscr{C}$-proper epimorphisms. Now let $M \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. Then by Lemma 4.10, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to K \to C \to M \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$. By Proposition 4.1(2), we have $K \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. It follows that $\mathscr{C}$ is a $\mathscr{C}$-proper generator for $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ and $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is a $\mathscr{C}$-resolving. □

As a consequence, we get the following

Corollary 4.13. *If $\mathscr{C}$ is a projective generator for $\mathscr{A}$, then $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is projectively resolving and injectively coresolving in $\mathscr{A}$.*

Proof. Let $\mathscr{C}$ be a projective generator for $\mathscr{A}$. Because $\mathcal{G}(\mathscr{C})^{\perp_1}$ is projectively resolving by Theorem 3.3(2), $\mathscr{C}$ is also a projective generator for $\mathcal{G}(\mathscr{C})^{\perp_1}$. It follows from Theorem 4.12 that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is projectively resolving. Now let I be an injective object in $\mathscr{A}$ and

$$0 \to K \to P \stackrel{f}{\to} I \to 0$$

an exact sequence in $\mathscr{A}$ with $P \in \mathscr{C}$. Then it is easy to see that $K \in \mathcal{G}(\mathscr{C})^{\perp_1}$ by Example 3.1(1) and Theorem 3.3(2). So f is a special $\mathcal{G}(\mathscr{C})$-precover of I and $I \in \mathrm{SPC}(\mathcal{G}(\mathscr{C}))$. On the other hand, by Lemma 4.11(2), we have that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is closed under cokernels of monomorphisms. Thus we conclude that $\mathrm{SPC}(\mathcal{G}(\mathscr{C}))$ is injectively coresolving. □

The following corollary is an immediate consequence of Corollary 4.13, in which the second assertion generalizes [15, Theorem 6.8(2)].

Corollary 4.14.

(1) $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{Mod}\,R)))$ *is projectively resolving and injectively coresolving in* $\mathrm{Mod}\,R$.

(2) If R is a left Noetherian ring, then $\mathrm{SPC}(\mathcal{G}(\mathscr{P}(\mathrm{mod}\,R)))$ *is projectively resolving and injectively coresolving in* $\mathrm{mod}\,R$.

Let SPE($\mathcal{G}(\mathscr{C})$) be the subcategory of $\mathscr{A}$ consisting of objects admitting special $\mathcal{G}(\mathscr{C})$-preenvelopes. We point out that the dual versions on ${}^{\perp_1}\mathcal{G}(\mathscr{C})$ and SPE($\mathcal{G}(\mathscr{C})$) of all of the above results also hold true by using completely dual arguments.

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