SOME PROPERTIES OF THE DERIVATIVES ON SIERPINSKI GASKET TYPE FRACTALS

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Abstract. In this paper, we focus on Strichartz's derivatives, a family of derivatives including the normal derivative, on p.c.f. (post critically finite) fractals, which are defined at vertices in the graphs that approximate the fractal. We obtain a weak continuity property of the derivatives for functions in the domain of the Laplacian. For a function with zero normal derivative at any fixed vertex, the derivatives, including the normal derivatives, of the neighboring vertices will decay to zero. The rates of approximations are described and several non-trivial examples are provided to illustrate that our estimates are optimal. We also study the boundedness property of derivatives for functions in the domain of the Laplacian. A necessary condition for a function having a weak tangent of order one at a vertex is provided. Furthermore, we give a counter-example of a conjecture of Strichartz on the existence of higher order weak tangents.

1. Introduction

The theory of analysis on fractals, analogous to that on manifolds, has been being well developed. The pioneering work is the analytic construction of the Laplacians, for a class of self-similar fractals that include the Sierpinski gasket as a typical example, developed by Kigami [15-20], in which the Laplacians are defined as renormalized limits of graph Laplacians. There are a lot of works in exploring some properties of these fractal Laplacians that are natural analogs of those of the usual Laplacian. See [1, 2, 7, 9, 13, 23-26, 28-36, 38] and the references therein. Especially, there were several works in developing a calculus on fractals [3, 8, 22, 27, 34, 39].

Since the fractal Laplacian acts as a differential operator with order greater than one, in analogy with the usual Laplacians on manifolds which are of second order (see [35, 37] for explanations), it is natural to make clear what is the first order derivative or gradient. Basically, there are two approaches. One is to regard the Dirichlet form as an integral of the inner product of gradients, see [4-6, 10-12, 14, 17, 22] for some works on this approach. Please see [11] for a survey on recent developments and more references therein. It seems that this could not provide any direct information for a pointwise gradient. The other is to define the pointwise gradient directly. Teplyaev [39] has made a satisfactory definition of the gradient at general points in fractals and obtained some properties. For the vertices (boundary

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points of cells) in fractals, starting from the normal derivative, Strichartz [34] has introduced a family of derivatives at any vertex x, and using which, he has made up a (local) gradient $df(x)$. See [39] to find a description of the relations between these different definitions and the results of Kigami, Kusuoka, Teplyaev and Strichartz.

In this paper, we continue to study the properties of Strichartz's derivatives at vertices in fractals.

We begin by assuming that a fractal K is an invariant set of a finite iterated function system (i.f.s.) of contractive similarities in some Euclidean space $\mathbb{R}^d$, which means K is the unique nonempty compact set satisfying

$$K = \bigcup_{i=1}^{N} F_i K,$$

where we denote the mappings by $\{F_i\}_{i=1,\cdots,N}$. We define $W_n = \{1, \cdots, N\}^n$, the set of words of length n, and write $F_w = F_{w_1} \circ \cdots \circ F_{w_n}$ for a word $w = w_1 \cdots w_n \in W_n$. We call $F_w K$ a *cell* of level n.

We use Strichartz's definition of the p.c.f. self-similar sets [37], which is simpler than Kigami's one [16], although all our results could be derived in his context. K is a *post critically finite (p.c.f.) self-similar set* if K is connected, and there is a finite set $V_0 \subset K$ called the *boundary*, such that for any two different words w and w' of the same length, the intersection of $F_w K$ and $F_{w'} K$ is contained in the intersection of their boundaries, i.e., $F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0$.

Denote by $V_n = \bigcup_{w \in W_n} F_w V_0$ and $V_* = \bigcup_{n \geq 0} V_n$. A point $x \in V_*$ is called a *junction vertex* if there are at least two different $w, w' \in W_n$ for some n such that $x \in F_w K \cap F_{w'} K$. Otherwise we call x a *nonjunction vertex*.

We assume that there is a regular harmonic structure on the p.c.f. self-similar set K. Thus there exists a self-similar *Dirichlet form* $\mathcal{E}$ on K such that for functions $f : K \to \mathbb{R}$, one has

$$\mathcal{E}(f) = \sum_{j=1}^{N} r_j^{-1} \mathcal{E}(f \circ F_j)$$

for some choice of *renormalization factors* $r_1, \cdots, r_N \in (0, 1)$. This quadratic form is obtained as the limit of $\mathcal{E}_m(f) := \mathcal{E}_m(f, f)$ on the m-level approximating graphs, where the m-level bilinear form $\mathcal{E}_m(\cdot, \cdot)$ is defined as

$$\mathcal{E}_m(f, g) = \sum_{|w|=m} r_w^{-1} \mathcal{E}_0(f \circ F_w, g \circ F_w),$$

with

$$\mathcal{E}_0(f, g) = \sum_{1 \leq i < j \leq N_0} c_{ij}(f(v_i) - f(v_j))(g(v_i) - g(v_j)),$$

for some positive *conductances* c_{ij}. Here we write $r_w = r_{w_1} \cdots r_{w_m}$ for $w = w_1 \cdots w_m$.

Let $\mathcal{H}_0$ denote the space of *harmonic functions* on K that minimize $\mathcal{E}_m$ at all levels for given boundary values on V_0. Let $S(\mathcal{H}_0, V_m)$ denote the space of continuous functions whose restrictions to each cell $F_w K$ of level m are harmonic (i.e., $h \circ F_w$ is harmonic for any $w \in W_m$).

Readers may refer to the books [21] and [37] for exact definitions and any unexplained notations.

In this paper, two additional assumptions are made, as Strichartz did in [34].

Hypothesis 1.1. *(a) Each point $v_j, j = 1, \cdots, N_0$ in the boundary set V_0 is the fixed point of a unique mapping in the i.f.s., which we denote F_j. Also, we assume that for any F_i and F_j in the i.f.s., $i \neq j$, the intersection $F_iK \cap F_jK$ consists of at most one point x with $x = F_iv_m = F_jv_n$ for some vertices v_m and v_n in V_0.*

(b) For each $v_j \in V_0$, let M_j denote the $N_0 \times N_0$ matrix that transforms the values $h|_{V_0}$ to $h|_{F_jV_0}$ for harmonic functions h, i.e.,

$$h(F_jv_k) = \sum_{l=1}^{N_0} (M_j)_{kl} h(v_l).$$

We assume that each M_j has a complete set of real left eigenvectors β_{jk} with real nonzero eigenvalues λ_{jk}, i.e.,

$$\beta_{jk} M_j = \lambda_{jk} \beta_{jk},$$

where for each j the eigenvalues λ_{jk} are labeled in decreasing order of absolute value.

We will list some basic properties of the eigenvalues and eigenvectors of the matrixes M_j in the next section. Here we only mention that the largest eigenvalue of M_j is $\lambda_{j1} = 1$, the second largest eigenvalue is $\lambda_{j2} = r_j$, the j-th renormalization factor of the harmonic structure, and $|\lambda_{jk}| < \lambda_{j2}$ for $k \geq 3$.

The following is the definition of Strichartz's derivatives at the boundary vertices.

Definition 1.2. *Let f be a continuous function defined in a neighborhood of $v_j \in V_0$. The derivatives $d_{jk}f(v_j)$ for $2 \leq k \leq N_0$ are defined as the following limits, if they exist,*

$$d_{jk}f(v_j) = \lim_{m\to\infty} \lambda_{jk}^{-m} \beta_{jk} f|_{F_j^m V_0}$$

where $\beta_{jk} f|_{F_j^m V_0}$ is

$$\sum_{l=1}^{N_0} (\beta_{jk})_l f(F_j^m v_l).$$

The derivative $d_{j2}f(v_j)$ is just the normal derivative of f at v_j with suitable choice of β_{j2}, and $d_{jk}f(v_j), k \geq 3$ could be viewed as derivatives of somewhat "higher order". If h is harmonic in a neighborhood of v_j, then all derivatives $d_{jk}h(v_j)$ exist and may be evaluated without taking the limit. See Lemma 3.3 in [34].

The above definition could be extended to all vertices in V_*. For a nonjunction vertex $x \in V_n \setminus V_{n-1}$, there is a unique word w of length n such that $x = F_wv_j$ for some $1 \leq j \leq N_0$. We write $U_m(x) = F_wF_j^mK$, and call $\{U_m(x)\}_{m\geq 0}$ a *standard system of neighborhoods* of x. For a junction vertex $x \in V_n \setminus V_{n-1}$, by Hypotheses 1.1(a), it is just an image under a mapping F_w of a junction vertex in V_1, where w is a word of length $n-1$. Let $J(x)$ denote the set of indices j such that there exist $1 \leq j' \leq N_0$ with $x = F_wF_jv_{j'}$. Obviously, $\sharp J(x) \geq 2$. We write $U_m(x) = \bigcup_{j\in J(x)} F_wF_jF_{j'}^mK$, and call $\{U_m(x)\}_{m\geq 0}$ a *standard system of neighborhoods* of x.

Definition 1.3. *Let f be a continuous function defined in a neighborhood of a vertex $x \in V_n \setminus V_{n-1}$.*

(a) If $x = F_wv_j$ is a nonjunction vertex, then the derivatives $d_{jk}f(x)$ for $2 \leq k \leq N_0$ are defined as the following limits, if they exist,

$$d_{jk}f(x) = \lim_{m\to\infty} r_w^{-1} \lambda_{jk}^{-m} \beta_{jk} f|_{F_wF_j^mV_0}. \tag{1.1}$$

(b) If x is a junction vertex, then the derivatives $d_{j'k}f(x)$ for $j \in J(x)$ and $2 \leq k \leq N_0$ are defined as the following limits, if they exist,

$$d_{j'k}f(x) = \lim_{m\to\infty} r_w^{-1} r_j^{-1} \lambda_{j'k}^{-m} \beta_{j'k} f|_{F_w F_j F_{j'}^m V_0}.$$

Furthermore, the normal derivatives of f at x are said to satisfy the compatibility condition if

$$\sum_{j\in J(x)} d_{j'2}f(x) = 0.$$

We write $df(x)$ for the collection of derivatives at x, and refer to it as the *gradient* of f at x. f is called *differentiable* at the vertex x if all the derivatives at x exist and the compatibility condition holds if x is a junction vertex. For example, if h is harmonic in a neighborhood of x, then h is differentiable at x and all the derivatives may be evaluated without taking the limit. See Lemma 3.6 in [34].

Let μ be a self-similar measure on K with weights $(\mu_1, \cdots, \mu_N)$. It is known that for any function f in $dom(\Delta_\mu)$, the normal derivatives $d_{j2}f(x)$ or $d_{j'2}f(x)$ always exist at any vertex x and satisfy the compatibility condition if x is a junction vertex, where the notation $dom(\Delta_\mu)$ denotes the domain of the Laplacian with respect to the measure μ. However, for other derivatives, in general, we need the assumption

$$r_j\mu_j < |\lambda_{jN_0}|, \forall 1 \leq j \leq N_0 \tag{1.2}$$

to guarantee the existence of $d_{jk}f(x)$ or $d_{j'k}f(x)$, $k \geq 3$. For fractals without symmetry, the condition is necessary. See Theorem 4.1 in [34] for details.

Remark 1.4. For "higher order" derivatives d_{jk} or $d_{j'k}(3 \leq k \leq N_0)$, there are two different scalings. Let $x = F_w v_j$ be a nonjunction vertex. Then for any word u, we have

$$d_{jk}(f \circ F_u^{-1})(F_u x) = r_u^{-1} d_{jk}f(x),$$

and for any $m \geq 0$, we have

$$d_{jk}(f \circ F_w F_j^m F_w^{-1})(x) = \lambda_{jk}^m d_{jk}f(x).$$

The case of junction vertices is very similar. We omit it.

It was proved in [34] that for a function $f \in dom(\Delta_\mu)$, the normal derivatives $d_{j2}f(x)$ and $d_{j'2}f(x)$ are uniformly bounded as x varies over all vertices. And for a harmonic function h with zero normal derivative at a vertex x, the normal derivatives of its neighboring vertices will decay to zero, which can be interpreted as a *weak continuity property* of the normal derivatives. See Theorem 4.3 in [34].

For general functions in $dom(\Delta_\mu)$, it is natural to expect the same properties. In fact, an easy observation is that if we assume (1.2) and $d_{j2}f(v_j) = 0$ for some vertex $v_j \in V_0$, by Lemma 6.4 of [29], we have

$$\|f - h\|_{L_\infty(F_j^m K)} \leq C_1 m (r_j\mu_j)^m \|\Delta_\mu f\|_\infty,$$

where h is the unique harmonic function defined on $F_j^m K$ satisfying $dh(v_j) = df(v_j)$ and $h(v_j) = f(v_j)$. Also, we have $\|h\|_{L_\infty(F_j^m K)} \leq C_2|\lambda_{j3}|^m$. Then using the local Gauss-Green's formula, letting ϕ be the unique harmonic function on $F_j^m K$ with value 1 at $F_j^m v_i$ and 0 at other vertices in $F_j^m V_0$ for some $i \neq j$, we have

$$\begin{aligned} |d_{i2}f(F_j^m v_i)| &\leq |\int_{F_j^m K} (\Delta_\mu f)\phi d\mu| + \sum_{l=1}^{N_0} |d_{l2}\phi(F_j^m v_l)| \cdot |f(F_j^m v_l)| \\ &\leq \mu_j^m \|\Delta_\mu f\|_\infty + C r_j^{-m} \|f\|_{L_\infty(F_j^m K)} = O((\lambda_{j3} r_j^{-1})^m). \end{aligned}$$

The above discussion is rough but shows that the expected weak continuity property is reasonable.

In this paper we will drop the assumption (1.2), and show that the normal derivative is continuous at any vertex x with vanished normal derivative, which means that the normal derivatives at all vertices in $U_m(x)$(not only in $\partial U_m(x)$) will go to zero as m goes to infinity. Nevertheless, for "higher order" derivatives, we will still obtain the boundedness property, and similar weak continuity property for functions in $dom(\Delta_\mu)$. For "higher order" derivatives, the assumption (1.2) is necessary, since it guarantees the existence of the derivatives. We will provide the optimal estimates for the rates of the all the above approximations.

We will prove the following three theorems. These results answer the question post by Strichartz in [34] positively.

Theorem 1.5. *Let $f \in dom(\Delta_\mu)$. Then the normal derivative of $f(x)$ is bounded by a multiple of $\|f\|_\infty + \|\Delta_\mu f\|_\infty$ as x varies over all vertices. Furthermore, for any fixed nonjunction vertex $x = F_w v_j$ (or junction vertex $x = F_w F_j v_{j'}$), if $d_{j2}f(x) = 0$ (or $d_{j'2}f(x) = 0$), we have the optimal estimate*

$$d_{i2}f(y)(or\ (d_{i'2}f(y)) = \begin{cases} O(\mu_j^m), & if\ r_j\mu_j > |\lambda_{j3}|, \\ O(m\mu_j^m), & if\ r_j\mu_j = |\lambda_{j3}|, \\ O((\lambda_{j3}r_j^{-1})^m), & if\ r_j\mu_j < |\lambda_{j3}|, \end{cases}$$

for all vertices $y \in U_m(x)$.

Theorem 1.6. *(a) Let h be a harmonic function. Then all the derivatives of $h(x)$ are uniformly bounded as x varies over all vertices.*

(b) Assume (1.2). Let $f \in dom(\Delta_\mu)$, then f is differentiable at all vertices and all the derivatives of f are uniformly bounded by a multiple of $\|f\|_\infty + \|\Delta_\mu f\|_\infty$.

Theorem 1.7. *(a) Let h be a harmonic function, $x = F_w v_j$ be a nonjunction vertex (or $x = F_w F_j v_{j'}$ be a junction vertex) with zero normal derivative. Then for any vertices $y \in U_m(x) \setminus \{x\}$, we have the optimal estimate*

$$d_{ik}h(y)(or\ d_{i'k}h(y)) = O((\lambda_{j3}r_j^{-1})^m), \forall k \geq 3.$$

(b) Assume (1.2). Let $f \in dom(\Delta_\mu)$, and x be a vertex with zero normal derivative, then the above estimate still holds, with f replaced by h.

Several non-trivial examples will be provided to illustrate that our estimates are optimal. There are some typical fractals, including the *Sierpinski gasket*, for which the condition (1.2) does not hold. However, for these fractals, the results in Theorem 1.6 and 1.7 are still valid, provided that $\Delta_\mu f$ satisfies an appropriate Hölder condition.

These results will be given in Section 3 and Section 4. We remark here that when we consider the *energy Laplacian* Δ_ν in replace of Δ_μ, where ν is the Kusuoka measure [22], we will have very similar results. The only difference is that for each $1 \leq j \leq N_0$, the order of $\nu(F_j^m K)$ should be r_j^m in replace of μ_j^m in the discussion.

We also study tangents in this paper. As in [34], for a function f differentiable at a vertex x, a *weak tangent of order one*, denoted as $T_1^x(f)$ at x, is defined as a harmonic function on $U_0(x)$, which assumes the same value and gradient at x as those of f.

For any function f which is differentiable at a vertex x, let h_m denote the harmonic function that assumes the same values as f at the boundary points of $U_m(x)$, extended to be harmonic on $U_0(x)$. In Theorem 3.11 in [34], it is proved that h_m

converges uniformly to $T_1^x(f)$ on $U_0(x)$ as $m \to \infty$. However, we will prove that it is not true in general, unless we assume some additional reasonable assumptions on the harmonic structure and the self-similar measure.

If we assume that $\sharp V_0 = 3$ and all structures have the full $D3$ symmetry, we could extend the definition of order one tangent to higher order. Here $D3$ symmetry means that all the structures are invariant under any homeomorphism of K. In this case, all r's and μ's should be the same. Denote ρ the value of $r_j\mu_j$ and λ_3 the value of λ_{j3} for $j = 1, 2, 3$ since they are the same, respectively. Then for a vertex x and a function f defined in a neighborhood of x, an n-harmonic function h is called a *weak tangent of order* n of f at x if

$$(f-h)|_{\partial U_m(x)} = o((\rho^{n-1}r)^m), \tag{1.3}$$

and

$$(f-h-(f-h)\circ g_x)|_{\partial U_m(x)} = o((\rho^{n-1}\lambda_3)^m),$$

where *n-harmonic functions* means those functions satisfying the equation $\Delta_\mu^n h = 0$, and g_x is a local point symmetry at x with reasonable understanding(we omit the exact definition).

In [34], there is a conjecture, Conjecture 6.7, saying that for a function $f \in dom(\Delta_\mu^{n-1})$, f has a weak tangent of order n at x if and only if $d\Delta_\mu^k f(x)$ exists with compatibility conditions holding at x for each $k \le n-1$. It is true for $n = 1$ since it is exact the definition of order one tangent. However, it is not true for $n \ge 2$. We will give a counter-example.

The results about tangents will be given in Section 5.

This paper can be regarded as a supplement of [34]. Before the end of this section, we mention a very useful result which could be obtained by an easy combination of the results in the appendix of [34] and the results in the appendix of [39], saying that, any function f in $dom(\Delta_\mu)$ satisfies a Hölder estimate that

$$|f(x) - f(y)| \le c r_w(\|f\|_\infty + \|\Delta_\mu f\|_\infty) \tag{1.4}$$

for any $x, y \in F_w K$ and any word w, where c is a positive constant.

2. Basic results of the eigenvectors of M_j

In this section, we will give some basic properties of the eigenvalues and eigenvectors of the transformation matrix M_j. Let $\{\lambda_{jk}\}_{1\le k\le N_0}$ be the set of eigenvalues labeled in decreasing order of absolute value. For each λ_{jk}, we denote β_{jk} and α_{jk} the left and right eigenvectors of λ_{jk} respectively. Additionally, we normalize that $\beta_{jk}\alpha_{jk} = 1$.

Proposition 2.1. *(a) The largest eigenvalue of M_j is $\lambda_{j1} = 1$. It has a right eigenvector $\alpha_{j1} = (1, \cdots, 1)^t$, and a left eigenvector β_{j1} with $(\beta_{j1})_l = \delta_{jl}$.*

(b) The second largest eigenvalue is $\lambda_{j2} = r_j < 1$, the j-th renormalization factor of the harmonic structure. It has a left eigenvector β_{j2} with $(\beta_{j2})_j = \sum_i c_{ij}$ and $(\beta_{j2})_l = -c_{lj}$ for $l \neq j$.

(c) The eigenspace of λ_{j2} is of one dimension and $|\lambda_{jk}| < \lambda_{j2}$ for $k \ge 3$.

(d) $\beta_{jk}\alpha_{jl} = \delta_{kl}$ for $1 \le k, l \le N_0$, where $\beta_{jk}\alpha_{jl}$ is $\sum_{s=1}^{N_0}(\beta_{jk})_s(\alpha_{jl})_s$.

(e) For $k \ge 2$, $\sum_{l=1}^{N_0}(\beta_{jk})_l = 0$ and $(\alpha_{jk})_j = 0$.

Proof. One could find the proofs of (a), (b), (c) from [34]. (d) is obvious. (e) follows from the combining of (a) and (d). $\square$

Let $\{h_{jk}\}_{1\leq k\leq N_0}$ be a collection of harmonic functions on K, where each h_{jk} assumes values α_{jk} on V_0, i.e., $h_{jk}(v_l) = (\alpha_{jk})_l$ for each l. Obviously, h_{j1} assumes constant value 1 on K.

Proposition 2.2. *(a)* $h_{jk}|_{F_jV_0} = \lambda_{jk}h_{jk}|_{V_0}$, $d_{jk}h_{jl}(v_j) = \delta_{kl}$.

(b) $h_{jk}(v_j) = 0$ *for* $k \geq 2$.

(c) $\{h_{jk}\}_{1\leq k\leq N_0}$ *spans the space of harmonic functions on* K. *For any harmonic function* h, *it could be written as a linear combination that*

$$h(\cdot) = h(v_j) + \sum_{k=2}^{N_0} d_{jk}h(v_j)h_{jk}(\cdot).$$

Proof. (a) follows from the definition of α_{jk} and β_{jk}. (b) follows from Proposition 2.1(e). (c) is a corollary of (a) and (b). □

In the rest of this section, we will give some necessary and sufficient conditions for $(\beta_{jk})_j = 0$ for all $k \geq 3$, which means that in this case the calculation of "higher order" derivatives of a function f at v_j will not involve the value $f(v_j)$. This will be useful in Section 5.

Proposition 2.3. *The following three conditions are equivalent.*

(a) $(\beta_{jk})_j = 0$ *for all* $k \geq 3$.

(b) $(\alpha_{j2})_l = c(1 - \delta_{jl})$ *for all* l, *where* c *is a nonzero constant.*

(c) The j*-th column of* M_j *assumes the values that* $(M_j)_{lj} = 1 - \lambda_{j2} + \lambda_{j2}\delta_{jl}$.

Proof. (a)⇒(b) Combining (a) and Proposition 2.1(e), we have that $\beta_{jk}, k \geq 3$ expand the linear space of dimension $N_0 - 2$ orthogonal to the constant vector and δ_{jl}. Since $\beta_{jk}\alpha_{j2} = 0, k \geq 3$, we conclude that

$$(\alpha_{j2})_l = s + t\delta_{jl}, l \geq 1,$$

for some constants s and t. Moreover, by Proposition 2.1(e), $(\alpha_{j2})_j = 0$. This determines that $s = -t$, which immediately yields (b).

(b)⇒(c) Taking α_{j2} into the characteristic equation, we have

$$M_j\alpha_{j2} = \lambda_{j2}\alpha_{j2},$$

which yields that

$$\sum_{k\neq j}(M_j)_{lk} = \lambda_{j2}, \text{ for all } l \neq j.$$

Noticing that all row sums of M_j are one and the j-th row of M_j is δ_{kj}, we then have

$$(M_j)_{lj} = 1 - \lambda_{j2} \text{ for } l \neq j, \text{ and } (M_j)_{jj} = 1,$$

which is what (c) says.

(c)⇒(a) For each $k \geq 3$, since $\beta_{jk}M_j = \lambda_{jk}\beta_{jk}$, by considering the j-th column of M_j, we have

$$\sum_{l\neq j}(1 - \lambda_{j2})(\beta_{jk})_l + (\beta_{jk})_j = \lambda_{jk}(\beta_{jk})_j.$$

Combining the above formula with Proposition 2.1(e), we obtain that $(\beta_{jk})_j = 0$. Thus (a) holds. □

Remark 2.4. In the $D3$ symmetry case, condition (c) automatically holds. Thus $(\beta_{j3})_j = 0$, which means that the tangential derivative of a function f at v_j does not involve the value $f(v_j)$.

Finally, we give an example which does not satisfy the conditions in Proposition 2.3.

Example 2.5. Let v_1, v_2, v_3 be the vertices of an equilateral triangle and let $F_i(x) = \frac{1}{2}(x + v_i)$, i=1,2,3. The *Sierpinski gasket*, $\mathcal{SG}$, is the unique compact set such that $\mathcal{SG} = \bigcup_{i=1}^{3} F_i\mathcal{SG}$. Then $V_0 = \{v_1, v_2, v_3\}$.

Consider a family of self-similar Dirichlet forms on $\mathcal{SG}$, that has a single bilateral symmetry. So we require $r_2 = r_3$ and

$$\mathcal{E}_0(f) = (f(v_1) - f(v_2))^2 + (f(v_1) - f(v_3))^2 + c(f(v_2) - f(v_3))^2$$

for some $c > 0$. We denote the conductances of $\mathcal{E}_0$ and $r_2\mathcal{E}_1$ on the edges of the graphs Γ_0 and Γ_1 in Figure 1, where $s = r_2/r_1$ is a constant to be determined.

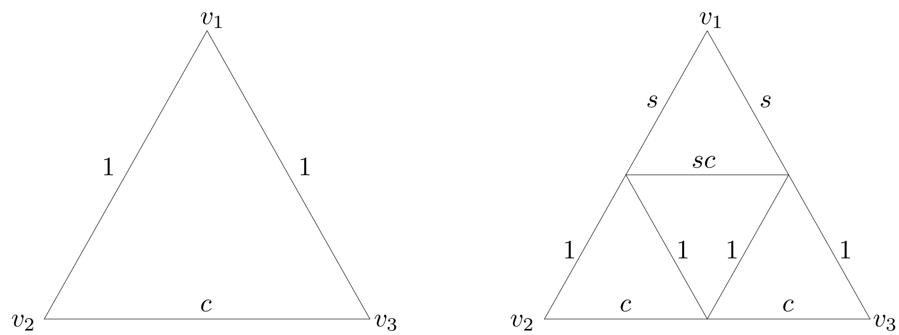

Figure 1. The conductances of $\mathcal{E}_0$ and $r_2\mathcal{E}_1$.

The renormalization equation requires s and c has the relationship

$$3s^2c^2 + 2s^2c - 2sc^2 - 2c - 1 = 0.$$

A detailed calculation could be found in Chapter 4 of [37].

Let h be a harmonic function on $\mathcal{SG}$ with respect to the above Dirichlet form. The mean value property of h at vertices F_2v_3, F_1v_3 and F_2v_1 give that

$$\begin{cases} (2+2c)h(F_2v_3) - h(F_1v_3) - h(F_2v_1) - ch(v_2) - ch(v_3) = 0, \\ (2+s+sc)h(F_1v_3) - h(F_2v_3) - sch(F_2v_1) - sh(v_1) - h(v_3) = 0, \\ (2+s+sc)h(F_2v_1) - h(F_2v_3) - sch(F_1v_3) - sh(v_1) - h(v_2) = 0. \end{cases}$$

This yields

$$\begin{pmatrix} h(F_2v_3) \\ h(F_1v_3) \\ h(F_2v_1) \end{pmatrix} = \begin{pmatrix} 1-2\eta & \eta & \eta \\ \frac{1+s-2\eta}{2+s} & \frac{\eta}{2+s} + \frac{sc}{(2+s)(2+s+2sc)} & \frac{\eta}{2+s} + \frac{2+s+sc}{(2+s)(2+s+2sc)} \\ \frac{1+s-2\eta}{2+s} & \frac{\eta}{2+s} + \frac{2+s+sc}{(2+s)(2+s+2sc)} & \frac{\eta}{2+s} + \frac{sc}{(2+s)(2+s+2sc)} \end{pmatrix} \begin{pmatrix} h(v_1) \\ h(v_2) \\ h(v_3) \end{pmatrix},$$

where $\eta = \frac{2c+sc+1}{2sc+2s+4c+2}$.

Thus the transformation matrix M_2 is

$$M_2 = \begin{pmatrix} \frac{1+s-2\eta}{2+s} & \frac{\eta}{2+s} + \frac{2+s+sc}{(2+s)(2+s+2sc)} & \frac{\eta}{2+s} + \frac{sc}{(2+s)(2+s+2sc)} \\ 0 & 1 & 0 \\ 1-2\eta & \eta & \eta \end{pmatrix}.$$

One can check that M_2 satisfies Hypothesis 1.1(b) when $|s-1|$ is sufficiently small. In fact, when $s = 1$, M_2 is diagonalizable with three different eigenvalues and all entries of M_2 are continuous functions of s.

Comparing $(M_2)_{12}$ and $(M_2)_{32}$, we can find they are not equal, since otherwise it leads to a different identity

$$2s^2c^2 + cs^2 + cs - 2c - s - 1 = 0$$

of s and c.

Hence M_2 does not satisfy the condition(c) in Proposition 2.3, at least for those s very close, but not equal to 1, which means $(\beta_{23})_2 \neq 0$.

3. Boundedness and weak continuity of normal derivatives

We prove Theorem 1.5 in this section, and provide some examples to show that our results are optimal.

Lemma 3.1. *Let $f \in dom(\Delta_\mu)$. Then the normal derivative of f over vertices of K is bounded by a multiple of $\|f\|_\infty + \|\Delta_\mu f\|_\infty$.*

This result is proved in [34], by using Gauss-Green's formula. For the convenience of readers, we still provide a proof. But our proof is somewhat different to that in [34], and could be extended to other derivatives.

Proof. Notice that from Proposition 2.1(e), for $1 \leq j \leq N_0$, we have $\sum_{l=1}^{N_0} (\beta_{j2})_l = 0$. Combining it with formula (1.4), the Hölder estimate of f, we obtain an estimate that

$$|r_w^{-1}\beta_{j2}f|_{F_wV_0}| \leq c(\|f\|_\infty + \|\Delta_\mu f\|_\infty)$$

for any word w and any j, with some constant $c > 0$. Since we have the existences of normal derivatives at all vertices, we get that for any $x = F_w v_j \in V_*$,

$$|d_{j2}f(x)| = |\lim_{m\to\infty} r_w^{-1}r_j^{-m}\beta_{j2}f|_{F_wF_j^mV_0}| \leq c(\|f\|_\infty + \|\Delta_\mu f\|_\infty). \qquad \square$$

Now we devote to prove the weak continuity property. For convenience, we only give the proof in the case of $x = v_j \in V_0$ for some $1 \leq j \leq N_0$, since for other vertices we just need to use scaling. First, we give some lemmas.

Lemma 3.2. *Let $1 \leq j \leq N_0$, $2 \leq k \leq N_0$, $m \geq 0$. For any $y \in F_j^m K$, we have*

$$d_{i2}h_{jk}(y)(or\ d_{i'2}h_{jk}(y)) = O((\lambda_{jk}r_j^{-1})^m).$$

Proof. By using Proposition 2.2(a) and Lemma 3.1, we get

$$\begin{aligned}|d_{i2}h_{jk}(y)| &= |r_j^{-m}d_{i2}(h_{jk}\circ F_j^m)(F_j^{-m}y)|\\ &= |(r_j^{-1}\lambda_{jk})^m d_{i2}h_{jk}(F_j^{-m}y)| \leq c(r_j^{-1}|\lambda_{jk}|)^m,\end{aligned}$$

for any nonjunction vertices $y \in F_j^m K$, where c is a positive constant. The same estimate holds for junction vertices. $\square$

Lemma 3.3. *Let $f \in dom(\Delta_\mu)$. Then for $1 \leq j \leq N_0$ and $m \geq 0$, we have*

$$d_{j2}f(v_j) = \int_{F_j^mK} H_j(F_j^{-m}z)\Delta_\mu f(z)d\mu(z) + r_j^{-m}\beta_{j2}f|_{F_j^mV_0}, \tag{3.1}$$

where H_j is the harmonic function on K with boundary values $H_j(v_l) = \delta_{jl}, 1 \leq l \leq N_0$.

Proof. First let $m = 0$. Applying the Gauss-Green's formula on K, we get

$$d_{j2}f(v_j) = \int_K H_j\Delta_\mu f d\mu + \sum_{l=1}^{N_0} f(v_l)d_{l2}H_j(v_l).$$

Replacing f by a harmonic function h in the above equality, we have

$$d_{j2}h(v_j) = \sum_{l=1}^{N_0} h(v_l) d_{l2} H_j(v_l),$$

which implies that $d_{l2}H_j(v_l) = (\beta_{j2})_l$ by the arbitrariness of h. Thus we have proved (3.1) in the case of $m = 0$.

For $m > 0$, we only need to apply the local Gauss-Green's formula on $F_j^m K$, and notice that $d_{l2}(H_j \circ F_j^{-m})(F_j^m v_l) = r_j^{-m}(\beta_{j2})_l$ by using scaling. □

We will need the *Green's function* $G(y, z)$ which solves the Dirichlet problem for the Poisson equations on K. Recall that $G(y, z)$ can be expressed as

$$G(y, z) = \sum_{|w| \geq 0} r_w \Psi(F_w^{-1} y, F_w^{-1} z),$$

where the summation is taken over all words, and Ψ is a linear combination of products $\psi_p(y)\psi_q(z)$ where ψ_p, ψ_q are tent functions in $S(\mathcal{H}_0, V_1)$, taking value 1 at p (or q) in $V_1 \setminus V_0$ and 0 at other vertices in V_1. For each term $\Psi(F_w^{-1}y, F_w^{-1}z)$, the understanding is that it assumes value 0 unless y and z both belong to the cell $F_w K$. See detailed explanations in [21] and [34].

For $1 \leq j \leq N_0$, $2 \leq k \leq N_0$, by the definition of the function Ψ, there exists a piecewise harmonic function $a_{jk} \in S(\mathcal{H}_0, V_1)$ satisfying

$$a_{jk}(z) = d_{jk}\Psi(\cdot, z)(v_j).$$

Obviously, $a_{jk}|_{V_0} = 0$ and $a_{jk} \neq 0$.

Lemma 3.4. *Let $f \in dom(\Delta_\mu)$. Then for $1 \leq j \leq N_0$, $2 \leq k \leq N_0$, $m \geq 0$,*

$$\lambda_{jk}^{-m} \beta_{jk} f|_{F_j^m V_0} = \beta_{jk} f|_{V_0} - \sum_{n=0}^{m-1} r_j^n \lambda_{jk}^{-n} \int_{F_j^n K} a_{jk}(F_j^{-n} z) \Delta_\mu f(z) d\mu(z). \tag{3.2}$$

Proof. Let h be a harmonic function which assumes the same values as f on V_0. Then

$$f = -\int_K G(\cdot, z)\Delta_\mu f(z) d\mu(z) + h.$$

Taking the above formula into the left side of (3.2), we obtain that it equals to

$$\begin{aligned}
& \lambda_{jk}^{-m}\beta_{jk} h|_{F_j^m V_0} - \lambda_{jk}^{-m} \int_K \beta_{jk} G(\cdot, z)|_{F_j^m V_0} \Delta_\mu f(z) d\mu(z) \\
= \quad & \beta_{jk} f|_{V_0} - \sum_{n=0}^{m-1} r_j^n \int_{F_j^n K} \lambda_{jk}^{-m} \beta_{jk} \Psi(F_j^{-n}\cdot, F_j^{-n} z)|_{F_j^m V_0} \Delta_\mu f(z) d\mu(z) \\
= \quad & \beta_{jk} f|_{V_0} - \sum_{n=0}^{m-1} r_j^n \lambda_{jk}^{-n} \int_{F_j^n K} \lambda_{jk}^{-(m-n)} \beta_{jk} \Psi(\cdot, F_j^{-n} z)|_{F_j^{m-n} V_0} \Delta_\mu f(z) d\mu(z) \\
= \quad & \beta_{jk} f|_{V_0} - \sum_{n=0}^{m-1} r_j^n \lambda_{jk}^{-n} \int_{F_j^n K} a_{jk}(F_j^{-n} z) \Delta_\mu f(z) d\mu(z),
\end{aligned}$$

where we use the fact that h is harmonic, $h|_{V_0} = f|_{V_0}$ and $\Psi(\cdot, \cdot)$ is piecewise harmonic with respect to the first variable. □

Proof of Theorem 1.5. The boundedness for the normal derivative of f has been shown in Lemma 3.1. So we only need to prove the weak continuity property.

As stated before, we only give the proof in the case of $x = v_j$. Split f on $F_j^m K$ into two functions,

$$f = f_1 + f_2,$$

where f_1 is the harmonic function on $F_j^m K$ assuming the same boundary values as f on $F_j^m V_0$. We will estimate the normal derivative of f_1 and f_2 in $F_j^m K$ separately.

First we look at f_1. By using Proposition 2.2(c) and Lemma 3.2, for any $y \in F_j^m K$, we have

$$|d_{i2} f_1(y)| = \Big|\sum_{k=2}^{N_0} d_{jk} f_1(v_j) \cdot d_{i2} h_{jk}(y)\Big| \leq \sum_{k=2}^{N_0} c(r_j^{-1}|\lambda_{jk}|)^m |d_{jk} f_1(v_j)| \tag{3.3}$$

for some positive constant c.

For $k = 2$, using Lemma 3.3, noticing that $d_{j2} f(v_j) = 0$, we have

$$d_{j2} f_1(v_j) = r_j^{-m} \beta_{j2} f|_{F_j^m V_0} = -\int_{F_j^m K} H_j(F_j^{-m} z) \Delta_\mu f(z) d\mu(z) = O(\mu_j^m). \tag{3.4}$$

For $k \geq 3$, using Lemma 3.4, we have

$$\begin{aligned} d_{jk} f_1(v_j) &= \lambda_{jk}^{-m} \beta_{jk} f|_{F_j^m V_0} \\ &= \beta_{jk} f|_{V_0} - \sum_{n=0}^{m-1} r_j^n \lambda_{jk}^{-n} \int_{F_j^n K} a_{jk}(F_j^{-n} z) \Delta_\mu f(z) d\mu(z) \\ &= \begin{cases} O(\mu_j^m r_j^m \lambda_{jk}^{-m}), & \text{if } r_j \mu_j > |\lambda_{jk}|, \\ O(m), & \text{if } r_j \mu_j = |\lambda_{jk}|, \\ O(1), & \text{if } r_j \mu_j < |\lambda_{jk}|. \end{cases} \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we have

$$d_{i2} f_1(y) = \begin{cases} O(\mu_j^m), & \text{if } r_j \mu_j > |\lambda_{j3}|, \\ O(m \mu_j^m), & \text{if } r_j \mu_j = |\lambda_{j3}|, \\ O((\lambda_{j3} r_j^{-1})^m), & \text{if } r_j \mu_j < |\lambda_{j3}|, \end{cases} \tag{3.6}$$

for any $y \in F_j^m K$.

Next, we estimate the normal derivatives of f_2 on $F_j^m K$. It is easy to check that $\Delta_\mu f|_{F_j^m K} = \Delta_\mu f_2|_{F_j^m K}$ and $f_2|_{F_j^m V_0} = 0$. Then by using Lemma 3.1, for any $y \in F_j^m K$, we have

$$\begin{aligned} |d_{i2} f_2(y)| &= r_j^{-m} |d_{i2}(f_2 \circ F_j^m)(F_j^{-m} y)| \\ &\leq c r_j^{-m} (\|\Delta_\mu (f_2 \circ F_j^m)\|_\infty + \|f_2 \circ F_j^m\|_\infty) \\ &= c r_j^{-m} (\|\Delta_\mu (f_2 \circ F_j^m)\|_\infty + \|\int_K G(\cdot, z) \Delta_\mu (f_2 \circ F_j^m)(z) d\mu(z)\|_\infty) \\ &\leq c' \mu_j^m \|\Delta_\mu f_2\|_{L_\infty(F_j^m K)} = c' \mu_j^m \|\Delta_\mu f\|_{L_\infty(F_j^m K)}, \end{aligned} \tag{3.7}$$

for some positive constants c, c'.

Combining (3.6) and (3.7), we have proved that

$$d_{i2}f(y) = d_{i2}f_1(y) + d_{i2}f_2(y) = \begin{cases} O(\mu_j^m), & \text{if } r_j\mu_j > |\lambda_{j3}|, \\ O(m\mu_j^m), & \text{if } r_j\mu_j = |\lambda_{j3}|, \\ O((\lambda_{j3}r_j^{-1})^m), & \text{if } r_j\mu_j < |\lambda_{j3}|. \end{cases}$$

It remains to show that our estimates are optimal. For convenience, we introduce the notation $a_m \asymp b_m$, which means that there exists a constant $C > 0$ such that $C^{-1}b_m \leq a_m \leq Cb_m$, $\forall m \geq 0$, for two sequences of numbers a_m, b_m.

For $r_j\mu_j < |\lambda_{j3}|$, consider the harmonic function h_{j3}. Obviously $d_{j2}h_{j3}(v_j) = 0$. Choose a vertex y_0 in K with nonzero normal derivative. Then by Proposition 2.2(a), we have

$$d_{i2}h_{j3}(F_j^m y_0) = (\lambda_{j3}r_j^{-1})^m d_{i2}h_{j3}(y_0), \forall m \geq 0,$$

which gives that

$$d_{i2}h_{j3}(F_j^m y_0) \asymp (\lambda_{j3}r_j^{-1})^m.$$

Thus the rate $O((\lambda_{j3}r_j^{-1})^m)$ is the optimal estimate in this case.

For $r_j\mu_j > |\lambda_{j3}|$, we take a function $f \in dom(\Delta_\mu)$, satisfying $\Delta_\mu f \equiv 1$ on K and $d_{j2}f(v_j) = 0$. By using the Gauss-Green's formula, we have

$$\sum_{l\neq j} d_{l2}f(F_j^m v_l) = \mu_j^m.$$

Thus there exists a sequence of $\{v_{l_m}\}_{m\geq 0}$ such that

$$d_{l_m 2}f(F_j^m v_{l_m}) \asymp \mu_j^m.$$

This shows that the rate $O(\mu_j^m)$ is the optimal estimate in this case.

As for $r_j\mu_j = |\lambda_{j3}|$ case, to find an optimal decay rate, we need that $\int_K a_{j3}(z)d\mu(z) \neq 0$ and $\lambda_{j3} > 0$. Obviously, this may happen. Still look at the function f satisfying that $\Delta_\mu f \equiv 1$ on K and $d_{j2}f(v_j) = 0$, and choose y_0 to be the vertice satisfying $d_{i2}h_{j3}(y_0) \neq 0$. Then

$$\begin{aligned} d_{i2}f(F_j^m y_0) &= d_{i2}f_1(F_j^m y_0) + d_{i2}f_2(F_j^m y_0) \\ &= (d_{j3}f_1(v_j))d_{i2}h_{j3}(F_j^m y_0) + O(\mu_j^m) \\ &= -\sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n}\mu_j^n(\int_K a_{j3}(z)d\mu(z))d_{i2}h_{j3}(F_j^m y_0) + O(\mu_j^m) \\ &= -mr_j^{-m}\lambda_{j3}^m(\int_K a_{j3}(z)d\mu(z))d_{i2}h_{j3}(y_0) + O(\mu_j^m) \\ &\asymp m\mu_j^m, \end{aligned}$$

since $\int_K a_{j3}(z)d\mu(z) \neq 0$, $d_{i2}h_{j3}(y_0) \neq 0$ and $r_j^{-1}\lambda_{j3} = \mu_j$. So the rate $O(m\mu_j^m)$ is the optimal estimate in this case. $\Box$

Remark 3.5. For $\mu_j r_j = |\lambda_{j3}|$, if it occurs that $\lambda_{j3} < 0$ or

$$\int_K a_{j3}(z)d\mu(z) = 0,$$

(This could happen, for example, see the Sierpinski gasket $\mathcal{SG}$ equipped with the standard Dirichlet form.) then the decay rate of normal derivatives is $o(m\mu_j^m)$ in

Theorem 1.5. In fact, we can rewrite the estimate in equality (3.5) for $k=3$ in this case that

$$\begin{aligned} d_{j3}f_1(v_j) &= \lambda_{j3}^{-m}\beta_{j3}f|_{F_j^m V_0} \\ &= \beta_{j3}f|_{V_0} - \sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n} \int_{F_j^n K} a_{j3}(F_j^{-n}z)\Delta_\mu f(z)d\mu(z) \\ &= O(1) - \sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n} \int_{F_j^n K} a_{j3}(F_j^{-n}z)(\Delta_\mu f(z) - \Delta_\mu f(v_j))d\mu(z) \\ &= o(m) \end{aligned}$$

since $\Delta_\mu f$ is continuous at v_j and $r_j|\lambda_{j3}|^{-1}\mu_j = 1$. Then following the same argument in the proof of Theorem 1.5, we get that

$$d_{i2}f(y) = o(m\mu_j^m)$$

for any $y \in F_j^m K$. The following example provides the nearest decay rate to $m\mu_j^m$ that we could find.

Example 3.6. Let $\{c_n\}_{n\geq 0}$ be a sequence of positive numbers which converge to 0, and ϕ be a nonnegative continuous function on $K \setminus \{v_j\}$ with values

$$\phi|_{F_j^n V_0} = c_n|\lambda_{j3}|^n r_j^{-n}, \forall n \geq 0, \tag{3.8}$$

and being harmonic in remaining regions. Let g be a function on K, defined as

$$g(x) = \phi(x)\sum_{n=0}^{\infty} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}x), \tag{3.9}$$

where we assume that $a_{j3}(F_j^{-n}x) = 0$ for $x \notin F_j^n K$.

Obviously, g is continuous on K, and $g(x) \to 0$ as $x \to v_j$, since $\|\phi\|_{L_\infty(F_j^n K)} = o((\lambda_{j3}r_j^{-1})^n)$. Define

$$f(x) = -\int_K (G(x,z) + h_{j2}(x)H_j(z))g(z)d\mu(z).$$

It is easy to check that $\Delta_\mu f = g$ and $d_{j2}f(v_j) = 0$, by Lemma 3.3.

Split $f = f_1 + f_2$ on $F_j^m K$ as we did in the proof of Theorem 1.5. Let $y \in F_j^m K$. We then have $d_{i2}f_2(y) = O(\mu_j^m)$. Expanding the harmonic function f_1 by Proposition 2.2(c), following the proof in Theorem 1.5, we write

$$f_1 = -h_{j3}\int_K \sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}z)g(z)d\mu(z) + R,$$

such that R is the summation of $d_{jk}f_1(v_j)h_{jk}$ for $k \neq 3$, satisfying $d_{i2}R(y) = O(\mu_j^m)$.

So it remains to estimate

$$d_{i2}h_{j3}(y)\int_K \sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}z)g(z)d\mu(z).$$

By Lemma 3.2, $d_{i2}h_{j3}(y) = O(\mu_j^m)$ for $y \in F_j^m K$. Moreover, for fixed vertex y_0 with $d_{i2}h_{j3}(y_0) \neq 0$, we have $d_{i2}h_{j3}(F_j^m y_0) \asymp \mu_j^m$.

As for $I := \int_K \sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}z)g(z)d\mu(z)$, we write $I = I_1 + I_2$, where

$$I_1 = \int_{F_j^m K} \sum_{n=0}^{m-1} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}z)g(z)d\mu(z)$$

and

$$I_2 = \sum_{l=0}^{m-1} \int_{F_j^l K \setminus F_j^{l+1} K} \sum_{n=0}^{l} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}z)g(z)d\mu(z).$$

It is easy to verify that $|I_1| = o(r_j^m \lambda_{j3}^{-m} \mu_j^m) = o(1)$, since $g(z) \to 0$ as $z \to v_j$.

Taking the expression (3.9) of g into I_2, we have

$$I_2 = \sum_{l=0}^{m-1} \int_{F_j^l K \setminus F_j^{l+1} K} (\sum_{n=0}^{l} r_j^n \lambda_{j3}^{-n} a_{j3}(F_j^{-n}z))^2 \phi(z)d\mu(z).$$

Since ϕ is bounded by the boundary values (3.8) on each $F_j^l K \setminus F_j^{l+1} K$, we get

$$I_2 \geq \sum_{l=0}^{m-1} c(r_j^l |\lambda_{j3}|^{-l})^2 c_l |\lambda_{j3}|^l r_j^{-l} \mu_j^l = c \sum_{n=0}^{m-1} c_n$$

for some constant $c > 0$.

Combining all the above estimates, we finally obtain that

$$|d_{i2} f(F_j^m y_0)| \geq c(\sum_{n=0}^{m-1} c_n)\mu_j^m$$

for some constant $c > 0$.

Looking at the choice of $\{c_n\}$, we have that the decay rate of $d_{i2} f(F_j^m y_0)$ could be very close to the rate of $m\mu_j^m$, but it still equals to $o(m\mu_j^m)$.

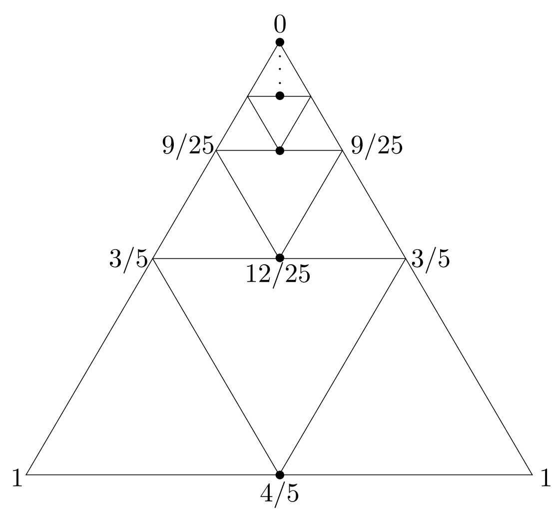

Figure 2. The values of $h = H_2 + H_3$.

Remark 3.7. The condition $d_{j2} f(x) = 0$ (or $d_{j'2} f(x) = 0$) is necessary. Otherwise, the continuity result in Theorem 1.5 is not true. For example, consider the harmonic function $h = H_2 + H_3$, which is a multiple of h_{12}, on the Sierpinski gasket, $\mathcal{SG}$, equipped with the standard Dirichlet form (In this case, $c_{ij} = 1$, $r_j = 3/5$ and $\lambda_{j3} = 1/5$ for all $i, j = 1, 2, 3$.). It is easy to calculate that $d_{12}h(v_1) \neq 0$ and $d_{32}h(F_1^m F_2 v_3) = 0$ for all $m \geq 0$. Thus $d_{32}h(F_1^m F_2 v_3)$ does not converge to $d_{12}h(v_1)$, although $F_1^m F_2 v_3$ converges to v_1, as $m \to \infty$. See Figure 2 for the values of h.

4. Boundness and weak continuity of other derivatives

In this section, we prove Theorem 1.6 and Theorem 1.7. Also, we provide some remarks and examples under the proofs.

Proof of Theorem 1.6. (a) From Proposition 2.1(e), we have $\sum_{l=1}^{N_0} (\beta_{jk})_l = 0$, $k \geq 2$. Combining it with the fact that h satisfies the Hölder estimate that $|h(x) - h(y)| \leq c\|h\|_\infty r_w$ for any $x, y \in F_w K$ and any word w, with some constant $c > 0$, we have

$$|d_{jk}h(x)| = |r_w^{-1}\beta_{jk}h|_{F_w V_0}| \leq c'\|h\|_\infty \text{ for any nonjunction vertex } x, \text{ and}$$
$$|d_{j'k}h(x)| = |r_w^{-1}r_j^{-1}\beta_{j'k}h|_{F_w F_j V_0}| \leq c'\|h\|_\infty \text{ for any junction vertex } x,$$

where $c' > 0$ be a constant.

(b) The differentiability of f at vertices in V_* is provided by Theorem 4.1 in [34]. We now estimate the bound of the derivatives. Let $x = F_w v_j$ be a nonjunction vertex. For $k \geq 2$, we still use a_{jk} to denote the piecewise harmonic function defined by $a_{jk}(z) = d_{jk}\Psi(v_j, z)$ as that in Section 3. Taking $m = 1$ in Lemma 3.4, we have

$$-\int_K a_{jk}(z)\Delta_\mu f(z)d\mu(z) = \lambda_{jk}^{-1}\beta_{jk}f|_{F_j V_0} - \beta_{jk}f|_{V_0}. \tag{4.1}$$

Scaling (4.1) down to $F_w F_j^n K$, $n \geq 0$, we get

$$\begin{aligned} &-\int_{F_w F_j^n K} r_j^n \lambda_{jk}^{-n} a_{jk} \circ F_j^{-n} \circ F_w^{-1}(z)\Delta_\mu f(z)d\mu(z) \\ =& r_w^{-1}\lambda_{jk}^{-(n+1)}\beta_{jk}f|_{F_w F_j^{n+1} V_0} - r_w^{-1}\lambda_{jk}^{-n}\beta_{jk}f|_{F_w F_j^n V_0}. \end{aligned} \tag{4.2}$$

Summing (4.2) from $n = 0$ to $m - 1$, we have

$$\begin{aligned} -\sum_{n=0}^{m-1}\int_{F_w F_j^n K} r_j^n \lambda_{jk}^{-n} a_{jk} \circ F_j^{-n} \circ F_w^{-1}(z)\Delta_\mu f(z)d\mu(z) = \\ r_w^{-1}\lambda_{jk}^{-m}\beta_{jk}f|_{F_w F_j^m V_0} - r_w^{-1}\beta_{jk}f|_{F_w V_0}. \end{aligned} \tag{4.3}$$

Since f is differentiable at x, the limit of the left side of (4.3) exists as $m \to \infty$. Moreover, by (1.2), the assumption that $r_j\mu_j < |\lambda_{jN_0}|$, it can be bounded by

$$\begin{aligned} &|\sum_{n=0}^{\infty}\int_{F_w F_j^n K} r_j^n \lambda_{jk}^{-n} a_{jk} \circ F_j^{-n} \circ F_w^{-1}(z)\Delta_\mu f(z)d\mu(z)| \\ \leq& \mu_w \sum_{n=0}^{\infty}(|\lambda_{jk}|^{-1}r_j\mu_j)^n\|a_{jk}\|_\infty\|\Delta_\mu f\|_\infty \leq \mu_w c_1\|\Delta_\mu f\|_\infty \end{aligned} \tag{4.4}$$

with some constant $c_1 > 0$ for all $k \geq 2$.

On the other hand, similar to the proof of (a) part, by using (1.4), the Hölder estimate of f, we also have $|r_w^{-1}\beta_{jk}f|_{F_wV_0}| \le c_2(\|f\|_\infty + \|\Delta_\mu f\|_\infty)$ for some constant $c_2 > 0$.

Thus

(4.5)
$$d_{jk}f(x) = -\sum_{n=0}^{\infty}\int_{F_wF_j^nK} r_j^n\lambda_{jk}^{-n}a_{jk}\circ F_j^{-n}\circ F_w^{-1}(z)\Delta_\mu f(z)d\mu(z) + r_w^{-1}\beta_{jk}f|_{F_wV_0}$$

is bounded by a multiple of $\|f\|_\infty + \|\Delta_\mu f\|_\infty$. For the junction vertices, the proof is same. Thus, all derivatives of f are uniformly bounded by a multiple of $\|f\|_\infty + \|\Delta_\mu f\|_\infty$. □

Remark 4.1. This boundedness property of derivatives can also be derived from Corollary 5.1 in [39], which says that under the same assumption (1.2), the gradient (in a different meaning) at any point in K exists and is continuous in the symbol space. In addition, the weak continuity property for "higher order" derivatives can be derived from it, although it could not provide the decay rate directly.

Proof of Theorem 1.7. The proof is analogous to that of Lemma 3.2 and Theorem 1.5, with suitable modifications. We still assume $x = v_j$, since for other vertices, we could use scaling.

(a) For any harmonic function h, and any vertex $y \in F_j^mK \setminus \{v_j\}$, we have the following equality using scaling,

$$d_{ik}(h\circ F_j^m)(F_j^{-m}y) = r_j^m d_{ik}h(y).$$

Since $d_{ik}h_{jl}$ is uniformly bounded by a constant $c > 0$ for all $l \ge 3$, as guaranteed by Theorem 1.6, we have

$$\begin{aligned} |d_{ik}h_{jl}(y)| &= |r_j^{-m}d_{ik}(h_{jl}\circ F_j^m)(F_j^{-m}y)| \\ &= |r_j^{-m}\lambda_{jl}^m d_{ik}h_{jl}(F_j^{-m}y)| \\ &\le cr_j^{-m}|\lambda_{jl}|^m \le c(|\lambda_{j3}|r_j^{-1})^m \end{aligned} \tag{4.6}$$

for all $y \in F_j^mK \setminus \{v_j\}$, where we use Proposition 2.2(a) for the second equality.

On the other hand, since h assumes 0 normal derivative at v_j, by using Proposition 2.2(c), we could write

$$h = h(v_j) + \sum_{l=3}^{N_0} d_{jl}h(v_j)h_{jl}.$$

Combining this with (4.6), we have $d_{ik}h(y) = O((\lambda_{j3}r_j^{-1})^m)$ for all $y \in F_j^mK\setminus\{v_j\}$.

(b) Similar to the proof of Theorem 1.5, we write

$$f = f_1 + f_2 \text{ on } F_j^mK$$

with f_1 and f_2 defined in the same manner. A similar argument yields that

$$d_{ik}f_1(y) = O((\lambda_{j3}r_j^{-1})^m)$$

and

$$d_{ik}f_2(y) = O(\mu_j^m)$$

for $y \in F_j^mK \setminus \{v_j\}$, since now we could use (4.6) and Theorem 1.6. Combining the above two estimates, noticing that $\mu_j < |\lambda_{j3}|r_j^{-1}$ from (1.2), we have proved that $d_{ik}f(y) = O((\lambda_{j3}r_j^{-1})^m)$. In addition, this estimate is optimal due to the same reason as in Theorem 1.5. $\Box$

Remark 4.2. Suppose $\sharp V_0 = 3$ and all structures have the full $D3$ symmetry. Theorem 1.6(b) and Theorem 1.7(b) are still valid without the hypothesis $r_j\mu_j < |\lambda_{j3}|$ (in this case, $N_0 = 3$), if we additionally assume that $g = \Delta_\mu f$ satisfies the Hölder estimate that

$$|g(x) - g(y)| \leq c\gamma^m \tag{4.7}$$

for all x, y belonging to the same m-cells, where γ is a constant satisfying

$$r_j\mu_j\gamma < |\lambda_{j3}|, \tag{4.8}$$

for all j.

The key observation is that a_{j3} is skew-symmetric with respect to the vertex v_j, which yields that in (4.4), each term in the summation could be rewrote as,

$$\int_{F_wF_j^nK} r_j^n\lambda_{j3}^{-n}a_{j3}\circ F_j^{-n}\circ F_w^{-1}(z)(\Delta_\mu f(z) - \Delta_\mu f(x))d\mu(z),$$

and this is bounded by a multiple of $\mu_w r_j^n|\lambda_{j3}|^{-n}\mu_j^n\gamma^n$. Since $r_j\mu_j\gamma < |\lambda_{j3}|$, we could still get the convergence of (4.4). In this setting, the existence of the derivatives also holds, which was proved in [34], due to the same reason.

Example 4.3. (1) The Sierpinski gasket, which has all $r_j = 3/5$, $\mu_j = 1/3$, $\lambda_{j3} = 1/5$ in the $D3$ symmetry case. Hence $r_j\mu_j = \lambda_{j3}$ for all j.

(2) The hexagasket, which can be generated by 6 mappings with simultaneously rotation and contraction by a ratio of 1/3 in the plane. In this case, we take all $r_j = 3/7$, $\mu_j = 1/6$ and $\lambda_{j3} = 1/7$, thus the condition $r_j\mu_j < |\lambda_{j3}|$ holds. See Figure 3 for the first two level graphs that approximate the hexagasket.

(3) The level 3 Sierpinski gasket, $\mathcal{SG}_3$, obtained by taking 6 contractive mappings of ratios 1/3, as shown in Figure 4. All $r_j = 7/15$, $\mu_j = 1/6$ and $\lambda_{j3} = 1/15$. Thus the condition $r_j\mu_j < |\lambda_{j3}|$ does not hold.

Please find the detail information of these examples in the book [37]. If $\Delta_\mu f \in dom(\Delta_\mu)$ then (4.7) holds with $\gamma = r_j$ as shown in (1.4). Then an easy calculation yields that the condition (4.8) holds for examples (1) and (3) above. Thus the conclusions in Theorem 1.6 and 1.7 are valid for these fractals.

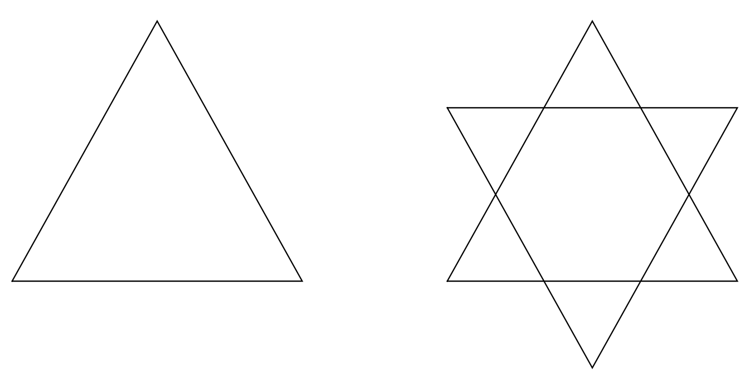

Figure 3. The first 2 graphs that approximate the hexagasket.

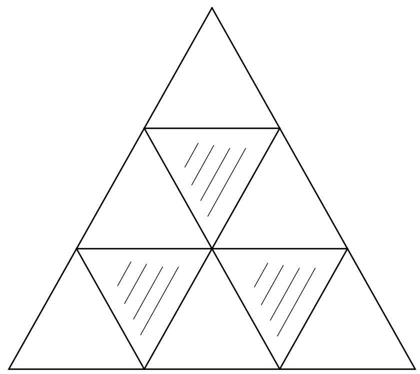

Figure 4. The first graph that approximates SG_3.

Remark 4.4. The condition $d_{j2}f(x) = 0$ in Theorem 1.7 could not be replaced by $d_{jk}f(x) = 0$, although it looks more "reasonable". For example, look at the Sierpinski gasket, $\mathcal{SG}$, equipped with the standard Dirichlet form. We consider the harmonic function $h = H_2 + H_3$, which is a multiple of h_{12}. It is easy to calculate that $d_{12}h(v_1) = -2$, $d_{13}h(v_1) = 0$, and $d_{13}h(F_1^m v_2) = 1/3$ for all $m \geq 1$. So $d_{13}h(F_1^m v_2)$ does not converge to $d_{13}h(v_1)$, although $F_1^m v_2$ converges to v_1, as $m \to \infty$. See Figure 5 for the values of h.

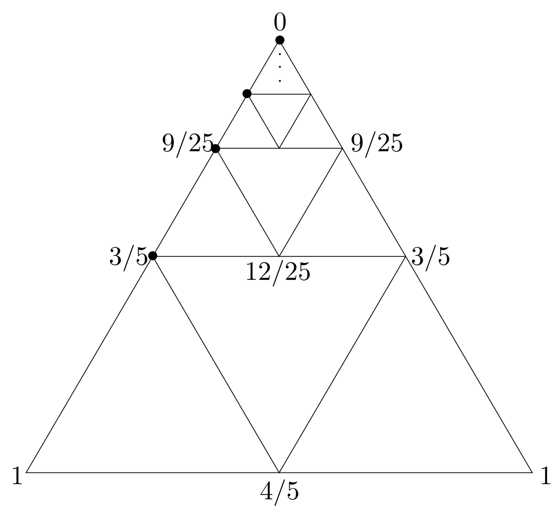

Figure 5. The values of h.

Remark 4.5. As we know, the assumption (1.2) in Theorem 1.6(b) is only a sufficient condition which guarantees the existence of all derivatives of f. It could be relaxed as stated in Remark 4.2 in the $D3$ symmetry case. One may ask a question that: Whether does Theorem 1.6(b) still hold as long as $f \in dom(\Delta_\mu)$ and f is differentiable at all vertices? We will give an example to illustrate that this is not true.

Example 4.6. Consider the Sierpinski gasket, $\mathcal{SG}$, equipped with the standard Dirichlet form and the standard self-similar measure. So all $r_i = 3/5, \mu_i = 1/3$.

First, we define a sequence of functions $g_l, l \geq 0$, satisfying

$$-\Delta_\mu g_l(x) = \sum_{n=0}^{l} a_{33}(F_3^{-n}x)$$

with the Dirichlet boundary condition, i.e., $g_l|_{V_0} = 0$. Here each term in the summation has the understanding that $a_{33}(F_3^{-n}x)$ is zero unless x belongs to $F_3^n\mathcal{SG}$.

It is easy to observe that $\|\Delta_\mu g_l\|_\infty$ is uniformly bounded and

$$d_{33}g_l(v_3) > (l+1)c > 0,$$

for all l with some constant $c > 0$. In fact, by using (4.5), noticing that a_{33} is skew-summery with respect to v_3 and $r_3\mu_3 = \lambda_{33}$, we have

(4.9)

$$\begin{aligned}
d_{33}g_l(v_3) =& -\sum_{m=0}^{\infty}\int_{F_3^m\mathcal{SG}} \lambda_{33}^{-m}r_3^m a_{33}(F_3^{-m}z)\Delta_\mu g_l(z)d\mu(z)\\
=&\sum_{n=0}^{l}\sum_{m=0}^{\infty}\int_{F_3^m\mathcal{SG}} \lambda_{33}^{-m}r_3^m a_{33}(F_3^{-m}z)a_{33}(F_3^{-n}z)d\mu(z)\\
\geq&\sum_{n=0}^{l}\sum_{m=n}^{\infty}\int_{F_3^m\mathcal{SG}} \lambda_{33}^{-m}r_3^m a_{33}(F_3^{-m}z)a_{33}(F_3^{-n}z)d\mu(z)\\
=&\sum_{n=0}^{l}\sum_{m=n}^{\infty}\int_{F_3^{m-n}\mathcal{SG}} \lambda_{33}^{-m}r_3^m \mu_3^n a_{33}(F_3^{-m+n}z)a_{33}(z)d\mu(z)\\
=&\sum_{n=0}^{l}\sum_{m=0}^{\infty}\int_{F_3^m\mathcal{SG}} \lambda_{33}^{-m}r_3^m a_{33}(F_3^{-m}z)a_{33}(z)d\mu(z) = (l+1)d_{33}g_0(v_3) > 0.
\end{aligned}$$

Now we define a function g, which is the solution of the following Dirichlet problem,

$$\begin{cases}\Delta_\mu g(x) = \sum_{l=0}^{\infty} 3^{-l}\Delta_\mu g_{3^{3l}}(F_1^{-1}F_2^{-l}x),\\ g|_{V_0} = 0.\end{cases}$$

See Figure 6 to find the support of $\Delta_\mu g(x)$.

Next we estimate the tangential derivatives of g at the vertices $F_2^lF_1v_3$. By using (4.5) and (4.9), we have

$$\begin{aligned}
d_{33}g(F_2^lF_1v_3) =& -\sum_{m=0}^{\infty}\int_{F_2^lF_1F_3^m\mathcal{SG}} r_3^m\lambda_{33}^{-m}a_{33}(F_3^{-m}F_1^{-1}F_2^{-l}z)\Delta_\mu g(z)d\mu(z)\\
&+ r_2^{-l}r_1^{-1}\beta_{33}g|_{F_2^lF_1V_0}\\
=& -\sum_{m=0}^{\infty}\int_{F_3^m\mathcal{SG}} 3^{-l}\mu_2^l\mu_1 r_3^m\lambda_{33}^{-m}a_{33}(F_3^{-m}z)\Delta_\mu g_{3^{3l}}(z)d\mu(z)\\
&+ r_2^{-l}r_1^{-1}\beta_{33}g|_{F_2^lF_1V_0}\\
=& -3^{-2l-1}\sum_{m=0}^{\infty}\int_{F_3^m\mathcal{SG}} r_3^m\lambda_{33}^{-m}a_{33}(F_3^{-m}z)\Delta_\mu g_{3^{3l}}(z)d\mu(z) + O(1)\\
=&3^{-2l-1}d_{33}g_{3^{3l}}(v_3) + O(1) \geq c3^{-2l-1}(3^{3l}+1) + O(1).
\end{aligned}$$

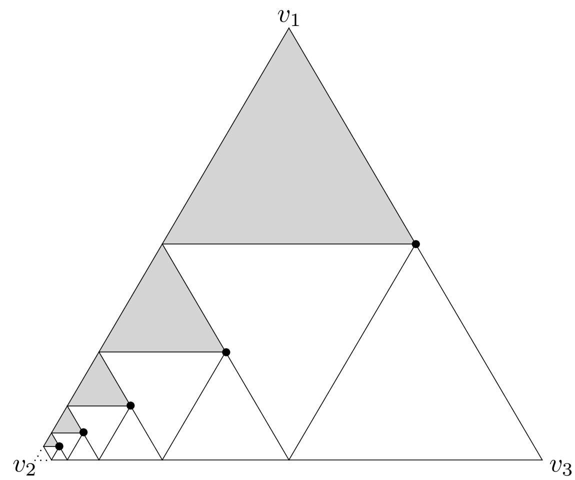

Figure 6. The support of $\Delta_\mu g(x)$.

Thus we have proved that $\{d_{33}g(F_2^l F_1 v_3)\}_{l\geq 0}$ is unbounded, although we have $g \in dom(\Delta_\mu)$ and is differentiable at all vertices in V_*. (The only vertex we need to check is v_3, where $\Delta_\mu g$ converges to 0 at an adequately large rate.)

We summarize this into the following theorem.

Theorem 4.7. *Let $f \in dom(\Delta_\mu)$ be differentiable at all vertices in V_*. The derivatives of f may not be uniformly bounded if the condition (1.2) does not hold.*

5. The weak tangent

Let f be a function which is differentiable at a vertex x. The weak tangent of order one of f at x, denoted as $T_1^x(f)$, is the harmonic function on $U_0(x)$ with the same value and the same gradient as f at x. Let h_m be the harmonic function assuming the same values as f at the boundary of $U_m(x)$, extended to be harmonic on $U_0(x)$. Theorem 3.11 in [34] says that h_m converges to $T_1^x(f)$ uniformly on $U_0(x)$ as m goes to infinity. However, the following example will show that this is not true.

Example 5.1. Consider the Sierpinski gasket $\mathcal{SG}$, equipped with a self-similar Dirichlet form which only has a single bilateral symmetry, as described in Example 2.5.

Define a function f on $\mathcal{SG}$ as following. We assume

$$\begin{cases} f(F_2F_3^m v_j) = \eta^m (\alpha_{32})_j \quad \text{for } j = 1,2 \text{ and } m \geq 0, \\ f(v_1) = 0, f(v_3) = 0, f(F_1 v_3) = 0, \end{cases}$$

where η is a constant such that $|\lambda_{23}| = |\lambda_{33}| < \eta < \lambda_{22} = \lambda_{32}$. For the values of f at other points, we take harmonic extension.

Choose $x = F_2 v_3$, it is easy to check that

$$d_{22}f(x) = d_{23}f(x) = d_{32}f(x) = d_{33}f(x) = 0.$$

Thus f is differentiable at x and $T_1^x(f) \equiv 0$ on $U_0(x)$.

On the other hand, using the bilateral symmetry, we could obtain that

$$h_m(x) = \frac{\sum_{y\sim_{m+1}x} c_{xy} f(y)}{\sum_{y\sim_{m+1}x} c_{xy}} = \eta^m \frac{\sum_{y\sim_1 x} c_{xy} f(y)}{\sum_{y\sim_1 x} c_{xy}} = \eta^m h_0(x),$$

which results that

$$d_{23}h_m(x) = r_3^{-1}\lambda_{23}^{-m}(\beta_{23})_2 h_m(x) = r_3^{-1}\lambda_{23}^{-m}\eta^m(\beta_{23})_2 h_0(x).$$

Thus $d_{23}h_m(x) \to \infty$ as $m \to \infty$ since $|\lambda_{23}| < \eta$ and $(\beta_{23})_2 \neq 0$ as shown in Example 2.5. So we have

$$\beta_{23}h_m|_{F_3V_0} \to \infty \text{ as } m \to \infty,$$

which means $\|h_m\|_\infty \to \infty$ as $m \to \infty$. Hence h_m does not converge to $T_1^x(f)$ as $m \to \infty$.

We need some extra assumption to make Theorem 3.11 in [34] holds.

Theorem 5.2. *Suppose one of the condition in Proposition 2.3 holds. Then for any f differentiable at x, h_m converges to $T_1^x(f)$ uniformly.*

Proof. The proof is essential the same as that of Theorem 3.11 in [34], where the condition $(\beta_{jk})_j = 0$ is misapplied. we omit it here. □

As pointed out below the proof of Proposition 2.3, in the D3 symmetry case, the assumption in Theorem 5.2 holds automatically.

Theorem 5.3. *Suppose*

$$r_j \max_{1\leq i\leq N_0} \mu_i < |\lambda_{jN_0}| \tag{5.1}$$

for every j. Then for any $f \in dom(\Delta_\mu)$, for any vertex x, h_m converges to $T_1^x(f)$ uniformly.

Proof. Condition (5.1) guarantees the differentiability of f at x by using Theorem 4.1 in [34].

For a nonjunction vertex x, $\forall k \geq 2$, we have

$$d_{jk}f(x) = \lim_{m\to\infty} d_{jk}h_m(x),$$

since on the right side of (1.1) we may replace f by h_m and h_m is harmonic on $U_0(x)$. In particular, this also shows the limit exists. We have $h_m(x) = f(x)$ for all m since x is a boundary point of $U_m(x)$. On the other hand, there is an estimate for harmonic functions, $|h(y)| \leq c(|h(x)| + \|dh(x)\|)$ uniformly for $y \in U_0(x)$, which is a result of Proposition 2.2(c). Using this estimate for $h_m - T_1^x(f)$, we obtain that h_m converges uniformly on $U_0(x)$ to $T_1^x(f)$.

If x is a junction vertex, $x = F_wF_jv_{j'}$, $\forall j \in J(x)$, x is no longer a boundary point of $U_m(x)$. We have to estimate $f(x) - h_m(x)$. Using the compatibility condition at x, we have $f(x) - h_m(x) = o(\lambda_{j'2}^m)$ for all j'. Furthermore, with the assumption (5.1), we can get a more precise estimate.

Let ψ_x^m denote the piecewise harmonic function in $S(\mathcal{H}_0, V_m)$ which takes value 1 at x and 0 at other vertices in V_m.

From the pointwise formula for $\Delta_\mu f$ at x, we have

$$
\begin{aligned}
\Delta_\mu f(x) &= \lim_{m\to\infty} \frac{\sum_{\sim_m} c_{xy}(f(y)-f(x))}{\int_K \psi_x^m d\mu} \\
&= -\lim_{m\to\infty} \frac{\sum_{j\in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} \beta_{j'2} f|_{F_w F_j F_{j'}^m V_0}}{\int_K \psi_x^{m+|w|+1} d\mu} \\
&= \lim_{m\to\infty} \frac{\sum_{j\in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} (\beta_{j'2})_{j'} (h_m(x)-f(x))}{\int_K \psi_x^{m+|w|+1} d\mu},
\end{aligned}
\tag{5.2}
$$

where for the third equality we use the compatibility condition

$$\sum_{j\in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} \beta_{j'2} h_m(x)|_{F_w F_j F_{j'}^m V_0} = 0,$$

since h_m is harmonic.

The integral $\int_K \psi_x^{m+|w|+1} d\mu$ in (5.2) can be calculated that

$$\int_K \psi_x^{m+|w|+1} d\mu = \sum_{j\in J(x)} \mu_w \mu_j \mu_{j'}^m \int_K H_{j'} d\mu$$

where H_j denotes the harmonic function taking 1 at v_j and 0 at other vertices in V_0. Thus the integral converges to zero with the rate $(\mu_{J(x)})^m$, where $\mu_{J(x)} = \max_{j\in J(x)} \mu_{j'}$. Denote $r_{J(x)} = \min_{j\in J(x)}\{r_{j'}\}$, we then have

$$f(x) - h_m(x) = O((r_{J(x)}\mu_{J(x)})^m)$$

from the convergence of (5.2).

Combining this estimate with the assumption (5.1), we get

$$f(x) - h_m(x) = o(\lambda_{j'k}),$$

for all j', $\forall k \geq 2$. So we have

$$
\begin{aligned}
d_{j'k} f(x) &= \lim_{m\to\infty} r_w^{-1} r_j^{-1} \lambda_{j'k}^{-m} \beta_{j'k} h_m|_{F_w F_j F_{j'}^m V_0} \\
&\quad + \lim_{m\to\infty} r_w^{-1} r_j^{-1} \lambda_{j'k}^{-m} (\beta_{j'k})_{j'} (f(x) - h_m(x)) \\
&= \lim_{m\to\infty} d_{j'k} h_m(x).
\end{aligned}
$$

Using a similar argument as the nonjunction case, we still obtain that h_m converges uniformly on $U_0(x)$ to $T_1^x(f)$. $\square$

At last, we will give an example which could serve as a counter-example of Conjecture 6.7 in [34] on the existence of weak tangents of higher order.

Example 5.4. For the Sierpinski gasket $\mathcal{SG}$, we assume all the structures satisfy the $D3$ symmetry. In this case, all $r_j = 3/5$, $\mu_j = 1/3$. Denote by $\rho = r_j \mu_j$ and r the common value of r_j. Define a function $f \in dom(\Delta_\mu)$ which satisfies

$$\begin{cases} \Delta_\mu f = \sum_{m=0}^{\infty} \eta^m \psi_{F_1^m F_2 v_3}^{m+1}, \\ f(v_1) = 0, df(v_1) = 0, \end{cases}$$

where η is a constant, $r < \eta < 1$, ψ_x^m is a piecewise harmonic function in $S(\mathcal{H}_0, V_m)$ satisfying $\psi_x^m(y) = \delta_{xy}$ for $y \in V_m$. One can easily verify that $d\Delta_\mu f(v_1) = 0$. We will show that f does not have a weak tangent at v_1 of order 2.

In fact, by using the Gauss-Green's formula, we have

$$f(v_2) + f(v_3) = \int_K H_1(x)\Delta_\mu f(x) d\mu(x),$$

where H_1 is the harmonic function satisfying $H_1(v_j) = \delta_{1j}$. Using scaling, we then have

$$\begin{aligned} f(F_1^m v_2) + f(F_1^m v_3) &= \rho^m \int_K H_1(x)(\Delta_\mu f)(F_1^m x) d\mu(x) \\ &= \rho^m \eta^m (f(v_2) + f(v_3)). \end{aligned} \tag{5.3}$$

But from the proof of Lemma 6.2 in [34], for any 2-harmonic function h, there exist constants $a, b, c \in \mathbb{R}$ such that

$$h(F_1^m v_2) + h(F_1^m v_3) = ar^m + b\rho^m + c(r\rho)^m. \tag{5.4}$$

Combining (5.3) and (5.4), we could claim that it is impossible to have any 2-harmonic function h satisfying (1.3), where n is replaced with 2, since $r < \eta < 1$. Thus f does not have a weak tangent of order 2 at v_1.

Before the end of this section, we would like to pose a problem that should be considered. The Hypothesis 1.1 requires the harmonic structure to be nondegenerate, i.e., all the transformation matrices to be nonsingular. This excludes some typical fractals such as the Vicsek set. Consider a square with corners $\{v_1, v_2, v_3, v_4\}$ and center v_5. For $1 \leq j \leq 5$, let F_j be a contractive mapping with ratio $1/3$ and fixed point v_j. The invariant set of this i.f.s. is called the Vicsek set, denoted by $\mathcal{V}$. Then $N = 5$, $N_0 = 4$ and $V_0 = \{v_1, v_2, v_3, v_4\}$. See Figure 7 for the second level graph of $\mathcal{V}$. This fractal has $D4$ symmetry. Equip $\mathcal{V}$ with the standard Dirichlet form and standard measure. Then all $r_j = 1/3$, $\mu_j = 1/5$, and all the transformation matrices M_j are permutations of

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{3}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}.$$

It is easy to calculate that $\lambda_{j2} = 1/3, \lambda_{j3} = \lambda_{j4} = 0$. Thus this harmonic structure of $\mathcal{V}$ is degenerate. Is there a satisfactory theory of derivatives or gradients on $\mathcal{V}$? Or even on other fractals in degenerate case?

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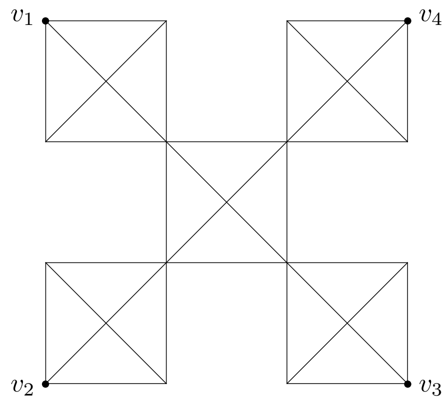

Figure 7. The second level graph of $\mathcal{V}$.

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