TILTING MODULES OVER AUSLANDER ALGEBRAS OF NAKAYAMA ALGEBRAS WITH RADICAL CUBE ZERO

ZONGZHEN XIE, HANPENG GAO, AND ZHAOYONG HUANG

Abstract. Let A' be the Auslander algebra of a finite dimensional basic connected Nakayama algebra A with radical cube zero and n simple modules. Then the cardinality $\#\mathrm{tilt}\,A'$ of the set consisting of isomorphism classes of basic tilting A'-modules is

$$\#\mathrm{tilt}\,A' = \begin{cases} \dfrac{(1+\sqrt{2})^{2n-2}-(1-\sqrt{2})^{2n-2}}{2\sqrt{2}}, & \text{if } A \text{ is non-self-injective with } n \geq 4; \\ \sqrt{[(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}]^2+4}, & \text{if } A \text{ is self-injective with } n \geq 2. \end{cases}$$

1. Introduction

Tilting theory is important in representation theory of artin algebras and homological algebra. There are many related works which made the theory fruitful, see [3, 5, 10] and references therein. In this theory, tilting modules play a central role. So it is fundamental and important to classify tilting modules for a given algebra. An effective method to construct tilting modules is given by mutation [18, 20]. However, the mutation of tilting modules is not always possible. To improve the behavior of mutation of tilting modules, Adachi, Iyama and Reiten [4] introduced support τ-tilting modules as a generalization of tilting modules. They showed that the mutation of support τ-tilting modules is always possible; in particular, τ-tilting modules share many nice properties of tilting modules.

It is showed by Auslander that there is a bijection between classes of representation-finite algebras and Auslander algebras [6]. There are many works on Auslander algebras. Brüstle, Hille, Ringel and Röhrle [8] classified tilting modules over the Auslander algebra of $K[x]/\langle x^n\rangle$ and showed that the number of tilting modules is $n!$. Iyama and Zhang [17] classified τ-tilting modules over the Auslander algebra of $K[x]/\langle x^n\rangle$. Recently, Zhang [21] gave a classification of tilting modules over Auslander algebras of Nakayama algebras with radical square zero. On the other hand, algebras with radical cube zero have gained a lot of attention. Hoshino [15] proved the Tachikawa version of the Nakayama conjecture for algebras with radical cube zero. Erdmann and Solberg [9] classified all the possible quivers of finite dimensional self-injective algebras with radical cube zero and finite complexity. Adachi and Aoki [2] calculated the number of two-term tilting complexes over symmetric algebras with radical cube zero.

In the literature, especially in mathematics and physics, there are a lot of integer numbers, which are used in almost every field of modern sciences. Admittedly, Pell numbers (sequence A000129 in OEIS) and Pell-Lucas numbers (sequence A002203 in OEIS) are very essential in the fields of combinatorics and number theory. The Pell sequence $\{\mathrm{P_n}\}$ are defined by recurrence $\mathrm{P_n} = 2\mathrm{P_{n-1}} + \mathrm{P_{n-2}}$ for any $n \geq 2$ with $\mathrm{P_0} = 0$ and $\mathrm{P_1} = 1$, and the Pell-Lucas sequence $\{\mathrm{Q_n}\}$ by the same recurrence but with initial conditions $\mathrm{Q_0} = \mathrm{Q_1} = 2$. Explicit Binet forms for $\{\mathrm{P_n}\}$ and $\{\mathrm{Q_n}\}$ are $\mathrm{P_n} = \frac{\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}}{\alpha-\beta}$ and $\mathrm{Q_n} = \alpha^{\mathrm{n}} + \beta^{\mathrm{n}}$, where α and β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. Then

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one gets $8\mathrm{P}_{\mathrm{n}}^2 = \mathrm{Q}_{\mathrm{n}}^2 - 4(-1)^{\mathrm{n}}$. Further details about Pell and Pell-Lucas sequences can be found in [7, 11, 12, 13, 14].

In this paper, by virtual of Pell and Pell-Lucas sequences, we will determine the number of isomorphism classes of basic tilting modules over Auslander algebras of Nakayama algebras with radical cube zero. Let A be a finite dimensional algebra over an algebraically closed field. We use $\operatorname{tilt} A$ to denote the set consisting of isomorphism classes of basic tilting modules. For a set X, and use $\#X$ to denote the cardinality of X. The following is our main result.

Theorem 1.1. (Theorem 3.8) *Let A be a Nakayama algebra with radical cube zero and n simple modules, and let A' be the Auslander algebra of A.*
(1) *If A is non-self-injective with $n \geq 4$, then $\#\operatorname{tilt} A' = \frac{(1+\sqrt{2})^{2n-2}-(1-\sqrt{2})^{2n-2}}{2\sqrt{2}}$.*
(2) *If A is self-injective with $n \geq 2$, then $\#\operatorname{tilt} A' = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}$.*

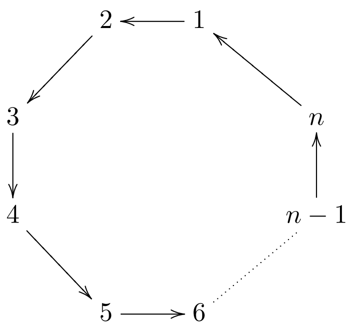
We also give two examples to illustrate this result.

2. Preliminaries

Throughout this paper, A is a finite dimensional algebra over an algebraically closed field K and τ the Auslander-Reiten translation. We use $\operatorname{mod} A$ to denote the category of finitely generated left A-modules and use $\operatorname{gl.dim} A$ to denote the global dimension of A. For a module $T \in \operatorname{mod} A$, we use $\mathsf{add}T$ to denote the subcategory of $\operatorname{mod} A$ consisting of direct summands of finite direct sums of T.

Recall that a module $T \in \operatorname{mod} A$ is called *(classical) tilting* if the projective dimension of T is at most one, $\operatorname{Ext}_A^1(T,T) = 0$ and there is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ in $\operatorname{mod} A$ with T_0 and T_1 in $\mathsf{add}T$. Also recall that A is called a *Nakayama algebra* if it is both right and left serial, that is, every indecomposable projective module and every indecomposable injective module in $\operatorname{mod} A$ are uniserial.

Proposition 2.1. ([5, Chapter V, Theorem 3.2]) *A basic and connected algebra A is a Nakayama algebra if and only if its ordinary quiver Q_A is one of the following two quivers:*
(1) $1 \to 2 \to 3 \to \cdots \to n-1 \to n$;
(2)



(with $n \geq 1$ vertices).

We use $|T|$ to denote the number of pairwise non-isomorphic indecomposable direct summands of T.

Definition 2.2. ([4, 19]) Let T be in $\operatorname{mod} A$.
(1) T is called *τ-rigid* if $\operatorname{Hom}_A(T, \tau T) = 0$, and T is called *τ-tilting* if T is τ-rigid and $|T| = |A|$.
(2) T is called *support τ-tilting* if there exists an idempotent e of A such that T is a τ-tilting $A/\langle e\rangle$-module.

We use $\operatorname{proj} A$ to denote the full subcategory of $\operatorname{mod} A$ consisting of projective modules. Sometimes, it is convenient to view support τ-tilting modules and τ-rigid modules as certain pairs of modules in $\operatorname{mod} A$.

Definition 2.3. Let (T, P) be a pair with $T \in \operatorname{mod} A$ and $P \in \operatorname{proj} A$.
(1) (T, P) is called a *τ-rigid pair* if T is τ-rigid and $\operatorname{Hom}_A(P, T) = 0$.
(2) (T, P) is called a *support τ-tilting pair* if (T, P) is τ-rigid and $|T| + |P| = |A|$.

We use $\operatorname{s\tau-tilt} A$ to denote the set of isomorphism classes of basic support τ-tilting modules in $\operatorname{mod} A$. For a module $M \in \operatorname{mod} A$, we use $\mathsf{Fac} M$ to denote the full subcategory of $\operatorname{mod} A$ consisting of modules isomorphic to factor modules of finite direct sums of copies of M.

Definition 2.4. ([4]) Let $T, U \in \operatorname{s\tau-tilt} A$. We call T a *mutation* of U if they have the same indecomposable direct summands except one. Precisely speaking, there are three cases:
(1) $T = V \oplus X$ and $U = V \oplus Y$ with $X \ncong Y$ indecomposable;
(2) $T = U \oplus X$ with X indecomposable;
(3) $U = T \oplus X$ with X indecomposable.
Moreover, we call T a *left mutation* (resp. *right*) *mutation* of U if $\mathsf{Fac} T \subsetneq \mathsf{Fac} U$ (resp. $\mathsf{Fac} T \supsetneq \mathsf{Fac} U$), and write $T = \mu_X^-(U)$ (resp. $T = \mu_X^+(U)$).

The following result [4, Theorem 2.30] gives a method for computing left mutations. For the convenience, we recall the definition of the Bongartz completion. For a τ-rigid A-module U, we have that $T := \mathrm{P}(^\perp(\tau U))$ is a τ-tilting A-module which is called a *Bongartz completion* of U satisfying $U \in \mathsf{add} T$ and $^\perp(\tau T) = \mathsf{Fac} T$, where $\mathrm{P}(^\perp(\tau U))$ is the direct sum of one copy of each of the indecomposable Ext-projective objects in $^\perp(\tau U)$ up to isomorphism.

Lemma 2.5. *Let $T = X \oplus U$ be a basic τ-tilting module which is the Bongartz completion of U with X indecomposable. Let*

$$X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$$

be an exact sequence with f the minimal left $\mathsf{add} U$-approximation. Then we have
(1) *If U is not sincere, then $Y = 0$. In this case, $U = \mu_X^-(T)$ holds and it is a basic support τ-tilting A-module that is not τ-tilting.*
(2) *If U is sincere, then Y is a direct sum of finite copies of an indecomposable A-module Y_1 and is not in $\mathsf{add} T$. In this case, $Y_1 \oplus U = \mu_X^-(T)$ holds and it is a basic τ-tilting A-module.*

We use $K^b(\operatorname{proj} A)$ to denote the bounded homotopy category of $\operatorname{proj} A$.

Definition 2.6. ([4]) Let P be a complex in $K^b(\operatorname{proj} A)$.
(1) P is called *presilting* if $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(P, P[n]) = 0$ for any $n \geq 1$.
(2) P is called *silting* if it is presilting and generates $K^b(\operatorname{proj} A)$ by taking direct sums, direct summands, shifts and mapping cones. In addition, it is called *tilting* if it is also satisfies $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(P, P[n]) = 0$ for all non-zero integers n.
(3) P is called *two-term silting* if it isomorphic to a complex concentrated in degree 0 and -1 in $K^b(\operatorname{proj} A)$.

We use $\operatorname{2-silt} A$ to denote the set of isomorphism classes of basic two-term silting complexes in $K^b(\operatorname{proj} A)$.

Lemma 2.7. ([4, Theorem 3.2]) *There exists a bijection*

$$\operatorname{2-silt} A \leftrightarrow \operatorname{s\tau-tilt} A$$

given by $\operatorname{2-silt} A \ni P \mapsto H^0(P) \in \operatorname{s\tau-tilt} A$ *and* $\operatorname{s\tau-tilt} A \ni (T, P) \mapsto (P_1 \oplus P \xrightarrow{(f\ 0)} P_0) \in \operatorname{2-silt} A$, *where $f : P_1 \to P_0$ is a minimal projective presentation of T.*

3. Main result

We begin with the following definition.

Definition 3.1. ([1, Definition 3.2]) Let $\Omega = (\Omega, \geq)$ be a poset and N a subposet of Ω.

(1) We define a new poset $\Omega^{\mathrm{N}} = (\Omega^{\mathrm{N}}, \geq_{\mathrm{N}})$ as follows, where $\mathrm{N}^+ := \{n^+ \mid n \in \mathrm{N}\}$ is a copy of N, and $\omega_1, \omega_2 \in \Omega \setminus \mathrm{N}$ and $n_1, n_2 \in \mathrm{N}$ are arbitrary elements:

$$\Omega^{\mathrm{N}} := \Omega \coprod \mathrm{N}^+,$$

$$\omega_1 \geq_{\mathrm{N}} \omega_2 :\Leftrightarrow \omega_1 \geq \omega_2,\ n_1 \geq_{\mathrm{N}} n_2 :\Leftrightarrow n_1 \geq n_2,$$

$$\omega_1 \geq_{\mathrm{N}} n_1 :\Leftrightarrow \omega_1 \geq n_1,\ n_1 \geq_{\mathrm{N}} \omega_1 :\Leftrightarrow n_1 \geq \omega_1,$$

$$n_1^+ \geq_{\mathrm{N}} \omega_1 :\Leftrightarrow n_1 \geq \omega_1,\ n_1^+ \geq_{\mathrm{N}} n_2 :\Leftrightarrow n_1 \geq n_2,$$

$$\omega_1 \geq_{\mathrm{N}} n_1^+ :\Leftrightarrow \omega_1 \geq n_1,\ n_1^+ \geq_{\mathrm{N}} n_2^+ :\Leftrightarrow n_1 \geq n_2.$$

In particular, $n_1 \geq_{\mathrm{N}} n_2^+$ never holds. It is easily to check that $(\Omega^{\mathrm{N}}, \geq_{\mathrm{N}})$ forms a poset.

(2) Let $\mathrm{H}(\Omega) := (\Omega, \mathrm{H_a})$ be the Hasse quiver of Ω. We define a new quiver $\mathrm{H}(\Omega)^{\mathrm{N}} := (\Omega^{\mathrm{N}}, \mathrm{H_a^N})$ as follows, where ω_1, ω_2 are arbitrary elements in $\Omega \setminus \mathrm{N}$ and n_1, n_2 are arbitrary elements in N:

$$\mathrm{H_a^N} = \{\omega_1 \to \omega_2 \mid \omega_1 \to \omega_2 \ \mathit{in}\ \mathrm{H_a}\} \coprod \{n_2 \to \omega_2 \mid n_2 \to \omega_2 \ \mathit{in}\ \mathrm{H_a}\}$$

$$\coprod \{n_1 \to n_2,\ n_1^+ \to n_2^+ \mid n_1 \to n_2 \ \mathit{in}\ \mathrm{H_a}\}$$

$$\coprod \{\omega_1 \to n_1^+ \mid \omega_1 \to n_1 \ \mathit{in}\ \mathrm{H_a}\} \coprod \{n_1^+ \to n_1 \mid n_1 \in \Omega\}.$$

It is easy to check that $\mathrm{H}(\Omega^{\mathrm{N}}) = \mathrm{H}(\Omega)^{\mathrm{N}}$ holds.

Assume that A has an indecomposable projective-injective summand L as an A-module. Moreover, let $S := \operatorname{soc} L$ and $\overline{A} := A/S$. Consider the functor

$$\overline{(-)} := - \otimes_A \overline{A} : \mathrm{mod} A \to \mathrm{mod} \overline{A}.$$

Then $\overline{L} = L/S$. Note that, for every indecomposable A-module $M \not\simeq L$, so we have an isomorphism $\overline{M} \simeq M$ as $\overline{A}$-modules.

Now let $\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{A} \mid \overline{L} \in \mathsf{add} N \text{ and } \mathrm{Hom}_A(N, L) = 0\}$. Applying Definition 3.1, we have a poset $(\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}}$. For any A-module M, we denote by $\alpha(M)$ a basic A-module satisfying $\mathsf{add}\,\alpha(M) = \mathsf{add}\,\overline{M}$.

Lemma 3.2. ([1, Theorem 3.3(1)]) *Let L be an indecomposable projective-injective summand of A as an A-module. Then there is an isomorphism of posets*

$$\mathrm{s}\tau\text{-tilt}\,A \to (\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}}$$

given by $M \mapsto \alpha(M)$. In particular, we have an isomorphism of Hasse quivers

$$\mathrm{H}(A) \simeq \mathrm{H}(\overline{A})^{\mathcal{N}}.$$

By the definition of $(\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}}$, we have

$$\#(\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}} = \#\mathrm{s}\tau\text{-tilt}\,\overline{A} + \#\mathcal{N}.$$

It follows from Lemma 3.2 that

$$\#\mathrm{s}\tau\text{-tilt}\,A = \#\mathrm{s}\tau\text{-tilt}\,\overline{A} + \#\mathcal{N}.$$

This equality will be crucial in proving our main result.

For an algebra A, assume that

$$0 \to A \to I^0(A) \to I^1(A) \to \cdots \to I^i(A) \to \cdots$$

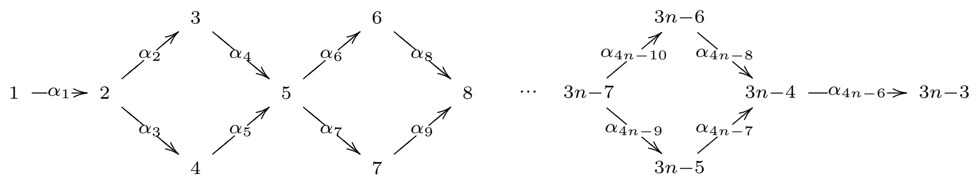
is the minimal injective resolution of ${}_A A$.

Lemma 3.3. ([16, Theorem 4.5]) *Let $I^0(A)$ be projective and e an idempotent of A such that* $\mathsf{add}\, eA = \mathsf{add}\, I^0(A)$. *Then the tensor functor $- \otimes_A A/\langle e\rangle$ induces a bijection from* $\operatorname{tilt} A$ *to* $\operatorname{s\tau\text{-}tilt} A/\langle e\rangle$.

Recall that A is called an *Auslander algebra* if $\operatorname{gl.dim} A \leq 2$ and both $I^0(A)$ and $I^1(A)$ are projective. Let A be representation-finite with M an additive generator for $\operatorname{mod} A$. Then $A' := \operatorname{End}_A(M)$ is an Auslander algebra [6]. In this case, A' is called the *Auslander algebra* of A.

In the rest of this section, A is a basic connected Nakayama algebra with radical cube zero and n simple modules, A' is the Auslander algebra of A and $\overline{A'} := A'/\langle e\rangle$ where $\mathsf{add}\, eA' = \mathsf{add}\, I^0(A')$ with e an idempotent of A'. The following result gives the structure of A', which is induced from Proposition 2.1 directly.

Proposition 3.4.
(1) *If A is non-self-injective with $n \geq 4$, then A' is given by the following quiver Q':*



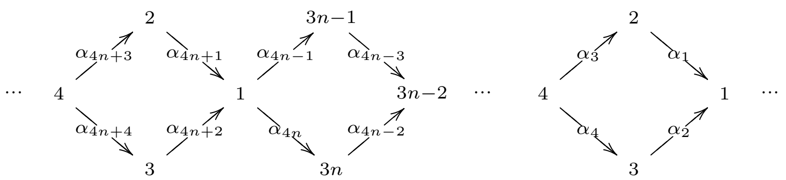
with relations

$$\alpha_{4i+2}\alpha_{4i+4} = \alpha_{4i+3}\alpha_{4i+5}, \quad \alpha_{4i+1}\alpha_{4i+3} = 0$$

for any $0 \leq i \leq n-3$ and

$$\alpha_{4n-7}\alpha_{4n-6} = 0.$$

(2) *If A is self-injective with $n \geq 2$, then A' is given by the following quiver Q':*



with relations

$$\alpha_{4i+3}\alpha_{4i+1} = \alpha_{4i+4}\alpha_{4i+2}, \alpha_{4i+5}\alpha_{4i+3} = 0$$

for any $i \geq 0$.

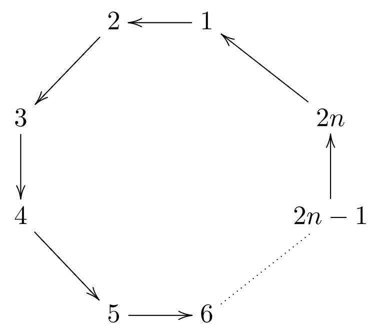
The following proposition is quite essential for the main result.

Proposition 3.5.
(1) *If A is non-self-injective with $n \geq 4$, then $\overline{A'}$ is given by the following quiver Q'':*

$$1 \to 2 \to 3 \to \cdots \to 2n-4 \to 2n-3$$

with $\operatorname{rad}^2 KQ'' = 0$.

(2) *If A is self-injective with $n \geq 2$, then $\overline{A'}$ is given by the following quiver Q'':*



with $\operatorname{rad}^2 KQ'' = 0$.

The following proposition gives some properties of indecomposable direct summands of tilting A'-modules.

Proposition 3.6. *Let T be a tilting module in $\operatorname{mod} A'$. Then we have*
(1) *The number of indecomposable projective-injective direct summands of T is n.*
(2) *The simple direct summand of T is either projective or a simple socle of an indecomposable projective A'-module.*
(3) *For any indecomposable non-projective-injective direct summand M of T, the Loewy length of M' which is the mutation of T on M is at most three.*

Proof. (1) By Proposition 3.4, we can easily get the number of indecomposable projective-injective direct summands of T. Since T is faithful, we have an epimorphism $T^n \twoheadrightarrow \mathbb{D}A'$, where $\mathbb{D} = \operatorname{Hom}_K(-, K)$ is the ordinary dual. If P is an indecomposable projective-injective module, then P is a direct summand of T.

If A is non-self-injective with $n(\geq 4)$ simple modules, then A' has $3n-3$ simple modules and the indecomposable projective-injective modules are $P(1),\ P(2),\ P(3),\ P(6),\ \cdots,\ P(3n-6)$. If A is self-injective with $n(\geq 2)$ simple modules, then A' has $3n$ simple modules and the indecomposable projective-injective modules are $P(3),\ P(6),\ \cdots,\ P(3n)$.

(2) If A is non-self-injective with $n(\geq 4)$ simple modules, then for any indecomposable projective module $P \in \operatorname{mod} A'$, $\operatorname{soc} P$ is either $S(3n-4)$ or $S(3i-3)$ with $2 \leq i \leq n$. Then by Lemma 2.5, we can verify directly that the simple direct summand of T is either projective or a simple socle of an indecomposable projective A'-module.

If A is self-injective with $n(\geq 2)$ simple modules, then for any indecomposable projective module $P \in \operatorname{mod} A'$, $\operatorname{soc} P = S(3i)$ with $1 \leq i \leq n$. Then by Lemma 2.5, we can verify directly that the simple direct summand of T is a simple socle of an indecomposable projective A'-module.

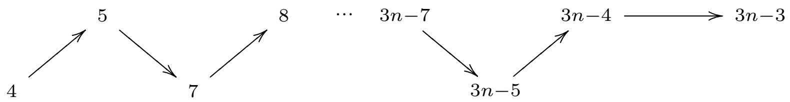
(3) If A is non-self-injective, then the quiver Q' of A' is as in Proposition 3.4(1). The indecomposable projective modules in $\operatorname{mod} A'$ are as follows:

$$P(1) = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix},\ P(2) = \begin{smallmatrix} & 2 & \\ 3 & 5 & 4 \\ & 6 & \end{smallmatrix},\ P(3) = \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \\ & 9 & \end{smallmatrix},\ P(4) = \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix},\ P(5) = \begin{smallmatrix} & 5 & \\ 6 & 8 & 7 \\ & 9 & \end{smallmatrix},\ P(6) = \begin{smallmatrix} & 6 & \\ & 8 & \\ 9 & & 10 \\ & 11 & \\ & 12 & \end{smallmatrix},\ P(7) = \begin{smallmatrix} 7 \\ 8 \\ 9 \end{smallmatrix},\ \cdots$$

$$P(3n-7) = \begin{smallmatrix} & 3n-7 & \\ 3n-6 & & 3n-5 \\ & 3n-4 & \end{smallmatrix},\ P(3n-6) = \begin{smallmatrix} 3n-6 \\ 3n-4 \\ 3n-3 \end{smallmatrix},\ P(3n-5) = \begin{smallmatrix} 3n-5 \\ 3n-4 \end{smallmatrix},$$

$$P(3n-4) = \begin{smallmatrix} 3n-4 \\ 3n-3 \end{smallmatrix},\ P(3n-3) = \begin{smallmatrix} 3n-3 \end{smallmatrix}.$$

By Proposition 3.5(1), the quiver Q'' of $\overline{A'}$ is as follows:



with the relation $\operatorname{rad}^2 KQ'' = 0$. The indecomposable projective modules in $\operatorname{mod}\overline{A'}$ are as follows:

$$P'(4) = \begin{smallmatrix} 4 \\ 5 \end{smallmatrix},\ P'(5) = \begin{smallmatrix} 5 \\ 7 \end{smallmatrix},\ P'(7) = \begin{smallmatrix} 7 \\ 8 \end{smallmatrix},\ \cdots,\ P'(3n-7) = \begin{smallmatrix} 3n-7 \\ 3n-5 \end{smallmatrix},\ P'(3n-5) = \begin{smallmatrix} 3n-5 \\ 3n-4 \end{smallmatrix},$$
$$P'(3n-4) = \begin{smallmatrix} 3n-4 \\ 3n-3 \end{smallmatrix},\ P'(3n-3) = \begin{smallmatrix} 3n-3 \end{smallmatrix}.$$

The maximal tilting A'-module is

$$T = P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(3n-3).$$

By (1), the indecomposable projective-injective direct summands of T are

$$P(1), P(2), P(3), P(6), \cdots, P(3n-6).$$

The maximal support τ-tilting $\overline{A'}$-module is

$$T' = P'(4) \oplus P'(5) \oplus P'(7) \oplus \cdots \oplus P'(3n-3).$$

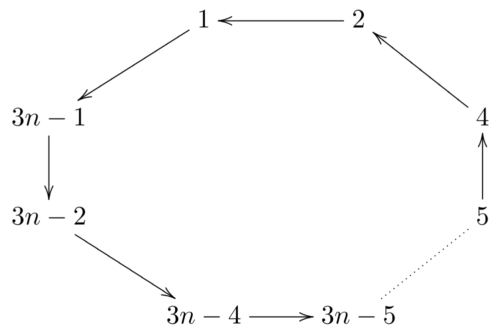
For any $i \in \{3j-2, 3j-1, 3n-3 \mid 2 \leq j \leq n-1\}$, we have a correspondence between $P(i)$ and $P'(i)$ by Lemma 3.3. Let L be an indecomposable direct summand of T'. Then there exists a module L' which is the mutation of T' on L by Lemma 2.5. We have that the Lowey length of L is at most two and the Lowey length of L' is at most one. Thus, if M is an indecomposable non-projective-injective direct summand of T. Then there exists a module M' which is the mutation of T on M. We have that the Lowey length of M is at most four and the Lowey length of M' is at most three.

If A is self-injective, then the quiver Q' of A' is as in Proposition 3.4(2). The indecomposable projective modules in $\operatorname{mod} A'$ are as follows:

$$P(1) = \begin{smallmatrix} & 1 & \\ 3n-1 & & 3n \\ & 3n-2 & \\ & 3n-3 & \end{smallmatrix},\ P(2) = \begin{smallmatrix} 2 \\ 1 \\ 3n \end{smallmatrix},\ P(3) = \begin{smallmatrix} & 3 & \\ & 1 & \\ 3n-1 & & 3n \\ & 3n-2 & \\ & 3n-3 & \end{smallmatrix},\ P(4) = \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \\ & 3n & \end{smallmatrix},\ P(5) = \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix},\ P(6) = \begin{smallmatrix} & 6 & \\ & 4 & \\ 2 & & 3 \\ & 1 & \\ & 3n & \end{smallmatrix},\ \cdots$$

$$P(3n-3) = \begin{smallmatrix} & 3n-3 & \\ & 3n-5 & \\ 3n-7 & & 3n-6 \\ & 3n-8 & \\ & 3n-9 & \end{smallmatrix},\ P(3n-2) = \begin{smallmatrix} & 3n-2 & \\ 3n-4 & & 3n-3 \\ & 3n-5 & \\ & 3n-6 & \end{smallmatrix},\ P(3n-1) = \begin{smallmatrix} 3n-1 \\ 3n-2 \\ 3n-3 \end{smallmatrix},\ P(3n) = \begin{smallmatrix} & 3n & \\ & 3n-2 & \\ 3n-4 & & 3n-3 \\ & 3n-5 & \\ & 3n-6 & \end{smallmatrix}.$$

By Proposition 3.5(2), the quiver Q'' of $\overline{A'}$ is as follows:



with the relation $\operatorname{rad}^2 KQ'' = 0$. The indecomposable projective modules in $\operatorname{mod}\overline{A'}$ are as follows:

$$P'(1) = \begin{smallmatrix} 1 \\ 3n-1 \end{smallmatrix},\ P'(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix},\ P'(4) = \begin{smallmatrix} 4 \\ 2 \end{smallmatrix},\ \cdots,\ P'(3n-2) = \begin{smallmatrix} 3n-2 \\ 3n-4 \end{smallmatrix},\ P'(3n-1) = \begin{smallmatrix} 3n-1 \\ 3n-2 \end{smallmatrix}.$$

The maximal tilting A'-module is

$$T = P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(3n).$$

By (1), the indecomposable projective-injective direct summands of T are as follows:

$$P(3),\ P(6),\ \cdots,\ P(3n).$$

The maximal support τ-tilting $\overline{A'}$-module is

$$T' = P'(1) \oplus P'(2) \oplus P'(4) \oplus \cdots \oplus P'(3n-1).$$

For any $i \in \{3j-2, 3j-1 \mid 1 \le j \le n\}$, we have a correspondence between $P(i)$ and $P'(i)$ by Lemma 3.3. Let L be an indecomposable direct summand of T'. Then there exists a module L' which is the mutation of T' on L by Lemma 2.5. We have that the Lowey length of L is two and the Lowey length of L' is at most one. Thus, if M is an indecomposable non-projective-injective direct summand of T and M' is the module which is the mutation of T on M, then the Lowey length of M is at most four and the Lowey length of M' is at most three. □

The following proposition calculates the number of support τ-tilting modules in $\operatorname{mod}\overline{A'}$.

Proposition 3.7.
(1) *If A is non-self-injective with $n \ge 4$, then* $\#\mathrm{s}\tau\text{-tilt}\,\overline{A'} = \frac{(1+\sqrt{2})^{2n-2}-(1-\sqrt{2})^{2n-2}}{2\sqrt{2}}$.
(2) *If A is self-injective with $n \ge 2$, then* $\#\mathrm{s}\tau\text{-tilt}\,\overline{A'} = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}$.

Proof. We only need to prove the case of radical square zero Nakayama algebra A by Lemma 3.3 and Proposition 3.5. Set $\mathrm{P_n} := \#\mathrm{s}\tau\text{-tilt}\,A$.

(1) If A is non-self-injective, then the quiver Q of A is

$$1 \to 2 \to 3 \to \cdots \to m-1 \to m$$

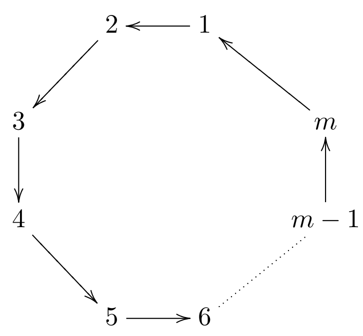
with the relation $\operatorname{rad}^2 KQ = 0$. Let $L = \begin{smallmatrix}1\\2\end{smallmatrix}$ be an indecomposable projective-injective summand of A. Then $\operatorname{soc} L = 2, \overline{L} = 1$ and $\overline{A} = A/\operatorname{soc} L$ is given by the following quiver:

$$1,\ 2 \to 3 \to \cdots \to m-1 \to m.$$

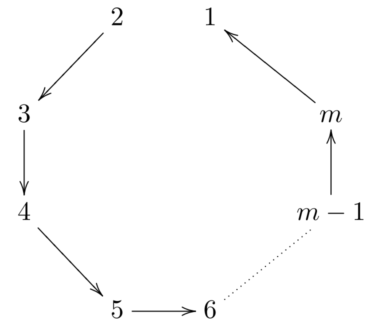
Thus $\#\mathrm{s}\tau\text{-tilt}\overline{A} = 2\mathrm{P_{m-1}}$.

By calculating $\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\overline{A} \mid \overline{L} \in \mathsf{add}N \text{ and } \operatorname{Hom}_A(N, L) = 0\}$, we get that the set $\mathcal{N}$ contains the module 1 but does not contain modules 2, $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}2\\3\end{smallmatrix}$. So we have $\#\mathcal{N} = \mathrm{P_{m-2}}$, and hence $\mathrm{P_m} = 2\mathrm{P_{m-1}} + \mathrm{P_{m-2}}$ by Lemma 3.2. It is a Pell-sequence (sequence A000129 in OEIS) and $\mathrm{P_m} = \frac{(1+\sqrt{2})^{m+1}-(1-\sqrt{2})^{m+1}}{2\sqrt{2}}$. By letting $m = 2n-3$, we get the desired assertion.

(2) If A is self-injective, then the quiver Q of A is



with the relation $\operatorname{rad}^2 KQ = 0$. Let $L = \frac{1}{2}$ be an indecomposable projective-injective summand of A. Then $\operatorname{soc} L = 2$, $\overline{L} = 1$ and $\overline{A} = A/\operatorname{soc} L$ is given by the following quiver:



Thus $\#\mathrm{s}\tau\text{-tilt}\,\overline{A} = \mathrm{P_m}$.

Similar to (1), we have $\#\mathcal{N} = \mathrm{P_{m-2}}$, and hence $\mathrm{Q_m} = \mathrm{P_m} + \mathrm{P_{m-2}}$. Applying $\mathrm{P_m} = 2\mathrm{P_{m-1}} + \mathrm{P_{m-2}}$ from (1), we get $\mathrm{Q_m} = 2\mathrm{Q_{m-1}} + \mathrm{Q_{m-2}}$. It is a Pell-Lucas sequence (sequence A002203 in OEIS) and

$$\mathrm{Q_m} = \sqrt{[(1+\sqrt{2})^m - (1-\sqrt{2})^m]^2 + 4(-1)^m}.$$

By letting $m = 2n$, we get the desired assertion. □

We now are in a position to give the main result.

Theorem 3.8.
(1) *If A is non-self-injective with $n \geq 4$, then* $\#\operatorname{tilt} A' = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}$.
(2) *If A is self-injective with $n \geq 2$, then* $\#\operatorname{tilt} A' = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}$.

Proof. Using the correspondence in Lemma 3.3, we can see that the number of tilting modules in $\operatorname{mod} A'$ is equal to the number of support τ-tilting modules in $\operatorname{mod} \overline{A'}$ which we have proved in Proposition 3.7. □

As a consequence, we have the following corollary.

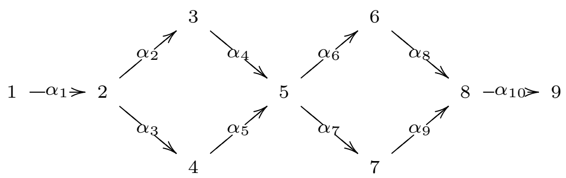
Corollary 3.9.
(1) *If A is non-self-injective with $n \geq 4$, then* $\#2\text{-silt}\,\overline{A'} = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}$.
(2) *If A is self-injective with $n \geq 2$, then* $\#2\text{-silt}\,\overline{A'} = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}$.

Proof. This follows from Lemma 2.7 and Proposition 3.7. □

4. Examples

In this section, we give two examples to illustrate the theorem in Section 3.

Example 4.1. Let A be an algebra given by the quiver Q: $1 \to 2 \to 3 \to 4$ with $\operatorname{rad}^3 KQ = 0$. The corresponding Auslander algebra A' is given by the quiver Q':



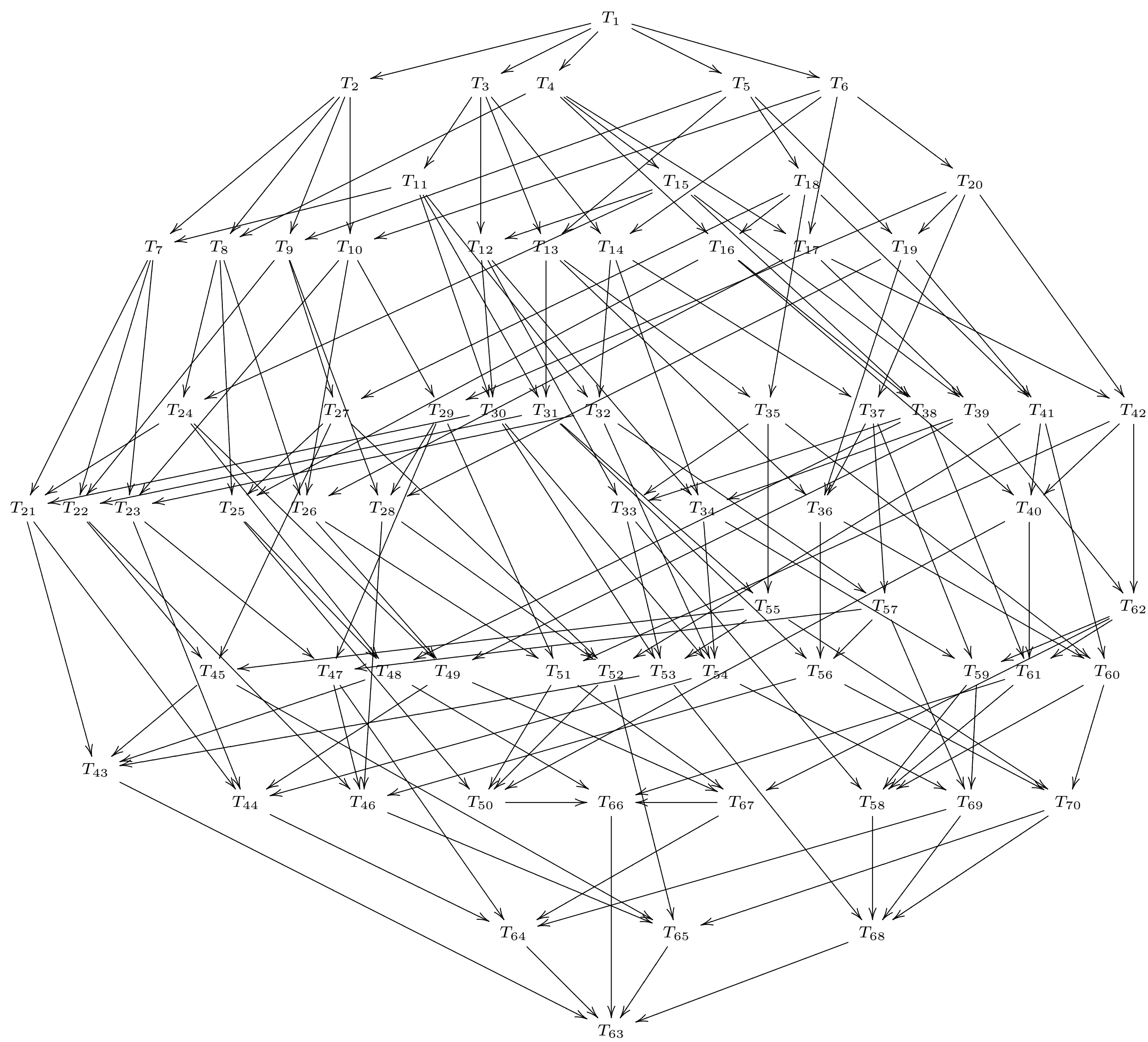
with relations

$$\alpha_{4i+2}\alpha_{4i+4} = \alpha_{4i+3}\alpha_{4i+5}, \quad \alpha_{4i+1}\alpha_{4i+3} = 0$$

for $i = 0, 1$, and

$$\alpha_9\alpha_{10} = 0.$$

Putting $n = 4$ in Theorem 3.8(1), we get $\#\mathrm{tilt}\,A' = 70$. The basic tilting A'-modules are presented by the following quiver Q'':



where

$$T_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9, \quad T_2 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_3 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9, \quad T_4 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_5 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9, T_6 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_7 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9, T_8 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_9 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9, T_{10} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{11} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9, T_{12} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{13} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9, T_{14} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{15} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9, T_{16} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{17} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8, T_{18} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{19} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix}, T_{20} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{21} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9, T_{22} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{23} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8, T_{24} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{25} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9, T_{26} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{27} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9, T_{28} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{29} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8, T_{30} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{31} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9, T_{32} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{33} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus 6 \oplus 9, T_{34} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{35} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9, T_{36} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{37} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8, T_{38} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{39}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus\begin{smallmatrix}8\\9\end{smallmatrix}\oplus 8,\ T_{40}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{41}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix},\ T_{42}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,$$

$$T_{43}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus 6\oplus 9,\ T_{44}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}8\\9\end{smallmatrix}\oplus 8,$$

$$T_{45}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus 9,\ T_{46}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}7\\8\end{smallmatrix}\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{47}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}7\\8\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,\ T_{48}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 6\oplus 9,$$

$$T_{49}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus\begin{smallmatrix}8\\9\end{smallmatrix}\oplus 8,\ T_{50}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{51}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,\ T_{52}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{53}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus 6\oplus 9,\ T_{54}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}8\\9\end{smallmatrix}\oplus 8,$$

$$T_{55}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus 9,\ T_{56}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}7\\8\end{smallmatrix}\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{57}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}7\\8\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,\ T_{58}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{59}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,\ T_{60}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{61}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},\ T_{62}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}4\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,$$

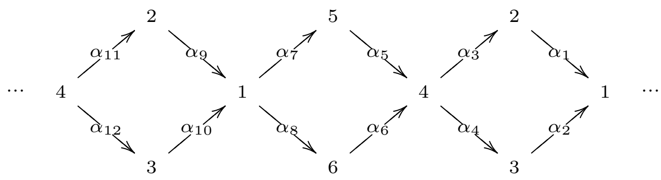
$$T_{63}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},\ T_{64}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,$$

$$T_{65}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix},\ T_{66}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{67}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,\ T_{68}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{69}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus\begin{smallmatrix}3\\5\\6\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}\oplus 8,\ T_{70}=\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\\9\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}6&&7\\&8&\end{smallmatrix}\oplus\begin{smallmatrix}6\\8\end{smallmatrix}.$$

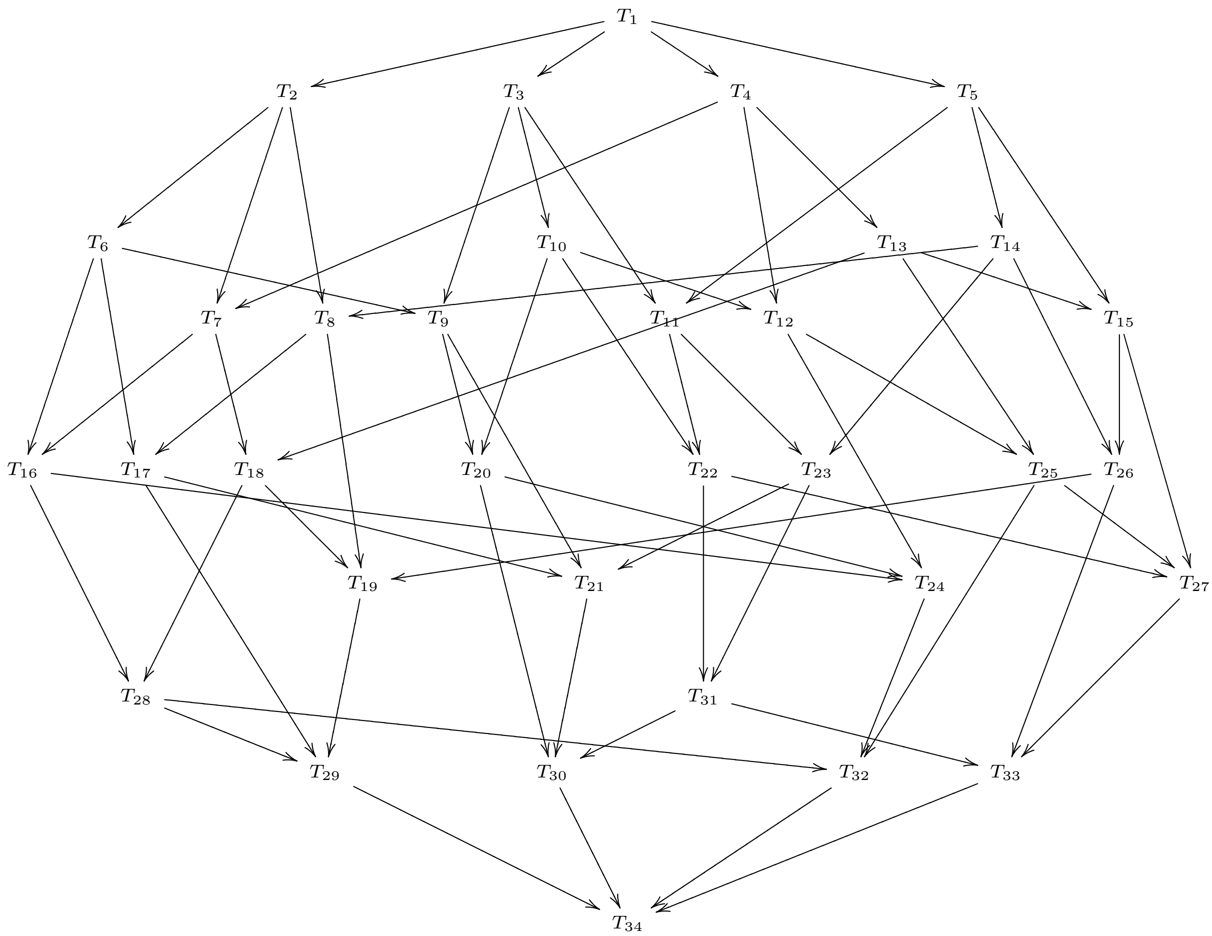
Example 4.2. Let A be an algebra given by the quiver Q: $1 \rightleftarrows 2$ with $\operatorname{rad}^3 KQ = 0$. The corresponding Auslander algebra A' is given by the quiver Q':



with relations

$$\alpha_{4i+3}\alpha_{4i+1} = \alpha_{4i+4}\alpha_{4i+2}, \quad \alpha_{4i+5}\alpha_{4i+3} = 0$$

for any $i \geq 0$. Putting $n = 2$ in Theorem 3.8(2), we get $\#\operatorname{tilt} A' = 34$. The basic tilting A'-modules are presented by the following quiver Q'':



where

$$T_1 = {\begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix}} \oplus {\begin{smallmatrix}2\\1\\6\end{smallmatrix}} \oplus {\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}} \oplus {\begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix}} \oplus {\begin{smallmatrix}5\\4\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}}, T_2 = {\begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix}} \oplus {\begin{smallmatrix}2\\1\\6\end{smallmatrix}} \oplus {\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}} \oplus {\begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix}} \oplus {\begin{smallmatrix}5\\4\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}},$$

$$T_3 = {\begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix}} \oplus {\begin{smallmatrix}4\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}} \oplus {\begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix}} \oplus {\begin{smallmatrix}5\\4\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}}, T_4 = {\begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix}} \oplus {\begin{smallmatrix}2\\1\\6\end{smallmatrix}} \oplus {\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}} \oplus {\begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix}} \oplus {\begin{smallmatrix}5\\4\\3\end{smallmatrix}} \oplus {\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}},$$

$$T_5 = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_6 = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_7 = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_8 = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_9 = 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{10} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{11} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{12} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{13} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{14} = \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{15} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{16} = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{17} = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{18} = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{19} = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{20} = 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{21} = 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{22} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{23} = \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{24} = 3 \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{25} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{26} = \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{27} = \begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{28} = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{29} = \begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{30} = 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{31} = \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{32} = 3 \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{33} = \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}, T_{34} = \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix}.$$

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Department of Mathematics and Computer Science, School of Biomedical Engineering and Informatics, Nanjing Medical University, Nanjing, 211166, P. R. China; Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China

E-mail address: zzhx@njmu.edu.cn, xiezongzhen3@163.com

Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China

E-mail address: hpgao07@163.com

Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China

E-mail address: huangzy@nju.edu.cn