Wakamatsu Tilting Modules over Noetherian Algebras [1]

Xi Tang
School of Science, Guilin University of Aerospace Technology, Guilin 541004, Guangxi Province, P.R. China
E-mail: tx5259@sina.com.cn
Zhaoyong Huang
School of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P.R. China
E-mail: huangzy@nju.edu.cn

Abstract

Let Λ be a Noetherian algebra and ${}_\Lambda C$ a Wakamatsu tilting module with $\Gamma = \operatorname{End}_\Lambda C$. We are concerned with some relative homological dimensions of certain modules with respect to C. Firstly, a new non-commutative generalization of the Auslander–Bridger formula for modules with finite C-Gorenstein projective dimensions is established. As an application, we prove under some conditions that if the Gorenstein projective dimension of C is finite, then C is isomorphic to Λ. This provides an evidence for the Wakamatsu tilting conjecture. Secondly, when Λ is semilocal, we show that the left and right injective dimension of ${}_\Lambda C_\Gamma$ is at most n if and only if the C-Gorenstein projective dimension of any simple left Λ-module (or right Γ-module) is at most n. Its dual version is also obtained. Finally, when $\Lambda = \Gamma$ is local, we give an equivalent characterization for the finiteness of the left and right injective dimensions of ${}_\Lambda C_\Gamma$ in terms of cotorsion pairs generated by the classes of C-∞-torsionfree modules.
2020 Mathematics Subject Classification: 16E65, 16L30, 18G25.
Keywords: Wakamatsu tilting modules, Noetherian algebras, (C)-Gorenstein projective dimension, Auslander projective dimension, Bass injective dimension, Jacobson radical, Semiocal rings.

1 Introduction

The study of generalized tilting modules, which were usually called Wakamatsu tilting modules, was initiated by Wakamatsu [45]. Foxby [14], Golod [17] and Vasconcelos [43] independently initiated the study of semidualizing modules (under different names) over commutative Noetherian rings, which were generalized to non-commutative rings in [22]. A Wakamatsu tilting module ${}_\Lambda C$ with $\Gamma = \operatorname{End}_\Lambda C$ is also known as a semidualizing bimodule ${}_\Lambda C_\Gamma$.

Of the relative homological invariants of modules, the Gorenstein projective dimension is the one of most interest in Gorenstein homological algebra [4, 13]. As a generalization of Gorenstein projective dimension, Golod [17] used a semidualizing module C to defined C-Gorenstein projective dimension for finitely generated modules. For an R-module M, we use $\operatorname{pd}_R M$ and G-$\operatorname{pd}_R M$ (G_C-$\operatorname{pd}_R M$) to denote the projective and (C-)Gorenstein projective dimensions of M, respectively. In fact, Golod proved the following well-known result for finitely generated modules over commutative Noetherian local rings.

Theorem 1.1. ([17]) *Let R be a commutative Noetherian local ring and C a semidualizing R-module. If M is a finitely generated R-module with finite C-Gorenstein projective dimension, then*

$$G_C\text{-}\operatorname{pd}_R M + \operatorname{depth}_R(M) = \operatorname{depth}_R(R).$$

[1] The research was partially supported by NSFC (Grant No. 12371038) and NSF of Guangxi Province of China (Grant No. 2026GXNSFAA00640581).

It is an extension of the Auslander–Bridge formula [4]. One aim of this paper is to give a non-commutative analogue of Theorem 1.1 for Noetherian algebras over a commutative Noetherian ring and give applications. A *Noetherian* R-algebra Λ is a ring endowed with a ring homomorphism $R \to \Lambda$ with R a commutative Noetherian ring, whose image is contained in the center of Λ and Λ is finitely generated as an R-module [33]. It follows that Λ is a left and right Noetherian ring. Moreover, the endomorphism ring $\Gamma := \operatorname{End}_\Lambda M$ of any finitely generated left Λ-module M is also a Noetherian R-algebra.

In contrast with the commutative case, the study of Gorenstein projective dimension with respect to a semidualizing bimodule is a very active area of research in non-commutative algebra, as evidenced by various works like [3, 27, 28, 29, 36, 50]. In the rest of this section, we always assume that Λ is a Noetherian R-algebra and ${}_\Lambda C$ is a Wakamatsu tilting (= semidualizing) module with $\Gamma = \operatorname{End}_\Lambda C$. Our first main result, which appears as Theorem 3.9, plays a crucial role in the study of module depth by relating it to C-Gorenstein projective dimension, thus provides an essential understanding of the structure and properties of modules over Noetherian algebras.

Theorem A. *Let $0 \neq M \in \operatorname{mod}\Lambda$ with $G_C\text{-pd}_\Lambda M < \infty$. If Λ is semilocal and C satisfies the condition $(hc_C^{\Gamma^{op}})$, then*

$$G_C\text{-pd}_\Lambda M + \operatorname{depth}_R(M) = \operatorname{depth}_R(C).$$

See Section 3 for the definition of the condition $(hc_C^{\Gamma^{op}})$. Motivated by [32, Proposition], we give an application of Theorem A concerning when a Wakamatsu tilting module C is isomorphic to Λ. The following theorem appears as Theorem 3.15.

Theorem B. *Let Λ be local and $\text{G-pd}_\Lambda C < \infty$. Then $C \cong \Lambda$ if one of the following two conditions is satisfied:*

(1) $\Lambda = \Gamma$.

(2) *R is a Henselian local ring and C is indecomposable as a Λ-module.*

The proof of the above theorem builds on two spectral sequences, which are of independent interest. In fact, this result is related to the *Wakamatsu tilting conjecture*, which states that, over an Artinian algebra Λ, if ${}_\Lambda C$ is a Wakamatsu tilting module with projective dimension finite, then C is a tilting Λ-module. Note that this long-standing conjecture can be stated over arbitrary rings and Gorenstein projective dimension is a refinement of projective dimension for modules. It is easily seen from Theorem B that the Wakamatsu tilting conjecture is true for special Wakamatsu tilting modules over Noetherian local algebras.

Suppose that R is a commutative Noetherian local ring and k is the residue field. The characterization of a local ring by the homological properties of its finite modules begins with the classical criteria for the regularity of R, which is due to Auslander, Buchsbaum and Serre [8, Theorem 2.2.7]. They proved that R is regular if and only if $\operatorname{pd}_R M < \infty$ for any finitely generated R-module M, and if and only if $\operatorname{pd}_R k < \infty$. Parallel to Auslander–Buchsbaum–Serre's regularity theorem, Christensen [11] showed that R is Gorenstein if and only if $\text{G-pd}_R M < \infty$ for any finitely generated R-module M, and if and only if $\text{G-pd}_R k < \infty$; and this was extended to the C-Gorenstein projective dimension by himself [12]. To be more precise, it was shown in [12, Proposition 8.4] that a semidualizing R-module C is dualizing if and only if $G_C\text{-pd}_R M < \infty$ for any finitely generated R-module M, and if and only if $G_C\text{-pd}_R k < \infty$. Inspired by the preceding studies in the commutative setting, we extend Christensen's result to the non-commutative case; moreover, the corresponding dual version is also obtained. Actually, we prove the following result (see Theorem 4.7).

Theorem C. *Let Λ be semilocal and $n \geqslant 0$.*

(i) *The following statements are equivalent.*

(1) *The left and right injective dimensions of ${}_{\Lambda}C_{\Gamma}$ are at most n.*

(2) G_C-$\mathrm{pd}_{\Lambda}\, S \leqslant n$ *for any simple left Λ-module S.*

(3) G_C-$\mathrm{pd}_{\Lambda}\, M \leqslant n$ *for any finitely generated left Λ-module M.*

(4) G_C-$\mathrm{pd}_{\Gamma^{op}}\, T \leqslant n$ *for any simple right Γ-module T.*

(5) G_C-$\mathrm{pd}_{\Gamma^{op}}\, N \leqslant n$ *for any finitely generated right Γ-module N.*

(ii) *The following statements are equivalent.*

($1'$) *The left and right projective dimensions of ${}_{\Lambda}C_{\Gamma}$ are at most n.*

($2'$) *The Bass injective dimension of any simple left Λ-module is at most n.*

($3'$) *The Bass injective dimension of any finitely generated left Λ-module is at most n.*

($4'$) *The Bass injective dimension of any simple right Γ-module is at most n.*

($5'$) *The Bass injective dimension of any finitely generated right Γ-module is at most n.*

($6'$) *The Auslander projective dimension of any simple left Γ-module is at most n.*

($7'$) *The Auslander projective dimension of any finitely generated left Γ-module is at most n.*

($8'$) *The Auslander projective dimension of any simple right Λ-module is at most n.*

($9'$) *The Auslander projective dimension of any finitely generated right Λ-module is at most n.*

Hovey proved that if R is a Gorenstein ring, then $(\mathcal{GP}(\mathrm{Mod}\, R), \mathcal{P}^{<\infty}(\mathrm{Mod}\, R))$ is a cotorsion pair [23, Theorem 8.3], where $\mathcal{GP}(\mathrm{Mod}\, R)$ is the class of Gorenstein projective left R-modules and $\mathcal{P}^{<\infty}(\mathrm{Mod}\, R)$ is the class of left R-modules with finite projective dimension. Note that any finitely generated Gorenstein projective module is ∞-torsionfree in the sense [4] (cf. Definition 4.13). It is natural to ask when the class $\mathcal{T}(\mathrm{mod}\, \Lambda)$ of finitely generated ∞-torsionfree left Λ-modules and the class $\mathcal{P}^{<\infty}(\mathrm{mod}\, \Lambda)$ of finitely generated left Λ-modules with finite projective dimension form a cotorsion pair. This question is investigated in Section 4, whose main result is the following theorem on Noetherian algebras.

Theorem D. *If Λ is local, then the following statements are equivalent.*

(1) *Λ is Gorenstein.*

(2) *Both $(\mathcal{T}(\mathrm{mod}\, \Lambda), \mathcal{P}^{<\infty}(\mathrm{mod}\, \Lambda))$ and $(\mathcal{T}(\mathrm{mod}\, \Lambda^{op}), \mathcal{P}^{<\infty}(\mathrm{mod}\, \Lambda^{op}))$ are cotorsion pairs.*

The theorem above, which is included in Corollary 4.15, is an immediate consequence of Theorem 4.14.

Now, we briefly describe the structure of this paper. In Section 2, we give some terminology and preliminary results. In Section 3, we prove Theorem A (see Theorem 3.9). A natural question concerning Wakamatsu tilting modules is: when is a Wakamatsu tilting module free? As an application of Theorem A, we provide some conditions for a Wakamatsu tilting module with finite Gorenstein projective dimension to be free (Theorem B; cf. Theorem 3.15).

In Section 4, when Λ is semilocal, we characterize the finiteness of the left and right injective dimensions of a Wakamatsu tilting module C in terms of the finiteness of the C-Gorenstein projective dimension of left or right simples modules. Surprisingly, we prove in a dual manner that the Auslander projective and Bass injective dimensions of left or right simple modules are useful in characterizing the projective dimensions of a Wakamatsu tilting module (Theorem C; cf. Theorem 4.7). Furthermore, we obtain a characterization of Wakamatsu tilting modules with finite injective dimensions in terms of cotorsion pairs generated by the class of C-∞-torsionfree modules (Theorem 4.14).

2 Preliminaries

We begin with our convention, which is valid throughout the paper.

Convention 2.1. All rings are associative rings with unit and all modules are unital. For a ring Λ, we use $\operatorname{Mod}\Lambda$ ($\operatorname{mod}\Lambda$) to denote the category of (finitely generated) left Λ-modules, and use $\mathbf{D}^b(\Lambda)$ to denote the bounded derived category of $\operatorname{Mod}\Lambda$. For a module $M \in \operatorname{Mod}\Lambda$, we use $\operatorname{add}_\Lambda M$ to denote the subcategory of $\operatorname{mod}\Lambda$ consisting of all direct summands of finite direct sums of copies of M, and use $\operatorname{pd}_\Lambda M$ and $\operatorname{id}_\Lambda M$ to denote the projective and injective dimensions of M, respectively. We use J to denote the Jacobson radical of Λ. Recall that Λ is *semilocal* if Λ/J is semisimple. Throughout this paper, R is a commutative Noetherian ring with the Jacobson radical $\mathfrak{m}$.

Let Λ, Γ be rings and ${}_\Lambda U_\Gamma$ a bimodule. Let $M \in \operatorname{Mod}\Lambda$ and let

$$\sigma_M^U : M \to \operatorname{Hom}_{\Gamma^{op}}(\operatorname{Hom}_\Lambda(M,U),U)$$

via $\sigma_M^U(x)(f) = f(x)$ for any $x \in M$ and $f \in \operatorname{Hom}_\Lambda(M,U)$ be the canonical evaluation homomorphism. Recall from [1] that M is called *U-torsionless* if σ_M^U is a monomorphism, and is called *U-reflexive* if σ_M^U is an isomorphism. The following observation is a non-commutative version of [11, Proposition 1.1.9(b)].

Lemma 2.2. *Let ${}_\Lambda U_\Gamma$ be a bimodule and let M be a Noetherian left Λ-module, and set $(-)^u := \operatorname{Hom}(-,U)$. Then the following statements are equivalent.*
(1) *M is U-reflexive.*
(2) *$M \cong M^{uu}$.*

Proof. The implication (1) $\Longrightarrow$ (2) is trivial. Suppose that (2) holds true. It follows from [1, Proposition 20.14] that $\sigma_M^U \cong \sigma_{M^{uu}}^U$ is a split monomorphism. So there is a module $X \in \operatorname{Mod} R$ such that $M \cong M^{uu} \cong M \oplus X$, and hence

$$M \cong M \oplus X \cong M \oplus X^{(2)} \cong \cdots \cong M \oplus X^{(n)} \cong \cdots .$$

If M is not U-reflexive, then $X \neq 0$, and thus

$$X \subsetneqq X^{(2)} \subsetneqq \cdots \subsetneqq X^{(n)} \subsetneqq \cdots$$

is a strictly increasing chain of submodules of M, which contradicts that M is Noetherian. It follows that M is U-reflexive and (1) holds true. □

Next we recall the definitions of semidualizing bimodule and Wakamatsu tilting module.

Definition 2.3. ([22]). A (Λ,Γ)-bimodule ${}_\Lambda C_\Gamma$ is called *semidualizing* if the following conditions are satisfied:
(a1) ${}_\Lambda C$ admits a degreewise finite Λ-projective resolution.
(a2) C_Γ admits a degreewise finite Γ^{op}-projective resolution.
(b1) The homothety map $\chi_C^\Lambda : {}_\Lambda\Lambda_\Lambda \to \operatorname{Hom}_{\Gamma^{op}}(C,C)$ is an isomorphism.
(b2) The homothety map $\chi_C^\Gamma : {}_\Gamma\Gamma_\Gamma \to \operatorname{Hom}_\Lambda(C,C)$ is an isomorphism.
(c1) $\operatorname{Ext}_\Lambda^{\geqslant 1}(C,C) = 0$.
(c2) $\operatorname{Ext}_{\Gamma^{op}}^{\geqslant 1}(C,C) = 0$.

We refer to [6, 22] for more general examples of semidualizing bimodules.

Definition 2.4. ([45, 46]). A module $C \in \operatorname{Mod}\Lambda$ is called *Wakamatsu tilting* if the following conditions are satisfied:

(a) ${}_\Lambda C$ admits a degreewise finite Λ-projective resolution.
(b) $\operatorname{Ext}_\Lambda^{\geqslant 1}(C, C) = 0$.
(c) There is an exact sequence

$$0 \to {}_\Lambda\Lambda \to C^0 \to C^1 \to \cdots \to C^i \to \cdots$$

in $\operatorname{Mod}\Lambda$ with all C^i in $\operatorname{add}_\Lambda C$, which remains exact after applying the functor $\operatorname{Hom}_\Lambda(-, C')$ for any $C' \in \operatorname{add}_\Lambda C$.

Remark 2.5. For a bimodule ${}_\Lambda C_\Gamma$, it follows from [46, Corollary 3.2] that ${}_\Lambda C$ is Wakamatsu tilting with $\Gamma = \operatorname{End}_\Lambda C$ if and only if C_Γ is Wakamatsu tilting with $\Lambda = \operatorname{End}_\Gamma C$, and if and only if ${}_\Lambda C_\Gamma$ is semidualizing bimodule.

Based on the work of Holm and Jørgensen [21], the following notions were introduced and studied in non-commutative rings by Liu, Huang and Xu [36].

Definition 2.6. ([36]). Let ${}_\Lambda C$ be a Wakamatsu tilting module with $\Gamma = \operatorname{End}_\Lambda C$.

(i) A module $M \in \operatorname{Mod}\Lambda$ is called *C-Gorenstein projective* if the following conditions are satisfied:
  (1) $\operatorname{Ext}_\Lambda^{\geqslant 1}(M, C \otimes_\Gamma P) = 0$ for any projective left Γ-module P.
  (2) There is an exact sequence

$$0 \to M \to C \otimes_\Gamma P^0 \to C \otimes_\Gamma P^1 \to \cdots$$

  in $\operatorname{Mod}\Lambda$ with all P^i projective, which remains exact after applying the functor $\operatorname{Hom}_\Lambda(-, C \otimes_\Gamma P)$ for any projective left Γ-module P.

(ii) A module $M \in \operatorname{Mod}\Gamma$ is called *C-Gorenstein injective* if the following conditions are satisfied:
  (1) $\operatorname{Ext}_\Gamma^{\geqslant 1}(\operatorname{Hom}_\Lambda(C, I), M) = 0$ for any injective left Λ-module I.
  (2) There is an exact sequence

$$\cdots \to \operatorname{Hom}_\Lambda(C, I_1) \to \operatorname{Hom}_\Lambda(C, I_0) \to M \to 0$$

  in $\operatorname{Mod}\Gamma$ with all I_i injective, which remains exact after applying the functor $\operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(C, I), -)$ for any injective left Λ-module I.

If ${}_\Lambda C_\Gamma = {}_\Lambda\Lambda_\Lambda$, then C-Gorenstein projective and injective modules are exactly Gorenstein projective and injective modules, respectively. We write

$$\mathcal{GP}_\mathcal{C}(\operatorname{Mod}\Lambda) := \{C\text{-Gorenstein projective left } \Lambda\text{-modules}\},$$

$$\mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda) := \mathcal{GP}_\mathcal{C}(\operatorname{Mod}\Lambda) \cap \operatorname{mod}\Lambda,$$

$$\mathcal{GP}(\operatorname{mod}\Lambda) := \{\text{finitely generated Gorenstein projective left } \Lambda\text{-modules}\},$$

$$\mathcal{GI}_\mathcal{C}(\operatorname{Mod}\Gamma) := \{C\text{-Gorenstein injective left } \Gamma\text{-modules}\}.$$

From now on, Λ is a Noetherian R-algebra, we fix a Wakamatsu tilting module ${}_\Lambda C$ with $\Gamma = \operatorname{End}_\Lambda C$. For convenience, we write

$$(-)^\dagger := \operatorname{Hom}_\Lambda(-, C) \text{ and } (-)^* := \operatorname{Hom}_\Lambda(-, \Lambda).$$

We state the definition of totally C-reflexive modules and the connection with C-Gorenstein projective modules.

Definition 2.7. We say that $M \in \operatorname{mod}\Lambda$ is *totally C-reflexive* if the following conditions are satisfied:
(1) $\operatorname{Ext}_{\Lambda}^{\geqslant 1}(M, C) = 0$.
(2) $\operatorname{Ext}_{\Gamma^{op}}^{\geqslant 1}(M^{\dagger}, C) = 0$.
(3) M is C-reflexive.

Note that these modules are called *C-reflexive* in [3]. For and $M \in \operatorname{mod}\Lambda$, it follows from [48, Theorem 4.4] that M is C-Gorenstein projective if and only if M is totally C-reflexive.

Let $\mathcal{X}$ be a subcategory of $\operatorname{Mod}\Lambda$. Recall that $\mathcal{X}$ is called *resolving* if it contains all projective left Λ-modules, and it is closed under extensions and kernels of epimorphisms. Dually, the notion of *coresolving subcategories* is defined. Let $M \in \operatorname{Mod}\Lambda$. The *$\mathcal{X}$-projective dimension* $\mathcal{X}\text{-}\operatorname{pd}_{\Lambda} M$ of M is defined as $\inf\{n \mid$ there is an exact sequence

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod}\Lambda$ with all X_i in $\mathcal{X}\}$. If no such an integer exists, then set $\mathcal{X}\text{-}\operatorname{pd}_{\Lambda} M = \infty$. Dually, the notion of *$\mathcal{X}$-injective dimension* $\mathcal{X}\text{-}\operatorname{id}_{\Lambda} M$ of M is defined (cf. [26, 27]). When $\mathcal{X} = \mathcal{GP}_{\mathcal{C}}(\operatorname{Mod}\Lambda)$, the $\mathcal{X}$-projective dimension of M, denoted by $G_C\text{-}\operatorname{pd}_{\Lambda} M$, is called the *$C$-Gorenstein projective dimension* of M. For any $M \in \operatorname{Mod}\Gamma$, when $\mathcal{X} = \mathcal{GI}_{\mathcal{C}}(\operatorname{Mod}\Gamma)$, the $\mathcal{X}$-injective dimension of M, denoted by $G_C\text{-}\operatorname{id}_{\Gamma} M$, is called the *$C$-Gorenstein injective dimension* of M. In particular, we use $\operatorname{G-pd}_{\Lambda} M$ and $\operatorname{G-id}_{\Lambda} M$ to denote the Gorenstein projective and injective dimensions of M, respectively.

Let I be an ideal of R and $M \in \operatorname{Mod} R$. Recall from [15, Definition 2.3] that

$$\operatorname{depth}_R(I, M) := \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\}$$

is called the *I-depth* of M over R. We write $\operatorname{depth}_R(M)$ for the $\mathfrak{m}$-depth of M. In [19, Corollary 3.2], Goto and Nishida established a characterization of the depth of a non-zero finitely generated Λ-module over local rings. Using similar arguments, we can extend this result to semilocal rings as follows.

Lemma 2.8. *Let R be a semilocal ring and $0 \neq M \in \operatorname{mod}\Lambda$. Then*

$$\operatorname{depth}_R(M) = \inf\{i \geqslant 0 \mid \operatorname{Ext}_{\Lambda}^i(\Lambda/J, M) \neq 0\}.$$

Proof. Suppose $\operatorname{depth}_R(M) = d$. We will make use of induction on d. If $d = 0$, then

$$\operatorname{Hom}_{\Lambda}(\Lambda/\mathfrak{m}\Lambda, M) \cong \operatorname{Hom}_R(R/\mathfrak{m}, M) \neq 0.$$

Since $m \operatorname{Hom}_{\Lambda}(\Lambda/\mathfrak{m}\Lambda, M) = 0$, it follows that $\operatorname{Hom}_{\Lambda}(\Lambda/\mathfrak{m}\Lambda, M)$ has finite length over R, and hence it also has finite length over Λ by [31, Lemma 2.2(a)]. This implies $\operatorname{Hom}_{\Lambda}(\Lambda/J, M) \neq 0$. Now suppose $d > 0$. It follows from [8, Theorem 1.2.5] that there is an M-regular element $x \in \mathfrak{m}$ such that $\operatorname{depth}_R(M/xM) = d - 1$. By the induction hypothesis, we have

$$\inf\{i \geqslant 0 \mid \operatorname{Ext}_{\Lambda}^i(\Lambda/J, M/xM) \neq 0\} = d - 1.$$

Note that $x \operatorname{Ext}_{\Lambda}^i(\Lambda/J, M) = 0$ for any $i \geqslant 0$. Thus applying the functor $\operatorname{Hom}_{\Lambda}(\Lambda/J, -)$ to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

yields the following exact sequence

$$0 \to \operatorname{Ext}_{\Lambda}^{i-1}(\Lambda/J, M) \to \operatorname{Ext}_{\Lambda}^{i-1}(\Lambda/J, M/xM) \to \operatorname{Ext}_{\Lambda}^{i}(\Lambda/J, M) \to 0$$

for any $i \geqslant 1$. Thus $\operatorname{Ext}_{\Lambda}^{i-1}(\Lambda/J, M/xM) = 0$ implies $\operatorname{Ext}_{\Lambda}^{i}(\Lambda/J, M)$ for $0 \leqslant i < d$ and $\operatorname{Ext}_{\Lambda}^{d}(\Lambda/J, M) \cong \operatorname{Ext}_{\Lambda}^{d-1}(\Lambda/J, M/xM) \neq 0$. □

Let $M \in \operatorname{Mod} R$. We use $\operatorname{Ass}(M)$ (respectively, $\operatorname{Supp}(M)$) to denote the set of associated (respectively, support) prime ideals of M, and use $\operatorname{ann}_R(M)$ to denote the annihilator of M in R. We use $\operatorname{Spec}(R)$ and $\operatorname{Max}(R)$ to denote the prime and maximal spectra of R, respectively.

For commutative algebra we use the same notation and terminology as in [8, 37]. We refer to [34] for the theory of non-commutative rings.

3 Auslander–Bridger formula for C-Gorenstein projective dimension

This section deals with the C-Gorenstein projective dimension of finitely generated modules. The main result (Theorem 3.9) gives a relative Gorenstein projective version of the classical Auslander–Bridger formula over Noetherian algebras.

For any $M \in \operatorname{Mod}\Lambda$ or $\operatorname{Mod}\Lambda^{op}$ and $x \in R$, we write $\overline{M} := M \otimes_R R/(x)$. In the following result, we use Lemma 2.2 to achieve a Wakamatsu tilting $\overline{\Lambda}$-module $\overline{C}$ with $\overline{\Gamma} = \operatorname{End}_{\overline{\Lambda}}\overline{C}$.

Proposition 3.1. *If $x \in \mathfrak{m}$ is C-regular, then $\overline{C}$ is a Wakamatsu tilting $\overline{\Lambda}$-module with $\overline{\Gamma} = \operatorname{End}_{\overline{\Lambda}}\overline{C}$.*

Proof. Since x is C-regular, there is an exact sequence

$$0 \to C \xrightarrow{x} C \to \overline{C} \to 0$$

in $\operatorname{mod}\Gamma^{op}$. Applying the functor $\operatorname{Hom}_{\Gamma^{op}}(-, C)$ to this sequence yields that $\overline{\Lambda} \to \operatorname{Ext}^1_{\Gamma^{op}}(\overline{C}, C)$ is an isomorphism and $\operatorname{Ext}^{\geqslant 2}_{\Gamma^{op}}(\overline{C}, C) = 0$. It follows from [40, Theorem 8.34] that

$$\operatorname{Ext}^i_{\overline{\Gamma^{op}}}(\overline{C}, \overline{C}) \cong \operatorname{Ext}^{i+1}_{\Gamma^{op}}(\overline{C}, C) = 0$$

for any $i \geqslant 0$. Thus

$$\overline{\Lambda} \cong \operatorname{Ext}^1_{\Gamma^{op}}(\overline{C}, C) \cong \operatorname{Hom}_{\overline{\Gamma^{op}}}(\overline{C}, \overline{C}) \cong \operatorname{Hom}_{\overline{\Gamma^{op}}}(\operatorname{Hom}_{\overline{\Lambda}}(\overline{\Lambda}, \overline{C}), \overline{C})$$

and $\operatorname{Ext}^{\geqslant 1}_{\overline{\Gamma^{op}}}(\overline{C}, \overline{C}) = 0$ for any $i \geqslant 1$. By Lemma 2.2, we have that $\overline{\Lambda}$ is $\overline{C}$-reflexive. Note that $\overline{\Lambda}$ is $\overline{C}$-reflexive if and only if the homothety map $\chi^{\overline{\Lambda}}_{\overline{C}} : {}_{\overline{\Lambda}}\overline{\Lambda}_{\overline{\Lambda}} \to \operatorname{Hom}_{\overline{\Gamma}^{op}}(\overline{C}, \overline{C})$ is an isomorphism. Similarly, we get that $\operatorname{Ext}^{\geqslant 1}_{\overline{\Lambda}}(\overline{C}, \overline{C}) = 0$ and the homothety map $\chi^{\overline{\Gamma}}_{\overline{C}} : {}_{\overline{\Gamma}}\overline{\Gamma}_{\overline{\Gamma}} \to \operatorname{Hom}_{\overline{\Lambda}}(\overline{C}, \overline{C})$ is an isomorphism. We conclude that $\overline{C}$ is a Wakamatsu tilting $\overline{\Lambda}$-module with $\overline{\Gamma} = \operatorname{End}_{\overline{\Lambda}}\overline{C}$ by Remark 2.5. □

We need the following two lemmas.

Lemma 3.2. *It holds that*

(1) *Let $0 \neq M \in \operatorname{mod}\Lambda$. If M is C-torsionless, then* $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(C)$.

(2) *If $\Lambda = \Gamma$ and Λ is local, then* $\operatorname{depth}_R(\Lambda) = \operatorname{depth}_R(C)$.

Proof. (1) Since M is C-torsionless, there is a monomorphism $0 \to M \to C^n$ in $\operatorname{mod}\Lambda$. So $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(C^n) = \operatorname{Ass}_R(C)$.

(2) For any $r \geqslant 1$, it suffices to show that $\operatorname{depth}_R(\Lambda) \geqslant r$ if and only if $\operatorname{depth}_R(C) \geqslant r$. Note that there are isomorphisms

$$\operatorname{Hom}_\Lambda(\Lambda/J, \Lambda) \cong \operatorname{Hom}_\Lambda(\Lambda/J, \operatorname{Hom}_\Lambda(C, C)) \cong \operatorname{Hom}_\Lambda(C \otimes_\Lambda \Lambda/J, C) \cong \operatorname{Hom}_\Lambda(C, \operatorname{Hom}_{\Lambda^{op}}(\Lambda/J, C)).$$

Thus $\operatorname{Hom}_{\Lambda^{op}}(\Lambda/J, C) = 0$ implies $\operatorname{Hom}_{\Lambda}(\Lambda/J, \Lambda) = 0$. On the contrary, if $\operatorname{Hom}_{\Lambda}(\Lambda/J, \Lambda) = 0$, since $\operatorname{Hom}_{\Lambda^{op}}(\Lambda/J, C)$ is annihilated by J and Λ is local, we obtain $\operatorname{Hom}_{\Lambda^{op}}(\Lambda/J, C) = 0$. It follows from Lemma 2.8 that $\operatorname{depth}_R(\Lambda) \geqslant 1$ if and only if $\operatorname{depth}_R(C) \geqslant 1$.

Let $r \geqslant 2$. Since

$$\mathfrak{m} \not\subset \cup_{\mathfrak{p} \in \operatorname{Ass}_R(C) \cup \operatorname{Ass}_R(\Lambda)} \mathfrak{p} = \cup_{\mathfrak{p} \in \operatorname{Ass}_R(C)} \mathfrak{p},$$

there is an element $x \in \mathfrak{m}$ which is both Λ-regular and C-regular by the induction hypothesis. It follows from Proposition 3.1 that $\overline{C}$ is a Wakamatsu tilting $\overline{\Lambda}$-module with $\overline{\Lambda} = \operatorname{End}_{\overline{\Lambda}} \overline{C}$. Thus by induction hypothesis, we have that $\operatorname{depth}_R(\overline{\Lambda}) \geqslant r-1$ if and only if $\operatorname{depth}_R(\overline{C}) \geqslant r-1$, and therefore $\operatorname{depth}_R(\Lambda) \geqslant r$ if and only if $\operatorname{depth}_R(C) \geqslant r$. □

Lemma 3.3. ([37, p.140, Lemma 2]) or [39, Lemma 2.1]) *Let Λ be a Noetherian R-algebra, and let $M, N \in \operatorname{mod} \Lambda$ and $x \in \mathfrak{m}$. Suppose that x is both Λ-regular and M-regular and that $xN = 0$. Then $\operatorname{Ext}^n_{\Lambda}(M, N) \cong \operatorname{Ext}^n_{\overline{\Lambda}}(\overline{M}, N)$ for any $n \geqslant 0$.*

Before proceeding, we need to investigate the totally C-reflexive property over a quotient ring modulo an ideal generated by a regular element.

Proposition 3.4. *Let $M \in \operatorname{mod} \Lambda$ such that $M \neq 0$ and let $x \in \mathfrak{m}$ be a C-regular element. If M is totally C-reflexive, then $\overline{M}$ is totally $\overline{C}$-reflexive.*

Proof. By assumption, M is C-torsionless. It follows from Lemma 3.2 that x is both Λ-regular and M-regular. Since x is C-regular, there is an exact sequence

$$0 \to C \xrightarrow{x} C \to \overline{C} \to 0$$

in $\operatorname{mod} \Lambda$. Because $\operatorname{Ext}^{\geqslant 1}_{\Lambda}(M, C) = 0$, applying the functor $\operatorname{Hom}_{\Lambda}(M, -)$ yields the following exact sequence

$$0 \to \operatorname{Hom}_{\Lambda}(M, C) \xrightarrow{x} \operatorname{Hom}_{\Lambda}(M, C) \to \operatorname{Hom}_{\Lambda}(M, \overline{C}) \to 0.$$

This shows that x is $\operatorname{Hom}_{\Lambda}(M, C)$-regular. In addition, we have $\operatorname{Ext}^i_{\overline{\Lambda}}(\overline{M}, \overline{C}) \cong \operatorname{Ext}^i_{\Lambda}(M, \overline{C}) = 0$ for any $i \geqslant 1$ and $\operatorname{Hom}_{\Lambda}(M, \overline{C}) \cong \operatorname{Hom}_{\overline{\Lambda}}(\overline{M}, \overline{C})$ by Lemma 3.3. Thus

$$\overline{\operatorname{Hom}_{\Lambda}(M, C)} \cong \operatorname{Hom}_{\Lambda}(M, \overline{C}) \cong \operatorname{Hom}_{\overline{\Lambda}}(\overline{M}, \overline{C}).$$

Similarly, since $\operatorname{Ext}^{\geqslant 1}_{\Gamma^{op}}(\operatorname{Hom}_{\Lambda}(M, C), C) = 0$, we have

$$\operatorname{Ext}^i_{\overline{\Gamma^{op}}}(\operatorname{Hom}_{\overline{\Lambda}}(\overline{M}, \overline{C}), \overline{C}) \cong \operatorname{Ext}^i_{\Gamma^{op}}(\operatorname{Hom}_{\Lambda}(M, C), \overline{C}) = 0$$

for any $i \geqslant 1$ and

$$\overline{\operatorname{Hom}_{\Gamma^{op}}(\operatorname{Hom}_{\Lambda}(M, C), C)} \cong \operatorname{Hom}_{\overline{\Gamma^{op}}}(\overline{\operatorname{Hom}_{\Lambda}(M, C)}, \overline{C}) \cong \operatorname{Hom}_{\overline{\Gamma^{op}}}(\operatorname{Hom}_{\overline{\Lambda}}(\overline{M}, \overline{C}), \overline{C}).$$

Then the following commutative diagram

$$\begin{array}{ccc}
\overline{M} & \xrightarrow{\sigma^{\overline{C}}_{\overline{M}}} & \operatorname{Hom}_{\overline{\Gamma^{op}}}(\operatorname{Hom}_{\overline{\Lambda}}(\overline{M}, \overline{C}), \overline{C}) \\
\downarrow{\scriptstyle \overline{\sigma^{C}_{M}}} & & \downarrow{\scriptstyle \cong} \\
\overline{\operatorname{Hom}_{\Gamma^{op}}(\operatorname{Hom}_{\Lambda}(M, C), C)} & \xrightarrow{\cong} & \operatorname{Hom}_{\overline{\Gamma^{op}}}(\overline{\operatorname{Hom}_{\Lambda}(M, C)}, \overline{C})
\end{array}$$

implies that $\sigma^{\overline{C}}_{\overline{M}}$ is an isomorphism. This completes the proof. □

Combining Lemma 3.2 with Proposition 3.4 we get the following result.

Corollary 3.5. *Let $M \in \operatorname{mod}\Lambda$. If $x \in \mathfrak{m}$ is both C-regular and M-regular, then $G_{\overline{C}}\text{-pd}_{\overline{\Lambda}}\,\overline{M} \leqslant G_C\text{-pd}_\Lambda\, M$.*

Proof. If $G_C\text{-pd}_\Lambda\, M = \infty$, then the assertion holds true obviously. Suppose $G_C\text{-pd}_\Lambda\, M = n < \infty$. By [3, Lemma 2.1], there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

in $\operatorname{mod}\Lambda$ with G_i totally C-reflexive. As x is C-regular, it is also Λ-regular by Lemma 3.2. Thus we deduce from the proof of [11, Lemma 1.3.4(1)] that $\operatorname{Tor}^\Lambda_{\geqslant 1}(\overline{\Lambda}, M) = 0$. So applying the functor $\overline{\Lambda} \otimes_\Lambda -$ to the above exact sequence yields the following exact sequence

$$0 \to \overline{G_n} \to \overline{G_{n-1}} \to \cdots \to \overline{G_1} \to \overline{G_0} \to \overline{M} \to 0.$$

Now $\overline{G_i}$ is totally $\overline{C}$-reflexive by Proposition 3.4, and thus $G_{\overline{C}}\text{-pd}_{\overline{\Lambda}}\,\overline{M} \leqslant n$ as desired. □

Next our aim is to establish Theorem A, advertised in the introduction. First we prove the following result.

Lemma 3.6. *Assume that $T^\dagger \neq 0$ for any simple Γ^{op}-module T. Then for any $M \in \operatorname{mod}\Lambda$, the following statements are equivalent.*
(1) $G_C\text{-pd}_\Lambda\, M = 0$.
(2) $G_C\text{-pd}_\Lambda\, M < \infty$.

Proof. (1) $\Longrightarrow$ (2) It is trivial.

(2) $\Longrightarrow$ (1) Suppose $G_C\text{-pd}_\Lambda\, M = n < \infty$. The case for $n = 0$ is trivial. If $n = 1$, there is an exact sequence

$$0 \to G_1 \to G_0 \to M \to 0$$

in $\operatorname{mod}\Lambda$ with G_0 and G_1 totally C-reflexive. Applying the functor $(-)^\dagger$ to it yields the following exact sequence

$$0 \to M^\dagger \to G_0^\dagger \to G_1^\dagger \to \operatorname{Ext}^1_\Lambda(M, C) \to 0$$

in $\operatorname{mod}\Gamma^{op}$ and $\operatorname{Ext}^{>1}_\Lambda(M, C) = 0$. Since both $\sigma^C_{G_0}$ and $\sigma^C_{G_1}$ are isomorphisms, it follows from the following commutative diagram

$$\begin{array}{ccccc} & & 0 \longrightarrow G_1 & \longrightarrow & G_0 \\ & & \big\downarrow {\scriptstyle \sigma^C_{G_1}} & & \big\downarrow {\scriptstyle \sigma^C_{G_0}} \\ 0 \longrightarrow \operatorname{Ext}^1_\Lambda(M, C)^\dagger & \longrightarrow & G_1^{\dagger\dagger} & \xrightarrow{\cong} & G_0^{\dagger\dagger} \end{array}$$

that $\operatorname{Ext}^1_\Lambda(M, C)^\dagger = 0$. If $\operatorname{Ext}^1_\Lambda(M, C) \neq 0$, there is an epimorphism $f : \operatorname{Ext}^1_\Lambda(M, C) \to T$ in $\operatorname{mod}\Gamma^{op}$ with T simple, and so $T^\dagger = 0$, contradicting the assumption. Thus $\operatorname{Ext}^1_\Lambda(M, C) = 0$, which implies $G_C\text{-pd}_\Lambda\, M = 0$ by [3, Theorem 2.2]. Now suppose that $n > 1$ and

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

is an exact sequence in $\operatorname{mod}\Lambda$ with all G_i totally C-reflexive. Set $M' := \operatorname{Im}(G_{n-1} \to G_{n-2})$. Since $G_C\text{-pd}_\Lambda\, M' \leqslant 1$, we have $G_C\text{-pd}_\Lambda\, M' = 0$ by the above argument. Repeating this process, we finally get $G_C\text{-pd}_\Lambda\, M = 0$. □

For a Noetherian R-algebra Λ, in order to study the problem when the injective dimension of a finitely generated module with finite injective dimension is equal to $\operatorname{depth}_R(\Lambda)$, Goto and Nishida in [18] introduced the following "homogeneity condition":

(hc) $\operatorname{Ext}^t_\Lambda(S, \Lambda) \neq 0$ for any simple left Λ-module S, where $t = \operatorname{depth}_R(\Lambda)$.

Following Lemma 3.6, it is natural to introduce the following condition, which extends the above condition to the relative setting.

For $0 \neq M \in \operatorname{mod}\Lambda$, we say that M satisfies the *condition* (hc^Λ_M) if $\operatorname{Ext}^t_\Lambda(S, C) \neq 0$ for any simple left Λ-module S, where $t = \operatorname{depth}_R(M)$. This condition will play an important role in the sequel. In addition, if Λ is local, then by Lemma 2.8 we infer that M satisfies the condition (hc^Λ_M) automatically. Now we provide an alternative characterization of the condition (hc^Λ_M).

Proposition 3.7. *Let $0 \neq M \in \operatorname{mod}\Lambda$ with $\operatorname{depth}_R(M) = t$. Then the following statements are equivalent.*

(1) *M satisfies the condition (hc^Λ_M).*

(2) *If*

$$I^\bullet := 0 \to M \to I^0_\Lambda(M) \xrightarrow{d^0} I^1_\Lambda(M) \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} I^i_\Lambda(M) \xrightarrow{d^i} \cdots$$

is a minimal injective resolution of M, then $E^t_\Lambda(M)$ contains S for any simple left Λ-module S.

Proof. We claim that the complex $\operatorname{Hom}_\Lambda(S, I^\bullet)$ has zero differentials for any simple left Λ-module S. If $f : S \to I^i_\Lambda(M)$ is some non-zero homomorphism, then f is monic since S is simple. Thus $I^i_\Lambda(M)$ contains S and the injective envelope $I_\Lambda(S)$ of S is a direct summand of $I^i_\Lambda(M)$. If $d^i f \neq 0$, then $d^i f$ is monic, and so $I^{i+1}_\Lambda(M)$ contains S and $I_\Lambda(S)$ is also a direct summand of $I^{i+1}_\Lambda(M)$. Thus $I_\Lambda(S)$ appears as a common direct summand of $I^i_\Lambda(M)$ and $I^{i+1}_\Lambda(M)$ under d^i, contradicting the minimality of $I^\bullet$. The claim is proved. Now the assertion follows easily from this claim. □

Proposition 3.8. *Let $0 \neq M \in \operatorname{mod}\Lambda$ with $G_C\text{-pd}_\Lambda M < \infty$. If Λ is semilocal and C satisfies the condition $(hc^{\Gamma^{op}}_C)$, then the following statements are equivalent.*

(1) $G_C\text{-pd}_\Lambda M = 0$.

(2) Every C-regular sequence is M-regular sequence.

(3) $\operatorname{depth}_R(M) \geqslant \operatorname{depth}_R(C)$.

(4) $\operatorname{depth}_R(M) = \operatorname{depth}_R(C)$.

Proof. By assumption, there is a commutative Noetherian semilocal ring R such that Λ is a Noetherian R-algebra.

(1) $\Longrightarrow$ (2) It follows from Lemma 3.2 and Proposition 3.4.

(2) $\Longrightarrow$ (3) It is obvious.

(3) $\Longrightarrow$ (4) We proceed by induction on $t := \operatorname{depth}_R(C)$. When $t = 0$, we have $G_C\text{-pd}_\Lambda M = 0$ by Lemma 3.6. As $M \neq 0$, we have $M^\dagger \neq 0$. Thus there is an epimorphism $f : M^\dagger \to T$ in $\operatorname{mod}\Gamma^{op}$ with T simple. By the condition $(hc^{\Gamma^{op}}_C)$, there is a monomorphism $0 \to T \to C$ in $\operatorname{mod}\Gamma^{op}$. Applying the functor $\operatorname{Hom}_{\Gamma^{op}}(M^\dagger, -)$ to it yields an exact sequence

$$0 \to \operatorname{Hom}_{\Gamma^{op}}(M^\dagger, T)(\neq 0) \to \operatorname{Hom}_{\Gamma^{op}}(M^\dagger, C) \cong M.$$

Thanks to [34, Proposition 20.6], we get $\mathfrak{m}\operatorname{Hom}_{\Gamma^{op}}(M^\dagger, T) = 0$, and thus $\operatorname{depth}_R(M) = 0$.

Now let $t \geqslant 1$. Since $\operatorname{depth}_R(M) \geqslant \operatorname{depth}_R(C) = t \geqslant 1$, there is an element $x \in \mathfrak{m}$ which is both C-regular and M-regular. By Corollary 3.5, we have $G_{\overline{C}}\text{-pd}_{\overline{\Lambda}} \overline{M} < \infty$. Since

$$\operatorname{depth}_R(\overline{M}) = \operatorname{depth}_R(M) - 1 \geqslant \operatorname{depth}_R(C) - 1 = \operatorname{depth}_R(\overline{C}),$$

we have $\operatorname{depth}_R(\overline{M}) = \operatorname{depth}_R(\overline{C})$ by the induction hypothesis. Thus $\operatorname{depth}_R(M) = \operatorname{depth}_R(C)$.

(4) $\Longrightarrow$ (1) By assumption, it suffices to prove $\operatorname{Ext}_\Lambda^{\geqslant 1}(M, C) = 0$. We also proceed by induction on $t := \operatorname{depth}_R(C)$. If $t = 0$, then $G_C\text{-pd}_\Lambda M = 0$ by Lemma 3.6. If $t \geqslant 1$, then $\operatorname{depth}_R(M) = \operatorname{depth}_R(C) \geqslant 1$, and hence there is an element $x \in \mathfrak{m}$ which is both C-regular and M-regular. Since

$$\operatorname{depth}_R(\overline{M}) = \operatorname{depth}_R(M) - 1 = \operatorname{depth}_R(C) - 1 = \operatorname{depth}_R(\overline{C}),$$

we have $\operatorname{Ext}_{\overline{\Lambda}}^{\geqslant 1}(\overline{M}, \overline{C}) = 0$ by the induction hypothesis. This gives $\operatorname{Ext}_\Lambda^{\geqslant 1}(M, \overline{C}) = 0$ by Lemma 3.3. From the exact sequence

$$0 \to C \xrightarrow{x} C \to \overline{C} \to 0$$

in $\operatorname{mod} \Lambda$, we get the following exact sequence

$$\operatorname{Ext}_\Lambda^i(M, C) \xrightarrow{x} \operatorname{Ext}_\Lambda^i(M, C) \to \operatorname{Ext}_\Lambda^i(M, \overline{C}) = 0$$

for any $i \geqslant 1$. It follows from the Nakayama lemma that $\operatorname{Ext}_\Lambda^{\geqslant 1}(M, C) = 0$. □

We are now in a position to prove the following theorem, which generalizes [38, Proposition 3.4].

Theorem 3.9. *Let $0 \neq M \in \operatorname{mod} \Lambda$ with $G_C\text{-pd}_\Lambda M < \infty$. If Λ is semilocal and C satisfies the condition $(hc_C^{\Gamma^{op}})$, then*

$$G_C\text{-pd}_\Lambda M + \operatorname{depth}_R(M) = \operatorname{depth}_R(C).$$

Proof. We proceed by induction on $G_C\text{-pd}_\Lambda M$. The case for $G_C\text{-pd}_\Lambda M = 0$ follows from Proposition 3.8. Now suppose $G_C\text{-pd}_\Lambda M = n \geqslant 1$. Then there is an exact sequence

$$0 \to K \to G \to M \to 0$$

in $\operatorname{mod} \Lambda$ with G being C-Gorenstein projective and $G_C\text{-pd}_\Lambda K = n - 1$. By the induction hypothesis, we have

$$G_C\text{- pd}_\Lambda K + \operatorname{depth}_R(K) = \operatorname{depth}_R(C).$$

Since $G_C\text{-pd}_\Lambda G = 0$ and $G_C\text{-pd}_\Lambda M = n \geqslant 1$, it follows from Proposition 3.8 that $\operatorname{depth}_R(G) = \operatorname{depth}_R(C)$ and $\operatorname{depth}_R(M) < \operatorname{depth}_R(C) = \operatorname{depth}_R(G)$. Thus $\operatorname{depth}_R(K) = \operatorname{depth}_R(M) + 1$ by [8, Proposition 1.2.9], and therefore the assertion follows. □

It should be pointed out that the condition $(hc_C^{\Gamma^{op}})$ is necessary for Theorem 3.9 as shown in the following example.

Example 3.10. Let Λ be a finite dimensional K-algebra over an algebraically closed field K given be the quiver

$$1 \longrightarrow 2.$$

For $i = 1, 2$, we use $P(i)$ and $S(i)$ to denote the projective and simple modules corresponding to the vertex i, respectively. Take $C := \Lambda$ and $M := S(1)$. Since $\operatorname{depth}_K(\Lambda) = 0$ and $\operatorname{Hom}_{\Lambda^{op}}(S(2), \Lambda) = 0$, it means that Λ does not satisfy the condition $(hc_\Lambda^{\Lambda^{op}})$. On the other hand, because $\operatorname{depth}_K(M) = 0$ and $\text{G-pd}_\Lambda M = \operatorname{pd}_\Lambda M = 1$, the equation in Theorem 3.9 does not hold true.

Corollary 3.11. *If Λ is semilocal and C satisfies the condition $(hc_C^{\Gamma^{op}})$, then* $\operatorname{depth}_R(\Lambda) = \operatorname{depth}_R(C)$.

Proof. Since G_C-$\operatorname{pd}_\Lambda \Lambda = 0$, we have $\operatorname{depth}_R(\Lambda) = \operatorname{depth}_R(C)$ by Theorem 3.9. □

The following corollary extends [39, Theorem 1.10].

Corollary 3.12. *Let $0 \neq M \in \operatorname{mod}\Lambda$ with G_C-$\operatorname{pd}_\Lambda M < \infty$. If Λ is semilocal and C satisfies the condition $(hc_C^{\Gamma^{op}})$, then*

$$G_C\text{-}\operatorname{pd}_\Lambda M + \operatorname{depth}_R(M) = \operatorname{depth}_R(\Lambda).$$

Proof. It follows from Theorem 3.9 and Corollary 3.11. □

Proposition 3.13. *Suppose that Λ is semilocal,* $\operatorname{depth}_R(\Lambda) = 0$ *and Λ satisfies the condition $(hc_\Lambda^{\Lambda^{op}})$. If* $\text{G-}\operatorname{pd}_\Lambda C < \infty$*, then $C \cong \Lambda$.*

Proof. Since $\operatorname{depth}_R(\Lambda) = 0$, we have $\text{G-}\operatorname{pd}_\Lambda C = 0$ by Theorem 3.9. In view of [44, Proposition 2.2], there are the following two spectral sequences with the same limit

$$E_2^{p,q} = \operatorname{Ext}^p_{\Gamma^{op}}(C, \operatorname{Ext}^q_{\Lambda^{op}}(C^*, \Lambda)) \Rightarrow H^n, \tag{3.1}$$

$$E_2^{p,q} = \operatorname{Ext}^p_{\Lambda^{op}}(\operatorname{Tor}^\Gamma_q(C, C^*), \Lambda) \Rightarrow H^n. \tag{3.2}$$

Then we have a filtration $\{\phi^j H^i\}$ of H^i satisfying

$$0 = \phi^{i+1}H^i \subseteq \phi^i H^i \subseteq \cdots \subseteq \phi^1 H^i \subseteq \phi^0 H^i = H^i$$

with $E_\infty^{j,i-j} \cong \phi^j H^i/\phi^{j+1}H^i$ for all i and j. Since C is Gorenstein projective, it follows that $\operatorname{Ext}^{\geqslant 1}_{\Lambda^{op}}(C^*, \Lambda) = 0$, and hence

$$0 = E_2^{0,1} = E_\infty^{0,1} \cong \phi^0 H^1/\phi^1 H^1$$

by (3.1). Since $E_2^{1,0} = \operatorname{Ext}^1_{\Gamma^{op}}(C, C) = 0$ by (3.1) again, we get

$$0 = E_2^{1,0} = E_\infty^{1,0} \cong \phi^1 H^1/\phi^2 H^1.$$

So $H^1 = 0$, and hence $\phi^1 H^1 = 0$. Furthermore, it follows from (3.2) that

$$\operatorname{Ext}^1_{\Lambda^{op}}(C \otimes_\Gamma C^*, \Lambda) = E_2^{1,0} = E_\infty^{1,0} \cong \phi^1 H^1/\phi^2 H^1 = 0.$$

Set $M := C \otimes_\Gamma C^*$. Then

$$M^* = (C \otimes_\Gamma C^*)^* \cong \operatorname{Hom}_{\Gamma^{op}}(C, C) \cong \Lambda.$$

Applying the functor $(-)^*$ to the following exact sequence

$$0 \to M_1 \to P \to M \to 0$$

in $\operatorname{mod}\Lambda^{op}$ with P projective yields an exact sequence

$$0 \to M^* \to P^* \to M_1^* \to 0. \tag{3.3}$$

Since M_1^* has finite projective dimension and $\operatorname{depth}_R(\Lambda) = 0$, we have that M_1^* is Gorenstein projective by Corollary 3.12, and hence the exact sequence (3.3) splits. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \cong \downarrow \sigma_P^\Lambda & & \downarrow \sigma_M^\Lambda & & \\
0 & \longrightarrow & M_1^{**} & \longrightarrow & P^{**} & \longrightarrow & M^{**} & \longrightarrow & 0.
\end{array}$$

It follows that $\sigma_M^\Lambda : M \to M^{**}$ is surjective. As $M^* \cong \Lambda$, this implies that there is a module $K \in \operatorname{mod} \Lambda^{op}$ such that $M \cong M^{**} \oplus K$, and hence

$$M^* \cong M^{***} \oplus K^* \cong M^* \oplus K^*.$$

Since Λ has invariant basis number, we have $K^* = 0$. If $K \neq 0$, then there is an epimorphism $f : K \to S$ in $\operatorname{mod} \Lambda^{op}$ with S simple. It follows that $S^* = 0$, which contradicts the assumption. Thus $K = 0$ and $M \cong M^{**} \cong \Lambda$. Therefore $C \cong \Lambda$ by [5, Proposition A.1]. □

Recalled from [35] that a local ring R is said to be *Henselian* provided, for every Noetherian R-algebra Λ (not necessarily commutative), each idempotent of Λ/J lifts to an idempotent of Λ. A slight modification of the proof of [35, Theorem 1.8] allows us to prove the following lemma.

Lemma 3.14. *If R is a Henselian local ring and $M \in \operatorname{mod} \Lambda$, then M is indecomposable if and only if $\Gamma' := \operatorname{End}_\Lambda M$ is a local ring.*

Proof. The sufficiency follows from [34, Section 19]. In the following, we prove the necessity.

We write $J(\Gamma')$ for the Jacobson radical of Γ'. Since $\mathfrak{m}\Gamma' \subseteq J(\Gamma')$ by [34, Proposition 20.6], we have that $\Gamma'/J(\Gamma')$ is a finite-dimensional k-algebra with $k = R/\mathfrak{m}$. Thus $\Gamma'/J(\Gamma')$ is semisimple by [1, Proposition 15.17]. Moreover, since M is indecomposable, it follows that Γ' has no non-trivial idempotents. Moreover, since R is Henselian local ring, each idempotent of $\Gamma'/J(\Gamma')$ lifts to an idempotent of Γ'. Thus $\Gamma'/J(\Gamma')$ has no non-trivial idempotents, which implies that $\Gamma'/J(\Gamma')$ is a division ring by Wedderburn-Artin Theorem, and therefore Γ' is a local ring by [34, Theorem 19.1]. □

As an application of Proposition 3.13, we give a non-commutative version of [32, Proposition].

Theorem 3.15. *Let Λ be local and $\operatorname{G-pd}_\Lambda C < \infty$. Then $C \cong \Lambda$ if one of the following two conditions is satisfied:*

(1) *$\Lambda = \Gamma$.*

(2) *R is a Henselian local ring and C is indecomposable as Λ-module.*

Proof. The assertion that $\operatorname{depth}_R(\Lambda) = \operatorname{depth}_R(C)$ follows from Lemma 3.2, Corollary 3.11 and Lemma 3.14 if either (1) or (2) holds.

If $\operatorname{depth}_R(\Lambda) \geqslant 1$, then by Lemma 3.2, there is an element x in $\mathfrak{m}$ such that it is both Λ-regular and C-regular. It follows from Proposition 3.1 and Corollary 3.5 that $\overline{C}$ is a Wakamatsu tilting $\overline{\Lambda}$-module with $\overline{\Lambda} = \operatorname{End}_{\overline{\Lambda}} \overline{C}$ and $\operatorname{G-pd}_{\overline{\Lambda}} \overline{C} \leqslant \operatorname{G-pd}_\Lambda C < \infty$. Thus, by using induction on $\operatorname{depth}_R(\Lambda)$, we may assume $\operatorname{depth}_R(\Lambda) = 0$. Then the result follows from Proposition 3.13. □

4 Injective and projective dimensions of C

In this section we discuss the projective (respectively, injective) dimension of a Wakamatsu tilting module C in connection with the homological behavior of the Auslander and Bass classes (respectively, C-Gorenstein projective modules).

Definition 4.1. ([22]). The *Auslander class* $\mathcal{A}_C(\Gamma)$ with respect to C consists of all left Γ-modules N satisfying the following conditions:

(A1) $\operatorname{Tor}^\Gamma_{\geqslant 1}(C, N) = 0$.

(A2) $\operatorname{Ext}_\Lambda^{\geqslant 1}(C, C \otimes_\Gamma N) = 0$.

(A3) The canonical evaluation homomorphism

$$\mu_N : N \to \mathrm{Hom}_\Lambda(C, C \otimes_\Gamma N)$$

defined by $\mu_N(x)(c) = c \otimes x$ for any $x \in N$ and $c \in C$ is an isomorphism in $\mathrm{Mod}\,\Gamma$.

The *Bass class* $\mathcal{B}_C(\Lambda)$ with respect to C consists of all left Λ-modules M satisfying the following conditions:

(B1) $\mathrm{Ext}_\Lambda^{\geqslant 1}(C, M) = 0$.

(B2) $\mathrm{Tor}^\Gamma_{\geqslant 1}(C, \mathrm{Hom}_\Lambda(C, M)) = 0$.

(B3) The canonical evaluation homomorphism

$$\theta_M : C \otimes_\Gamma \mathrm{Hom}_\Lambda(C, M) \to M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in \mathrm{Hom}_\Lambda(C, M)$ is an isomorphism in $\mathrm{Mod}\,\Lambda$.

Symmetrically, the Auslander class in $\mathrm{Mod}\,\Lambda^{op}$ and the Bass class in $\mathrm{Mod}\,\Gamma^{op}$ are defined.

Definition 4.2. A full subcategory $\mathcal{A}_C^\bullet(\Gamma)$ of $\mathbf{D}^b(\Gamma)$ consisting of complexes $N^\bullet$ is called the *Auslander class* with respect to C if the following conditions are satisfied:

(1) $C \otimes_\Gamma^{\mathbf{L}} N^\bullet \in \mathbf{D}^b(\Lambda)$.

(2) The natural morphism $N^\bullet \to \mathbf{R}\,\mathrm{Hom}_\Lambda(C, C \otimes_\Gamma^{\mathbf{L}} N^\bullet)$ is an isomorphism in $\mathbf{D}^b(\Gamma)$.

The proof of the following lemma is similar to that of [41, Lemma 5.3].

Lemma 4.3. *For any $N \in \mathrm{Mod}\,\Gamma$, the following statements are equivalent.*

(1) $\mathcal{A}_C(\Gamma)\text{-pd}_\Gamma N < \infty$.

(2) $N \in \mathcal{A}_C^\bullet(\Gamma)$.

Proof. (1) $\Longrightarrow$ (2) Since $\mathcal{A}_C(\Gamma)\text{-pd}_\Gamma N < \infty$, there is an exact sequence

$$0 \to A_n \to \cdots \to A_1 \to A_0 \to N \to 0$$

in $\mathrm{Mod}\,\Gamma$ with A_i in $\mathcal{A}_C(\Gamma)$. Note from the definition that any Γ-module in $\mathcal{A}_C(\Gamma)$ belongs to $\mathcal{A}_C^\bullet(\Gamma)$. Since $\mathcal{A}_C^\bullet(\Gamma)$ is closed under taking triangles, we have $N \in \mathcal{A}_C^\bullet(\Gamma)$.

(2) $\Longrightarrow$ (1) Suppose $N \in \mathcal{A}_C^\bullet(\Gamma)$ and $P^\bullet$ is a projective resolution of N. Since $C \otimes_\Gamma^{\mathbf{L}} N \in \mathbf{D}^b(\Lambda)$, the supremum $s := \sup\{n \mid H_n(C \otimes_\Gamma^{\mathbf{L}} N) \neq 0\}$ is finite. Consider the truncated complex

$$\tau P^\bullet := \cdots \to P_{s+2} \to P_{s+1} \to P_s \to 0 \to 0 \to \cdots.$$

It is easy to verify that

$$H_i(C \otimes_\Gamma^{\mathbf{L}} \tau P^\bullet[-s]) \cong H_{i+s}(C \otimes_\Gamma^{\mathbf{L}} N) = 0$$

for any $i \geqslant 1$. Since $\tau P^\bullet[-s]$ is isomorphic to the s-th syzygy module $\Omega^s N$ of N, it follows that $\mathrm{Tor}^\Gamma_{\geqslant 1}(C, \Omega^s N) = 0$. Since $N \in \mathcal{A}_C^\bullet(\Gamma)$, we have $\tau P^\bullet[-s] \in \mathcal{A}_C^\bullet(\Gamma)$ as well. Thus the natural morphism $\Omega^s N \to \mathbf{R}\,\mathrm{Hom}_\Lambda(C, C \otimes_\Gamma \Omega^s N)$ is an isomorphism, equivalently $\Omega^s N \cong \mathrm{Hom}_\Lambda(C, C \otimes_\Gamma \Omega^s N)$ and $\mathrm{Ext}_\Lambda^{\geqslant 1}(C, C \otimes_\Gamma \Omega^s N) = 0$. Consequently, $\Omega^s N$ is in $\mathcal{A}_C(\Gamma)$ and $\mathcal{A}_C(\Gamma)\text{-pd}_\Gamma N \leqslant s < \infty$. □

Proposition 4.4. ([25, Theorem 4.4]) *Let $N \in \mathrm{Mod}\,\Gamma$ with $\mathcal{A}_C(\Gamma)\text{-pd}_\Gamma N < \infty$ and $n \geqslant 0$. Then the following statements are equivalent.*

(1) $\mathcal{A}_C(\Gamma)\text{-pd}_\Gamma N \leqslant n$.

(2) $\operatorname{Tor}^{\Gamma}_{\geqslant n+1}(C,N)=0$.

Lemma 4.5. *Let M be a finite length Λ-module.*

(1) *Assume that $\mathcal{X}$ is a resolving subcategory of* $\operatorname{Mod}\Lambda$. *If $\mathcal{X}\text{-pd}_\Lambda S<\infty$ for any simple Λ-module S, then $\mathcal{X}\text{-pd}_\Lambda M<\infty$.*

(2) *Assume that $\mathcal{Y}$ is a coresolving subcategory of* $\operatorname{Mod}\Lambda$. *If $\mathcal{Y}\text{-id}_\Lambda S<\infty$ for any simple Λ-module S, then $\mathcal{Y}\text{-id}_\Lambda M<\infty$.*

Proof. We only prove (1) since (2) is dual. Setting $l:=\operatorname{length}_\Lambda M<\infty$, we verify the finiteness of $\mathcal{X}\text{-pd}_\Lambda M$ by induction on l. If $l=1$, then M is simple and thus $\mathcal{X}\text{-pd}_\Lambda M<\infty$ by assumption. If $l>1$, then there is an exact sequence

$$0\to M_1\to M\to M_2\to 0$$

in $\operatorname{Mod}\Lambda$ with $\operatorname{length}_\Lambda M_1=l-1$ and $\operatorname{length}_\Lambda M_2=1$. By the induction hypothesis, we obtain $\mathcal{X}\text{-pd}_\Lambda M_1<\infty$ and $\mathcal{X}\text{-pd}_\Lambda M_2<\infty$. In view of [26, Theorem 3.2], we have $\mathcal{X}\text{-pd}_\Lambda M<\infty$. □

Now we prove Theorem C from Section 1; our proof hinges on the following lemma.

Lemma 4.6. *Let Λ be semilocal.*

(i) *Assume that $\operatorname{Ext}^i_\Lambda(-,C)$ vanishes on* $\operatorname{mod}\Lambda$ *for $i\gg 0$. Then the following statements are equivalent.*
  (1) *$G_C\text{-pd}_\Lambda S<\infty$ for any simple Λ-module S.*
  (2) *$G_C\text{-pd}_\Lambda M<\infty$ for any $M\in\operatorname{mod}\Lambda$.*

(ii) *Assume that $\operatorname{Ext}^i_\Lambda(C,-)$ vanishes on* $\operatorname{mod}\Lambda$ *for $i\gg 0$. Then the following statements are equivalent.*
  (1′) *$\mathcal{B}_C(\Lambda)\text{-id}_\Lambda S<\infty$ for any simple Λ-module S.*
  (2′) *$\mathcal{B}_C(\Lambda)\text{-id}_\Lambda M<\infty$ for any $M\in\operatorname{mod}\Lambda$.*

(iii) *Assume that $\operatorname{Tor}^\Gamma_i(C,-)$ vanishes on* $\operatorname{mod}\Gamma$ *for $i\gg 0$. Then the following statements are equivalent.*
  (1″) *$\mathcal{A}_C(\Gamma)\text{-pd}_\Gamma T<\infty$ for any simple Γ-module T.*
  (2″) *$\mathcal{A}_C(\Gamma)\text{-id}_\Gamma N<\infty$ for any $N\in\operatorname{mod}\Gamma$.*

Proof. By assumption, there is a commutative Noetherian semilocal ring R such that Λ is a Noetherian R-algebra. Let $M\in\operatorname{mod}\Lambda$ and set $d:=\dim_R M$, the Krull dimension of M as an R-module. The set

$$M':=\{x\in M\mid \mathfrak{m}^n x=0 \text{ for some } n>0\}$$

is a Λ-submodule of M, which has finite length over R, and hence it also has finite length over Λ by [31, Lemma 2.2(a)]. Then we get an exact sequence

$$0\to M'\to M\to M''\to 0$$

in $\operatorname{mod}\Lambda$ with $M''\cong M/M'$. Since the R-module M'' has a positive depth, there is $r\in\mathfrak{m}$ such that it is not a zero-divisor on M''. Then we get an exact sequence

$$0\to M''\xrightarrow{x} M''\to\overline{M''}\to 0$$

in $\operatorname{mod}\Lambda$ with $\dim_R(\overline{M''})=\dim_R(M'')-1$. Moreover, It follows from [20, Corollary 7.22] that there is a triangle in $\mathbf{D}^b(\operatorname{mod}\Lambda)$ of the form

$$M''\xrightarrow{x} M''\to\overline{M''}\to M''[1].$$

The implications (2) $\Longrightarrow$ (1) and (2$'$) $\Longrightarrow$ (1$'$) are trivial.

(1) $\Longrightarrow$ (2) We verify the finiteness of $G_C\text{-pd}_\Lambda M$ by induction on d. Notice that M is a finitely generated R-module, so $d = 0$ means M has finite length as an R-module, and hence it also has finite length as a Λ-module. Since $\mathcal{GP}_\mathcal{C}(\operatorname{Mod}\Lambda)$ is a resolving subcategory of $\operatorname{Mod}\Lambda$ by [26, Remark 4.4], we have $G_C\text{-pd}_\Lambda M < \infty$ by Lemma 4.5.

Now suppose that $d \geqslant 1$ and the assertion (2) holds for all finitely generated Λ-modules of Krull dimension smaller than d. By assumption and [3, Lemma 3.3], we have that $G_C\text{-pd}_\Lambda M < \infty$ if and only if the natural morphism $\sigma_M^C : M \to \mathbf{R}\operatorname{Hom}_{\Gamma^{op}}(\mathbf{R}\operatorname{Hom}_\Lambda(M, C), C)$ is an isomorphism in $\mathbf{D}^b(\operatorname{mod}\Lambda)$, and if and only if its mapping cone $\operatorname{Cone}(\sigma_M^C)$ is acyclic.

By the above argument, it suffices to prove $G_C\text{-pd}_\Lambda M'' < \infty$. Note that $\dim_R(\overline{M''}) < d$. By the induction hypothesis, one has that $G_C\text{-pd}_\Lambda \overline{M''} < \infty$ and hence $\operatorname{Cone}(\sigma_{\overline{M''}}^C)$ is acyclic. Set $F(-) := \mathbf{R}\operatorname{Hom}_{\Gamma^{op}}(\mathbf{R}\operatorname{Hom}_\Lambda(-, C), C)$. By [47, Exercise 10.2.6], we get the following diagram whose rows and columns are triangles

$$\begin{array}{ccccccc}
M'' & \xrightarrow{x} & M'' & \longrightarrow & \overline{M''} & \longrightarrow & M''[1] \\
\downarrow{\scriptstyle \sigma_{M''}^C} & & \downarrow{\scriptstyle \sigma_{M''}^C} & & \downarrow{\scriptstyle \sigma_{\overline{M''}}^C} & & \downarrow \\
F(M'') & \xrightarrow{x} & F(M'') & \longrightarrow & F(\overline{M''}) & \longrightarrow & F(M'')[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Cone}(\sigma_{M''}^C) & \xrightarrow{x} & \operatorname{Cone}(\sigma_{M''}^C) & \longrightarrow & \operatorname{Cone}(\sigma_{\overline{M''}}^C) & \longrightarrow & \operatorname{Cone}(\sigma_{M''}^C)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M''[1] & \longrightarrow & M''[1] & \longrightarrow & \overline{M''}[1] & \longrightarrow & M''[2]
\end{array}$$

in $\mathbf{D}^b(\operatorname{mod}\Lambda)$. Observe that the source and target of $\sigma_{M''}^C$ are complexes that have finitely generated cohomology in each degree and $r \in \mathfrak{m}$. Thus the Nakayama lemma implies that the homology $\operatorname{Cone}(\sigma_{M''}^C)$ is zero, as desired.

(1$'$) $\Longrightarrow$ (2$'$) It is dual to the argument in the proof of (1) $\Longrightarrow$ (2). We verify the finiteness of $\mathcal{B}_C(\Lambda)\text{-id}_\Lambda M$ by induction on d. Notice that M is a finitely generated R-module, so $d = 0$ means M has finite length as an R-module, and hence it also has finite length as a Λ-module. Since $\mathcal{B}_C(\Lambda)$ is a coresolving subcategory of $\operatorname{Mod}\Lambda$ by [26, Remark 4.4], we have $\mathcal{B}_C(\Lambda)\text{-id}_\Lambda M < \infty$ by Lemma 4.5.

Now suppose that $d \geqslant 1$ and the assertion (2$'$) holds for all finitely generated Λ-modules of Krull dimension smaller than d. Note that both Λ and Γ are Noetherian R-algebras. By assumption and [41, Lemma 5.3], we have that $\mathcal{B}_C(\Lambda)\text{-id}_\Lambda M < \infty$ if and only if the natural morphism $\theta_M : C \otimes_\Gamma^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_\Lambda(C, M) \to M$ is an isomorphism in $\mathbf{D}^b(\operatorname{mod}\Lambda)$, and if and only if its mapping cone $\operatorname{Cone}(\theta_M)$ is acyclic.

By the above argument, it suffices to prove that $\mathcal{B}_C(\Lambda)\text{-id}_\Lambda M'' < \infty$. Note that $\dim_R(\overline{M''}) < d$. By the induction hypothesis, one has that $\mathcal{B}_C(\Lambda)\text{-id}_\Lambda \overline{M''} < \infty$ and hence $\operatorname{Cone}(\theta_{\overline{M''}})$ is acyclic. Set $G(-) := C \otimes_\Gamma^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_\Lambda(C, -)$. By [47, Exercise 10.2.6], we get the following diagram whose rows and columns are triangles

$$\begin{array}{ccccccc}
G(M'') & \xrightarrow{x} & G(M'') & \longrightarrow & G(\overline{M''}) & \longrightarrow & G(M'')[1] \\
\downarrow{\scriptstyle \theta_{M''}} & & \downarrow{\scriptstyle \theta_{M''}} & & \downarrow{\scriptstyle \theta_{\overline{M''}}} & & \downarrow \\
M'' & \xrightarrow{x} & M'' & \longrightarrow & \overline{M''} & \longrightarrow & M''[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Cone}(\theta_{M''}) & \xrightarrow{x} & \mathrm{Cone}(\theta_{M''}) & \longrightarrow & \mathrm{Cone}(\theta_{\overline{M''}}) & \longrightarrow & \mathrm{Cone}(\theta_{M''})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G(M'')[1] & \longrightarrow & G(M'')[1] & \longrightarrow & G(\overline{M''})[1] & \longrightarrow & G(M'')[2]
\end{array}$$

in $\mathbf{D}^b(\operatorname{mod}\Lambda)$. Observe that the source and target of θ''_M are complexes that have finitely generated cohomology in each degree and $r \in \mathfrak{m}$. Thus the Nakayama lemma implies that the homology $\mathrm{Cone}(\theta_{M''})$ is zero, as desired.

Similarly, we can prove the equivalence of $(1'') \Longleftrightarrow (2'')$ by using Lemma 4.3. □

Now we are ready to prove the following theorem.

Theorem 4.7. *Let Λ be semilocal and $n \geqslant 0$.*

(i) *The following statements are equivalent.*

(1) $\mathrm{id}_\Lambda C = \mathrm{id}_{\Gamma^{op}} C \leqslant n$.

(2) G_C-$\mathrm{pd}_\Lambda S \leqslant n$ *for any simple left Λ-module S.*

(3) G_C-$\mathrm{pd}_\Lambda M \leqslant n$ *for any $M \in \operatorname{mod}\Lambda$.*

(4) G_C-$\mathrm{pd}_{\Gamma^{op}} T \leqslant n$ *for any simple right Γ-module T.*

(5) G_C-$\mathrm{pd}_{\Gamma^{op}} N \leqslant n$ *for any $N \in \operatorname{mod}\Gamma^{op}$.*

(ii) *The following statements are equivalent.*

(1′) $\mathrm{pd}_\Lambda C = \mathrm{pd}_{\Gamma^{op}} C \leqslant n$.

(2′) $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_\Lambda S \leqslant n$ *for any simple left Λ-module S.*

(3′) $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_\Lambda M \leqslant n$ *for any $M \in \operatorname{mod}\Lambda$.*

(4′) $\mathcal{B}_C(\Gamma^{op})$-$\mathrm{id}_{\Gamma^{op}} T \leqslant n$ *for any simple right Γ-module T.*

(5′) $\mathcal{B}_C(\Gamma^{op})$-$\mathrm{id}_{\Gamma^{op}} N \leqslant n$ *for any $N \in \operatorname{mod}\Gamma^{op}$.*

(6′) $\mathcal{A}_C(\Gamma)$-$\mathrm{pd}_\Gamma T' \leqslant n$ *for any simple left Γ-module T'.*

(7′) $\mathcal{A}_C(\Gamma)$-$\mathrm{pd}_\Gamma N' \leqslant n$ *for any $N' \in \operatorname{mod}\Gamma$.*

(8′) $\mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} S' \leqslant n$ *for any simple right Λ-module S'.*

(9′) $\mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} M' \leqslant n$ *for any $M' \in \operatorname{mod}\Lambda^{op}$.*

Proof. The implications $(1) \Longrightarrow (3)$ and $(3) \Longrightarrow (1)$ follow from [28, Theorem 5.20] and [50, Theorem 3.6], respectively. The implication $(3) \Longrightarrow (2)$ is trivial.

$(2) \Longrightarrow (3)$ By [39, Proposition.2.7], we have $\mathrm{id}_\Lambda C \leqslant n$. Thus for any $M \in \operatorname{mod}\Lambda$, we have G_C-$\mathrm{pd}_\Lambda M < \infty$ by Lemma 4.6, and therefore G_C-$\mathrm{pd}_\Lambda M \leqslant n$ by [3, Theorem 2.2].

The implications $(1') \Longrightarrow (3')$ and and $(1') \Longrightarrow (7')$ follow from [41, Corollary 4.9] and [27, Theorem 4.15], respectively. Both $(3') \Longrightarrow (2')$ and $(7') \Longrightarrow (6')$ are trivial.

$(3') \Longrightarrow (1')$ Since $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_\Lambda \Lambda \leqslant n$, we have $\mathrm{pd}_{\Gamma^{op}} C \leqslant n$ by [41, Theorem 4.8(2)]. For any $X \in \operatorname{Mod}\Lambda$, we have $X = \varinjlim_{i\in I} X_i$ with all X_i in $\operatorname{mod}\Lambda$, and thus

$$\mathrm{Ext}^j_\Lambda(C, X) \cong \mathrm{Ext}^j_\Lambda(C, \varinjlim_{i\in I} X_i) \cong \varinjlim_{i\in I} \mathrm{Ext}^j_\Lambda(C, X_i)$$

by [16, Lemma 6.6]. It follows from [41, Theorem 4.2] that $\mathrm{Ext}_\Lambda^{\geqslant n+1}(C, X_i) = 0$, which implies $\mathrm{Ext}_\Lambda^{\geqslant n+1}(C, X) = 0$. Thus $\mathrm{pd}_\Lambda C \leqslant n$, and therefore $\mathrm{pd}_\Lambda C = \mathrm{pd}_{\Gamma^{op}} C \leqslant n$ by [42, Proposition 4.1].

$(2') \Longrightarrow (3')$ Since Λ is semilocal, we have that Λ/J is a direct sum of simple Λ-modules and each simple Λ-module occurs in the sum, up to isomorphism. By $(2')$, we have $\mathrm{Ext}_\Lambda^{n+1}(C, \Lambda/J) = 0$. It follows from [49, Corollary 2] that $\mathrm{pd}_\Lambda C \leqslant n$. Thus for any $M \in \mathrm{mod}\,\Lambda$, we have $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_\Lambda M < \infty$ by Lemma 4.6, and therefore $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_\Lambda M \leqslant n$ by [41, Theorem 4.2].

$(6') \Longrightarrow (7')$ By $(6')$ and Proposition 4.4, we get $\mathrm{Tor}_{n+1}^\Gamma(C, \Gamma/J(\Gamma)) = 0$. It follows from [49, Corollary 2] that $\mathrm{pd}_{\Gamma^{op}} C \leqslant n$. Then for any $N' \in \mathrm{mod}\,\Gamma$, we have $\mathcal{A}_C(\Gamma)$-$\mathrm{pd}_\Gamma N' < \infty$ by Lemma 4.6, and thus $\mathcal{A}_C(\Gamma)$-$\mathrm{pd}_\Gamma N' \leqslant n$ by Proposition 4.4 again.

The implication $(6') \Longleftrightarrow (4')$ follows from [25, Theorem 3.3].

According to the symmetry, the proof is finished. □

The following corollary is a consequence of Theorem 4.7, in which the assertion (i) is a C-version of [10, Theorem 3.1].

Corollary 4.8. *Let $\Lambda = \Gamma$ be semilocal and $n \geqslant 0$.*

(i) *The following statements are equivalent.*
  (1) $\mathrm{id}_\Lambda C = \mathrm{id}_{\Lambda^{op}} C \leqslant n$.
  (2) G_C-$\mathrm{pd}_\Lambda \Lambda/J \leqslant n$.
  (3) G_C-$\mathrm{pd}_{\Lambda^{op}} \Lambda/J \leqslant n$.

(ii) *The following statements are equivalent.*
  (1′) $\mathrm{pd}_\Lambda C = \mathrm{pd}_{\Lambda^{op}} C \leqslant n$.
  (2′) $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_\Lambda \Lambda/J \leqslant n$.
  (3′) $\mathcal{B}_C(\Lambda)$-$\mathrm{id}_{\Lambda^{op}} \Lambda/J \leqslant n$.
  (4′) $\mathcal{A}_C(\Lambda)$-$\mathrm{pd}_\Lambda \Lambda/J \leqslant n$.
  (5′) $\mathcal{A}_C(\Lambda)$-$\mathrm{pd}_{\Lambda^{op}} \Lambda/J \leqslant n$.

Proof. Since Λ is semilocal, we have that Λ/J is a direct sum of simple Λ-modules and each simple Λ-module occurs in the sum, up to isomorphism. Now both assertions follow immediately from Theorem 4.7. □

The following result gives a sufficient condition for the C-Gorenstein projective and $\mathcal{A}_C(\Lambda)$-projective dimensions of J to be left-right symmetric, in which the assertion (1) is a C-version of [10, Corollary 3.3].

Corollary 4.9. *If $\Lambda = \Gamma$ is semilocal, then it holds that*
  (1) *G_C-$\mathrm{pd}_\Lambda J = G_C$-$\mathrm{pd}_{\Lambda^{op}} J$;*
  (2) *$\mathcal{A}_C(\Lambda)$-$\mathrm{pd}_\Lambda J = \mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} J$.*

Proof. (1) It suffices to prove the inequality G_C-$\mathrm{pd}_\Lambda J \leqslant G_C$-$\mathrm{pd}_{\Lambda^{op}} J$. When G_C-$\mathrm{pd}_{\Lambda^{op}} J = \infty$, the assertion follows trivially. Now suppose G_C-$\mathrm{pd}_{\Lambda^{op}} J = n < \infty$. Note that G_C-$\mathrm{pd}_{\Lambda^{op}} \Lambda = G_C$-$\mathrm{pd}_\Lambda \Lambda = 0$ (cf. [26, Remark 4.4(3)(a)]). It follows from [36, Lemma 3.2(1)] that

$$G_C\text{--}\,\mathrm{pd}_{\Lambda^{op}} \Lambda/J \leqslant \max\{G_C\text{-}\,\mathrm{pd}_{\Lambda^{op}} \Lambda, G_C\text{-}\,\mathrm{pd}_{\Lambda^{op}} J + 1\} = n + 1.$$

In light of Corollary 4.8(i), we have G_C-$\mathrm{pd}_\Lambda \Lambda/J = n + 1$. It follows from [36, Lemma 3.2(2)] that

$$G_C\text{-}\,\mathrm{pd}_\Lambda J \leqslant \max\{G_C\text{-}\,\mathrm{pd}_\Lambda \Lambda, G_C\text{-}\,\mathrm{pd}_\Lambda \Lambda/J - 1\} = n.$$

The assertion follows.

(2) It suffices to prove the inequality $\mathcal{A}_C(\Lambda)$-$\mathrm{pd}_\Lambda J \leqslant \mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} J$. When $\mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} J = \infty$, the assertion follows trivially. Now suppose $\mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} J = n < \infty$. Note that $\mathcal{A}_C(\Lambda^{op})$-$\mathrm{pd}_{\Lambda^{op}} \Lambda = \mathcal{A}_C(\Lambda)$-$\mathrm{pd}_\Lambda \Lambda = 0$ by [22, Lemma 4.1], it follows from [26, Theorem 3.2] that

$$\mathcal{A}_C(\Lambda^{op})\text{- pd}_{\Lambda^{op}} \Lambda/J \leqslant \max\{\mathcal{A}_C(\Lambda^{op})\text{- pd}_{\Lambda^{op}} \Lambda, \mathcal{A}_C(\Lambda^{op})\text{- pd}_{\Lambda^{op}} J + 1\} = n + 1.$$

In light of Corollary 4.8(ii), we have $\mathcal{A}_C(\Lambda)$-$\mathrm{pd}_\Lambda \Lambda/J = n+1$. Applying [26, Theorem 3.2] again, we have

$$\mathcal{A}_C(\Lambda)\text{- pd}_\Lambda J \leqslant \max\{\mathcal{A}_C(\Lambda)\text{- pd}_\Lambda \Lambda, \mathcal{A}_C(\Lambda)\text{- pd}_\Lambda \Lambda/J - 1\} = n.$$

The assertion follows. □

Recall that a module $M \in \mathrm{Mod}\,\Lambda$ is called *Gorenstein flat* if the following conditions are satisfied:

(1) $\mathrm{Tor}^\Lambda_{\geqslant 1}(I, M) = 0$ for any injective right Λ-module I.

(2) There is an exact sequence

$$0 \to M \to F^0 \to F^1 \to \cdots$$

in $\mathrm{Mod}\,\Lambda$ with all F^i flat, which remains exact after applying the functor $I \otimes_\Lambda -$ for any injective right Λ-module I.

When $\mathcal{X} = \{$Gorenstein flat left Λ-modules$\}$, the $\mathcal{X}$-projective dimension of M is called the *Gorenstein flat dimension* of M ([13]).

Recall that a left and right Noetherian ring is called *Gorenstein* if its left and right self-injective dimensions are finite. For any $M \in \mathrm{Mod}\,\Lambda$, recall that $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called its *character module*, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. As a consequence of Corollary 4.8(i), we obtain some new characterizations of Gorenstein algebras, which extends [10, Theorem 3.1].

Corollary 4.10. *If Λ is a semilocal ring, then the following statements are equivalent.*

(1) *Λ is Gorenstein.*

(2) G-$\mathrm{pd}_\Lambda \Lambda/J < \infty$.

(2′) G-$\mathrm{pd}_{\Lambda^{op}} \Lambda/J < \infty$.

If the character module of any Gorenstein injective left Λ-module is Gorenstein flat, then the above three conditions and the below condition are equivalent.

(3) G-$\mathrm{id}_\Lambda \Lambda/J < \infty$.

Proof. Putting ${}_\Lambda C_\Gamma = {}_\Lambda\Lambda_\Lambda$ in Corollary 4.8(i), we get (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (2′). By [28, Theorem 5.20], we have (1) $\Longrightarrow$ (3).

Now suppose that the character module of any Gorenstein injective left Λ-module is Gorenstein flat, we prove (3) $\Longrightarrow$ (2′).

By (3), we may assume that G-$\mathrm{id}_\Lambda \Lambda/J = n < \infty$. Then for any simple left Λ-module S, we have G-$\mathrm{id}_\Lambda S \leqslant n$ since S is a direct summand of Λ/J. As $(\Lambda/J)^+$ is semisimple, it follows that $(\Lambda/J)^+ \cong \oplus_{i\in I} S_i$, where all S_i are simple left Λ-module. Thus by [30, Theorem 2], G-$\mathrm{id}_\Lambda(\Lambda/J)^+ \cong \mathrm{G}$-$\mathrm{id}_\Lambda(\oplus_{i\in I} S_i) \leqslant n$ and the Gorenstein flat dimension of $(\Lambda/J)^{++}$ is at most n. Since $(\Lambda/J)^{++}$ is semisimple, we have that Λ/J is a direct summand of $(\Lambda/J)^{++}$. It follows from [7, Proposition 1.3] that G-$\mathrm{pd}_{\Lambda^{op}} \Lambda/J \leqslant n$. □

If Λ is one of the following rings: (1) a commutative Noetherian ring with a dualizing complex; (2) a left Noetherian ring having finite right self-injective dimension; (3) a left Artinian ring such that the injective envelope of any simple left module is finitely generated (in particular, an Artinian algebra), then the character module of any Gorenstein injective left Λ-module is a Gorenstein flat ([30, Example 3.1]).

Let Γ be semilocal and $n \geqslant 0$. In view of Theorem 4.7, it is interesting to know whether the following statements are equivalent.

(1) $\mathrm{id}_\Lambda C = \mathrm{id}_{\Gamma^{op}} C \leqslant n$.
(2) G_C-$\mathrm{id}_\Gamma T' \leqslant n$ for any simple left Γ-module T'.
(3) G_C-$\mathrm{id}_\Gamma N' \leqslant n$ for any $N' \in \mathrm{mod}\,\Gamma$.
(4) G_C-$\mathrm{id}_{\Gamma^{op}} T \leqslant n$ for any simple right Γ-module T.
(5) G_C-$\mathrm{id}_{\Gamma^{op}} N \leqslant n$ for any $N \in \mathrm{mod}\,\Gamma^{op}$.

In general, we have (1) $\iff$ (3) $\iff$ (5) $\implies$ (2) + (4) by [28, Theorem 5.20]. When ${}_\Lambda C_\Gamma = {}_\Lambda \Lambda_\Lambda$ and Λ is either a commutative Noetherian ring with a dualizing complex or an Artinian algebra, all these conditions are equivalent by Corollary 4.10.

In the sequel we aim at giving some equivalent characterizations of Wakamatsu tilting modules with finite injective dimension in terms of certain cotorsion pairs. The following result will be useful, and the idea of the proof comes from that of [2, Lemma 2.2].

Lemma 4.11. *Let R be semilocal and $M \in \mathrm{mod}\,\Gamma$. If $\mathrm{depth}_R(C) = t$ and M is free as a $\Gamma_\mathfrak{p}$-module for any $\mathfrak{p} \in \mathrm{Spec}(R)\backslash \mathrm{Max}(R)$, then it holds that*

(1) $\mathrm{Ext}^i_\Lambda(C \otimes_\Gamma M, C) \cong \mathrm{Ext}^i_\Gamma(M, \Gamma)$ *for any* $i \leqslant t$.
(2) *There is an injection* $\mathrm{Ext}^{t+1}_\Lambda(C \otimes_\Gamma M, C) \to \mathrm{Ext}^{t+1}_\Gamma(M, \Gamma)$.

Proof. (1) Since ${}_\Lambda C$ is a Wakamatsu tilting module, by [40, Theorem 10.62] there is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_\Lambda(\mathrm{Tor}^\Gamma_q(C, M), C) \Rightarrow \mathrm{Ext}^n_\Gamma(M, \Gamma).$$

Then we have a filtration $\{\phi^j H^i\}$ of $H^i = \mathrm{Ext}^i_\Gamma(M, \Gamma)$ satisfying

$$0 = \phi^{i+1} H^i \subseteq \phi^i H^i \subseteq \cdots \subseteq \phi^1 H^i \subseteq \phi^0 H^i = H^i$$

with $E_\infty^{j,i-j} \cong \phi^j H^i/\phi^{j+1} H^i$ for all i and j. For any $\mathfrak{p} \in \mathrm{Spec}(R)\backslash \mathrm{Max}(R)$, since $M_\mathfrak{p}$ is a free $\Gamma_\mathfrak{p}$-module by assumption, we get that $\mathrm{Tor}^\Gamma_i(C, M)$ has finite length as a Λ-module for any $i > 0$. Thus by Lemma 2.8 we have $E_2^{p,q} = 0$ for any $p < t$ and $q > 0$. For any $i \leqslant t$, we have $E_2^{i,0} \cong E_\infty^{i,0} \cong \phi^i H^i/\phi^{i+1} H^i$. So $\mathrm{Ext}^i_\Lambda(C \otimes_\Gamma M, C) \cong \mathrm{Ext}^i_\Gamma(M, \Gamma)$.

(2) Since $E_2^{t+1,0} \cong E_\infty^{t+1,0}$, we obtain an injection $\mathrm{Ext}^{t+1}_\Lambda(C \otimes_\Gamma M, C) \to \mathrm{Ext}^{t+1}_\Gamma(M, \Gamma)$. $\square$

We recall the notion of cotorsion pairs.

Definition 4.12. (cf. [16]) Let $\mathcal{U}, \mathcal{V}$ be subcategories of $\mathrm{mod}\,\Lambda$. The pair $(\mathcal{U}, \mathcal{V})$ is called a *cotorsion pair* if

$$\mathcal{U} = {}^{\perp_1}\mathcal{V} := \{U \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(U, V) = 0 \text{ for any } V \in \mathcal{V}\}$$

and

$$\mathcal{V} = \mathcal{U}^{\perp_1} := \{V \in \mathrm{Mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(U, V) = 0 \text{ for any } U \in \mathcal{U}\}.$$

Let $M \in \mathrm{mod}\,\Lambda$ and let

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

be a projective presentation of M in $\mathrm{mod}\,\Lambda$. Recall from [29] that $\mathrm{Tr}_C M := \mathrm{Coker}\, f^\dagger$ is called the *transpose* of M with respect to C. Note that $\mathrm{Tr}_C M$ is defined up to projective summands.

Definition 4.13. A module $M \in \mathrm{mod}\,\Lambda$ is called *C-∞-torsionfree* if $\mathrm{Ext}^{\geqslant 1}_{\Gamma^{op}}(\mathrm{Tr}_C M, C) = 0$.

When ${}_\Lambda C_\Gamma = {}_\Lambda\Lambda_\Lambda$, a C-∞-torsionfree module is exactly a ∞-torsionfree module defined in [4]. We write

$$\mathcal{T}_C(\operatorname{mod}\Lambda) := \{M \in \operatorname{mod}\Lambda \mid M \text{ is } C\text{-}\infty\text{-torsionfree}\},$$
$$\mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda) := \{X \in \operatorname{mod}\Lambda \mid \operatorname{add} C\text{-}\operatorname{pd}_\Lambda X < \infty\},$$
$$\mathcal{X}_C(\operatorname{mod}\Lambda) := \{M \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(M, X) = 0 \text{ for any } X \in \mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda)\}.$$

In particular, when ${}_\Lambda C_\Gamma = {}_\Lambda\Lambda_\Lambda$, we write

$$\mathcal{T}(\operatorname{mod}\Lambda) := \mathcal{T}_\Lambda(\operatorname{mod}\Lambda),\ \mathcal{P}^{<\infty}(\operatorname{mod}\Lambda) := \mathcal{P}_\Lambda^{<\infty}(\operatorname{mod}\Lambda) \ \text{ and } \ \mathcal{X}(\operatorname{mod}\Lambda) := \mathcal{X}_\Lambda(\operatorname{mod}\Lambda).$$

The right-sided versions of these natation are defined symmetrically.

Part of [2, Theorem 1.1] are included in the following theorem.

Theorem 4.14. *Let $\Lambda = \Gamma$ be local. Then the following statements are equivalent.*

(1) $\operatorname{id}_\Lambda C = \operatorname{id}_{\Lambda^{op}} C < \infty$.
(2) $\mathcal{X}_C(\operatorname{mod}\Lambda) = \mathcal{T}_C(\operatorname{mod}\Lambda)$ *and* $\mathcal{X}_C(\operatorname{mod}\Lambda^{op}) = \mathcal{T}_C(\operatorname{mod}\Lambda^{op})$.
(3) $\mathcal{X}_C(\operatorname{mod}\Lambda) = \mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda)$ *and* $\mathcal{X}_C(\operatorname{mod}\Lambda^{op}) = \mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda^{op})$.
(4) *Both* $(\mathcal{T}_C(\operatorname{mod}\Lambda), \mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda))$ *and* $(\mathcal{T}_C(\operatorname{mod}\Lambda^{op}), \mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda^{op}))$ *are cotorsion pairs.*
(5) *Both* $(\mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda), \mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda))$ *and* $(\mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda^{op}), \mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda^{op}))$ *are cotorsion pairs.*

Proof. By Lemma 3.2, we may suppose $\operatorname{depth}_R(\Lambda) = \operatorname{depth}_R(C) = t$.

(2) $\Longrightarrow$ (1) Since $\operatorname{Ext}^i_{\Lambda^{op}}(\Lambda/J, \Lambda)$ has finite length as a Λ-module for each i, we have $\operatorname{Ext}^j_\Lambda(\operatorname{Ext}^i_{\Lambda^{op}}(\Lambda/J, \Lambda), \Lambda) = 0$ for any $1 \leqslant i \leqslant t+1$ and $0 \leqslant j \leqslant i-2$. It follows from [4, Proposition 2.26] that $\Omega^{t+1}(\Lambda/J)$ is $(t+1)$-torsionfree. Thus $\operatorname{Ext}^{1\leqslant i\leqslant t+1}_\Lambda(\operatorname{Tr}\Omega^{t+1}(\Lambda/J), \Lambda) = 0$ for any $1 \leqslant i \leqslant t+1$. Since $\operatorname{Tr}\Omega^{t+1}(\Lambda/J)$ is free as a $\Lambda_\mathfrak{p}$-module for any $\mathfrak{p} \in \operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$, we get from Lemma 4.11 that $\operatorname{Ext}^{1\leqslant i\leqslant t+1}_\Lambda(C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J), C) = 0$.

Let $0 \neq X \in \mathcal{P}_C^{<\infty}(\operatorname{mod}\Lambda)$. Then Corollary 3.12 yields $\operatorname{add} C\text{-}\operatorname{pd}_\Lambda X \leqslant t$, and hence there is an exact sequence

$$0 \to C_t \xrightarrow{f_t} C_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0$$

in $\operatorname{mod}\Lambda$ with all C_i in $\operatorname{add} C$. Set $X_i := \operatorname{Im} f_i$. Then we have

$$\begin{aligned}&\operatorname{Ext}^1_\Lambda(C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J), X) \cong \operatorname{Ext}^2_\Lambda(C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J), X_1) \cong \cdots \\ \cong &\operatorname{Ext}^{t+1}_\Lambda(C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J), X_t) = 0.\end{aligned}$$

It means $C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J) \in \mathcal{X}_C(\operatorname{mod}\Lambda)$, whence, by assumption $C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J) \in \mathcal{T}_C(\operatorname{mod}\Lambda)$. Since there are isomorphisms

$$\begin{aligned}&\operatorname{Hom}_\Lambda(C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J), C)\\ \cong &\operatorname{Hom}_\Lambda(\operatorname{Tr}\Omega^{t+1}(\Lambda/J), \Lambda)\\ \cong &\Omega^2\Omega^{t+1}(\Lambda/J) \cong \Omega^{t+3}(\Lambda/J),\end{aligned}$$

the fact $C \otimes_\Lambda \operatorname{Tr}\Omega^{t+1}(\Lambda/J) \in \mathcal{T}_C(\operatorname{mod}\Lambda)$ implies

$$\operatorname{Ext}^{i+t+3}_{\Lambda^{op}}(\Lambda/J, C) \cong \operatorname{Ext}^i_{\Lambda^{op}}(\Omega^{t+3}(\Lambda/J), C) = 0$$

for any $i \geqslant 1$. So $\operatorname{id}_{\Lambda^{op}} C < \infty$. Similarly, we get that $\operatorname{id}_\Lambda C < \infty$ by the equality $\mathcal{X}_C(\operatorname{mod}\Lambda^{op}) = \mathcal{T}_C(\operatorname{mod}\Lambda^{op})$. Therefore (1) holds by [24, Theorem 2.7].

Since $\mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda) \subseteq \mathcal{T}_C(\operatorname{mod}\Lambda)$ and $\mathcal{GP}_\mathcal{C}(\operatorname{mod}\Lambda^{op}) \subseteq \mathcal{T}_C(\operatorname{mod}\Lambda^{op})$, the proof of (3) $\Longrightarrow$ (1) is similar to that of (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (3) The inclusion $\mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda) \subseteq \mathcal{X}_C(\mathrm{mod}\,\Lambda)$ is obvious. As for the reverse inclusion, let $M \in \mathcal{X}_C(\mathrm{mod}\,\Lambda)$. It follows from [28, Theorem 1.4] that $G_C\text{-pd}_\Lambda M < \infty$. Then by [36, Lemma 2.7], there is an exact sequence

$$0 \to X \to G \to M \to 0$$

in $\mathrm{mod}\,\Lambda$ with $\mathrm{add}\,C\text{-}\mathrm{pd}_\Lambda X < \infty$ and $G \in \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$. So this sequence splits, and hence $M \in \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$ and $\mathcal{X}_C(\mathrm{mod}\,\Lambda) \subseteq \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$. This proves $\mathcal{X}_C(\mathrm{mod}\,\Lambda) = \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$. Symmetrically, we have $\mathcal{X}_C(\mathrm{mod}\,\Lambda^{op}) = \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda^{op})$.

(1) $\Longrightarrow$ (2) Since (1) implies (3) and $\mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda) \subseteq \mathcal{T}_C(\mathrm{mod}\,\Lambda)$, it suffices to show $\mathcal{T}_C(\mathrm{mod}\,\Lambda) \subseteq \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$. Given $M \in \mathcal{T}_C(\mathrm{mod}\,\Lambda)$, then there is an exact sequence

$$0 \to M \to C_0 \to C_1 \to \cdots \to C_n \to \dots$$

in $\mathrm{mod}\,\Lambda$ with all C_i in $\mathrm{add}\,C$. It follows from [3, Lemma 2.1] that $M \in \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$ and $\mathcal{T}_C(\mathrm{mod}\,\Lambda) \subseteq \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$. Symmetrically, we have $\mathcal{X}_C(\mathrm{mod}\,\Lambda^{op}) = \mathcal{T}_C(\mathrm{mod}\,\Lambda^{op})$.

The implications (4) $\Longrightarrow$ (2) and (5) $\Longrightarrow$ (3) are obvious.

(1) $\Longrightarrow$ (5) Since (1) implies (3) and $\mathcal{P}_C^{<\infty}(\mathrm{mod}\,\Lambda) \subseteq \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)^{\perp_1}$, it suffices to show $\mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)^{\perp_1} \subseteq \mathcal{P}_C^{<\infty}(\mathrm{mod}\,\Lambda)$. Given $M \in \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)^{\perp_1}$, we have $G_C\text{-pd}_\Lambda M < \infty$ by [28, Theorem 1.4]. It follows from [36, Corollary 3.4] that there is an exact sequence

$$0 \to M \to X \to G \to 0$$

in $\mathrm{mod}\,\Lambda$ with $\mathrm{add}\,C\text{-}\mathrm{pd}_\Lambda X < \infty$ and $G \in \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$. Then this sequence splits and hence $M \in \mathcal{P}_C^{<\infty}(\mathrm{mod}\,\Lambda)$. It means that $(\mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda), \mathcal{P}_C^{<\infty}(\mathrm{mod}\,\Lambda))$ is a cotorsion pair. Symmetrically, we have that $(\mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda^{op}), \mathcal{P}_C^{<\infty}(\mathrm{mod}\,\Lambda^{op}))$ is a cotorsion pair.

(1) $\Longrightarrow$ (4) Since (1) implies (2) and (3), we get

$$\mathcal{X}_C(\mathrm{mod}\,\Lambda) = \mathcal{T}_C(\mathrm{mod}\,\Lambda) = \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda)$$

and

$$\mathcal{X}_C(\mathrm{mod}\,\Lambda^{op}) = \mathcal{T}_C(\mathrm{mod}\,\Lambda^{op}) = \mathcal{GP}_{\mathcal{C}}(\mathrm{mod}\,\Lambda^{op}).$$

□

If we specialize the preceding theorem to the case for ${}_\Lambda C_\Gamma = {}_\Lambda\Lambda_\Lambda$, we obtain the following equivalent characterizations of Gorenstein algebras in terms of certain cotorsion pairs.

Corollary 4.15. *If Λ is local, then the following statements are equivalent.*

(1) *Λ is Gorenstein.*
(2) *$\mathcal{X}(\mathrm{mod}\,\Lambda) = \mathcal{T}(\mathrm{mod}\,\Lambda)$ and $\mathcal{X}(\mathrm{mod}\,\Lambda^{op}) = \mathcal{T}(\mathrm{mod}\,\Lambda^{op})$.*
(3) *$\mathcal{X}(\mathrm{mod}\,\Lambda) = \mathcal{GP}(\mathrm{mod}\,\Lambda)$ and $\mathcal{X}(\mathrm{mod}\,\Lambda^{op}) = \mathcal{GP}(\mathrm{mod}\,\Lambda^{op})$.*
(4) *Both $(\mathcal{T}(\mathrm{mod}\,\Lambda), \mathcal{P}^{<\infty}(\mathrm{mod}\,\Lambda))$ and $(\mathcal{T}(\mathrm{mod}\,\Lambda^{op}), \mathcal{P}^{<\infty}(\mathrm{mod}\,\Lambda^{op}))$ are cotorsion pairs.*
(5) *Both $(\mathcal{GP}(\mathrm{mod}\,\Lambda), \mathcal{P}^{<\infty}(\mathrm{mod}\,\Lambda))$ and $(\mathcal{GP}(\mathrm{mod}\,\Lambda^{op}), \mathcal{P}^{<\infty}(\mathrm{mod}\,\Lambda^{op}))$ are cotorsion pairs.*

Data availability

No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Grad. Texts in Math. **13**, Springer-Verlag, New York, 1992.

[2] T. Araya, K.I. Iima and R. Takahashi, *On the left perpendicular category of the modules of finite projective dimension*, Comm. Algebra **40** (2012), 2693–2699.

[3] T. Araya, R. Takahashi and Y. Yoshino, *Homological invariants associated to semi-dualizing bimodules*, J. Math. Kyoto Univ. **45** (2005), 287–306.

[4] M. Auslander and M. Bridger, *Stable module theory*, Memoirs Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969.

[5] M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. **97** (1960), 1–24.

[6] A. Bahlekeh, S. Salarian and Z. Toosi, *Cohen-Macaulay Noetherian algebras*, Kyoto J. Math. **63** (2023), 1–22.

[7] D. Bennis and N. Mahdou, *Strongly Gorenstein projective, injective, and flat modules*, J. Pure Appl. Algebra **210** (2007), 437–445.

[8] W. Bruns and J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. in Adv. Math. **39**, Cambridge Univ. Press, Cambridge, 1993.

[9] R.O. Buchweitz, Maximal Cohen–Macaulay Modules and Tate ohomology, with appendices and an introduction by L. L. Avramov, B. Briggs, S. B. Iyengar and J. C. Letz., Math. Surveys and Monographs **262**, Amer. Math. Soc., Providence, RI, 2021.

[10] X.-W Chen, S.B. Iyengar and R. Marczinzik, *Homological dimensions of the Jacobson radical*, Proc. Amer. Math. Soc., Ser. B **11** (2024), 211–223.

[11] L.W. Christensen, Gorenstein Dimensions, Lecture Notes in Math. **1747**, Springer, Berlin-Heidelberg, 2000.

[12] L.W. Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), 1839–1883.

[13] E.E. Enochs and O.M.G. Jenda, Relative Homological Algebra, Vol. 1, the second revised and extended edition, de Gruyter Expositions in Math. **30**, Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

[14] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284.

[15] H.-B. Foxby and S. Iyengar, *Depth and amplitude for unbounded complexes*, In: Commutative Algebra (Grenoble/Lyon, 2001), Contemp. Math., **331**, Amer. Math. Soc., Providence, RI, 2003, pp.119–137.

[16] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, de Gruyter Exp. in Math. **41**, Walter de Gruyter GmbH & Co. KG, Berlin, 2006.

[17] E.S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66; English translation in Proc. Steklov Inst. Math. **165**(1985).

[18] S. Goto and K. Nishida, *Finite modules of finite injective dimension over a Noetherian algebra*, J. London. Math. Soc. **63** (2001), 319–335.

[19] S. Goto and K. Nishida, *Towards a theory of Bass numbers with application to Gorenstein algebras*, Colloq. Math. **91** (2002), 191–253.

[20] T. Holm, P. Jørgensen and R. Rouquier, Triangulated categories, London Math. Soc. Lecture Note Ser. **375**, Cambridge Univ. Press, Cambridge, 2010.

[21] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), 423–445.

[22] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), 781–808.

[23] M. Hovey, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002), 553–592.

[24] Z.Y. Huang, *Generalized tilting modules with finite injective dimension.* J. Algebra **311** (2007), 619–634.

[25] Z.Y. Huang, *Duality pairs induced by Auslander and Bass classes*, Georgian Math. J. **28** (2021), 867–882.

[26] Z.Y. Huang, *Homological dimensions relative to preresolving subcategories II*, Forum Math. **34** (2022), 507–530.

[27] Z.Y. Huang, *Generalized Gorenstein categories*, J. Algebra **697** (2026), 106–144.

[28] Z.Y. Huang and M.Z. Sheng, *Gorenstein modules induced by Foxby equivalence*, Publ. Res. Inst. Math. Sci. **62** (2026), 27–73.

[29] Z.Y. Huang and G.H. Tang, *Self-orthogonal modules over coherent rings*, J. Pure Appl. Algebra **161** (2001), 167–176.

[30] A. Iacob, *Direct limits of Gorenstein injective modules*, Osaka J. Math. **63** (2026), 83–89.

[31] O. Iyama and Y. Kimura, *Classifying subcategories of modules over Noetherian algebras*, Adv. Math. **446** (2024), Paper No. 109631.

[32] M. I. Jinnah, *A note on PG-modules*, Math. Scand. **37** (1975), 27–28.

[33] R. Kanda R and T. Nakamura *Flat cotorsion modules over Noether algebras*, Doc. Math. **27**(2022), 1101–1167.

[34] T.Y. Lam, A First Course in Noncommutative Rings, Grad. Texts in Math. **131**, Springer-Verlag, New York, 1991.

[35] G.J. Leuschke and R. Wiegand, Cohen–Macaulay Representations, Math. Surveys and Monographs **181**, Amer. Math. Soc., Providence, RI, 2012.

[36] Z.F. Liu, Z.Y. Huang and A.M. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra **41** (2013), 1–18.

[37] H. Matsumura, Commutative Ring Theory, Cambridge Stud. in Adv. Math. **8**, Cambridge University Press, Cambridge, 1989.

[38] K. Nishida, *Linkage and duality of modules*, Math. J. Okayama Univ. **51** (2009), 71–81.

[39] M. Ramras, *Maximal orders over regular local rings of dimension two*, Trans. Amer. Math. Soc. **142** (1969), 457–479.

[40] J.J. Rotman, An Introduction to Homological Algebra, Second ed., Universitext, Springer, New York, 2009.

[41] X. Tang and Z.Y. Huang, *Homological aspects of the dual Auslander transpose II*, Kyoto J. Math. **57** (2017), 17–53.

[42] X. Tang and Z.Y. Huang, *Homological aspects of the adjoint cotranspose*, Colloq. Math. **150** (2017), 293–311.

[43] W. V. Vasconcelos, Divisor Theory in Module Categories, North-Holland Publishing Co., Amsterdam, 1974.

[44] R. Vyas, *A homological reformulation of the link condition*, J. Algebra **394** (2013), 223–244.

[45] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra **114** (1988), 106–114.

[46] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra **275** (2004), 3–39.

[47] C.A. Weibel, An Introduction to Homological Algebra, Cambridge Stud. in Adv. Math. **38**, Cambridge University Press, Cambridge, 1994.

[48] D. White, *Gorenstein projective dimension with respect to a semidualizing module*. J. Commut. Algebra **2** (2010), 111–137.

[49] J.Z. Xu and F.C. Cheng, *Homological dimensions over non-commutative semi-local rings*, J. Algebra **169** (1994), 679–685.

[50] G.Q. Zhao and L.R. Yin, *Wakamatsu tilting modules with finite injective dimension*, Czechoslovak Math. J. **63** (2013), 865–876.