Thick subcategories of extriangulated categories *†

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Abstract

We introduce the spectrum of an extriangulated category with respect to a class of thick subcategories as well as supports of thick subcategories. Then we give a classification of radical thick subcategories of an extriangulated category in terms of supports of thick subcategories. Moreover, we introduce the notions of prime thick subcategories and classifying support data, and show that if (X, σ) is a classifying support data on an extriangulated category $\mathcal{K}$, then there is a homeomorphism from the topological space X to the spectrum of $\mathcal{K}$ with respect to the set of prime thick subcategories. This result generalizes a result of Matsui.

1 Introduction

The notion of extriangulated categories was introduced by Nakaoka and Palu [11], which is a simultaneous generalization of exact categories and extension-closed subcategories of triangulated categories. After that, the study of extriangulated categories has become an active topic, and many results on exact categories and triangulated categories have been gotten realization in the setting of extriangulated categories ([4–7, 11–14]). Recently, Nakaoka, Ogawa and Sakai [10] showed that the localization of an extriangulated category by a multiplicative system satisfying mild assumptions can be equipped with a natural, universal structure of an extriangulated category. In this process, the notion of thick subcategories was introduced, which is very important in constructing a suitable multiplicative system. A classification of thick subcategories is very useful to understand structural information about the ambient category. For example, Benson, Carlson and Rickard [2] classified the thick subcategories of the stable module category of the group algebra kG for a p-group G in terms of closed subvarietes of the maximal ideal spectrum of the group cohomology ring. Balmer [1] established a theory in an essentially small triangulated category, called the tensor triangular geometry, which includes the notions of thick tensor-ideals, radical thick-tensor ideals, and prime thick tensor-ideals, and then he defined a topology on the set of prime thick tensor-ideals. Recently, Matsui and Takahashi [8, 9] introduced the notion of prime thick subcategories in a triangulated category (which is not necessarily tensor triangulated) and defined the spectrum of this triangulated category. Inspired by Matsui's work, we will study thick subcategories of extriangulated categories in terms of spectrum theory, which

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provides a framework for exact categories and extension-closed subcategories of triangulated categories.

In Section 2, we recall some terminologies and preliminary results needed in this paper. Let $\mathcal{K}$ be an extriangulated category. We define a topology on a subset $\mathscr{C}$ of thick subcategories of $\mathcal{K}$, and get a topological space $\operatorname{Esp}_{\mathscr{C}}\mathcal{K}$, called the *spectrum* of $\mathcal{K}$ with respect to $\mathscr{C}$. We show that the topological space $\operatorname{Esp}_{\mathscr{C}}\mathcal{K}$ is a T_0-space (Proposition 2.12). We also introduce the support of objects and subcategories, which are useful in the next section.

In Section 3, we introduce the notion of radical thick subcategories, and show that there is a one-to-one correspondence from the set of radical thick subcategories to that of support of thick subcategories of $\mathcal{K}$ (Theorem 3.3). Moreover, we introduce the notions of prime thick subcategories and classifying support data. We show that if (X, σ) is a classifying support data, then there is a homeomorphism from the topological space X to the topological space $\operatorname{Esp}_{\mathscr{P}}\mathcal{K}$, where $\mathscr{P}$ is the set of prime thick subcategories (Theorem 3.14). This generalizes a result of Matsui [8].

2 Preliminary

2.1 Extriangulated categories

We first recall some notions from [11].

In this subsection, $\mathcal{K}$ is an additive category and $\mathbb{E} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathfrak{Ab}$ is a biadditive functor, where $\mathfrak{Ab}$ is the category of abelian groups.

Let $A, C \in \mathcal{K}$. An element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Two sequences of morphisms

$$A \xrightarrow{x} B \xrightarrow{y} C \text{ and } A \xrightarrow{x'} B' \xrightarrow{y'} C$$

in $\mathcal{K}$ are said to be *equivalent* if there exists an isomorphism $b \in \operatorname{Hom}_{\mathcal{K}}(B, B')$ such that $x' = bx$ and $y = y'b$. We denote by $[\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$. In particular, we write $0 := [\, A \xrightarrow{\binom{\mathrm{id}_A}{0}} A \oplus C \xrightarrow{(0\ \mathrm{id}_C)} C \,]$.

For an $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, we briefly write

$$a_\star\delta := \mathbb{E}(C, a)(\delta) \text{ and } c^\star\delta := \mathbb{E}(c, A)(\delta).$$

For two $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, a *morphism* from δ to δ' is a pair (a, c) of morphisms with $a \in \operatorname{Hom}_{\mathcal{K}}(A, A')$ and $c \in \operatorname{Hom}_{\mathcal{K}}(C, C')$ such that $a_\star\delta = c^\star\delta'$.

Definition 2.1. ([11, Definition 2.9]) Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ to each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ provided that it satisfies the following condition.

(R) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,] \text{ and } \mathfrak{s}(\delta') = [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,].$$

Then for any morphism $(a,c) : \delta \to \delta'$, there exists $b \in \mathrm{Hom}_{\mathcal{K}}(B,B')$ such that the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

commutes.

Let $\mathfrak{s}$ be a realization of $\mathbb{E}$. If $\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ for some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$, then one says that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ *realizes* δ; and in the condition (R), the triple (a,b,c) *realizes* the morphism (a,c).

For any two equivalence classes $[\, A \xrightarrow{x} B \xrightarrow{y} C \,]$ and $[\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,]$, we define

$$[\, A \xrightarrow{x} B \xrightarrow{y} C \,] \oplus [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,] := [\, A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C' \,].$$

Definition 2.2. ([11, Definition 2.10]) A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions.

(1) For any $A, C \in \mathcal{K}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C,A)$ satisfies $\mathfrak{s}(0) = 0$.
(2) For any pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$, we have $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Definition 2.3. ([11, Definition 2.12]) The triple $(\mathcal{K}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated* (or *extriangulated* for short) category if it satisfies the following conditions.

(ET1) $\mathbb{E} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathfrak{Ab}$ is a biadditive functor.
(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
(ET3) Let $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,] \text{ and } \mathfrak{s}(\delta') = [\, A' \xrightarrow{x'} B' \xrightarrow{y'} C' \,].$$

For any commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in $\mathcal{K}$, there exists a morphism $(a,c) : \delta \to \delta'$ which is realized by the triple (a,b,c).

(ET3)$^{\mathrm{op}}$ Dual of (ET3).
(ET4) Let $\delta \in \mathbb{E}(C,A)$ and $\rho \in \mathbb{E}(F,B)$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\, A \xrightarrow{x} B \xrightarrow{y} C \,] \text{ and } \mathfrak{s}(\rho) = [\, B \xrightarrow{u} D \xrightarrow{v} F \,].$$

Then there exist an object $E \in \mathcal{K}$, an $\mathbb{E}$-extension ξ with $\mathfrak{s}(\xi) = [\, A \xrightarrow{z} D \xrightarrow{w} E \,]$, and a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle s} \\ A & \xrightarrow{z} & D & \xrightarrow{w} & E \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle t} \\ & & F & = & F \end{array}$$

in $\mathcal{K}$, which satisfy the following compatibilities.

(i) $\mathfrak{s}(y_\star\rho) = [\ C \xrightarrow{s} E \xrightarrow{t} F\ ]$.

(ii) $s^\star\xi = \delta$.

(iii) $x_\star\xi = t^\star\rho$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

Definition 2.4. ([11, Definitions 2.15 and 2.19]) Let $\mathcal{K}$ be an extriangulated category.

- A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{K}$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation*.
- If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{K}$ realizes $\delta \in \mathbb{E}(C, A)$, the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ is called an $\mathbb{E}$-*triangle*, and write it in the following way.
$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$
We usually do not write this "δ" if it is not used in the argument.
- Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c) : \delta \to \delta'$, then we write it as
$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array}$$
and call (a, b, c) a *morphism* of $\mathbb{E}$-triangles.

  If a, b, c above are isomorphisms, then $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ are said to be *isomorphic*.

Example 2.5. Both exact categories and triangulated categories are extriangulated categories (see [11, Proposition 3.22]) and extension closed subcategories of extriangulated categories are again extriangulated (see [11, Remark 2.18]). Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories (see [11, Proposition 3.30], [12, Corollary 4.12 and Remark 4.13]).

2.2 Spectrum and support

From now on, $\mathcal{K}$ is an essentially small extriangulated category.

Definition 2.6. (c.f. [10, Definition 4.1]) A *thick* subcategory $\mathcal{X}$ of $\mathcal{K}$ is a non-empty full subcategory such that the following conditions are satisfied:

(a) $\mathcal{X}$ satisfies the two out of three property: Given any $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ in $\mathcal{K}$, if any two out of A, B, C belong to $\mathcal{X}$, then so does the third.

(b) $\mathcal{X}$ is closed under direct summands.

Remark 2.7. Assume that $\mathcal{X}$ is a thick subcategory of $\mathcal{K}$.

(1) For any $A \in \mathcal{X}$, consider the trivial $\mathbb{E}$-triangle $A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \dashrightarrow$. By Definition 2.6(a), $0 \in \mathcal{X}$.

(2) Let $A \in \mathcal{X}$, and suppose $A \cong B$ in $\mathcal{K}$. We may assume that $f : A \to B$ is an isomorphism, then there is a commutative diagram:
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle f^{-1}} & & \| \\ A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & 0. \end{array}$$

Since the second row is an $\mathbb{E}$-triangle, so does the first row. By (1), $0 \in \mathcal{X}$, and hence $B \in \mathcal{X}$ since $\mathcal{X}$ is closed under extensions. This shows that $\mathcal{X}$ is closed under isomorphisms.

(3) $\mathcal{X}$ is additive.

(4) The intersection of any collection of thick subcategories is again a thick subcategory.

Given a collection $\mathcal{S}$ of objects of $\mathcal{K}$, we denote by $\langle \mathcal{S} \rangle$ the smallest thick subcategory of $\mathcal{K}$ which contains $\mathcal{S}$. By Remark 2.7(4), $\langle \mathcal{S} \rangle$ is exactly the intersection of all thick subcategories which contain $\mathcal{S}$.

We denote by $\mathscr{T}(\mathcal{K})$ the set of all thick subcategories of $\mathcal{K}$. In the sequel, unless otherwise stated, we always fix a subset $\mathscr{C}$ of $\mathscr{T}(\mathcal{K})$. Now we define a topology on $\mathscr{C}$.

Definition 2.8. For a class $\mathcal{S}$ of objects of $\mathcal{K}$, we write

$$\mathbf{Z}(\mathcal{S}) := \{\mathcal{P} \in \mathscr{C} \mid \mathcal{P} \cap \mathcal{S} = \emptyset\}.$$

Remark 2.9.

(1) $\bigcap_{i\in I} \mathbf{Z}(\mathcal{S}_i) = \mathbf{Z}(\bigcup_{i\in I} \mathcal{S}_i)$ for any indexed set I.

(2) $\mathbf{Z}(\mathcal{S}_1) \cup \mathbf{Z}(\mathcal{S}_2) = \mathbf{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2)$, where $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{A_1 \oplus A_2 \mid A_1 \in \mathcal{S}_1, A_2 \in \mathcal{S}_2\}$.

(3) $\mathbf{Z}(\mathcal{K}) = \emptyset$.

(4) $\mathbf{Z}(\emptyset) = \mathscr{C}$.

Thus the collection $\{\mathbf{Z}(\mathcal{S}) \mid \forall \mathcal{S} \subseteq \mathcal{K}\}$ defines the closed subsets of a topology on $\mathscr{C}$. We denote this topological space by $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$, and call it the *spectrum* of $\mathcal{K}$ with respect to $\mathscr{C}$.

We write

$$\mathbf{U}(\mathcal{S}) := \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \backslash \mathbf{Z}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{S} \cap \mathcal{P} \neq \emptyset\},$$

which is the open complement of $\mathbf{Z}(\mathcal{S})$ for any $\mathcal{S} \subseteq \mathcal{K}$. For any object $A \in \mathcal{K}$, we write

$$\mathrm{Supp}(A) := \mathbf{Z}(\{A\}) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid A \notin \mathcal{P}\},$$

which is the closed subset of the topology space $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$, and call it the *support* of A. On the other hand, for any object $A \in \mathcal{K}$, we have

$$\mathbf{U}(A) := \mathbf{U}(\{A\}) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid A \in \mathcal{P}\}.$$

By the definition, $\mathbf{Z}(\mathcal{S}) = \bigcap_{A\in\mathcal{S}} \mathrm{Supp}(A)$ for any $\mathcal{S} \subseteq \mathcal{K}$, and thus the family $\{\mathrm{Supp}(A)\}_{A\in\mathcal{K}}$ of closed subsets forms a closed basis of the topological space $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$. Similarly, $\mathbf{U}(\mathcal{S}) = \bigcup_{A\in\mathcal{S}} \mathbf{U}(A)$ for any $\mathcal{S} \subseteq \mathcal{K}$, and thus the family $\{\mathbf{U}(A)\}_{A\in\mathcal{K}}$ of open subsets forms an open basis of the topological space $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$.

For a topological space X, we denote by $\mathcal{X}_{\mathrm{cl}}(X)$ the collection of all closed subsets of X, and by $\mathcal{X}_{\mathrm{op}}(X)$ the collection of all open subsets of X. We have following easy observation.

Lemma 2.10. *Let $\mathcal{K}$ be an extriangulated category. The assignment* $\mathrm{Supp}(-) : \mathcal{K} \to \mathcal{X}_{\mathrm{cl}}(\mathrm{Esp}_{\mathscr{C}}\mathcal{K})$ *given by* $A \mapsto \mathrm{Supp}(A)$ *satisfies the following properties:*

(1) $\mathrm{Supp}(0) = \emptyset$.

(2) $\mathrm{Supp}(A \oplus B) = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$.

(3) Given any $\mathbb{E}$*-triangle* $A_1 \longrightarrow A_2 \longrightarrow A_3 \dashrightarrow$ *in* $\mathcal{K}$*, we have* $\mathrm{Supp}(A_i) \subseteq \mathrm{Supp}(A_j) \cup \mathrm{Supp}(A_k)$ *for any* $\{i, j, k\} \in \{1, 2, 3\}$.

Dually, we have the following result.

Lemma 2.11. *Let $\mathcal{K}$ be an extriangulated category. The assignment $\mathbf{U}(-) : \mathcal{K} \to \mathcal{X}_{\rm op}(\mathrm{Esp}_{\mathscr{C}}\mathcal{K})$ given by $A \mapsto \mathbf{U}(A)$ satisfies the following properties:*

(1) $\mathbf{U}(0) = \mathrm{Esp}_{\mathscr{C}}\mathcal{K}$.

(2) $\mathbf{U}(A \oplus B) = \mathbf{U}(A) \cap \mathbf{U}(B)$.

(3) Given any $\mathbb{E}$-triangle $A_1 \longrightarrow A_2 \longrightarrow A_3 \dashrightarrow$ in $\mathcal{K}$, it holds that $\mathbf{U}(A_i) \supseteq \mathbf{U}(A_j) \cap \mathbf{U}(A_k)$ for any $\{i, j, k\} \in \{1, 2, 3\}$.

Let W be a subset of $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$, by the above lemma, the full subcategory

$$\mathrm{Supp}^{-1}W := \{A \in \mathcal{K} \mid \mathrm{Supp}(A) \subseteq W\}$$

of $\mathcal{K}$ is a thick subcategory, which implies that $\mathrm{Supp}(\mathcal{X}) = \mathrm{Supp}(\langle\mathcal{X}\rangle)$ for each subcategory $\mathcal{X}$ of $\mathcal{K}$.

Let ω be a subset of $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$. It is easy to see that the closure of ω in $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$ is

$$\overline{\omega} = \bigcap_{\substack{A \in \mathcal{K} \text{ s.t.} \\ \omega \subseteq \mathrm{Supp}(A)}} \mathrm{Supp}(A).$$

The following result shows that the topological space $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$ is a T_0-space.

Proposition 2.12. *Let $\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K}$. Then its closure is $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{Q} \subseteq \mathcal{P}\}$. In particular, for any $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K}$, if $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$, then $\mathcal{P}_1 = \mathcal{P}_2$.*

Proof. Let $\mathcal{S}_0 = \mathcal{K} \setminus \mathcal{P}$. Then

$$\mathbf{Z}(\mathcal{S}_0) = \{\mathcal{Q} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{Q} \cap \mathcal{S}_0 = \emptyset\} = \{\mathcal{Q} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

Clearly, $\mathcal{P} \in \mathbf{Z}(\mathcal{S}_0)$. If $\mathcal{P} \in \mathbf{Z}(\mathcal{S})$ for some subset $\mathcal{S}$ of $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$, then $\mathcal{S} \subseteq \mathcal{S}_0$, and hence $\mathbf{Z}(\mathcal{S}_0) \subseteq \mathbf{Z}(\mathcal{S})$. This means that $\mathbf{Z}(\mathcal{S}_0)$ is the smallest closed subset which contains $\mathcal{P}$, thus $\overline{\{\mathcal{P}\}} = \mathbf{Z}(\mathcal{S}_0)$.

Assume that $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$. Then $\mathcal{P}_1 \in \overline{\{\mathcal{P}_1\}}$ implies $\mathcal{P}_1 \in \overline{\{\mathcal{P}_2\}}$, and hence $\mathcal{P}_1 \subseteq \mathcal{P}_2$ by the former statement. Similarly, we have $\mathcal{P}_2 \subseteq \mathcal{P}_1$, and thus $\mathcal{P}_1 = \mathcal{P}_2$. □

For a subcategory $\mathcal{X}$ of $\mathcal{K}$, we define its *support* as follows:

$$\mathrm{Supp}(\mathcal{X}) := \bigcup_{A \in \mathcal{X}} \mathrm{Supp}(A) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{X} \nsubseteq \mathcal{P}\}$$

which is a specialization-closed subset of $\mathrm{Esp}_{\mathscr{C}}\mathcal{K}$. Indeed, $\mathcal{P} \in \mathrm{Supp}(\mathcal{X})$ if and only if there exists an object $A \in \mathcal{K}$ such that $\mathcal{P} \in \mathrm{Supp}(A)$, and if and only if there exists an object $A \in \mathcal{X}$ such that $A \notin \mathcal{P}$, if and only if $\mathcal{X} \nsubseteq \mathcal{P}$.

Note that $\mathbf{Z}(\mathcal{X}) \subsetneqq \mathrm{Supp}(\mathcal{X})$ in general. For example, let $A = kQ$ be a path algebra with k a field and Q the quiver $1 \to 2 \to 3$, and let $\mathcal{K} = \mathrm{mod}A$. Take $\mathscr{C} = \{\mathrm{add}\{3, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\}\}$ and $\mathcal{X} = \mathrm{add}\{3, 1\}$. Then $\mathbf{Z}(\mathcal{X}) = \emptyset$, but $\mathrm{Supp}(\mathcal{X}) = \mathscr{C}$.

3 Radical and prime thick subcategories

3.1 Radical thick subcategories

Definition 3.1. Let $\mathcal{X}$ be a thick subcategory of $\mathcal{K}$. We define the *radical* of $\mathcal{X}$ by

$$\sqrt{\mathcal{X}} := \bigcap_{\substack{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \\ \text{s.t. } \mathcal{X} \subseteq \mathcal{P}}} \mathcal{P}.$$

A thick subcategory $\mathcal{X}$ of $\mathcal{K}$ is *radical* if $\sqrt{\mathcal{X}} = \mathcal{X}$.

We denote by $\mathrm{Rad}_{\mathscr{C}}\mathcal{K}$ the set of all radical thick subcategories of $\mathcal{K}$. Obviously, $\mathscr{C} \subseteq \mathrm{Rad}_{\mathscr{C}}\mathcal{K}$.

Proposition 3.2. *Let $\mathcal{K}$ be an extriangulated category.*

(1) If A is an object of $\mathcal{K}$, then $\mathrm{Supp}(A) = \emptyset$ if and only if $A \in \sqrt{0}$.

(2) If $\mathcal{X} \in \mathscr{T}(\mathcal{K})$, then $\mathrm{Supp}(\sqrt{\mathcal{X}}) = \mathrm{Supp}(\mathcal{X})$ and $\sqrt{\sqrt{\mathcal{X}}} = \sqrt{\mathcal{X}}$ (that is, $\sqrt{\mathcal{X}}$ is a radical thick subcategory).

Proof. (1) Clearly, $\sqrt{0} = \bigcap_{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K}} \mathcal{P}$. Thus

$$\begin{aligned} \mathrm{Supp}(A) = \emptyset &\Leftrightarrow \ \forall \mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K},\ A \in \mathcal{P} \\ &\Leftrightarrow \ A \in \bigcap_{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K}} \mathcal{P} \\ &\Leftrightarrow \ A \in \sqrt{0}. \end{aligned}$$

(2) Let $\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K}$. By the definition of $\sqrt{\mathcal{X}}$, we have $\mathcal{X} \subseteq \sqrt{\mathcal{X}}$. Thus $\sqrt{\mathcal{X}} \subseteq \mathcal{P}$ implies $\mathcal{X} \subseteq \mathcal{P}$. On the other hand, if $\mathcal{X} \subseteq \mathcal{P}$, then

$$\sqrt{\mathcal{X}} = \mathcal{P} \cap \bigcap_{\substack{\mathcal{P}' \in (\mathrm{Esp}_{\mathscr{C}}\mathcal{K}) \backslash \mathcal{P} \\ \text{s.t. } \mathcal{X} \subseteq \mathcal{P}'}} \mathcal{P}',$$

and hence $\sqrt{\mathcal{X}} \subseteq \mathcal{P}$. It follows that $\sqrt{\mathcal{X}} \subseteq \mathcal{P}$ if and only if $\mathcal{X} \subseteq \mathcal{P}$. Thus

$$\sqrt{\sqrt{\mathcal{X}}} = \bigcap_{\substack{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \\ \text{s.t. } \sqrt{\mathcal{X}} \subseteq \mathcal{P}}} \mathcal{P} = \bigcap_{\substack{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \\ \text{s.t. } \mathcal{X} \subseteq \mathcal{P}}} \mathcal{P} = \sqrt{\mathcal{X}}.$$

Moreover, we have

$$\mathrm{Supp}(\sqrt{\mathcal{X}}) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \sqrt{\mathcal{X}} \nsubseteq \mathcal{P}\} = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{X} \nsubseteq \mathcal{P}\} = \mathrm{Supp}(\mathcal{X}).$$

□

We define the *parameter set* $\mathrm{Param}_{\mathscr{C}}\mathcal{K}$ as the set of all supports of thick subcategories of $\mathcal{K}$:

$$\mathrm{Param}_{\mathscr{C}}\mathcal{K} := \{\mathrm{Supp}(\mathcal{X}) \mid \mathcal{X} \in \mathscr{T}(\mathcal{K})\}.$$

Theorem 3.3. *Keep the notations as above. There is a one-to-one correspondence:*

$$\mathrm{Rad}_{\mathscr{C}}\mathcal{K} \underset{\mathrm{Supp}^{-1}}{\overset{\mathrm{Supp}}{\rightleftarrows}} \mathrm{Param}_{\mathscr{C}}\mathcal{K}$$

Proof. Let $\mathcal{X} \in \mathscr{T}(\mathcal{K})$. Note that $\mathrm{Supp}^{-1}(\mathrm{Supp}(\mathcal{X})) = \{A \in \mathcal{K} \mid \mathrm{Supp}(A) \subseteq \mathrm{Supp}(\mathcal{X})\}$. Since $\mathrm{Supp}(A) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid A \notin \mathcal{P}\}$ and $\mathrm{Supp}(\mathcal{X}) = \{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \mid \mathcal{X} \nsubseteq \mathcal{P}\}$, we have that $\mathrm{Supp}(A) \subseteq \mathrm{Supp}(\mathcal{X})$ if and only if for any $\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K}$, $\mathcal{X} \subseteq \mathcal{P}$ implies $A \in \mathcal{P}$. This yields

$$\mathrm{Supp}^{-1}(\mathrm{Supp}(\mathcal{X})) \subseteq \bigcap_{\substack{\mathcal{P} \in \mathrm{Esp}_{\mathscr{C}}\mathcal{K} \\ \text{s.t. } \mathcal{X} \subseteq \mathcal{P}}} \mathcal{P} = \sqrt{\mathcal{X}}.$$

Conversely, $\sqrt{\mathcal{X}} \subseteq \mathrm{Supp}^{-1}(\mathrm{Supp}(\mathcal{X}))$ by roposition 3.2(2). Thus $\mathrm{Supp}^{-1}(\mathrm{Supp}(\mathcal{X})) = \sqrt{\mathcal{X}}$ is a radical thick subcategory, which means that Supp^{-1} is a well-defined map, and meanwhile, $\mathrm{Supp}^{-1} \circ \mathrm{Supp} = \mathrm{Id}_{\mathrm{Rad}_{\mathscr{C}}\mathcal{K}}$.

On the other hand, by Proposition 3.2 again, we have

$$\mathrm{Supp}(\mathrm{Supp}^{-1}(\mathrm{Supp}(\mathcal{X}))) = \mathrm{Supp}(\sqrt{\mathcal{X}}) = \mathrm{Supp}(\mathcal{X}),$$

and hence $\mathrm{Supp} \circ \mathrm{Supp}^{-1} = \mathrm{Id}_{\mathrm{Param}_{\mathscr{C}}\mathcal{K}}$. This completes the proof. □

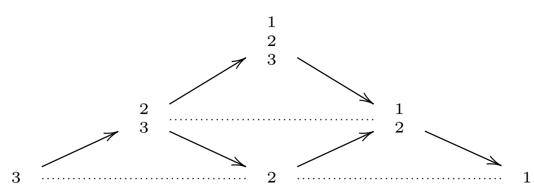
Example 3.4. (1) ([9, Theorem 2.9]) Let $\mathcal{K} = \mathcal{T}$ be a triangulated category, [1] the shift functor and $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(-, -[1])$. There is a one-to-one correspondence:

$$\mathrm{Rad}_{\mathscr{C}}\mathcal{T} \underset{\mathrm{Supp}^{-1}}{\overset{\mathrm{Supp}}{\rightleftarrows}} \mathrm{Param}_{\mathscr{C}}\mathcal{T}.$$

(2) Let $\mathcal{K} = \mathcal{E}$ be an exact category and $\mathbb{E} = \mathrm{Ext}^1_{\mathcal{E}}(-, -)$. There is a one-to-one correspondence:

$$\mathrm{Rad}_{\mathscr{C}}\mathcal{E} \underset{\mathrm{Supp}^{-1}}{\overset{\mathrm{Supp}}{\rightleftarrows}} \mathrm{Param}_{\mathscr{C}}\mathcal{E}.$$

Example 3.5. Let $A = kQ$ be a path algebra with k a field and Q the quiver $1 \to 2 \to 3$, and let $\mathcal{K} = \mathrm{mod}A$. The Auslander-Reiten quiver of $\mathrm{mod}A$ is as follows:



All thick subcategories of $\mathrm{mod}A$:

$$T_0 = \{0\}$$

$$T_1 = \mathrm{add}\{3\},\quad T_2 = \mathrm{add}\{\begin{smallmatrix}2\\3\end{smallmatrix}\},\quad T_3 = \mathrm{add}\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}\},\quad T_4 = \mathrm{add}\{2\},\quad T_5 = \mathrm{add}\{\begin{smallmatrix}1\\2\end{smallmatrix}\},\quad T_6 = \mathrm{add}\{1\}$$

$$T_7 = \mathrm{add}\{3, 1\},\quad T_8 = \mathrm{add}\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, 2\}$$

$$T_9 = \mathrm{add}\{3, \begin{smallmatrix}2\\3\end{smallmatrix}, 2\},\quad T_{10} = \mathrm{add}\{3, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\},\quad T_{11} = \mathrm{add}\{\begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, 1\},\quad T_{12} = \{2, \begin{smallmatrix}1\\2\end{smallmatrix}, 1\}$$

$$T_{13} = \mathrm{add}\{3, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, 2, \begin{smallmatrix}1\\2\end{smallmatrix}, 1\}.$$

Take $\mathscr{C} = \{T_9, T_{10}, T_{11}, T_{12}, T_{13}\}$. By a direct computation, we have

$$\mathrm{Rad}_{\mathscr{C}}\mathcal{K} = \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_9, T_{10}, T_{11}, T_{12}, T_{13}\}$$

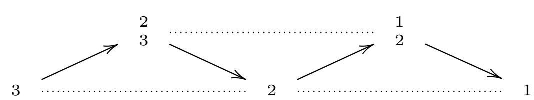
and

$$\begin{aligned}\mathrm{Param}_{\mathscr{C}}\mathcal{K} = \{&\emptyset,\ \{T_{11}, T_{12}\},\ \{T_{10}, T_{12}\},\ \{T_9, T_{12}\},\ \{T_{10}, T_{11}\},\ \{T_9, T_{11}\},\ \{T_9, T_{10}\},\\ &\{T_{10}, T_{11}, T_{12}\}, \{T_9, T_{11}, T_{12}\},\ \{T_9, T_{10}, T_{12}\},\ \{T_9, T_{10}, T_{11}\},\ \{T_9, T_{10}, T_{11}, T_{12}\}\}\,.\end{aligned}$$

The one-to-one correspondence $\mathrm{Rad}_{\mathscr{C}}\mathcal{K} \underset{\mathrm{Supp}^{-1}}{\overset{\mathrm{Supp}}{\rightleftarrows}} \mathrm{Param}_{\mathscr{C}}\mathcal{K}$ is given by

$$\begin{gathered} T_0 \ \leftrightarrow\ \emptyset,\ \ T_1 \ \leftrightarrow\ \{T_{11}, T_{12}\},\ \ T_2 \ \leftrightarrow\ \{T_{10}, T_{12}\},\ \ T_3 \ \leftrightarrow\ \{T_9, T_{12}\},\\ T_4 \ \leftrightarrow\ \{T_{10}, T_{11}\},\ \ T_5 \ \leftrightarrow\ \{T_9, T_{11}\},\ \ T_6 \ \leftrightarrow\ \{T_9, T_{10}\},\\ T_9 \ \leftrightarrow\ \{T_{10}, T_{11}, T_{12}\},\ \ T_{10} \ \leftrightarrow\ \{T_9, T_{11}, T_{12}\},\ \ T_{11} \ \leftrightarrow\ \{T_9, T_{10}, T_{12}\},\\ T_{12} \ \leftrightarrow\ \{T_9, T_{10}, T_{11}\},\ \ T_{13} \ \leftrightarrow\ \{T_9, T_{10}, T_{11}, T_{12}\}. \end{gathered}$$

Example 3.6. Let $A = kQ$ be a path algebra with k a field and Q the quiver $1 \to 2 \to 3$, and let $\mathcal{K} = \dfrac{\mathrm{mod}A}{\mathrm{add}\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}\}}$. Then $\mathcal{K}$ is an extriangulated category which is neither an exact category nor an triangulated category. The Auslander-Reiten quiver of $\mathcal{K}$ is as follows:



All thick subcategories of $\mathcal{K}$:

$$\begin{gathered} T_0 = \{0\}\\ T_1 = \mathrm{add}\{3\},\ \ T_2 = \mathrm{add}\{\begin{smallmatrix}2\\3\end{smallmatrix}\},\ \ T_3 = \mathrm{add}\{2\},\ \ T_4 = \mathrm{add}\{\begin{smallmatrix}1\\2\end{smallmatrix}\},\ \ T_5 = \mathrm{add}\{1\}\\ T_6 = \mathrm{add}\{3, \begin{smallmatrix}1\\2\end{smallmatrix}\},\ \ T_7 = \mathrm{add}\{3, 1\},\ \ T_8 = \mathrm{add}\{\begin{smallmatrix}2\\3\end{smallmatrix}, 1\}\\ T_9 = \mathrm{add}\{3, \begin{smallmatrix}2\\3\end{smallmatrix}, 2, \begin{smallmatrix}1\\2\end{smallmatrix}, 1\}. \end{gathered}$$

Take $\mathscr{C} = \{T_6, T_7, T_8, T_9\}$. By a direct computation, we have

$$\mathrm{Rad}_{\mathscr{C}}\mathcal{K} = \{T_0, T_1, T_5, T_6, T_7, T_8, T_9\}$$

and

$$\mathrm{Param}_{\mathscr{C}}\mathcal{K} = \{\emptyset,\ \{T_8\},\ \{T_6\},\ \{T_7, T_8\},\ \{T_6, T_8\},\ \{T_6, T_7\},\ \{T_6, T_7, T_8\}\}\,.$$

The one-to-one correspondence $\mathrm{Rad}_{\mathscr{C}}\mathcal{K} \underset{\mathrm{Supp}^{-1}}{\overset{\mathrm{Supp}}{\rightleftarrows}} \mathrm{Param}_{\mathscr{C}}\mathcal{K}$ is given by

$$\begin{gathered} T_0 \ \leftrightarrow\ \emptyset,\ \ T_1 \ \leftrightarrow\ \{T_8\},\ \ T_5 \ \leftrightarrow\ \{T_6\},\ \ T_6 \ \leftrightarrow\ \{T_7, T_8\},\\ T_7 \ \leftrightarrow\ \{T_6, T_8\},\ \ T_8 \ \leftrightarrow\ \{T_6, T_7\},\ \ T_9 \ \leftrightarrow\ \{T_6, T_7, T_8\}. \end{gathered}$$

3.2 Prime thick subcategories

Definition 3.7. Let $\mathcal{P} \in \mathscr{T}(\mathcal{K})$. We say that $\mathcal{P}$ is *prime* if there is a unique thick subcategory which is minimal among all thick subcategories $\mathcal{X}$ of $\mathcal{K}$ satisfying $\mathcal{P} \subseteq \mathcal{X}$.

We denote by $\mathscr{P}(\mathcal{K})$ the set of all prime thick subcategories of $\mathcal{K}$ and $\mathrm{Esp}_{\mathscr{P}}(\mathcal{K})$ the spectrum of $\mathcal{K}$ with respect to $\mathscr{P}(\mathcal{K})$.

Example 3.8.

(1) In Example 3.5, $\mathscr{P}(\mathcal{K}) = \{T_7, T_8, T_9, T_{10}, T_{11}, T_{12}\}$.

(2) In Example 3.6, $\mathscr{P}(\mathcal{K}) = \{T_2, T_3, T_4, T_6, T_7, T_8\}$.

Definition 3.9. A *support data* on an extriangulated category $\mathcal{K}$ is a pair (X, σ), where X is a topological space and σ is an assignment from $\mathcal{K}$ to $\mathcal{X}_{\mathrm{cl}}(X)$ which satisfies the following rules:

(SD1) $\sigma(0) = \emptyset$.

(SD2) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$.

(SD3) Given an $\mathbb{E}$-triangle $a_1 \longrightarrow a_2 \longrightarrow a_3 \dashrightarrow$ in $\mathcal{K}$, it holds that $\sigma(a_i) \subseteq \sigma(a_j) \cup \sigma(a_k)$ for any $\{i, j, k\} \in \{1, 2, 3\}$.

Example 3.10. By Lemma 2.10, $(\mathrm{Esp}_{\mathscr{C}}(\mathcal{K}), \mathrm{Supp})$ is a support data on $\mathcal{K}$ for any $\mathscr{C} \subseteq \mathscr{T}(\mathcal{K})$. In particular, $(\mathrm{Esp}_{\mathscr{P}}(\mathcal{K}), \mathrm{Supp})$ is a support data on $\mathcal{K}$.

Lemma 3.11. *Let (X, σ) be a support data on $\mathcal{K}$. For any subset W of X, define $\sigma^{-1}(W) := \{A \in \mathcal{K} \mid \sigma(A) \subseteq W\}$. Then $\sigma^{-1}(W)$ is a thick subcategory of $\mathcal{K}$.*

Proof. It follows from Definition 3.9 directly. Indeed,

- The condition (SD1) yields $0 \in \sigma^{-1}(W)$.
- The condition (SD2) shows that $\sigma^{-1}(W)$ is closed under direct summands.
- The condition (SD3) shows that $\sigma^{-1}(W)$ satisfies the 2-out-of-3 property.

□

Let X be a topological space. Recall from [3, 1.1.3] that a subset W of X is *specialization-closed* if it is a union of closed subsets of X. It is easy to see that W is specialization-closed if and only if $\overline{\{x\}} \subseteq W$ for any $x \in W$.

We denote by $\mathbf{Spcl}(X)$ the set of all specialization-closed subsets of a topological space X.

Definition 3.12. A support data (X, σ) on an extriangulated category $\mathcal{K}$ is said to be *classifying* if it satisfies the following conditions:

(CSD1) The topological space X is a noetherian sober space, that is, X is noetherian and each irreducible closed subset of X has a unique generic point.

(CSD2) There is a mutually inverse lattice isomorphism

$$\mathscr{T}(\mathcal{K}) \underset{\sigma^{-1}}{\overset{\sigma}{\rightleftarrows}} \mathbf{Spcl}(X),$$

where $\sigma(\mathcal{X}) := \bigcup_{A \in \mathcal{X}} \sigma(A)$ and $\sigma^{-1}(W) := \{A \in \mathcal{K} \mid \sigma(A) \subseteq W\}$.

We write

$$\mathbf{Spcl}_1(X) = \{W \in \mathbf{Spcl}(X) \mid \text{ there is a unique minimal } T \in \mathbf{Spcl}(X) \text{ such that } W \subsetneqq T\}.$$

Lemma 3.13. *Let (X,σ) be a classifying support data on an extriangulated category $\mathcal{K}$. Then σ induces a mutually inverse lattice isomorphism*

$$\mathscr{P}(\mathcal{K}) \underset{\sigma^{-1}}{\overset{\sigma}{\rightleftarrows}} \mathbf{Spcl}_1(X) .$$

Proof. It follows from Definitions 3.7 and 3.12 directly. □

Let (X,σ) be a classifying support data on an extriangulated category $\mathcal{K}$. By [8, Lemma 2.15], a specialization-closed subset W of X belongs to $\mathbf{Spcl}_1(X)$ if and only if there is a unique element $x \in X$ such that $W = \{x' \in X \mid x \notin \overline{\{x'\}}\}$, which induces a well-defined bijection

$$\begin{aligned} \varphi :\ & X \to \mathrm{Esp}_{\mathscr{P}}(\mathcal{K}) \\ & x \mapsto \sigma^{-1}(\{x' \in X \mid x \notin \overline{\{x'\}}\}). \end{aligned}$$

Moreover, this map has the following properties, which are given originally in the proof of [8, Theorem 2.16].

Fact I. For any $x \in X$, we have $\varphi(x) = \{A \in \mathcal{K} \mid x \notin \sigma(A)\}$. Indeed, by the definition, $A \in \varphi(x)$ if and only if $\sigma(A) \subseteq \{x' \in X \mid x \notin \overline{\{x'\}}\}$. Since X is a noetherian sober space and $\sigma(A)$ is a closed subset of X, we can decompose $\sigma(A) = \overline{\{x_1\}} \cup \cdots \overline{\{x_n\}}$. Then $A \in \varphi(x)$ if and only if $x_i \in \{x' \in X \mid x \notin \overline{\{x'\}}\}$ for any $1 \leq i \leq n$, if and only if $x \notin \overline{\{x_i\}}$ for any $1 \leq i \leq n$, which means that $x \notin \sigma(A)$.

Fact II. For any $x, x' \in X$, we have that $x \in \overline{\{x'\}}$ if and only if $\varphi(x) \subseteq \varphi(x')$. Indeed, If $x \in X$, then $\sigma(\varphi(x)) = \{x' \in X \mid x \notin \overline{\{x'\}}\}$ by the definition. Thus for any $x, x' \in X$, we have that $x \in \overline{\{x'\}}$ if and only if $x' \notin \sigma(\varphi(x))$. Moreover, since $\sigma(\varphi(x)) = \bigcup_{A \in \varphi(x)} \sigma(A)$, it follows that $x' \notin \sigma(\varphi(x))$ if and only if $x' \notin \sigma(A)$ for any $A \in \varphi(x)$, and if and only if $A \in \varphi(x')$ for any $A \in \varphi(x)$, that is, $\varphi(x) \subseteq \varphi(x')$.

Now we sated the main result in this section as follows. It generalizes [8, Theorem 2.16].

Theorem 3.14. *Let (X,σ) be a classifying support data on an extriangulated category $\mathcal{K}$. Then the map $\varphi : X \to \mathrm{Esp}_{\mathscr{P}}(\mathcal{K})$ is a homeomorphism.*

Proof. We only need to show that φ and φ^{-1} are continuous.

- φ is continuous. Indeed, $\varphi(x) \in \mathrm{Supp}(A)$ if and only if $A \notin \varphi(x)$, which is equivalent to that $x \in \sigma(A)$ by Fact I. Thus $\varphi^{-1}(\mathrm{Supp}(A)) = \sigma(A)$ for any $A \in \mathcal{K}$, and hence φ is continuous.
- φ^{-1} is continuous. Indeed, for any $x' \in X$, we have $\overline{\{\varphi(x')\}} = \{\varphi(x) \in \mathrm{Esp}_{\mathscr{P}}\mathcal{K} \mid \varphi(x) \subseteq \varphi(x') \text{ for some } x \in X\}$ by Proposition 2.12, and hence $\overline{\{\varphi(x')\}} = \{\varphi(x) \mid x \in \overline{\{x'\}}\}$ by Fact II, that is, $\overline{\{\varphi(x')\}} = \varphi(\overline{\{x'\}})$.

□

Example 3.15. (1) Let $\mathcal{K} = \mathcal{T}$ be a triangulated category, [1] the shift functor and $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(-,-[1])$. If (X,σ) is a classifying support data on $\mathcal{T}$, then the map $\varphi : X \to \mathrm{Esp}_{\mathscr{P}}(\mathcal{T})$ is a homeomorphism ([8, Theorem 2.16]).

(2) Let $\mathcal{K} = \mathcal{E}$ be an exact category and $\mathbb{E} = \mathrm{Ext}^1_{\mathcal{E}}(-,-)$. If (X,σ) is a classifying support data on $\mathcal{E}$, then the map $\varphi : X \to \mathrm{Esp}_{\mathscr{P}}(\mathcal{E})$ is a homeomorphism.

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