SPECTRAL ANALYSIS BEYOND ℓ^2 ON SIERPINSKI LATTICES

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Abstract. We study the spectrum of the Laplacian on the Sierpinski lattices. First, we show that the spectrum of the Laplacian, as a subset of $\mathbb{C}$, remains the same for any ℓ^p spaces. Second, we characterize all the spectral points on the lattices with a boundary point.

1. introduction

In this note, we study the spectrum of the Laplacian on the Sierpinski Lattice $\widetilde{\mathcal{SG}}$. This problem was fully investigated by A. Teplyaev [9] in the ℓ^2 setting. We will continue his study for the ℓ^p case, and the ℓ^1 and ℓ^∞ cases are of special interest.

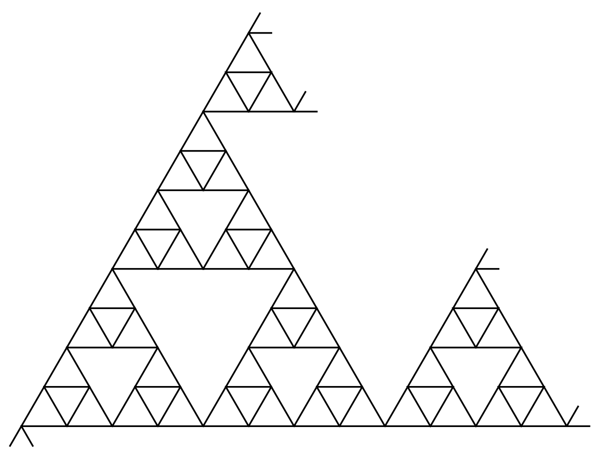

Figure 1. The Sierpinski lattices.

The *Sierpinski lattice* is an infinite graph defined as follows. Let

$$F_0(x) = \frac{1}{2}x + (\frac{1}{4}, \frac{\sqrt{3}}{4}), \quad F_1(x) = \frac{1}{2}x, \quad F_2(x) = \frac{1}{2}x + (\frac{1}{2}, 0),$$

and $q_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), q_1 = (0,0), q_2 = (1,0)$ be the fixed points of F_i respectively. Given an infinite word $\omega = \omega_1\omega_2\cdots \in \{0,1,2\}^\infty$, the corresponding Sierpinski lattice $\widetilde{\mathcal{SG}}$ is

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constructed as

$$\widetilde{\mathcal{SG}} = \bigcup_{m=1}^{\infty} F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1}(V_m),$$

where $V_0 = \{q_0, q_1, q_2\}$ and $V_m = \bigcup_{i=0}^{2} F_i(V_{m-1})$ are defined iteratively. See Figure 1. For $x, y \in \widetilde{\mathcal{SG}}$, we write $x \sim y$ if $x, y \in F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1} F_{l_m} F_{l_{m-1}} \cdots F_{l_1}(V_0)$ for a sequence $l_1, \cdots, l_m \in \{0,1,2\}$. Call y a *neighbouring vertex* of x. All vertices in the Sierpinski lattice have four neighbouring vertices, except at most one vertex called the *boundary vertex*, which admits only two neighbouring vertices. The boundary vertex exists if and only if there exists $M \in \mathbb{N}$ and $i \in \{0,1,2\}$ such that $\omega_m = i$ for all $m \geq M$, see [9].

The Laplacian Δ on $\widetilde{\mathcal{SG}}$ is defined as

$$\Delta f(x) = \begin{cases} \sum_{y \sim x} f(y) - 4f(x), & \text{if } x \text{ is not a boundary point,} \\ \sum_{y \sim x} 2f(y) - 4f(x), & \text{if } x \text{ is a boundary point.} \end{cases} \tag{1.1}$$

The celebrated result of A. Teplyaev [9] showed that the Lapalacian Δ, viewed as an operator $\ell^2(\widetilde{\mathcal{SG}}) \to \ell^2(\widetilde{\mathcal{SG}})$ has pure point spectrum. What's more, $\ell^2(\widetilde{\mathcal{SG}})$ admits a basis of localized eigenfunctions of the Laplacian, which can be generated using the spectral decimation recipe [1, 5]. Read the book [6] for an introduction to the spectral decimation. For the spectral analysis on other fractal graphs and related fractalfolds, see [3, 7, 8].

In this work, we will describe the spectrum of the Laplacian $\Delta : \ell^p(\widetilde{\mathcal{SG}}) \to \ell^p(\widetilde{\mathcal{SG}})$. In Section 2, we will show that the spectrum is a union of a Julia set and a discrete set named 6-series eigenvalues. The spectrum remains the same for any $1 \leq p \leq \infty$ and any lattice. On the other hand, in Section 3, we will see that for $1 < p < \infty$, the Laplacian has only point spectrum and continuous spectrum, while $\Delta : \ell^1 \to \ell^1$ has all three kinds of spectral points. This phenomenon is a consequence of the existence of the 4-eigenfunctions in $\ell^\infty(\widetilde{\mathcal{SG}})$, which does not live in other $\ell^p(\widetilde{\mathcal{SG}})$ spaces. In addition, we get a full description of the ℓ^1 spectrum for the lattices with one boundary vertex.

2. The spectrum of Δ on $\widetilde{\mathcal{SG}}$

In this section, we compute the spectrum of the Laplacian $\Delta : \ell^p(\widetilde{\mathcal{SG}}) \to \ell^p(\widetilde{\mathcal{SG}})$, which is stated in the following Theorem 2.2.

Definition 2.1. *The Julia set corresponding to the polynomial $R(\lambda) = \lambda(5-\lambda)$ is defined as*

$$\mathcal{J} = \{x \in \mathbb{C} : \{R^{\circ k}(x)\}_{k=0}^{\infty} \in l^\infty\}.$$

Remark. The Hausdorff dimension of $\mathcal{J}$ is the unique zero of the Bowen's function

$$B(t) = \lim_{k \to \infty} \frac{1}{k} \ln \sum_{w \in R^{-k}(x)} |(R^{\circ k})'(w)|^{-t},$$

where x is any chosen point in $\mathcal{J}$. See Section 9.1 in the book [4].

Theorem 2.2. *Let $\sigma(\Delta) = \mathcal{J} \cup \Sigma_6$, where $\Sigma_6 = \{6\} \cup (\bigcup_{m=0}^{\infty} R^{\circ -m}\{3\})$. The spectrum of $\Delta : \ell^p(\widetilde{\mathcal{SG}}) \to \ell^p(\widetilde{\mathcal{SG}})$ is equal to $\sigma(\Delta)$ for all $1 \leq p \leq \infty$.*

Remark. Theorem 2.2 is proved for the $\ell^2(\widetilde{\mathcal{SG}})$ case in [9]. Here a different approach will be used to deal with general ℓ^p cases. The theorem is also valid for $\Delta : C_0(\widetilde{\mathcal{SG}}) \to C_0(\widetilde{\mathcal{SG}})$.

We will prove Theorem 2.2 with several lemmas. For a fixed infinite word ω, we consider a sequence of sparse lattices $\widetilde{\mathcal{SG}}^{(-k)}$ defined as

$$\widetilde{\mathcal{SG}}^{(-k)} = \bigcup_{m=k}^{\infty} F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1} V_{m-k},$$

and we say $x \sim_{-k} y$ if $x, y \in F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1} F_{l_{m-k}} F_{l_{m-k-1}} \cdots F_{l_1}(V_0)$ for a sequence $l_1, \cdots, l_{m-k} \in \{0, 1, 2\}$. The Laplacian $\Delta_{(-k)}$ on $\widetilde{\mathcal{SG}}^{(-k)}$ can be defined in a similar manner as (1.1). The eigenvalues and eigenfunctions on $\widetilde{\mathcal{SG}}^{(-k)}$ for different k's are related by the spectral decimation method [1, 8, 9].

To understand the spectral decimation, we only need to focus on a small neighbourhood of a point y_0 in $\widetilde{\mathcal{SG}}^{(-k-1)}$. For convenience, we only consider the case $k = 0$ and a point $y_0 \in \widetilde{\mathcal{SG}}^{(-1)}$ with four neighbouring vertices, noticing that the boundary vertices and $k \geq 1$ cases can be dealt with in an essentially same way. Let $\{x_i\}_{i=1}^4$ be the neighbouring vertices of y_0 in $\widetilde{\mathcal{SG}}^{(-1)}$, and let $\{y_i\}_{i=1}^6$ be the vertices in $\widetilde{\mathcal{SG}}$ bounded by $\{x_i\}_{i=1}^4$. The induced subgraph in $\widetilde{\mathcal{SG}}$ is denoted by Γ in the following context, see Figure 2. Clearly, the definition of Δ and $\Delta_{(-1)}$ on Γ are naturally inherited from those on the graphs $\widetilde{\mathcal{SG}}$ and $\widetilde{\mathcal{SG}}^{(-1)}$ as follows,

$$\Delta f(y_i) = \sum_{z \sim y_i} f(z) - 4f(y_i), \quad 0 \leq i \leq 6, \tag{2.1}$$

$$\Delta_{(-1)} f(y_0) = \sum_{i=1}^{4} f(x_i) - 4f(y_0). \tag{2.2}$$

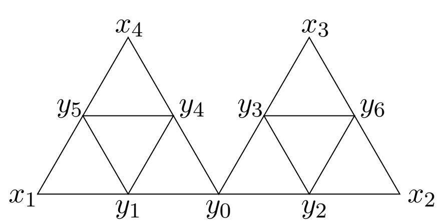

FIGURE 2. The graph Γ.

Proposition 2.3 (Spectral Decimation). *Let $\lambda \notin \{2, 5, 6\}$ and $R(\lambda) = \lambda(5 - \lambda)$. Let Δ and $\Delta_{(-1)}$ on Γ be defined in (2.1) and (2.2).*

(a). Let $f \in l(\Gamma)$ and $-\Delta f(y_i) = \lambda f(y_i), \forall 0 \leq i \leq 6$. Then $-\Delta_{(-1)} f(y_0) = R(\lambda) f(y_0)$.

(b). Given any values $f(x_i), i = 1, 2, 3, 4$ and $f(y_0)$ such that $-\Delta_{(-1)} f(y_0) = R(\lambda) f(y_0)$, there is a unique extension $f \in l(\Gamma)$ such that f satisfies the eigenvalue equations $-\Delta f(y_i) = \lambda f(y_i), \forall 0 \leq i \leq 6$.

Proof. This can be done by direct computation. See Section 3.2 in the book [6] for details. □

Proposition 2.3 can be easily applied to the eigenvalue problems on the Sierpinski lattices. However, to deal with the spectrum, we need somewhat stronger versions. We will do this in the following two lemmas.

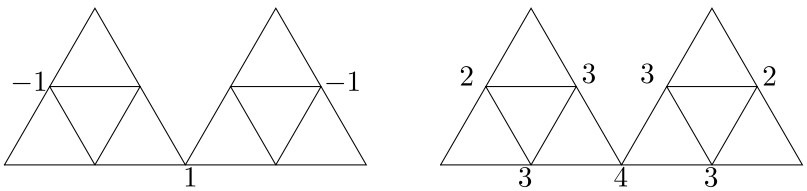

FIGURE 3. The 4-Dirichlet eigenfunction and 1-Dirichlet eigenfunction on Γ (with only non-zero values marked).

Lemma 2.4. *Consider the Dirichlet eigenvalue problem on* Γ*:*

$$\begin{cases} f(x_i) = 0, & 1 \leq i \leq 4, \\ -\Delta f(y_i) = \lambda f(y_i), & 1 \leq i \leq 6. \end{cases}$$

All the Dirichlet eigenvalues are $\{1, 2, 4, 5, 6\}$.

Proof. We can easily find one 2-eigenfunction, three 5-eigenfunctions and one 6-eigenfunction, see Section 3.2, 3.3 in [6] for an illustration of such eigenfunctions. In addition, we find one 4-eigenfunction and one 1-eigenfunction as shown in Figure 2. All the above give 7 linearly independent eigenfunctions. □

Lemma 2.5. *Let* $\lambda \notin \{1, 2, 4, 5, 6\}$. *There exist constants* $\{c_{i,\lambda}\}_{i=0}^{6}$ *such that for any* $f \in l(\Gamma)$ *we have*

$$(R(\lambda) + \Delta_{(-1)})f(y_0) = \sum_{i=0}^{6} c_{i,\lambda}(\lambda + \Delta)f(y_i).$$

In addition, $c_{0,\lambda} \neq 0$.

Proof. Let $f \in l(\Gamma)$. By the assumption and using Lemma 2.4, we know that λ is not a Dirichlet eigenvalue. So there is a unique solution to each of the following boundary value problems.

$$\begin{cases} u(x_i) = 0, & 1 \leq i \leq 4, \\ (\lambda + \Delta)u(y_i) = (\lambda + \Delta)f(y_i), & 0 \leq i \leq 6, \end{cases} \tag{2.3}$$

$$\begin{cases} v(x_i) = f(x_i), & 1 \leq i \leq 4 \\ (\lambda + \Delta)v(y_i) = 0, & 0 \leq i \leq 6. \end{cases} \tag{2.4}$$

Clearly, we have

$$f = u + v.$$

In addition, since v is an eigenfunction of Δ, we have

$$(R(\lambda) + \Delta_{(-1)})v(y_0) = 0,$$

by using Proposition 2.3. As a consequence, we get $(R(\lambda)+\Delta_{(-1)})f(y_0) = (R(\lambda)+\Delta_{(-1)})u(y_0)$. On the other hand, since u is uniquely determined by the linear equations (2.3), we conclude there are constants $c_{i,\lambda}$ such that

$$(R(\lambda)+\Delta_{(-1)})f(y_0) = (R(\lambda)+\Delta_{(-1)})u(y_0) = \sum_{i=0}^{6} c_{i,\lambda}(\lambda+\Delta)f(y_i).$$

Lastly, we prove $c_{0,\lambda}\neq 0$ by contradiction. Assume $c_{0,\lambda}=0$, then we have

$$f(y_0) = \big(R(\lambda)-4\big)^{-1}\big(-\sum_{i=1}^{4} f(x_i) + \sum_{i=1}^{6} c_{i,\lambda}(\lambda+\Delta)f(y_i)\big).$$

In addition, it is easy to see that $\{f(y_i)\}_{i=1}^6$ are uniquely determined by $\{f(x_i)\}_{i=1}^4, \{(\Delta+\lambda)f(y_i)\}_{i=1}^6$ and $f(y_0)$. As a consequence, f is uniquely determined by the 10 numbers $\{f(x_i)\}_{i=1}^4, \{(\Delta+\lambda)f(y_i)\}_{i=1}^6$, which contradicts the fact that $l(\Gamma)$ is 11 dimensional. □

Now, we return to investigate the Sierpinski lattice $\widetilde{\mathcal{SG}}$, applying the above two lemmas locally.

Lemma 2.6. *Let $\lambda\notin\{1,2,4,5,6\}$, and consider $\Delta_{(-k)}:\ell^p(\widetilde{\mathcal{SG}}^{(-k)})\to\ell^p(\widetilde{\mathcal{SG}}^{(-k)})$, $1\le p\le\infty$. Then $\lambda+\Delta_{(-k)}$ is invertible if and only if $R(\lambda)+\Delta_{(-k-1)}$ is invertible.*

Proof. Without loss of generality, we consider the $k=0$ case. Fix any point $y_0\in\widetilde{\mathcal{SG}}^{(-1)}$, and choose a neighbourhood of y_0 in $\widetilde{\mathcal{SG}}$ that is isomorphic to Γ. For each $g\in l(\widetilde{\mathcal{SG}})$, we define

$$Tg(y_0) = \sum_{i=0}^{6} c_{i,\lambda}g(y_i),$$

where $c_{i,\lambda}$ is defined in Lemma 2.5. It is clear that T is bounded from $\ell^p(\widetilde{\mathcal{SG}})$ to $\ell^p(\widetilde{\mathcal{SG}}^{(-1)})$. In addition, since $c_{0,\lambda}\neq 0$, T is surjective.

Then by using Lemma 2.5 at each point of $\widetilde{\mathcal{SG}}^{(-1)}$, the following two systems of equations give the same solutions on $\widetilde{\mathcal{SG}}$,

$$(\lambda+\Delta)f(x) = g(x), \text{ on } \widetilde{\mathcal{SG}}, \tag{2.5}$$

and

$$\begin{cases} (\lambda+\Delta)f(x) = g(x), & \text{for } x\in\widetilde{\mathcal{SG}}\setminus\widetilde{\mathcal{SG}}^{(-1)}, \\ (R(\lambda)+\Delta_{(-1)})f(x) = Tg(x), & \text{for } x\in\widetilde{\mathcal{SG}}^{(-1)}, \end{cases} \tag{2.6}$$

where $g\in\ell^p(\widetilde{\mathcal{SG}})$. But (2.6) has a unique solution in $\ell^p(\widetilde{\mathcal{SG}})$ if and only if

$$(R(\lambda)+\Delta_{(-1)})f(x) = Tg(x), \text{ on } \widetilde{\mathcal{SG}}^{(-1)}, \tag{2.7}$$

has a unique solution in $\ell^p(\widetilde{\mathcal{SG}}^{(-1)})$. The lemma follows immediately from the equivalence of solvabilty and uniqueness of solutions to (2.5) and (2.7). □

In fact, in Lemma 2.6, the only exceptions are $\lambda = \{2, 6\}$. The cases for $\lambda = 1, 4, 5, 6$ are easy to check, while the case $\lambda = 2$ needs a little more work. Luckily, by a same idea as the proof of Lemma 2.4, 2.5 and 2.6. We will get the following lemma.

Lemma 2.7. *Let $\lambda \notin \{1, 3, 4, 5, 6\} \cup R^{\circ -1}\{1, 2, 5\}$, and consider $\Delta_{(-k)} : \ell^p(\widetilde{\mathcal{SG}}^{(-k)}) \to \ell^p(\widetilde{\mathcal{SG}}^{(-k)}), 1 \leq p \leq \infty$. Then $(\lambda + \Delta_{(-k)})^{-1}$ is invertible if and only if $(R^{\circ 2}(\lambda) + \Delta_{(-k-2)})^{-1}$ is invertible.*

In particular, $(2 + \Delta_{(-k)})^{-1}$ is invertible if and only if $(-6 + \Delta_{(-k-2)})^{-1}$ is invertible.

We end this section with the proof of Theorem 2.2.

Proof of Theorem 2.2. Clearly, $\Sigma_6 = \{6\} \cup (\bigcup_{m=0}^{\infty} R^{\circ -m}\{3\}) \subset \sigma(\Delta)$. See [9] for the eigenfunctions for $\lambda \in \Sigma_6$. Thus, it is easy to see that

$$\mathcal{J} \cup \Sigma_6 = \overline{\Sigma_6} \subset \sigma(\Delta).$$

It remains to show that $\sigma(\Delta) \subset \mathcal{J} \cup \Sigma_6$. It suffices to show that if $\lambda \notin \mathcal{J} \cup \Sigma_6$, then $\lambda + \Delta$ is invertible. We consider two cases below.

First, consider $\lambda \notin \mathcal{J} \cup \Sigma_6 \cup (\bigcup_{m=0}^{\infty} R^{\circ -m}\{2\})$. It is easy to see that

$$\|\Delta_{(-k)}\|_{op} \leq 8.$$

On the other hand, by the definition of $\mathcal{J}$, there exists $k \geq 0$ such that

$$|R^{\circ k}(\lambda)| > 8 \geq \|\Delta_{(-k)}\|_{op},$$

which implies that $R^{\circ k}(\lambda) + \Delta_{(-k)}$ is invertible. By using Lemma 2.6 repeatedly, we see that $\lambda + \Delta$ is invertible.

Second, if $\lambda \in \bigcup_{m=0}^{\infty} R^{\circ -m}\{2\}$, by using Lemma 2.6 and 2.7, and a same argument as the first case, we can show that $\lambda + \Delta$ is also invertible. $\square$

3. A spectral analysis on lattices with one boundary

In this section, we focus on characterizing each point in the spectrum. We will point out that the $\ell^p, 1 < p < \infty$ cases and ℓ^1 case are very different. A full description of the spectral points in the $\ell^p, 1 < p < \infty$ cases is easy with the method developed by A. Teplyaev [9], while the ℓ^1 case is much complicated and we only give a full answer for the lattices with a boundary point.

As preparation, we define the *inner product* of real functions on $\widetilde{\mathcal{SG}}$ as follows,

$$< f, g >= \sum_{x \in \widetilde{\mathcal{SG}}} \mu_x f(x) g(x),$$

where $\mu_x = \begin{cases} 1, & \text{if } x \text{ is not a boundary point}, \\ 1/2, & \text{if } x \text{ is a boundary point}. \end{cases}$

Lemma 3.1. *Let $f \in \ell^p(\widetilde{\mathcal{SG}})$ and $g \in \ell^q(\widetilde{\mathcal{SG}})$ with $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$, then we have $< \Delta f, g >=< f, \Delta g >$.*

Proof. It is easy to see that the summation below converges absolutely, so we can rearrange the order,

$$\sum_{x\in\widetilde{\mathcal{SG}}}\sum_{y:y\sim x} g(x)f(y) = \sum_{x\in\widetilde{\mathcal{SG}}}\sum_{y:y\sim x} f(x)g(y).$$

As a result,

$$\begin{aligned}
<\Delta f, g> &= \sum_{x\in\widetilde{\mathcal{SG}}} \mu_x g(x) \sum_{y:y\sim x} \frac{1}{\mu_x}\big(f(y)-f(x)\big)\\
&= -4 < f,g > + \sum_{x\in\widetilde{\mathcal{SG}}}\sum_{y:y\sim x} g(x)f(y)\\
&= -4 < f,g > + \sum_{x\in\widetilde{\mathcal{SG}}}\sum_{y:y\sim x} f(x)g(y) =< f,\Delta g > .
\end{aligned}$$ □

In the following, we use $\sigma_c(\Delta)$ to denote the continuous spectrum of the Laplacian, and $\sigma_p(\Delta)$ for the point spectrum, $\sigma_r(\Delta)$ for the residue spectrum. As a consequence of Lemma 3.1, we have the following criterion for λ to be a residue spectral point.

Lemma 3.2. *Let $1\leq p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, and assume $\lambda\notin\sigma_p(\Delta)$ for $\Delta:\ell^p(\widetilde{\mathcal{SG}})\to\ell^p(\widetilde{\mathcal{SG}})$. Then $\lambda\in\sigma_r(\Delta)$ for $\Delta:\ell^p(\widetilde{\mathcal{SG}})\to\ell^p(\widetilde{\mathcal{SG}})$ if and only if $\lambda\in\sigma_p(\Delta)$ for $\Delta:\ell^q(\widetilde{\mathcal{SG}})\to\ell^q(\widetilde{\mathcal{SG}})$.*

Proof. The lemma is an easy application of Lemma 3.1. In fact, if $(\lambda+\Delta)(\ell^p(\widetilde{\mathcal{SG}}))$ is not dense in $\ell^p(\widetilde{\mathcal{SG}})$, there exists a non-zero $f_\lambda\in\ell^q(\widetilde{\mathcal{SG}})$ such that $< f,(\lambda+\Delta)f_\lambda >=< (\lambda+\Delta)f, f_\lambda >= 0$ for any $f\in\ell^p(\widetilde{\mathcal{SG}})$. This shows that $(\lambda+\Delta)f_\lambda=0$. Conversely, it is clear that if λ is an eigenvalue of $\Delta:\ell^q(\widetilde{\mathcal{SG}})\to\ell^q(\widetilde{\mathcal{SG}})$ with a corresponding eigenfunction $f_\lambda\in\ell^q(\widetilde{\mathcal{SG}})$, then $< (\lambda+\Delta)f, f_\lambda >= 0$ for any $f\in\ell^p(\widetilde{\mathcal{SG}})$. □

3.1. **Lattices with a boundary point.** In this part, we will characterize each spectral point for $\Delta:\ell^p(\widetilde{\mathcal{SG}})\to\ell^p(\widetilde{\mathcal{SG}})$, given the condition that $\widetilde{\mathcal{SG}}$ is a lattice with a boundary point. The result is stated as follows.

Theorem 3.3. *Write*

$$\Sigma_4=\bigcup_{m=0}^{\infty} R^{\circ -m}\{4\},\quad \Sigma_5=\bigcup_{m=0}^{\infty} R^{\circ -m}\{5\}\quad \textit{and}\quad \Sigma_6=\{6\}\bigcup\Big(\bigcup_{m=0}^{\infty} R^{\circ -m}\{3\}\Big).$$

(a). For $\Delta:\ell^p(\widetilde{\mathcal{SG}})\to\ell^p(\widetilde{\mathcal{SG}})$ with $1<p<\infty$, we have $\sigma_p(\Delta)=\Sigma_5\cup\Sigma_6$ and $\sigma_c(\Delta)=\mathcal{J}\setminus\Sigma_5$. There is no residue spectral point.

(b). For $\Delta:\ell^1(\widetilde{\mathcal{SG}})\to\ell^1(\widetilde{\mathcal{SG}})$, we have all three types of spectral points as follows,

$$\sigma_p(\Delta)=\Sigma_5\cup\Sigma_6,\quad \sigma_c(\Delta)=\mathcal{J}\setminus(\Sigma_4\cup\Sigma_5\cup\{0\}),\quad \sigma_r(\Delta)=\{0\}\cup\Sigma_4.$$

By Lemma 3.2, Theorem 3.3 is an immediate consequence of the following proposition.

Proposition 3.4. *For $\Delta:\ell^\infty(\widetilde{\mathcal{SG}})\to\ell^\infty(\widetilde{\mathcal{SG}})$, we have $\sigma_p(\Delta)=\{0\}\cup\Sigma_4\cup\Sigma_5\cup\Sigma_6$.*

Proof. Without loss of generality, we take $\omega = 0000\cdots$ and $\widetilde{\mathcal{SG}} = \bigcup_{m=0}^{\infty} F_0^{-m} V_m$, since any two lattices with a boundary point are isomorphic to each other [9]. The following proof relies on Lemma 3.5 and Lemma 3.6, which will be stated later.

The existence of 5-series and 6-series eigenfunctions is a well-known result of the spectral decimation, see [9]. See Appendix for the existence of the 4-series eigenfunctions. So it suffices to show that there are no other ℓ^∞ eigenvalues.

Take $\lambda \in \mathcal{J} \setminus \big(\{0\} \cup \Sigma_4 \cup \Sigma_5\big)$, and let f be a λ-eigenfunction. For convenience, we write $q_i^{(-m)} = F_0^{-m} q_i, i = 0, 1, 2$. By direct computation, and using the decimation method, we get

$$\begin{aligned}
\begin{pmatrix} f(q_0) \\ f(q_1) \\ f(q_2) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{4-\lambda}{(2-\lambda)(5-\lambda)} & \frac{4-\lambda}{(2-\lambda)(5-\lambda)} & \frac{2}{(2-\lambda)(5-\lambda)} \\ \frac{4-\lambda}{(2-\lambda)(5-\lambda)} & \frac{2}{(2-\lambda)(5-\lambda)} & \frac{4-\lambda}{(2-\lambda)(5-\lambda)} \end{pmatrix} \begin{pmatrix} f(q_0) \\ f(q_1^{(-1)}) \\ f(q_2^{(-1)}) \end{pmatrix} \\
&= f(q_0) \begin{pmatrix} 1 \\ 1-\frac{\lambda}{4} \\ 1-\frac{\lambda}{4} \end{pmatrix} + \frac{\lambda\big(f(q_1^{(-1)}) - f(q_2^{(-1)})\big)}{2R(\lambda)} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
&+ \Big(\frac{f(q_1^{(-1)}) + f(q_2^{(-1)})}{2} - (1 - \frac{R(\lambda)}{4}) f(q_0)\Big) \frac{6-\lambda}{(2-\lambda)(5-\lambda)} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},
\end{aligned} \tag{3.1}$$

as f is an $R(\lambda)$-eigenfunction on $\widetilde{\mathcal{SG}}^{(-1)}$. In addition,

$$f(q_1^{(-m)}) + f(q_2^{(-m)}) - (2 - \frac{R^{\circ m}(\lambda)}{2}) f(q_0) = 0, \forall m \geq 0. \tag{3.2}$$

By using (3.1) and (3.2) repeatedly and using symmetry, we get

$$\begin{pmatrix} f(q_0) \\ f(q_1) \\ f(q_2) \end{pmatrix} = f(q_0) \begin{pmatrix} 1 \\ 1-\frac{\lambda}{4} \\ 1-\frac{\lambda}{4} \end{pmatrix} + \frac{\lambda\big(f(q_1^{(-m)}) - f(q_2^{(-m)})\big)}{2R^{\circ m}(\lambda)} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{3.3}$$

and

$$\begin{aligned}
\begin{pmatrix} f(F_0^{-m} F_1^m q_0) \\ f(q_1^{(-m)}) \\ f(F_0^{-m} F_1^m q_2) \end{pmatrix} &= f(q_1^{(-m)}) \begin{pmatrix} 1-\frac{\lambda}{4} \\ 1 \\ 1-\frac{\lambda}{4} \end{pmatrix} + \frac{\lambda\big(f(q_0) - f(q_2^{(-m)})\big)}{2R^{\circ m}(\lambda)} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
+ \Big(\frac{f(q_0) + f(q_2^{(-m)})}{2} &- (1 - \frac{R^{\circ m}(\lambda)}{4}) f(q_1^{(-m)})\Big) \prod_{l=0}^{m-1} \frac{6 - R^{\circ l}(\lambda)}{(2 - R^{\circ l}(\lambda))(5 - R^{\circ l}(\lambda))} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned} \tag{3.4}$$

As a consequence of (3.2) and (3.3), we get the estimate

$$\max\{|f(q_0)|, |f(q_1)|, |f(q_2)|\} \leq C(1 \vee \frac{\lambda}{R^{\circ m}(\lambda)}) \cdot \max\{|f(q_1^{(-m)})|, |f(q_2^{(-m)})|\}. \tag{3.5}$$

On the other hand, by using (3.2) and (3.4), we get the equation

$$\begin{aligned} f(F_0^{-m}F_1^m q_0) + f(F_0^{-m}F_1^m q_2) &= (2-\frac{\lambda}{2})f(q_1^{(-m)}) \\ &+\big((\frac{2}{4-R^{\circ m}(\lambda)} - 2 + \frac{R^{\circ m}(\lambda)}{2})f(q_1^{(-m)}) + (\frac{2}{4-R^{\circ m}(\lambda)}+1)f(q_2^{(-m)})\big)P_m \\ &= (2-\frac{\lambda}{2})f(q_1^{(-m)}) + P_m(\lambda)(a_m f(q_1^{(-m)}) + b_m f(q_2^{(-m)})). \end{aligned} \tag{3.6}$$

where

$$P_m(\lambda) = \prod_{l=0}^{m-1} \frac{6-R^{\circ l}(\lambda)}{(2-R^{\circ l}(\lambda))(5-R^{\circ l}(\lambda)}, \tag{3.7}$$

$a_m = \frac{2}{4-R^{\circ m}(\lambda)} - 2 + \frac{R^{\circ m}(\lambda)}{2}$ and $b_m = \frac{2}{4-R^{\circ m}(\lambda)} + 1$. By symmetry, we also have

$$\begin{aligned} &f(F_0^{-m}F_2^m q_0) + f(F_0^{-m}F_2^m q_1) \\ &\qquad = (2-\frac{\lambda}{2})f(q_2^{(-m)}) + P_m(\lambda)(a_m f(q_2^{(-m)}) + b_m f(q_1^{(-m)})). \end{aligned} \tag{3.8}$$

It is easy to check that $a_m^2 \neq b_m^2$ if $R^{\circ m}(\lambda) \in \mathcal{J} \setminus \{0,4\}$. By using Lemma 3.5 below, we can find an increasing sequence $\{m_k\}_{k=1}^\infty$ such that $R^{\circ m_k}(\lambda)$ is bounded away from 0 and 4. Thus, we have

$$\begin{aligned} &\max\{|f(F_0^{-m_k}F_1^{m_k}q_0) + f(F_0^{-m_k}F_1^{m_k}q_2)|, |f(F_0^{-m_k}F_2^{m_k}q_0) + f(F_0^{-m_k}F_2^{m_k}q_1)|\} \\ \geq &(2C|P_{m_k}(\lambda)| - |2-\frac{\lambda}{2}|)\max\{|f(q_1^{(-m_k)})|, |f(q_2^{(-m_k)})|\}, \end{aligned} \tag{3.9}$$

where C is independent of k. However, according to Lemma 3.6, we can see that $\lim_{k\to\infty} |P_{m_k}(\lambda)| = +\infty$. Combining the estimates (3.5) and (3.9), and letting $k \to \infty$, we see that f is unbounded. □

At the end of this subsection, we prove the lemmas that are used in the proof of Proposition 3.4.

First, let's recall some basic facts about the totally disconnected Julia set, which can be found in many textbooks, see for example [2]. Denote $\varphi_-(x) = \frac{5-\sqrt{25-4x}}{2}$ and $\varphi_+(x) = \frac{5+\sqrt{25-4x}}{2}$. There is a natural homeomorphism π from the Cantor set $\mathcal{C} = \{-,+\}^{\mathbb{N}}$ to $\mathcal{J}$ defined as follows

$$\pi(\eta) = \bigcap_{m=1}^{\infty} \varphi_{\eta_1}\varphi_{\eta_2}\cdots\varphi_{\eta_m}(\mathcal{J}),$$

for $\eta = \eta_1\eta_2\cdots \in \mathcal{C}$. In particular, $\pi(---\cdots) = 0$ and $\pi(+++\cdots) = 4$. In addition, define the left shift operator $\iota : \mathcal{C} \to \mathcal{C}$ by $\iota(\eta_1\eta_2\eta_3\cdots) = \eta_2\eta_3\cdots$. Then

$$R \circ \pi(\eta) = \pi \circ \iota(\eta), \forall \eta \in \mathcal{C}.$$

Lemma 3.5. *Let $\lambda \in \mathcal{J} \setminus \big(\{0\} \cup \Sigma_4 \cup \Sigma_5\big)$.*

(a). There is a sequence $m_1 < m_2 < \cdots$ such that $\min\{|R^{\circ m_k}(\lambda)|, |R^{\circ m_k}(\lambda) - 4|\} > C$, where C is a positive constant independent of λ.

(b). There is an infinite sequence $n_1 < n_2 < \cdots$ such that $R^{\circ n_k}(\lambda) \in \varphi_-(\mathcal{J})$.

Proof. (a). Clearly, we can find countably infinite different positive integers m such that $\iota^{\circ m}\pi^{-1}(\lambda)$ are of the form $+-\cdots$ or $-+\cdots$. This means $R^{\circ m}(\lambda) \in \varphi_-\varphi_+(\mathcal{J}) \cup \varphi_+\varphi_-(\mathcal{J})$, and thus $R^{\circ m}(\lambda)$ is bounded away from $\{0, 4\}$.

(b). Clearly, we can find countably infinite different integers n such that $\big(\iota^{\circ n}\pi^{-1}(\lambda)\big)_1 = -$. $\square$

Next, we give an estimate for $P_m(\lambda)$ in (3.7).

Lemma 3.6. *For $\lambda \in \mathcal{J} \setminus \Sigma_4$, we have*

$$\lim_{m\to\infty} |P_m(\lambda)| = \lim_{m\to\infty} \prod_{l=0}^{m-1} \Big|\frac{6 - R^{\circ l}(\lambda)}{(2 - R^{\circ l}(\lambda))(5 - R^{\circ l}(\lambda)}\Big| = \infty.$$

Proof. Using the fact that $R(x) = x(5 - x)$ and by direct computation, we can show

$$|P_m(\lambda)| = \Big|\frac{\lambda(6-\lambda)}{R^{\circ m}(\lambda)\big(2 - R^{\circ m-1}(\lambda)\big)}\Big| \cdot \prod_{l=0}^{m-2} |3 - R^{\circ l}(\lambda)|.$$

Thus it suffices to show $\lim\limits_{m\to\infty} \prod_{l=0}^{m} |3 - R^{\circ l}(\lambda)| = \infty$.

FIGURE 4. An illustration of the A, B, C areas and approximated values of the end points.

Let $A = \varphi_-(\mathcal{J})$, $B = \varphi_+(\mathcal{J}) \cap (0, 4]$ and $C = \varphi_+(\mathcal{J}) \cap (4, 5]$, so that $\mathcal{J} = A \cup B \cup C$. See Figure 4 for an illustration. For $x \in A$, we have $|3 - x| > 1.5$; for $x \in C$, we have $|3 - x| > 1$; For $x \in B$, we have $R(x) \in C$ and $(3 - x)\big(3 - R(x)\big) > 1$ by an easy estimate.

As a consequence, we have the estimates

$$\begin{aligned}
\prod_{l=0}^{m} |3 - R^{\circ l}(\lambda)| &\geq c\Big(\prod_{l\in I_{m,A}} |3 - R^{\circ l}(\lambda)|\Big) \cdot \Big(\prod_{l\in I_{m,B}} |(3 - R^{\circ l}(\lambda))\big(3 - R^{\circ l+1}(\lambda)\big)|\Big) \\
&\geq c(\frac{3}{2})^{\# I_{m,A}},
\end{aligned}$$

where $c = \min\{|x - 3| : x \in \mathcal{J}\}$, $I_{m,A} = \{0 \leq l \leq m : R^{\circ l}(\lambda) \in A\}$ and $I_{m,B} = \{0 \leq l \leq m - 1 : R^{\circ l}(\lambda) \in B\}$. The lemma follows immediately from Lemma 3.5 (b). $\square$

3.2. **Lattices with no boundary.** In A. Teplyaev's work [9], it was shown that the localized eigenfunctions form a complete basis of the $\ell^2(\widetilde{\mathcal{SG}})$ space. The basic idea is to find a localized eigenfunction f_λ such that $< f, f_\lambda > \neq 0$ for each nonzero $f \in \ell^2(\widetilde{\mathcal{SG}})$. This same proof can be easily extended to $C_0(\widetilde{\mathcal{SG}})$ case, where $C_0(\widetilde{\mathcal{SG}}) = \{f \in \ell^\infty(\widetilde{\mathcal{SG}}) : \lim_{x\to\infty} f(x) = 0\}$. We state the result as follows.

Lemma 3.7. *For any $f \in C_0(\widetilde{\mathcal{SG}})$, there exists a localized eigenfunction f_λ of Δ such that $< f_\lambda, f > \neq 0$.*

As an immediate consequence, there is no eigenvalue of Δ on $C_0(\widetilde{\mathcal{SG}})$ other than the 5 or 6 series.

Proposition 3.8. *Let* $\Delta : C_0(\widetilde{\mathcal{SG}}) \to C_0(\widetilde{\mathcal{SG}})$, *we have* $\sigma_p(\Delta) = \Sigma_5 \cup \Sigma_6$.

Proof. Assume there exists an eigenvalue $\lambda \notin \Sigma_5 \cup \Sigma_6$, and let f_λ be the corresponding eigenfunction. By Lemma 3.7, there is a localized eigenfunction $f_{\lambda'}$ such that $< f_\lambda, f_{\lambda'} > \neq 0$. On the other hand, $\lambda \neq 0$, and

$$\lambda^{-1} < -\Delta f_\lambda, f_{\lambda'} > = < f_\lambda, f_{\lambda'} > = \lambda'^{-1} < f_\lambda, -\Delta f_{\lambda'} > .$$

Then by Lemma 3.1, this implies that $< f_\lambda, f_{\lambda'} > = 0$, a contradiction. □

Theorem 3.9. *For* $1 < p < \infty$, $\Delta : \ell^p(\widetilde{\mathcal{SG}}) \to \ell^p(\widetilde{\mathcal{SG}})$ *has point spectrum* $\sigma_p(\Delta) = \Sigma_5 \cup \Sigma_6$, *and continuous spectrum* $\sigma_c(\Delta) = \mathcal{J} \setminus \Sigma_5$. *There is no residue spectrum.*

Proof. As a direct consequence of Proposition 3.8, $\sigma_p(\Delta) = \Sigma_5 \cup \Sigma_6$. In addition, we see that there is no residue spectral point by Lemma 3.2. □

However, the ℓ^1 spectrum of the Laplacian is much complicated on the lattices without boundary, and it seems possible that the eigenvalues for $\Delta : \ell^\infty(\widetilde{\mathcal{SG}}) \to \ell^\infty(\widetilde{\mathcal{SG}})$ depend on the generating sequence ω of the lattice. Further researches are suggested in the future.

4. Appendix

In this appendix, we construct the 4-eigenfunctions on $\widetilde{\mathcal{SG}}$. Note that this induces a class of eigenvalues $\Sigma_4 = \bigcup_{m=0}^{\infty} R^{\circ -m}\{4\}$.

We introduce the following orthogonal matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

Recall that if we fix an infinite word $\omega = \omega_1\omega_2\cdots$, then there is a Sierpinski lattice defined by $\widetilde{\mathcal{SG}} = \bigcup_{m=0}^{\infty} F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1} V_m$. For convenience, we write

$$q_i^{(-m)} = F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1}(q_i), \quad i = 0, 1, 2,$$

and we write

$$q_{li}^{(-m)} = F_{\omega_1}^{-1} F_{\omega_2}^{-1} \cdots F_{\omega_m}^{-1}(F_l q_i), \quad l \in W_m = \{0, 1, 2\}^m.$$

Clearly $q_i = q^{(-m)}_{\omega_m\omega_{m-1}\cdots\omega_1 i}$.

Proposition 4.1. *(a). If* $\widetilde{\mathcal{SG}}$ *has no boundary, then there is a three dimensional* ℓ^∞ *eigenspace of* Δ *corresponding to* 4. *(b). If* $\widetilde{\mathcal{SG}}$ *has a boundary point, then there is a two dimensional* ℓ^∞ *eigenspace of* Δ *corresponding to* 4.

Proof. (a). Let $f(q_0) = a, f(q_1) = b, f(q_2) = c$, where a, b, c are arbitrary real number. Define

$$\begin{pmatrix} f(q_0^{(-m)}) \\ f(q_1^{(-m)}) \\ f(q_2^{(-m)}) \end{pmatrix} = A_{\omega_m}^{-1} \cdots A_{\omega_2}^{-1} A_{\omega_1}^{-1} \begin{pmatrix} f(q_0) \\ f(q_1) \\ f(q_2) \end{pmatrix}, \tag{4.2}$$

and

$$\begin{pmatrix} f(q_{l0}^{(-m)}) \\ f(q_{l1}^{(-m)}) \\ f(q_{l2}^{(-m)}) \end{pmatrix} = A_{l_1} A_{l_2} \cdots A_{l_m} \begin{pmatrix} f(q_0^{(-m)}) \\ f(q_1^{(-m)}) \\ f(q_2^{(-m)}) \end{pmatrix}, \quad \forall l \in W_m. \tag{4.3}$$

See Figure 5 for an example of the extension of f. One can easily check that f is a 4-eigenfunction of Δ on $\widetilde{\mathcal{SG}}$ and f is bounded. By the above construction, we get a three dimensional eigenspace to 4.

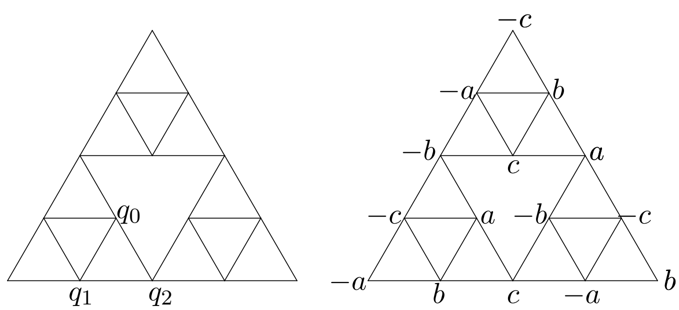

Figure 5. An illustration for extending f to be a 4-eigenfunction.(We take $\omega_1 = 2, \omega_2 = 1$ as shown in the left picture.)

On the other hand, noticing that 4 is not a forbidden eigenvalue, a 4-eigenfunction f is unqiuely determined by $f|_{V_0}$.

(b). The proof of (b) is essentially the same. The eigenspace is 2 dimensional as a consequence of the eigenvalue equation at the boundary point. □

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