TILTING MODULES OVER AUSLANDER ALGEBRAS OF NAKAYAMA ALGEBRAS WITH RADICAL CUBE ZERO

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Abstract. Let $A'$ be the Auslander algebra of a finite dimensional basic connected Nakayama algebra $A$ with radical cube zero and $n$ simple modules. Then the cardinality $\#\text{tilt} A'$ of the set consisting of isomorphism classes of basic tilting $A'$-modules is

$$\#\text{tilt} A' = \begin{cases} 
\frac{(1 + \sqrt{2})^{2n-2} - (1 - \sqrt{2})^{2n-2}}{2\sqrt{2}}, & \text{if } A \text{ is non-self-injective with } n \geq 4; \\
\sqrt{[(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}]^2 + 4}, & \text{if } A \text{ is self-injective with } n \geq 2.
\end{cases}$$

1. Introduction

Tilting theory is important in representation theory of artin algebras and homological algebra. There are many related works which made the theory fruitful, see [3, 5, 10] and references therein. In this theory, tilting modules play a central role. So it is fundamental and important to classify tilting modules for a given algebra. An effective method to construct tilting modules is given by mutation [18, 20]. However, the mutation of tilting modules is not always possible. To improve the behavior of mutation of tilting modules, Adachi, Iyama and Reiten [4] introduced support $\tau$-tilting modules as a generalization of tilting modules. They showed that the mutation of support $\tau$-tilting modules is always possible; in particular, $\tau$-tilting modules share many nice properties of tilting modules.

It is showed by Auslander that there is a bijection between classes of representation-finite algebras and Auslander algebras [6]. There are many works on Auslander algebras. Brüstle, Hille, Ringel and Röhrle [8] classified tilting modules over the Auslander algebra of $K[x]/(x^n)$ and showed that the number of tilting modules is $n!$. Iyama and Zhang [17] classified $\tau$-tilting modules over the Auslander algebra of $K[x]/(x^n)$. Recently, Zhang [21] gave a classification of tilting modules over Auslander algebras of Nakayama algebras with radical square zero. On the other hand, algebras with radical cube zero have gained a lot of attention. Hoshino [15] proved the Tachikawa version of the Nakayama conjecture for algebras with radical cube zero. Erdmann and Solberg [9] classified all the possible quivers of finite dimensional self-injective algebras with radical cube zero and finite complexity. Adachi and Aoki [2] calculated the number of two-term tilting complexes over symmetric algebras with radical cube zero.

In the literature, especially in mathematics and physics, there are a lot of integer numbers, which are used in almost every field of modern sciences. Admittedly, Pell numbers (sequence A000129 in OEIS) and Pell-Lucas numbers (sequence A002203 in OEIS) are very essential in the fields of combinatorics and number theory. The Pell sequence $\{P_n\}$ are defined by recurrence $P_n = 2P_{n-1} + P_{n-2}$ for any $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$, and the Pell-Lucas sequence $\{Q_n\}$ by the same recurrence but with initial conditions $Q_0 = Q_1 = 2$. Explicit Binet forms for $\{P_n\}$ and $\{Q_n\}$ are $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $Q_n = \alpha^n + \beta^n$, where $\alpha$ and $\beta$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. Then

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one gets $8P_n^2 = Q_n^2 - 4(-1)^n$. Further details about Pell and Pell-Lucas sequences can be found in [7, 11, 12, 13, 14].

In this paper, by virtue of Pell and Pell-Lucas sequences, we will determine the number of isomorphism classes of basic tilting modules over Auslander algebras of Nakayama algebras with radical cube zero. Let $A$ be a finite dimensional algebra over an algebraically closed field. We use tilt $A$ to denote the set consisting of isomorphism classes of basic tilting modules. For a set $X$, and use $\#X$ to denote the cardinality of $X$. The following is our main result.

Theorem 1.1. (Theorem 3.8) Let $A$ be a Nakayama algebra with radical cube zero and $n$ simple modules, and let $A'$ be the Auslander algebra of $A$.

(1) If $A$ is non-self-injective with $n \geq 4$, then $\#\text{tilt} A' = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}$.

(2) If $A$ is self-injective with $n \geq 2$, then $\#\text{tilt} A' = \sqrt{[(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}]^2 + 4}$.

We also give two examples to illustrate this result.

2. Preliminaries

Throughout this paper, $A$ is a finite dimensional algebra over an algebraically closed field $K$ and $\tau$ the Auslander-Reiten translation. We use mod $A$ to denote the category of finitely generated left $A$-modules and use gl.dim $A$ to denote the global dimension of $A$. For a module $T \in \text{mod} A$, we use add $T$ to denote the subcategory of mod $A$ consisting of direct summands of finite direct sums of $T$.

Recall that a module $T \in \text{mod} A$ is called (classical) tilting if the projective dimension of $T$ is at most one, Ext$_A^1(T; T) = 0$ and there is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ in mod $A$ with $T_0$ and $T_1$ in add $T$. Also recall that $A$ is called a Nakayama algebra if it is both right and left serial, that is, every indecomposable projective module and every indecomposable injective module in mod $A$ are uniserial.

Proposition 2.1. ([5, Chapter V, Theorem 3.2]) A basic and connected algebra $A$ is a Nakayama algebra if and only if its ordinary quiver $Q_A$ is one of the following two quivers:

(1) $1 \to 2 \to 3 \to \cdots \to n - 1 \to n$;

(2) \[ \begin{array}{ccc}
2 & \cdots & 1 \\
\vdots & & \vdots \\
3 & \cdots & n \\
4 & \cdots & n-1 \\
5 & \cdots & 6 \\
\end{array} \]

(with $n \geq 1$ vertices).

We use $|T|$ to denote the number of pairwise non-isomorphic indecomposable direct summands of $T$.

Definition 2.2. ([4, 19]) Let $T$ be in mod $A$.

(1) $T$ is called $\tau$-rigid if Hom$_A(T, \tau T) = 0$, and $T$ is called $\tau$-tilting if $T$ is $\tau$-rigid and $|T| = |A|$.

(2) $T$ is called support $\tau$-tilting if there exists an idempotent $e$ of $A$ such that $T$ is a $\tau$-tilting $A/(e)$-module.
We use \( \text{proj} \ A \) to denote the full subcategory of \( \text{mod} \ A \) consisting of projective modules. Sometimes, it is convenient to view support \( \tau \)-tilting modules and \( \tau \)-rigid modules as certain pairs of modules in \( \text{mod} \ A \).

**Definition 2.3.** Let \((T, P)\) be a pair with \( T \in \text{mod} \ A \) and \( P \in \text{proj} \ A \).

1. \((T, P)\) is called a \( \tau \)-rigid pair if \( T \) is \( \tau \)-rigid and \( \text{Hom}_A(P, T) = 0 \).
2. \((T, P)\) is called a support \( \tau \)-tilting pair if \( (T, P) \) is \( \tau \)-rigid and \(|T| + |P| = |A|\).

We use \( \text{sr-tilt} \ A \) to denote the set of isomorphism classes of basic support \( \tau \)-tilting modules in \( \text{mod} \ A \). For a module \( M \in \text{mod} \ A \), we use \( \text{Fac} M \) to denote the full subcategory of \( \text{mod} \ A \) consisting of modules isomorphic to factor modules of finite direct sums of copies of \( M \).

**Definition 2.4.** ([4]) Let \( T, U \in \text{sr-tilt} \ A \). We call \( T \) a mutation of \( U \) if they have the same indecomposable direct summands except one. Precisely speaking, there are three cases:

1. \( T = V \oplus X \) and \( U = V \oplus Y \) with \( X \ncong Y \) indecomposable;
2. \( T = U \oplus X \) with \( X \) indecomposable;
3. \( U = T \oplus X \) with \( X \) indecomposable.

Moreover, we call \( T \) a left mutation (resp. right) mutation of \( U \) if \( \text{Fac} T \subseteq \text{Fac} U \) (resp. \( \text{Fac} T \supseteq \text{Fac} U \)), and write \( T = \mu_X(U) \) (resp. \( T = \mu_X^+(U) \)).

The following result ([4, Theorem 2.30]) gives a method for computing left mutations. For the convenience, we recall the definition of the Bongartz completion. For a \( \tau \)-rigid \( A \)-module \( U \), we have that \( T := P(\tau(U)) \) is a \( \tau \)-tilting \( A \)-module which is called a Bongartz completion of \( U \) satisfying \( U \in \text{add} \ T \) and \( \tau(T) = \text{Fac} T \), where \( P(\tau(U)) \) is the direct sum of one copy of each of the indecomposable Ext-projective objects in \( \tau(U) \) up to isomorphism.

**Lemma 2.5.** Let \( T = X \oplus U \) be a basic \( \tau \)-tilting module which is the Bongartz completion of \( U \) with \( X \) indecomposable. Let

\[
X \xrightarrow{f} U' \xrightarrow{g} Y \rightarrow 0
\]

be an exact sequence with \( f \) the minimal left \( \text{add} U \)-approximation. Then we have

1. If \( U \) is not sincere, then \( Y = 0 \). In this case, \( U = \mu_X(T) \) holds and it is a basic support \( \tau \)-tilting \( A \)-module that is not \( \tau \)-tilting.
2. If \( U \) is sincere, then \( Y \) is a direct sum of finite copies of an indecomposable \( A \)-module \( Y_1 \) and is not in \( \text{add} \ T \). In this case, \( Y_1 \oplus U = \mu_X(T) \) holds and it is a basic \( \tau \)-tilting \( A \)-module.

We use \( K^b(\text{proj} \ A) \) to denote the bounded homotopy category of \( \text{proj} \ A \).

**Definition 2.6.** ([4]) Let \( P \) be a complex in \( K^b(\text{proj} \ A) \).

1. \( P \) is called presilting if \( \text{Hom}_{K^b(\text{proj} \ A)}(P, P[n]) = 0 \) for any \( n \geq 1 \).
2. \( P \) is called silting if it is presilting and generates \( K^b(\text{proj} \ A) \) by taking direct sums, direct summands, shifts and mapping cones. In addition, it is called tilting if it also satisfies \( \text{Hom}_{K^b(\text{proj} \ A)}(P, P[n]) = 0 \) for all non-zero integers \( n \).
3. \( P \) is called two-term silting if it isomorphic to a complex concentrated in degree 0 and \(-1\) in \( K^b(\text{proj} \ A) \).

We use \( 2\text{-silt} \ A \) to denote the set of isomorphism classes of basic two-term silting complexes in \( K^b(\text{proj} \ A) \).

**Lemma 2.7.** ([4, Theorem 3.2]) There exists a bijection \( 2\text{-silt} \ A \leftrightarrow \text{sr-tilt} \ A \)

given by \( 2\text{-silt} \ A \ni \ P \mapsto \ H^0(\ P) \in \text{sr-tilt} \ A \) and \( \text{sr-tilt} \ A \ni (T, P) \mapsto (P_1 \oplus P \xrightarrow{f \ 0} P_0) \in 2\text{-silt} \ A \), where \( f : P_1 \rightarrow P_0 \) is a minimal projective presentation of \( T \).
3. Main result

We begin with the following definition.

**Definition 3.1.** ([1, Definition 3.2]) Let $\Omega = (\Omega, \geq)$ be a poset and $N$ a subposet of $\Omega$.

1. We define a new poset $\Omega^N = (\Omega^N, \geq_N)$ as follows, where $N^+ := \{n^+ | n \in N\}$ is a copy of $N$, and $\omega_1, \omega_2 \in \Omega \setminus N$ and $n_1, n_2 \in N$ are arbitrary elements:
   
   $\Omega^N := \Omega \bigsqcup N^+$,
   
   $\omega_1 \geq_N \omega_2 \iff \omega_1 \geq \omega_2, n_1 \geq n_2 \iff n_1 \geq n_2,$
   
   $\omega_1 \geq_N n_1 \iff \omega_1 \geq n_1, n_1 \geq n_2 \iff n_1 \geq n_2,$
   
   $n_1 \geq_N \omega_1 \iff n_1 \geq \omega_1, n_1 \geq n_2 \iff n_1 \geq n_2,$
   
   $\omega_1 \geq n_1 \iff \omega_1 \geq n_1, n_1 \geq n_2 \iff n_1 \geq n_2.$

In particular, $n_1 \geq_N n_2^+$ never holds. It is easily to check that $(\Omega^N, \geq_N)$ forms a poset.

2. Let $H(\Omega) := (\Omega, H_\Omega)$ be the Hasse quiver of $\Omega$. We define a new quiver $H(\Omega)^N := (\Omega^N, H_\Omega^N)$ as follows, where $\omega_1, \omega_2$ are arbitrary elements in $\Omega \setminus N$ and $n_1, n_2$ are arbitrary elements in $N$:
   
   $H^N_a = \{\omega_1 \to \omega_2 | \omega_1 \to \omega_2 \text{ in } H_\Omega\} \bigsqcup \{n_2 \to \omega_2 \text{ in } H_\Omega\}$
   
   $\bigsqcup \{n_1 \to n_2, n_1^+ \to n_2^+ | n_1 \to n_2 \text{ in } H_\Omega\}$
   
   $\bigsqcup \{\omega_1 \to n_1^+ | \omega_1 \to n_1 \text{ in } H_\Omega\} \bigsqcup \{n_1^+ \to n_1 | n_1 \in \Omega\}.$

It is easy to check that $H(\Omega^N) = H(\Omega)^N$ holds.

Assume that $A$ has an indecomposable projective-injective summand $L$ as an $A$-module. Moreover, let $S := \text{soc } L$ and $\overline{A} := A/S$. Consider the functor

$(-) := - \otimes_A \overline{A} : \text{mod } A \to \text{mod } \overline{A}.$

Then $\overline{L} = L/S$. Note that, for every indecomposable $A$-module $M \neq L$, so we have an isomorphism $\overline{M} \simeq M$ as $\overline{A}$-modules.

Now let $\mathcal{N} := \{N \in \text{sr-tilt } \overline{A} | L \in \text{add } N \text{ and } \text{Hom}_A(N, L) = 0\}$. Applying Definition 3.1, we have a poset $(\text{sr-tilt } \overline{A})^\mathcal{N}$. For any $A$-module $M$, we denote by $\alpha(M)$ a basic $A$-module satisfying $\text{add } \alpha(M) = \text{add } \overline{M}$.

**Lemma 3.2.** ([1, Theorem 3.3(1)]) Let $L$ be an indecomposable projective-injective summand of $A$ as an $A$-module. Then there is an isomorphism of posets

$\text{sr-tilt } A \to (\text{sr-tilt } \overline{A})^\mathcal{N}$

given by $M \mapsto \alpha(M)$. In particular, we have an isomorphism of Hasse quivers

$H(A) \simeq H(\overline{A})^\mathcal{N}.$

By the definition of $(\text{sr-tilt } \overline{A})^\mathcal{N}$, we have

$\#(\text{sr-tilt } \overline{A})^\mathcal{N} = \#\text{sr-tilt } \overline{A} + \#\mathcal{N}.$

It follows from Lemma 3.2 that

$\#\text{sr-tilt } A = \#\text{sr-tilt } \overline{A} + \#\mathcal{N}.$

This equality will be crucial in proving our main result.

For an algebra $A$, assume that

$0 \to A \to I^0(A) \to I^1(A) \to \cdots \to I^i(A) \to \cdots$

is the minimal injective resolution of $A A$. 


Lemma 3.3. ([16, Theorem 4.5]) Let \( I^0(A) \) be projective and \( e \) an idempotent of \( A \) such that \( \text{add}eA = \text{add}I^0(A) \). Then the tensor functor \(- \otimes_A A/\langle e \rangle\) induces a bijection from \( \text{tilt}A \) to \( \text{srt-tilt}A/\langle e \rangle \).

Recall that \( A \) is called an Auslander algebra if \( \text{gl.dim}A \leq 2 \) and both \( I^0(A) \) and \( I^1(A) \) are projective. Let \( A \) be representation-finite with \( M \) an additive generator for \( \mod A \). Then \( A' := \text{End}_A(M) \) is an Auslander algebra [6]. In this case, \( A' \) is called the Auslander algebra of \( A \).

In the rest of this section, \( A \) is a basic connected Nakayama algebra with radical cube zero and \( n \) simple modules, \( A' \) is the Auslander algebra of \( A \) and \( \overline{A'} := A'/\langle e \rangle \) where \( \text{add}eA' = \text{add}I^0(A') \) with \( e \) an idempotent of \( A' \). The following result gives the structure of \( A' \), which is induced from Proposition 2.1 directly.

Proposition 3.4.

1. If \( A \) is non-self-injective with \( n \geq 4 \), then \( A' \) is given by the following quiver \( Q' \):

   ![Quiver Diagram](attachment:quiver.png)

   with relations

   \[ \alpha_{4i+2}\alpha_{4i+4} = \alpha_{4i+3}\alpha_{4i+5}, \quad \alpha_{4i+1}\alpha_{4i+3} = 0 \]

   for any \( 0 \leq i \leq n-3 \) and

   \[ \alpha_{4n-7}\alpha_{4n-6} = 0. \]

2. If \( A \) is self-injective with \( n \geq 2 \), then \( A' \) is given by the following quiver \( Q' \):

   ![Quiver Diagram](attachment:quiver.png)

   with relations

   \[ \alpha_{4i+3}\alpha_{4i+1} = \alpha_{4i+4}\alpha_{4i+2}, \quad \alpha_{4i+5}\alpha_{4i+3} = 0 \]

   for any \( i \geq 0 \).

The following proposition is quite essential for the main result.

Proposition 3.5.

1. If \( A \) is non-self-injective with \( n \geq 4 \), then \( \overline{A} \) is given by the following quiver \( Q'' \):

   \[ 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow 2n-4 \rightarrow 2n-3 \]

   with \( \text{rad}^2KQ'' = 0 \).
(2) If $A$ is self-injective with $n \geq 2$, then $A'$ is given by the following quiver $Q''$:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \\
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \\
4
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
5 \\
6
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2n \\
2n-1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3n-7 \\
3n-4 \\
3n-5
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3n-6 \\
3n-3
\end{array}
\end{array}
\end{array}
\end{array}
\]

with $\text{rad}^2 KQ'' = 0$.

The following proposition gives some properties of indecomposable direct summands of tilting $A'$-modules.

**Proposition 3.6.** Let $T$ be a tilting module in $\text{mod} \ A'$. Then we have

1. The number of indecomposable projective-injective direct summands of $T$ is $n$.
2. The simple direct summand of $T$ is either projective or a simple socle of an indecomposable projective $A'$-module.
3. For any indecomposable non-projective-injective direct summand $M$ of $T$, the Loewy length of $M'$ which is the mutation of $T$ on $M$ is at most three.

**Proof.** (1) By Proposition 3.4, we can easily get the number of indecomposable projective-injective direct summands of $T$. Since $T$ is faithful, we have an epimorphism $T^n \twoheadrightarrow \mathbb{D}A'$, where $\mathbb{D} = \text{Hom}_K(-, K)$ is the ordinary dual. If $P$ is an indecomposable projective-injective module, then $P$ is a direct summand of $T$.

If $A$ is non-self-injective with $n(\geq 4)$ simple modules, then $A'$ has $3n - 3$ simple modules and the indecomposable projective-injective modules are $P(1)$, $P(2)$, $P(3)$, $P(6)$, $\cdots$, $P(3n - 6)$. If $A$ is self-injective with $n(\geq 2)$ simple modules, then $A'$ has $3n$ simple modules and the indecomposable projective-injective modules are $P(3)$, $P(6)$, $\cdots$, $P(3n)$.

(2) If $A$ is non-self-injective with $n(\geq 4)$ simple modules, then for any indecomposable projective module $P \in \text{mod} \ A'$, soc $P$ is either $S(3n - 4)$ or $S(3i - 3)$ with $2 \leq i \leq n$. Then by Lemma 2.5, we can verify directly that the simple direct summand of $T$ is either projective or a simple socle of an indecomposable projective $A'$-module.

If $A$ is self-injective with $n(\geq 2)$ simple modules, then for any indecomposable projective module $P \in \text{mod} \ A'$, soc $P = S(3i)$ with $1 \leq i \leq n$. Then by Lemma 2.5, we can verify directly that the simple direct summand of $T$ is a simple socle of an indecomposable projective $A'$-module.

(3) If $A$ is non-self-injective, then the quiver $Q'$ of $A'$ is as in Proposition 3.4(1). The indecomposable projective modules in $\text{mod} \ A'$ are as follows:

$$P(1) = \frac{1}{3}, \ P(2) = \frac{3}{6}, \ P(3) = \frac{3}{8}, \ P(4) = \frac{5}{6}, \ P(5) = \frac{5}{9}, \ P(6) = \frac{8}{9}, \ P(7) = \frac{8}{12}, \ P(8) = \frac{8}{10}, \ P(9) = \frac{7}{9}, \ \cdots$$

$$P(3n - 7) = \frac{3n-7}{3n-4}, \ P(3n - 6) = \frac{3n-6}{3n-3}, \ P(3n - 5) = \frac{3n-5}{3n-4},$$

$$P(3n - 4) = \frac{3n-4}{3n-3}, \ P(3n - 3) = 3n-3.$$
By Proposition 3.5(1), the quiver $Q''$ of $\overline{A}$ is as follows:

```
4 5 7 8 3n-7 3n-4 3n-3
```

with the relation $\text{rad}^2 KQ'' = 0$. The indecomposable projective modules in mod $\overline{A}$ are as follows:

$$P'(4) = \frac{4}{3}, \quad P'(5) = \frac{5}{3}, \quad P'(7) = \frac{7}{3}, \quad \cdots, \quad P'(3n-7) = \frac{3n-7}{3n-1}, \quad P'(3n-5) = \frac{3n-5}{3n-4},$$

$$P'(3n-4) = \frac{3n-4}{3n-3}, \quad P'(3n-3) = \frac{3n-3}{3n-4}.$$

The maximal tilting $A'$-module is

$$T = P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(3n-3).$$

By (1), the indecomposable projective-injective direct summands of $T$ are

$$P(1), P(2), P(3), P(6), \cdots, P(3n-6).$$

The maximal support $\tau$-tilting $\overline{A}$-module is

$$T' = P'(4) \oplus P'(5) \oplus P'(7) \oplus \cdots \oplus P'(3n-3).$$

For any $i \in \{3j-2, 3j-1, 3n-3 \mid 2 \leq j \leq n-1\}$, we have a correspondence between $P(i)$ and $P'(i)$ by Lemma 3.3. Let $L$ be an indecomposable direct summand of $T'$. Then there exists a module $L'$ which is the mutation of $T'$ on $L$ by Lemma 2.5. We have that the Lowey length of $L$ is at most two and the Lowey length of $L'$ is at most one. Thus, if $M$ is an indecomposable non-projective-injective direct summand of $T$. Then there exists a module $M'$ which is the mutation of $T$ on $M$. We have that the Lowey length of $M$ is at most four and the Lowey length of $M'$ is at most three.

If $A$ is self-injective, then the quiver $Q'$ of $A'$ is as in Proposition 3.4(2). The indecomposable projective modules in mod $A'$ are as follows:

$$P(1) = \frac{3n-1}{3n-3}, P(2) = \frac{2}{3n}, P(3) = \frac{3n-1}{3n-2}, P(4) = 2, P(5) = 3, P(6) = 2, \cdots$$

$$P(3n-3) = \frac{3n-3}{3n-5}, P(3n-2) = \frac{3n-2}{3n-4}, P(3n-1) = \frac{3n-1}{3n-4}, P(3n) = \frac{3n}{3n-4}, \cdots$$

By Proposition 3.5(2), the quiver $Q''$ of $\overline{A}$ is as follows:

```
1 ---- 2
3n-1 ---- 4
3n-2 ---- 5
3n-4 ---- 3n-5
```

with the relation $\text{rad}^2 KQ'' = 0$. The indecomposable projective modules in mod $\overline{A}$ are as follows:

$$P'(1) = \frac{1}{3n-1}, \quad P'(2) = \frac{2}{3n}, \quad P'(4) = \frac{4}{3n}, \quad \cdots, \quad P'(3n-2) = \frac{3n-2}{3n-4}, \quad P'(3n-1) = \frac{3n-1}{3n-2}.$$

The maximal tilting $A'$-module is

$$T = P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(3n).$$
By (1), the indecomposable projective-injective direct summands of $T$ are as follows:

$$P(3), P(6), \ldots, P(3n).$$

The maximal support $\tau$-tilting $A$-module is

$$T' = P'(1) \oplus P'(2) \oplus P'(4) \oplus \cdots \oplus P'(3n - 1).$$

For any $i \in \{3j - 2, 3j - 1 \mid 1 \leq j \leq n\}$, we have a correspondence between $P(i)$ and $P'(i)$ by Lemma 3.3. Let $L$ be an indecomposable direct summand of $T'$. Then there exists a module $L'$ which is the mutation of $T'$ on $L$ by Lemma 2.5. We have that the Loewy length of $L$ is two and the Loewy length of $L'$ is at most one. Thus, if $M$ is an indecomposable non-projective-injective direct summand of $T$ and $M'$ is the module which is the mutation of $T$ on $M$, then the Loewy length of $M$ is at most four and the Loewy length of $M'$ is at most three. \qed

The following proposition calculates the number of support $\tau$-tilting modules in $\text{mod } A$.

**Proposition 3.7.**

1. If $A$ is non-self-injective with $n \geq 4$, then $\# \text{sr-tilt } A = \frac{(1 + \sqrt{2})^{2n-2} - (1 - \sqrt{2})^{2n-2}}{2\sqrt{2}}$.

2. If $A$ is self-injective with $n \geq 2$, then $\# \text{sr-tilt } A = \sqrt{[(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}]^2 + 4}$.

**Proof.** We only need to prove the case of radical square zero Nakayama algebra $A$ by Lemma 3.3 and Proposition 3.5. Set $P_n := \# \text{sr-tilt } A$.

(1) If $A$ is non-self-injective, then the quiver $Q$ of $A$ is

$$1 \to 2 \to 3 \to \cdots \to m - 1 \to m$$

with the relation $\text{rad}^2 KQ = 0$. Let $L = \frac{1}{2}$ be an indecomposable projective-injective summand of $A$. Then $\text{soc } L = 2$, $L = 1$ and $A/\text{soc } L$ is given by the following quiver:

$$1, \ 2 \to 3 \to \cdots \to m - 1 \to m.$$

Thus $\# \text{sr-tilt } A = 2P_{m-1}$.

By calculating $N := \{N \in \text{sr-tilt } A \mid L \in \text{add } N \text{ and } \text{Hom}_A(N, L) = 0\}$, we get that the set $N$ contains the module 1 but does not contain modules 2, $\frac{1}{2}$ and $\frac{3}{2}$. So we have $\# N = P_{m-2}$, and hence $P_m = 2P_{m-1} + P_{m-2}$ by Lemma 3.2. It is a Pell-sequence (sequence A000129 in OEIS) and $P_m = \frac{(1 + \sqrt{2})^{m+1} - (1 - \sqrt{2})^{m+1}}{2\sqrt{2}}$. By letting $m = 2n - 3$, we get the desired assertion.

(2) If $A$ is self-injective, then the quiver $Q$ of $A$ is
with the relation rad^2 KQ = 0. Let L = \frac{1}{2} be an indecomposable projective-injective summand of A. Then soc L = 2, \overline{L} = 1 and \overline{A} = A/socL is given by the following quiver:

Thus \#\overline{\text{tilt}} \overline{A} = P_m.

Similar to (1), we have \#A = P_m - 2, and hence Q_m = P_m + P_{m-2}. Applying P_m = 2P_{m-1} + P_{m-2} from (1), we get Q_m = 2Q_{m-1} + Q_{m-2}. It is a Pell-Lucas sequence (sequence A002203 in OEIS) and

\[ Q_m = \sqrt{[(1 + \sqrt{2})^m - (1 - \sqrt{2})^m]^2 + 4(-1)^m}. \]

By letting \( m = 2n \), we get the desired assertion.

We now are in a position to give the main result.

**Theorem 3.8.**

(1) If A is non-self-injective with \( n \geq 4 \), then \#\text{tilt} \overline{A}' = \frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}.

(2) If A is self-injective with \( n \geq 2 \), then \#\text{tilt} \overline{A}' = \sqrt{[(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}]^2 + 4}.

**Proof.** Using the correspondence in Lemma 3.3, we can see that the number of tilting modules in mod \overline{A}' is equal to the number of support \tau-tilting modules in mod \overline{A} which we have proved in Proposition 3.7.

As a consequence, we have the following corollary.

**Corollary 3.9.**

(1) If A is non-self-injective with \( n \geq 4 \), then \#2-\text{silt} \overline{A}' = \frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}.

(2) If A is self-injective with \( n \geq 2 \), then \#2-\text{silt} \overline{A}' = \sqrt{[(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}]^2 + 4}.

**Proof.** This follows from Lemma 2.7 and Proposition 3.7.

4. **Examples**

In this section, we give two examples to illustrate the theorem in Section 3.

**Example 4.1.** Let A be an algebra given by the quiver Q: 1 → 2 → 3 → 4 with \text{rad}^3 KQ = 0. The corresponding Auslander algebra A' is given by the quiver Q':

with relations

\[ \alpha_{4i+2}\alpha_{4i+4} = \alpha_{4i+3}\alpha_{4i+5}, \quad \alpha_{4i+1}\alpha_{4i+3} = 0 \]
for $i = 0, 1$, and

$$\alpha_9\alpha_{10} = 0.$$ 

Putting $n = 4$ in Theorem 3.8(1), we get $\#\text{tilt} A' = 70$. The basic tilting $A'$-modules are presented by the following quiver $Q'$:

![Quiver Diagram](image)

where

$$T_1 = \frac{1}{3} \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9, \quad T_2 = \frac{1}{3} \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9, \quad T_3 = \frac{1}{3} \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9, \quad T_4 = \frac{1}{3} \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9.$$
\[ T_5 = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_6 = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_7 = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_8 = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_9 = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{10} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{11} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{12} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{13} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{14} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{15} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{16} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{17} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{18} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{19} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{20} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{21} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{22} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{23} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{24} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{25} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{26} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{27} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{28} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{29} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{30} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{31} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{32} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{33} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{34} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{35} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{36} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[ T_{37} = \frac{1}{3} + \frac{2}{6} + \frac{4}{8} + \frac{2}{7} + \frac{5}{6} + \frac{6}{3} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \]
\[
T_{39} = \frac{1}{3} + \frac{2}{5} + \frac{3}{6} + \frac{4}{7} + \frac{5}{8} + \frac{6}{9} + \frac{7}{10} + \frac{8}{11} + \frac{9}{12}, T_{40} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{41} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{42} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{43} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{44} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{45} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{46} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{47} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{48} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{49} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{50} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10}, T_{51} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{52} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{53} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{54} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{55} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{56} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{57} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{58} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{59} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{60} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{61} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{62} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{63} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{64} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{65} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{66} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{67} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{68} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}, T_{69} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9} + \frac{8}{10} + \frac{9}{11}.
Example 4.2. Let $A$ be an algebra given by the quiver $Q$: $1 \rightarrow 2$ with $\text{rad}^3 KQ = 0$. The corresponding Auslander algebra $A'$ is given by the quiver $Q'$:

\[
\begin{array}{c}
\vdots \\
\alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
5 & 4 & 3 & 2 \\
\end{array}
\]

with relations

\[\alpha_{4i+3}\alpha_{4i+1} = \alpha_{4i+4}\alpha_{4i+2}, \quad \alpha_{4i+5}\alpha_{4i+3} = 0\]

for any $i \geq 0$. Putting $n = 2$ in Theorem 3.8(2), we get $\#\text{tilt}A' = 34$. The basic tilting $A'$-modules are presented by the following quiver $Q''$:

where

\[
T_1 = 5 \frac{1}{3} \oplus 6 \frac{1}{6} \oplus 2 \frac{1}{6} \oplus 5 \frac{3}{4} \oplus 2 \frac{4}{3} \oplus 3 \frac{2}{6} \oplus 6 \frac{1}{1}, T_2 = 2 \frac{1}{3} \oplus 6 \frac{2}{6} \oplus 5 \frac{3}{4} \oplus 2 \frac{4}{3} \oplus 6 \frac{1}{1},
\]

\[
T_3 = 5 \frac{1}{3} \oplus 6 \frac{1}{6} \oplus 2 \frac{4}{3} \oplus 3 \frac{2}{6} \oplus 6 \frac{1}{1}, T_4 = 5 \frac{1}{3} \oplus 6 \frac{1}{6} \oplus 5 \frac{3}{4} \oplus 2 \frac{4}{3} \oplus 6 \frac{1}{1},
\]
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TILTING MODULES OVER AUSLANDER ALGEBRAS OF NAKAYAMA ALGEBRAS WITH RADICAL CUBE ZERO

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