RESISTANCE FORMS ON SELF-SIMILAR SETS WITH FINITE RAMIFICATION OF FINITE TYPE

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Abstract. In this paper, we introduce the finite neighboring type and the finite chain length conditions for a connected self-similar set K. We show that with these two conditions, K is a finitely ramified graph directed (f.r.g.d.) fractal defined by Hambly and Nyberg[17]. We give some nontrivial examples and compute the harmonic structures on them explicitly. Furthermore, for a f.r.g.d. self-similar set K, we provide an equivalent description, the finitely ramified of finite type (f.r.f.t.) cell structure of K, and investigate the relationship of harmonic structures associated with different f.r.f.t. cell structures of K.

1. Introduction

The construction of Laplacians is a core topic in analysis on fractals. One of the most important approaches is Kigami's construction on p.c.f. self-similar sets[18, 19], using graph approximations. See books [20, 34] for details, and [21, 23] for deep discussions on closely related concepts such as resistance forms and harmonic structures. Beyond Kigami's construction, the only other approaches to obtaining Laplacians are indirect and nonconstructive by probability techniques, see [8, 9, 10, 11, 15].

It is desirable to enlarge the class of self-similar sets on which Kigami's spirit works. Hambly and Nyberg[17] introduced a class of fractals, named finitely ramified graph directed (f.r.g.d. for short) fractals, which generalized the class of p.c.f. self-similar sets and admit very natural graph approximations. The technique of Kigami continues to work through simple extensions. See [16, 27, 29] for a discussion on the existence of harmonic structures and Laplacians on f.r.g.d. fractals.

The graph directed constructions provide a more general setting of fractals which are no longer exactly self-similar but do inherit self-similar features. See [26] for a detailed analysis on geometry structures and dimension estimates of such fractals. The finitely ramified assumption is essentially necessary for Kigami's technique so as to provide suitable graph approximations. One of the well known examples of f.r.g.d. fractals is the Hanoi attractor, see Figure 1. The Laplacians and their properties on the Hanoi attractor have been well investigated in a sequence of papers [2, 3, 4, 5]. Besides Hambly and Nyberg's extension, there is another direction of extension of Kigami's approach, see [6, 14, 32, 33] for the construction of Laplacians on a class of Julia sets for quadratic or cubic polynomials, which are finitely ramified.

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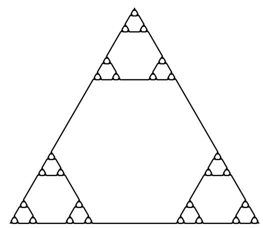


FIGURE 1. The Hanoi attractor.

In certain cases, a self-similar set can be an f.r.g.d. fractal(we always assume it to be connected). Trivial examples are p.c.f. self-similar sets, which have f.r.g.d. constructions with the directed-graphs being singletons. A more interesting example is the diamond fractal(see Figure 2), on which the Laplacian was constructed and well studied[24, 28]. See [1] for an investigation of the heat kernel on the diamond fractal. This sheds light on defining

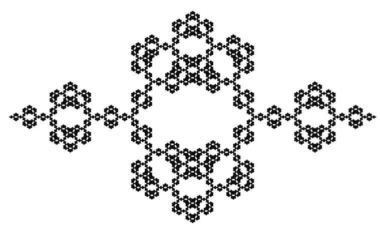


FIGURE 2. The diamond fractal.

Laplacians in a direct and constructive way on certain non p.c.f. self-similar sets.

In this work, we will give conditions that guarantee a self-similar set K to be an f.r.g.d. fractal. Intuitively, the graph directed requirement allows overlaps when the similitudes are iterated, but it seems that the overlapping types among distinct comparable similar copies of K should be finite. Basing on this observation, together with the finitely ramified requirement, we present two conditions for K, called the finite neighboring type and finite chain length conditions, denoted by (**F1**) and (**F2**), which do not imply each other. We will prove that K is an f.r.g.d. fractal providing it satisfies both the two conditions.

The "finite type" assumption is quite useful for calculating the Hausdorff dimension of certain self-similar sets which do not necessarily satisfy the open set condition. See [25, 30, 31] for detail discussions on this topic. Our condition (**F1**) is a variation of their assumptions. On the other hand, the consideration of chains of copies is also frequently used in analysis of fractals, for example see [13, 22, 23] to find investigations on heat kernel estimates. In our setting, we require the overlap to be infinite.

We will provide some interesting examples satisfying the above two conditions, see Figure 3. Harmonic structures on them will be computed in detail. Notice that there may exist multiple f.r.g.d. constructions associated with one self-similar set. We will have a discussion on when harmonic structures on different f.r.g.d. constructions lead to a same resistance form. In particular, we will introduce the concept of homogeneous harmonic structures.

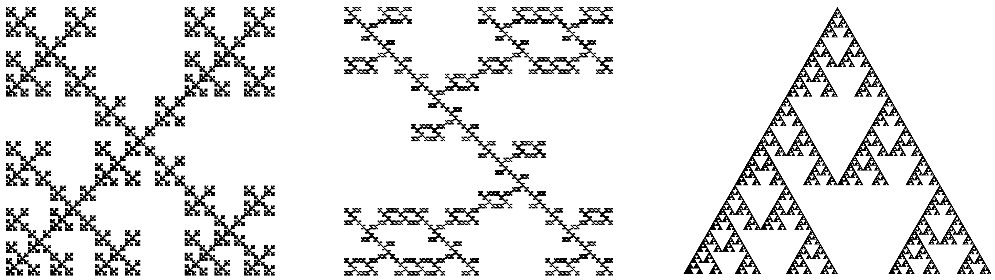


FIGURE 3. Examples of f.r.g.d. self-similar sets.

However, we will point out that an f.r.g.d. self-similar set K does not necessarily admit the conditions (**F1**) and (**F2**). We will use the idea of finitely ramified cell structures introduced by Teplyaev [35] to get an equivalent condition for K to be f.r.g.d. To be more precise, we will introduce a finitely ramified of finite type (f.r.f.t. for short) cell structure of K basing on its f.r.g.d. construction. This structure provides some convenience when we are interested in K itself rather than the f.r.g.d. fractal family including K. We will use this new setting to study homogeneous harmonic structures.

At the end of this introduction, let's look at the organization of the paper. In Section 2, we will introduce (**F1**) and (**F2**), and show that they lead to f.r.g.d. constructions. In Section 3, we will show some f.r.g.d. constructions associated with the examples in Figure 3, and will provide details on computing the harmonic structures. In Section 4, we will introduce the concept of f.r.f.t. cell structures. In Section 5, we will deal with homogeneous harmonic structures.

2. (F1), (F2) AND F.R.G.D. CONSTRUCTIONS

We consider a connected self-similar set K in $\mathbb{R}^n$, which is the attractor of an i.f.s. $\{F_i\}_{i=1}^N$, i.e.

$$K = \bigcup_{i=1}^{N} F_i K.$$

We denote c_i the *similarity ratio* of F_i, $1 \leq i \leq N$, and $c_* = \min\{c_1, c_2, \cdots, c_N\}$.

For $n \geq 1$, denote $W_n = \{1, 2, \cdots, N\}^n$ the set of *words* of length n. Together with $W_0 = \{\emptyset\}$, we write $W_* = \bigcup_{n\geq 0} W_n$. For $w = w_1 w_2 \cdots w_n \in W_*$, write $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$ and $c_w = c_{w_1} c_{w_2} \cdots c_{w_n}$ for short, with the convention that $c_\emptyset = 1$.

2.1. **The conditions (F1) and (F2).** First, the finite neighboring type condition is defined as follows.

(**F1**). *There are only finitely many similitudes $h = F_w^{-1} F_u$ with $w, u \in W_*$ and $F_w K \cap F_u K \neq \emptyset$, and with similarity ratio $c_h \in (c_*, 1/c_*)$.*

This condition, formulated in algebraic terms, was introduced in [7] by Bandt and Rao to describe algorithms to verify the open set condition. It is also related with the finite type concept as mentioned in the Introduction.

Next, we introduce the *finite chain length condition*. Before that, we first note that it is possible that $F_w = F_u$ with $w \neq u \in W_*$. By removing all but the smallest words in the lexicographical order (or any fixed order), we obtain a word set $W_\# \subset W_*$ such that $\{F_w K\}_{w\in W_\#}$ consists of distinct copies of K and $\{F_w K\}_{w\in W_\#} = \{F_w K\}_{w\in W_*}$. For $0<\lambda<1$, define

$$W_\lambda = \{w = w_1 w_2 \cdots w_n \in W_\# : c_w \leq \lambda < c_{w_1} c_{w_2} \cdots c_{w_{n-1}}\},$$

and call it a *partition* with respect to λ. Note that the set W_λ is finite; for distinct words $w, u \in W_\lambda$, $F_w K$ and $F_u K$ can not contain each other; and $\lambda c_* < c_w \leq \lambda$, $\forall w \in W_\lambda$.

Definition 2.1. *(a). We call a finite collection of words in $W_\#$*

$$\gamma = (w^{(1)}, w^{(2)}, \cdots, w^{(n)})$$

an overlapping chain if

$$\#\big(F_{w^{(i+1)}} K \cap (\bigcup_{1\leq j\leq i} F_{w^{(j)}} K)\big) = \infty, \forall 1 \leq i \leq n-1,$$

and $F_{w^{(i)}} K \nsubseteq F_{w^{(j)}} K$ for distinct $1 \leq i, j \leq n$, and call n the length of γ.

(b). Moreover, for $0<\delta<1$, we call the chain γ a δ-overlapping chain if $\delta \leq c_{w^{(i)}} c^{-1}_{w^{(j)}} \leq \delta^{-1}$ for any $w^{(i)}$ and $w^{(j)}$ in γ.

(c). Denote $\mathcal{L}_\delta(K)$ the supremum of the lengths of δ-overlapping chains in K, $\forall 0<\delta<1$.

The finite chain length condition is defined as follows.

(**F2**). *$\mathcal{L}_\delta(K) < \infty$ for any $0<\delta<1$.*

The following proposition shows that (**F2**) is equivalent to a seemingly weaker version.

($\tilde{\mathbf{F}}$**2**). *$\mathcal{L}_\delta(K) < \infty$ for some $0<\delta\leq c_*$.*

Proposition 2.2. *(**F2**) is equivalent to ($\tilde{\mathbf{F}}$**2**).*

Proof. It is enough to show that ($\tilde{\mathbf{F}}$**2**) implies (**F2**). Fix a $0<\delta\leq c_*$ such that $\mathcal{L}_\delta(K)<\infty$.

First, we can see that $\mathcal{L}_{\delta'}(K) \leq \mathcal{L}_\delta(K) < \infty$ for any $1 > \delta' \geq \delta$. This is due to the fact that any δ'-overlapping chain is automatically a δ-overlapping chain.

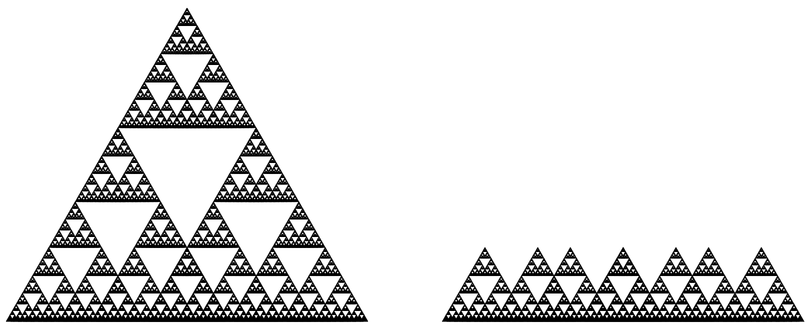
Next, we show $\mathcal{L}_{\delta'}(K) < \infty$ for any $0<\delta'<\delta$. Let γ be a δ'-overlapping chain with $\gamma = (w^{(1)}, w^{(2)}, \cdots, w^{(n)})$. We denote by $\lambda = \max\{c_{w^{(i)}} : 1 \leq i \leq n\}$, then for each $1 \leq i \leq n$, we could choose $\widetilde{w}^{(i)}$ in W_λ such that $F_{w^{(i)}} K \subset F_{\widetilde{w}^{(i)}} K$. It is clear that, after deleting the repeated ones if necessary, $(\widetilde{w}^{(1)}, \widetilde{w}^{(2)}, \cdots, \widetilde{w}^{(n)})$ forms a c_*-overlapping chain, we denote it by $\widetilde{\gamma}$. Since $\delta \leq c_*$, $\widetilde{\gamma}$ is also a δ-overlapping chain, and hence the length of $\widetilde{\gamma}$ is no more than $\mathcal{L}_\delta(K)$. Notice that the similarity ratio of elements in γ has a lower bound $\lambda\delta'$, and the similarity ratio of elements in $\widetilde{\gamma}$ has an upper bound λ. Then an easy calculation shows that the length of γ is no more than $\# W_{\delta'} \mathcal{L}_\delta(K)$. From the arbitrariness of γ, we have proved that $\mathcal{L}_{\delta'}(K) < \infty$. □

Remark. *(**F1**) and (**F2**) can not imply each other, which can be illustrated by the following two examples.*

Example 1.(*Golden ratio Sierpinski gasket*) Let $\{q_i\}_{i=1}^3$ be the three vertices of an equilateral triangle, and $\{F_i\}_{i=1}^3$ be the three contractive similitudes,

$$F_1 : x \to \rho^2(x - q_1) + q_1,$$
$$F_2 : x \to \rho(x - q_2) + q_2, \quad F_3 : x \to \rho(x - q_3) + q_3,$$

with $\rho = \frac{\sqrt{5}-1}{2}$. The *golden ratio Sierpinski gasket* $\mathcal{SG}^g$ is the invariant set of the *i.f.s.* $\{F_i\}_{i=1}^3$, i.e., $\mathcal{SG}^g = \bigcup_{i=1}^3 F_i(\mathcal{SG}^g)$, see Figure 4 (a). It is a slight variant of Example 5.4 in [30].



(a) (b)

FIGURE 4. The golden ratio Sierpinski gasket $\mathcal{SG}^g$ (left) and an overlapping chain (right).

Obviously, $\mathcal{SG}^g$ satisfies (**F1**), see a detailed discussion in [30]. However, $\mathcal{SG}^g$ does not satisfy (**F2**). In fact, consider the collection of copies $\{F_w\mathcal{SG}^g | w \in \{2,3\}^n\}$, $n \geq 1$, located along the bottom line of $\mathcal{SG}^g$. By ordering the words in lexicographical order, i.e., letting $w^{(1)} = 22\cdots 2, w^{(2)} = 22\cdots 23, \cdots, w^{(2^n)} = 33\cdots 3$, and removing the completely overlapping ones, we can find that the collection $\gamma_n = (w^{(1)}, w^{(2)}, \cdots, w^{(2^n)})$ provides a δ-overlapping chain for any $0 < \delta < 1$. See Figure 4 (b) for such a chain with $n = 3$. Since n can be arbitrarily large, $\mathcal{SG}^g$ does not satisfy (**F2**).

Example 2.(λ-*gaskets with irrational moving sliders*) Let $\lambda \in [0,1]$. Define the following *i.f.s.*$\{F_i\}_{i=0}^4$ in $\mathbb{R}^2$,

$$F_0 : x \to \frac{1}{3}x, \quad F_1 : x \to \frac{1}{3}x + (\frac{1}{3}, 0), \quad F_2 : x \to \frac{1}{3}x + (\frac{2}{3}, 0),$$
$$F_3 : x \to \frac{1}{3}x + (\frac{1}{3}, \frac{\sqrt{3}}{3}), \quad F_4 : x \to \frac{1}{3}x + (\frac{1}{6} + \frac{\lambda}{3}, \frac{\sqrt{3}}{6}).$$

Let K_λ be the invariant set of this *i.f.s.* See Figure 5 for an illustration.

Obviously, K_λ satisfies (**F2**). In fact, given any two copies F_wK_λ and F_uK_λ, either they intersect each other by at most one point, or one contains the other. So any overlapping chain has length at most 1.

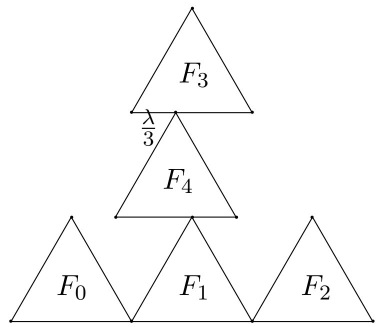


FIGURE 5. An illustration for K_λ.

On the other hand, K_λ does not satisfy (**F1**) when λ is an irrational number. In fact, write the ternary expansion of λ,

$$\lambda = \sum_{i=1}^{\infty} l_i 3^{-i},$$

with $l_i \in \{0,1,2\}, \forall i \geq 1$. By shifting the coefficients in this expansion, we get a sequence of irrational numbers

$$\lambda_k = \sum_{i=1}^{\infty} l_{i+k} 3^{-i}, k \geq 1.$$

For the sake of uniformity, write $\lambda_0 = \lambda$. It is not hard to see that for any $k \geq 0$, $F_4 F_3^k K_\lambda \cap F_3 F_{[l]_k} K_\lambda \neq \emptyset$, where $[l]_k = l_1 l_2 \cdots l_k \in \{0,1,2\}^k \subset W_*$. Moreover, a calculation yields that

$$(F_3 F_{[l]_k})^{-1} \circ F_4 F_3^k : x \to x - (\frac{1}{2}, \frac{\sqrt{3}}{2}) + (\lambda_k, 0).$$

Notice that $\lambda_k \neq \lambda_{k'}$ when $k \neq k'$ since λ is irrational, and thus K_λ does not satisfy (**F1**).

This example was introduced in [36](Example 3) for a different purpose. Some adjustment is made in our setting.

2.2. **Relation with f.r.g.d. constructions.** Recall the concept of graph-directed construction and f.r.g.d. fractals, which can be found in detail in [17, 26]. Let $G = (S, E)$ be a *directed-graph*, where S is the set of *states*(call vertices in this graph states to avoid confusion) and E is the set of *edges* of the graph. Note that multiple edges and loops are allowed. For an edge $e \in E$, denote by $i(e)$ the *initial state* and $f(e)$ the *final state* of e.

Definition 2.3. *Let $G = (S, E)$ be a directed-graph. Assign each $e \in E$ a similitude ψ_e with similarity ratio l_e, and each $s \in S$ a compact connected set J_s. Call $\mathcal{G} = (S, E, \{\psi_e\}_{e\in E})$ a graph-directed construction if the following conditions are satisfied,*

1. $\forall s \in S$, there is at least one edge $e \in E$, such that $s = i(e)$;

2. $\forall s \in S$, $\bigcup_{i(e)=s} \psi_e J_{f(e)} \subset J_s$;

3. For a cycle $e_1 e_2 \cdots e_n$, where cycle means $f(e_k) = i(e_{k+1}), \forall k = 1, 2, \cdots, n-1$ and $f(e_n) = i(e_1)$, we have $\prod_{k=1}^n l_{e_k} < 1$.

Condition 2 in Definition 2.3 means that ψ_e maps from $J_{f(e)}$ into $J_{i(e)}$. To see why we need Condition 3, one can consider the case $S=\{1\}$, and then condition 3 means that each similitude ψ_e is a contraction, so by standard theory, there exists a unique self-similar set $K_1 \subset J_1$ such that $K_1 = \cup_{e\in E}\psi_e(K_1)$.

It is well-known that there is a unique vector of compact sets $\mathcal{K}=\{K_s\}_{s\in S}$ such that for each $s\in S$, K_s is contained in J_s, and

$$K_s = \bigcup_{i(e)=s} \psi_e K_{f(e)}.$$

We call them the *invariant sets* of the graph-directed construction $\mathcal{G}$. We will always assume that each K_s is connected.

We define a *shift space* associated with $\mathcal{G}$ to address points in $K_s, s\in S$. A finite sequence of edges in G, denoted by $\boldsymbol{e}=e_1e_2\cdots e_n$, is called a *walk* if $f(e_k)=i(e_{k+1}), \forall 1\le k\le n-1$. We write $|\boldsymbol{e}|=n$ for the length of the walk. An infinite sequence of edges is called an *infinite walk*, denoted by $\boldsymbol{\epsilon}=\epsilon_1\epsilon_2\cdots$, if for any $n\ge 1$, the first n steps $[\boldsymbol{\epsilon}]_n=\epsilon_1\epsilon_2\cdots\epsilon_n$ is a walk of length n. Denote by E_* the collection of finite walks in G, and E^∞ the space of infinite walks. For convenience, let $i(\boldsymbol{e})=i(e_1)$ or $i(\boldsymbol{\epsilon})=i(\epsilon_1)$ the initial state of a walk, and let $f(\boldsymbol{e})=f(\boldsymbol{e}_{|\boldsymbol{e}|})$ the final state of a walk. Then we define a projection $\pi: E^\infty\to\bigcup_{s\in S}K_s$ by

$$\{\pi(\boldsymbol{\epsilon})\} = \bigcap_{n=1}^{\infty} \psi_{[\boldsymbol{\epsilon}]_n} K_{f([\boldsymbol{\epsilon}]_n)},$$

where we use the notation $\psi_{\boldsymbol{e}}=\psi_{e_1}\circ\psi_{e_2}\circ\cdots\circ\psi_{e_n}$. Noticing that $\psi_{[\boldsymbol{\epsilon}]_n}K_{f([\boldsymbol{\epsilon}]_n)}$ is a decreasing sequence of subsets in $K_{i(\boldsymbol{\epsilon})}$, with diameter converging to 0 by Condition 3 in Definition 2.3, the right side of the above identity determines a unique point $\pi(\boldsymbol{\epsilon})$ in $K_{i(\boldsymbol{\epsilon})}$.

Analogous to p.c.f. self-similar sets, for each $s\in S$, we introduce the set of level 1 intersection $C_s=\bigcup_{e\neq e'\in i^{-1}(s)}\psi_eK_{f(e)}\cap\psi_{e'}K_{f(e')}$, where $e\in i^{-1}(s)$ means $i(e)=s$, the *critical set* $\mathcal{C}_{\mathcal{G}}=\bigcup_{s\in S}\pi^{-1}(C_s)$, and the *post-critical set* $\mathcal{P}_{\mathcal{G}}=\bigcup_{n=1}^\infty\sigma^n(\mathcal{C}_{\mathcal{G}})$, where σ is the *shift map* on E^∞, i.e., $\sigma(\epsilon_1\epsilon_2\cdots)=\epsilon_2\epsilon_3\cdots$. For each $s\in S$, we write $V_s=\{\pi(\boldsymbol{\epsilon}): i(\boldsymbol{\epsilon})=s, \boldsymbol{\epsilon}\in\mathcal{P}_{\mathcal{G}}\}$. We can see that

$$\psi_eK_{f(e)}\cap\psi_{e'}K_{f(e')}=\psi_eV_{f(e)}\cap\psi_{e'}V_{f(e')},\ \text{if } i(e)=i(e').$$

In particular, if V_s is finite for each $s\in S$, each pair of cells $\psi_eK_{f(e)}$ and $\psi_{e'}K_{f(e')}$ intersect at finitely many points. This leads to the definition of f.r.g.d. fractals [17].

Definition 2.4. *A family $\mathcal{K}=\{K_s\}_{s\in S}$ constructed by the graph-directed construction $\mathcal{G}=(S,E,\{\psi_e\}_{e\in E})$ is called a finitely ramified graph-directed (f.r.g.d. for short) fractal family if V_s is finite for each $s\in S$. Each member $K_s\in\mathcal{K}$ is called an f.r.g.d. fractal.*

We will briefly recall the construction of resistance forms on f.r.g.d. fractals in Subsection 3.1. See [17] for more details.

In the rest of this section, we construct an f.r.g.d. construction for a self-similar set K under the assumptions (**F1**) and (**F2**). The idea is to break K into a finite union of c_*-overlapping chains of finite types. We need to overcome the difficulty of showing $\#V_s<\infty$ for each state. We will achieve this idea by using *maximal chains*.

Definition 2.5. *(1). Say a c_*-overlapping chain $\gamma = (w^{(1)}, w^{(2)}, \cdots, w^{(n)})$ a maximal c_*-overlapping chain if and only if there is no another $w \in W_\#$ such that $\tilde{\gamma} = (w^{(1)}, w^{(2)}, \cdots, w^{(n)}, w)$ is a c_*-overlapping chain.*

(2). Denote $\Lambda = \{\gamma : \gamma$ is a maximal c_*-overlapping chain$\}$, and write $K_\gamma = \bigcup_{w\in\gamma} F_w K$ for each $\gamma \in \Lambda$. In particular, $K_\theta = K$, where $\theta := (\emptyset) \in \Lambda$ is a trivial chain.

(3). For $\gamma, \eta \in \Lambda$, say $\gamma \sim \eta$ if there exists a similitude $\phi_{\gamma,\eta}$ such that $\forall w \in \gamma$ and $w' \in W_*$ with $F_{w'}K = F_w K$, there are $u \in \eta$ and $u' \in W_*$ with $F_{u'}K = F_u K$, satisfying that $\phi_{\gamma,\eta} \circ F_{w'} = F_{u'}$, and a same condition holds conversely for $\phi_{\eta,\gamma} = \phi_{\gamma,\eta}^{-1}$.

Clearly, $\phi_{\gamma,\eta}(K_\gamma) = K_\eta$ for $\gamma \sim \eta$. Obviously, "$\sim$" is an equivalent relationship on Λ.

Lemma 2.6. *Assume **(F1)** and **(F2)**. Then $\#(\Lambda/\sim) < \infty$.*

Proof. We enlarge Λ to be $\Lambda' = \{\gamma : \gamma$ is a c_*-overlapping chain$\}$, and define $\sim$ on Λ' as in Definition 2.5 (3). Write $\Lambda'_n = \{\gamma : \gamma$ is a c_*-overlapping chain of length $n\}$ for $n \geq 1$.

One can check that $\#(\Lambda'_n/\sim) < \infty$, $\forall n \geq 1$. This can be done by induction hypothesis. First, $\#(\Lambda'_1/\sim) = 1$. Next, assume $\#(\Lambda'_{n-1}/\sim) < \infty$. By (**F1**), there are at most finitely many ways to add a new word $w \in W_\#$ to a fixed $\gamma \in \Lambda'_{n-1}$. So each type in $\Lambda'_{n-1}/\sim$ will only lead to finitely many different types in $\Lambda'_n/\sim$, which implies $\#(\Lambda'_n/\sim) < \infty$.

Lastly, by (**F2**), we have $\#(\Lambda/\sim) \leq \#(\Lambda'/\sim) = \sum_{n=1}^{\mathcal{L}_{c_*}(K)} \#(\Lambda'_n/\sim) < \infty$. □

The following lemma shows how to get an f.r.g.d. construction of K by using maximal c_*-overlapping chains.

Lemma 2.7. *Assume **(F1)** and **(F2)**. For $\gamma \in \Lambda$, there is a finite number of maximal c_*-overlapping chains $\{\gamma_i\}_{i=1}^k$ with $k > 1$ satisfying the following properties:*

1. $K_\gamma = \bigcup_{i=1}^k K_{\gamma_i}$;

2. $\#K_{\gamma_i} \cap K_{\gamma_j} < \infty$ for $i \neq j$;

3. For each $\gamma_i = (w_i^{(1)}, w_i^{(2)}, \cdots w_i^{(n)})$, and for any $\gamma' \sim \gamma$ with a similitude $\phi_{\gamma,\gamma'}$ as defined in Definition 2.5 (3), write $\gamma'_i = (w_i'^{(1)}, \cdots, w_i'^{(n)})$ with $F_{w_i'^{(j)}}K = \phi_{\gamma,\gamma'}(F_{w_i^{(j)}}K)$. Then γ'_i is still a maximal c_*-chain and $\gamma'_i \sim \gamma_i$.

Proof. Let $c_\gamma = \min_{v\in\gamma} c_v$. Let $0 < \lambda \leq c_* c_\gamma$, and write $W_{\lambda,\gamma} = \{w \in W_\lambda : \#K_w \cap K_\gamma = \infty\}$.For any $w \in W_{\lambda,\gamma}$ and $w' \in W_*$ with $F_{w'}K = F_w K$, there exist $v \in \gamma$ and $v', u \in W_*$, such that $F_{v'}K = F_v K$ and $F_{w'} = F_{v'}F_u$, since otherwise we can lengthen the chain γ with a new word $\tilde{w}$ satisfying $F_w K \subset F_{\tilde{w}}K$. In particular, we have $F_w K \subset K_\gamma$.

For each $w \in W_{\lambda,\gamma}$, there exists a unique maximal c_*-overlapping chain γ_w containing w such that $\gamma_w \subset W_{\lambda,\gamma}$. In fact, we choose γ_w to be the largest c_*-overlapping chain consisting of words in $W_{\lambda,\gamma}$ that contains w. If γ_w is not maximal, then there exists a word $u \in W_\# \setminus W_{\lambda,\gamma}$ such that $\gamma_w \cup \{u\}$ is still a c_*-overlapping chain. Let's consider two cases of c_u:

1. $c_u \leq \lambda$. Then there is a word $u' \in W_\lambda$ such that $F_u K \subset F_{u'}K$, and thus $\#F_{u'}K \cap K_{\gamma_w} = \infty$. This gives that $u' \in W_{\lambda,\gamma}$, and thus $F_{u'}K \subset K_\gamma$. This contradicts the definition of γ_w.

2. $c_u > \lambda$. Then we can choose a word $u' \in W_{\lambda,\gamma}$ such that $F_{u'}K \subset F_u K$ and $\#F_{u'}K \cap K_{\gamma_w} = \infty$. A same contradiction as case 1 happens.

Thus, we have proved that γ_w is a maximal c_*-overlapping chain. Obviously, $K_\gamma = \bigcup_{w\in W_{\lambda,\gamma}} K_{\gamma_w}$ and $\#K_{\gamma_w} \cap K_{\gamma_u} < \infty$ if $\gamma_w \neq \gamma_u$. Moreover, due to (**F2**), by choosing λ small enough, we can ensure the existence of at least two chains of this form.

Conclusion 3 is easy to get by choosing $\lambda' = \lambda \cdot \frac{c_{\gamma'}}{c_\gamma}$, and repeating the same operation on $K_{\gamma'}$ as we did on K_γ, noticing the one to one correspondence between $W_{\lambda,\gamma}$ and $W_{\lambda',\gamma'}$ by the discussion in the first paragraph. $\square$

As a consequence of Lemma 2.7, we are able to define an f.r.g.d. construction of K with state set $S := \Lambda/\sim$. For each $s \in S$, we choose a representative maximal c_*-overlapping chain γ_s in s, and denote $K_s = K_{\gamma_s}$. In particular, for the equivalent class of $\theta = (\emptyset)$, we denoted it by ϑ, and always choose $\gamma_\vartheta = \theta$ and $K_\vartheta = K$.

For each $s \in S$, by Lemma 2.7, we can find finitely many maximal c_*-overlapping chains $\gamma_i^{(s)}, 1 \le i \le n_s$, such that $K_s = \bigcup_{i=1}^{n_s} K_{\gamma_i^{(s)}}$ and $\# K_{\gamma_i^{(s)}} \cap K_{\gamma_j^{(s)}} < \infty$ for any $i \ne j$. For each $\gamma_i^{(s)}$, denote its equivalent class by s_i, and define an edge e such that $i(e) = s$ and $f(e) = s_i$. In addition, for this edge e, we assign a similitude ψ_e by

$$\psi_e = \phi_{\gamma_{s_i}, \gamma_i^{(s)}}.$$

Let E be the collection of edges defined above. We then get a graph-directed construction $(S, E, \{\psi_e\}_{e\in E})$. It remains to show that it is finitely ramified.

Theorem 2.8. *Each self-similar set K satisfying (**F1**) and (**F2**) is a f.r.g.d. fractal.*

Proof. Let $(S, E, \{\psi_e\}_{e\in E})$ be defined as above. For $s \in S$, let V_s be the same as introduced above Definition 2.4. We only need to prove that $\# V_s < \infty$, $\forall s \in S$.

In fact, for each $x \in V_s$, we can find a γ in s and a walk $\boldsymbol{e}$ so that $K_\gamma = \psi_{\boldsymbol{e}} K_s$ and $\psi_{\boldsymbol{e}} x \in K_\gamma \cap \overline{K \setminus K_\gamma}$. In addition, assume $x \in F_w K$ where $w \in \gamma_s$, and let $v \in \gamma$ so that $F_v K = \psi_{\boldsymbol{e}} F_w K$. By Definition 2.5(3), we can find $v' \in W_*$ with $F_{v'} = \psi_{\boldsymbol{e}} F_w$ and $F_{v'} K = F_v K$.

Let $\lambda = c_* \min_{u\in\gamma} c_u$. Then we can find $u \in W_\lambda \setminus W_{\lambda,\gamma}$ so that $\psi_{\boldsymbol{e}} x \in F_u K \cap F_v K$. Since γ is maximal, we have $\# F_u K \cap F_v K < \infty$. In other words, if we define $h = F_{v'}^{-1} F_u$, then $\psi_{\boldsymbol{e}} x \in F_{v'} h K \cap F_{v'} K$, which shows that $x \in F_w h K \cap F_w K$.

The above discussions shows that for each $x \in V_s$, we can find a word $w \in \gamma_s$, a similitude h of the form $F_v^{-1} F_u$ with $c_v^{-1} c_u \in (c_*^3, 1)$ such that $x \in F_w h K \cap F_w K$ and $\# F_w h K \cap F_w K < \infty$. By (**F1**), we know that there are only finitely many different choices of h, so V_s is a finite set. $\square$

3. Examples

In this section, we will introduce some examples of self-similar sets which are not p.c.f. but satisfying (**F1**) and (**F2**). We will provide f.r.g.d. constructions and construct resistance forms for these examples.

3.1. **Harmonic structures on f.r.g.d. fractals.** First, let's briefly recall the harmonic structures on f.r.g.d. fractal families(Definition 2.4) introduced by Hambly and Nyberg[17]. The idea is based on Kigami's compatible sequence of resistance networks[20, 21].

Let V be a finite set. A symmetric linear operator(matrix) $H : l(V) \to l(V)$ is called a *(discrete) Laplacian* on V if H is non-positive definite, $Hu = 0$ if and only if u is a constant on V, and $H_{xy} \ge 0$ for all $x \ne y \in V$. The pair (V, H) is called a *(finite) resistance network.*

Write $x \sim y$ if $H_{xy} > 0$. There is a symmetric bilinear form $\mathcal{E}_H(\cdot,\cdot)$ on $l(V)$, called the *(discrete) resistance form* associated with H, written as

$$\mathcal{E}_H(u,v) = \sum_{x\sim y} c_{x,y}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big), \quad \forall u,v \in l(V),$$

with $c_{x,y} = H_{xy}$ called the *conductance* between x, y. We write $\mathcal{E}_H(u) = \mathcal{E}_H(u,u)$ for short.

Now, consider an f.r.g.d. construction $\mathcal{G} = (S, E, \{\psi_e\}_{e\in E})$. Let $\mathcal{K} = \{K_s\}_{s\in S}$ be the associated f.r.g.d. fractal family. Recall that for each $s \in S$, V_s is the boundary of K_s. For $n \geq 0$, let $V_s^{(n)} = \bigcup_{|\boldsymbol{e}|=n, i(\boldsymbol{e})=s} \psi_{\boldsymbol{e}} V_{f(\boldsymbol{e})}$ be the level n vertices. Let $\boldsymbol{D} = \{D_s\}_{s\in S}$ where D_s is a Laplacian on V_s, and let $\boldsymbol{r} = \{r_e\}_{e\in E} \in \mathbb{R}_+^E$. Inductively, we define a resistance form on $V_s^{(n)}$ by

$$\mathcal{E}_s^{(n)}(u,v) = \sum_{|\boldsymbol{e}|=n, i(\boldsymbol{e})=s} r_{\boldsymbol{e}}^{-1} \mathcal{E}_{D_{f(\boldsymbol{e})}}(u\circ\psi_{\boldsymbol{e}}^{-1}, v\circ\psi_{\boldsymbol{e}}^{-1}), \quad \forall u,v \in l(V_s^{(n)}),$$

where $r_{\boldsymbol{e}} = r_{e_1} r_{e_2}\cdots r_{e_n}$ for $\boldsymbol{e} = e_1e_2\cdots e_n$. In particular $\mathcal{E}_s^{(0)} = \mathcal{E}_{D_s}$.

Write $H_s^{(n)}$ the Laplacian corresponding to $\mathcal{E}_s^{(n)}$. For $s \in S$, we require the sequence $\{(V_s^{(n)}, H_s^{(n)})\}_{n\geq 0}$ to be compatible in the following sense.

Definition 3.1. *If (V, H_1) and (U, H_2) are two pairs of resistance networks satisfying that $V \subset U$ and*

$$\mathcal{E}_{H_1}(v) = \min\{\mathcal{E}_{H_2}(u) : u \in l(U), u|_V = v\}, \quad \forall v \in l(V), \tag{3.1}$$

we say they are compatible and write $(V, H_1) \leq (U, H_2)$.

This was introduced by Kigami in the construction of harmonic structures and resistance forms on p.c.f. self-similar sets [18, 19]. Note that if $(V, H_1) \leq (U, H_2)$, then for any $v \in l(V)$, there exists a unique function $u \in l(U)$ attaining the minimum in (3.1). Essentially using the same idea, Hambly and Nyberg[17] extend the concept of *harmonic structure* to f.r.g.d. fractal families in the following way.

Definition 3.2. *(a). The pair $(\boldsymbol{D}, \boldsymbol{r})$, where $\boldsymbol{D} = \{D_s\}_{s\in S}$ is a set of Laplacians on V_s and $\boldsymbol{r} \in \mathbb{R}_+^E$, is called a harmonic structure if*

$$(V_s, D_s) \leq (V_s^{(1)}, H_s^{(1)}), \quad \forall s \in S. \tag{3.2}$$

(b). Say $(\boldsymbol{D}, \boldsymbol{r})$ is a regular harmonic structure if $r_{\boldsymbol{e}} < 1$ for each cycle $\boldsymbol{e}$($\boldsymbol{e} \in E_*$ with $i(\boldsymbol{e}) = f(\boldsymbol{e})$).

Using a standard proof, for each $s \in S$, we see that

$$(V_s, D_s) \leq (V_s^{(1)}, H_s^{(1)}) \leq (V_s^{(2)}, H_s^{(2)}) \leq \cdots,$$

which gives a compatible sequence of networks on approximation graphs of K_s. Furthermore, if $(\boldsymbol{D}, \boldsymbol{r})$ is regular, then for any Borel measure μ_s on K_s, there is a limit form $(\mathcal{E}_s, \mathcal{F}_s)$, which is a local regular Dirichlet form on $L_{\mu_s}(K_s)$.

Before ending this subsection, we will mention an interesting fact about cut-points in the fractals that will simplify the calculation. Let $\mathcal{K} = \{K_s\}_{s\in S}$ be an f.r.g.d. fractal family. For $s \in S$, we call p a *cut-point* of K_s if $K_s \setminus \{p\}$ is disconnected. Note that since K_s is connected, $K_s \setminus \{p\}$ is a locally arcwise connected set.

Proposition 3.3. *Let $\mathcal{G}=(S,E,\{\psi_e\}_{e\in E})$ be an f.r.g.d. construction, and $\mathcal{K}=\{K_s\}_{s\in S}$ be the corresponding f.r.g.d. fractal family. For $s\in S$, let p be a cut-point of K_s, and $\{p_k\}_{k=1}^m$ be a finite set in $K_s\setminus\{p\}$. Write the restriction of $\mathcal{E}_s$ onto $V=\{p\}\cup\{p_k\}_{k=1}^m$ by*

$$\mathcal{E}_s|_V(u,v)=\sum_{x\neq y\in V}c_{x,y}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big),\quad \forall u,v\in l(V),$$

then $c_{p_k,p_l}>0$ only if p_k,p_l belong to a same connected component of $K_s\setminus\{p\}$.

Proof. The proof is obvious and routine by a standard discussion of harmonic structures. We omit it. □

3.2. **Overlapping Vicsek set.** Let $\{q_i\}_{i=1}^4$ be the four vertices of a square, and q_5 be the center. The *Overlapping Vicsek set*(see Figure 6), denoted by $\mathcal{OV}$, is the invariant set of the i.f.s. $\{F_i\}_{i=1}^5$,

$$F_1: x\to\frac{1}{2}x+\frac{1}{2}q_1,\quad F_i: x\to\frac{1}{3}x+\frac{2}{3}q_i, i=2,3,4,5.$$

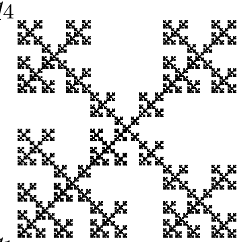


FIGURE 6. The set $\mathcal{OV}$.

We can easily check that both (**F1**) and (**F2**) hold for $\mathcal{OV}$, and thus by Theorem 2.8, $\mathcal{OV}$ is an f.r.g.d. fractal. We provide an f.r.g.d. construction of $\mathcal{OV}$ as follows.

Let $K_1=\mathcal{OV}$ and $K_2=\bigcup_{i=1,2,4,5}F_i\mathcal{OV}$, then

$$\begin{cases}K_1=K_2\cup F_3K_1,\\ K_2=F_1K_2\cup F_2K_1\cup F_4K_1\cup F_5K_1.\end{cases}\tag{3.3}$$

Then $\{K_1,K_2\}$ can be viewed as invariant sets of an f.r.g.d. construction $\mathcal{G}=(S,E,\Gamma)$ with $S=\{1,2\}$ being the state set, E being the edge set consisting of 6 edges, and Γ being the collection of similitudes associated with E, see Figure 7. In an obvious way, we could rewrite (3.3) into

$$\begin{cases}K_1=\psi_1K_2\cup\psi_2K_1,\\ K_2=\psi_3K_2\cup\psi_4K_1\cup\psi_5K_1\cup\psi_6K_1.\end{cases}\tag{3.4}$$

The sets $V_1=\{q_1,q_2,q_3,q_4\}$ and $V_2=\{q_1,q_2,q_4,F_3q_1\}$ are boundaries of K_1 and K_2 respectively as defined in the last section. Since q_5 is a cut-point that divides both K_1 and K_2 into pieces, and in view of Proposition 3.3, it is more convenient to look at the harmonic structures involving q_5. Let

$$\tilde{V}_1=\{q_i\}_{i=1}^5,\quad \tilde{V}_2=\{q_1,q_2,q_4,F_3q_1,q_5\},$$

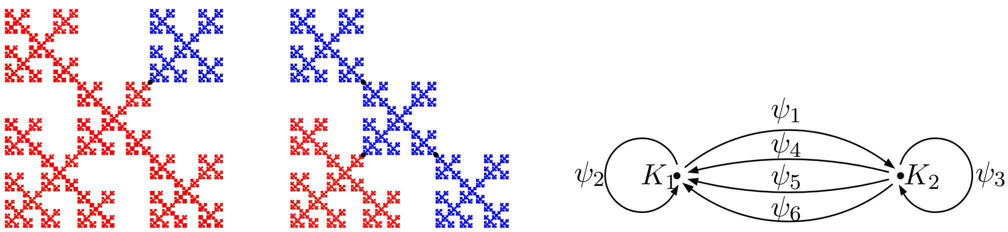


FIGURE 7. An f.r.g.d. construction of $\mathcal{OV}$.

and for $s=1,2$, $n\geq 0$, write $\tilde{V}_s^{(n)}=\bigcup_{|\boldsymbol{e}|=n,i(\boldsymbol{e})=s}\psi_{\boldsymbol{e}}\tilde{V}_{f(\boldsymbol{e})}$. Given $\tilde{\boldsymbol{D}}=\{\tilde{D}_s\}_{s\in S}$ and $\boldsymbol{r}=\{r_e\}_{e\in E}$, we can define $\tilde{H}_s^{(n)}$ on $\tilde{V}_s^{(n)}$ in a same manner as that for the pair $(\boldsymbol{D},\boldsymbol{r})$, which is the restriction of $(\tilde{\boldsymbol{D}},\boldsymbol{r})$ to $\{V_s\}_{s\in S}$. The resulting resistance forms on $\mathcal{OV}$ are the same as that of $(\boldsymbol{D},\boldsymbol{r})$.

Using Proposition 3.3, we start from resistance networks on $\{\tilde{V}_1,\tilde{V}_2\}$ as shown in Figure 8. To simplify the notations, we write $r^{(i)}_{p,q}=1/(\tilde{D}_i)_{pq}$ the *resistance* between p,q in the network $(\tilde{V}_i,\tilde{D}_i)$. Without loss of generality, we denote

$$r^{(1)}_{q_1,q_5}=1,\quad r^{(1)}_{q_2,q_5}=a,\quad r^{(1)}_{q_3,q_5}=b,\quad r^{(1)}_{q_4,q_5}=c,$$

and

$$r^{(2)}_{q_1,q_5}=1,\quad r^{(2)}_{q_2,q_5}=a',\quad r^{(2)}_{F_3q_1,q_5}=d,\quad r^{(2)}_{q_4,q_5}=c'.$$

We write $r_1,r_2,\cdots,r_6$ for the renormalization factors, see Figure 9.

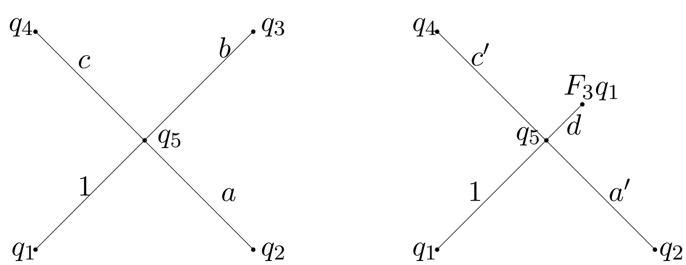


FIGURE 8. The resistance network $(\tilde{V}_i,\tilde{D}_i)$, $i=1,2$.

By operating on resistors in series, the renormalization formulas (see (3.1) and (3.2)) are equivalent to the following equations,

$$\begin{cases} a=a', c=c', r_1=1, r_6b=d,\\ r_2(1+b)+d=b,\\ r_3(1+d)+r_6=1,\\ r_4(a+c)+r_6a=a,\\ r_5(a+c)+r_6c=c. \end{cases}\tag{3.5}$$

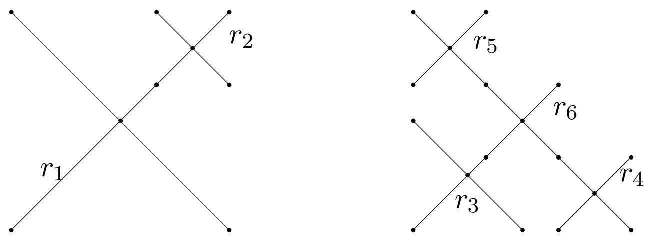


FIGURE 9. The renormalization factors $r_1, r_2, \cdots, r_6$.

It is easy to get that

$$r_1 = 1,\ r_2 = \frac{b-d}{1+b},\ r_3 = \frac{b-d}{b+bd},\ r_4 = \frac{ab-ad}{ab+cb},\ r_5 = \frac{cb-cd}{ab+cb},\ r_6 = \frac{d}{b}. \tag{3.6}$$

To make the harmonic structure regular, we only need to assume $a, b, c, d > 0$ and $b > d$. The solution depends on 4 parameters.

At the end of this example, we point out that the f.r.g.d. construction of $\mathcal{OV}$ is not unique. For example, if we insist on using the maximal c_*-overlapping chains as illustrated in the proof of Lemma 2.7 and Theorem 2.8, we can get another f.r.g.d. construction as shown in Figure 10. Furthermore, if we impose some good symmetry on the resulting resistance forms, we will see that these two constructions provide the same resistance forms on $\mathcal{OV}$. See discussions in Section 5.

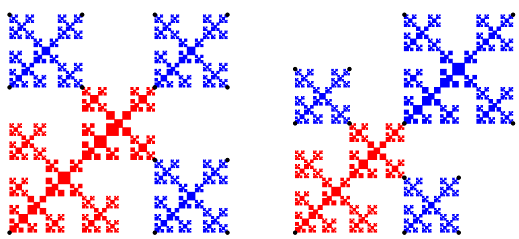


FIGURE 10. Another f.r.g.d. construction of $\mathcal{OV}$.

3.3. **Overlapping gasket.** Let $\{q_i\}_{i=1}^3$ be the vertices of an equilateral triangle, and $\{q_4, q_5\}$ be the midpoints of the line segments $\overline{q_1q_2}$ and $\overline{q_1q_3}$. The *Overlapping gasket*, denoted by $\mathcal{OG}$, is the invariant set of the i.f.s. $\{F_i\}_{i=1}^5$,

$$F_2 : x \to \frac{1}{2}x + \frac{1}{2}q_2, \quad F_i : x \to \frac{1}{3}x + \frac{2}{3}q_i, i = 1, 3, 4, 5,$$

see the left picture of Figure 11 for $\mathcal{OG}$.

Let $K_1 = \mathcal{OG}$ and $K_2 = (\mathcal{OG} \setminus F_2\mathcal{OG}) \cup \{q_4\}$, see Figure 11. We have

$$\begin{cases} K_1 = F_2K_1 \cup K_2, \\ K_2 = F_1K_1 \cup F_3K_1 \cup F_5K_1 \cup F_4K_2, \end{cases} \tag{3.7}$$

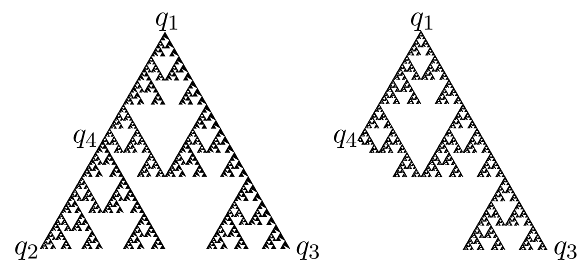


FIGURE 11. The set $\mathcal{OG}$ (left) and K_2 (right).

and thus $\{K_1, K_2\}$ can be viewed as invariant sets of an f.r.g.d. construction $\mathcal{G} = (S, E, \Gamma)$ with $S = \{1, 2\}$ being the state set, E being the edge set consisting of 6 edges, and Γ being the collection of similitudes associated with E. In an obvious way, we rewrite (3.7) into

$$\begin{cases} K_1 = \psi_1 K_1 \cup \psi_2 K_2, \\ K_2 = \psi_3 K_1 \cup \psi_4 K_1 \cup \psi_5 K_1 \cup \psi_6 K_2. \end{cases} \tag{3.8}$$

See Figure 12 for an illustration.

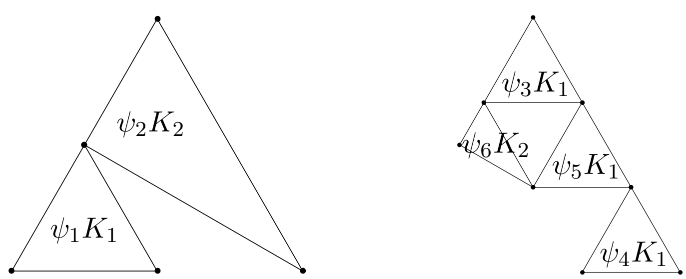


FIGURE 12. An f.r.g.d. construction of $\mathcal{OG}$.

Now, let's compute the harmonic structures. For simplicity, take

$$r^{(1)}_{q_1,q_2} = a, r^{(1)}_{q_1,q_3} = b, r^{(1)}_{q_2,q_3} = c, r^{(2)}_{q_1,q_4} = d, r^{(2)}_{q_1,q_3} = e, r^{(2)}_{q_3,q_4} = f,$$

where $r^{(i)}_{p,q}$'s are the resistances in the resistance networks $(V_i, D_i), i = 1, 2$, see Figure 13.

For the renormalization factors, we set $r_2 = 1$ as $\psi_2 = id$. Furthermore, to simplify the computation, we demand that cells of same size have the same energy. To be precise, for any two cells $K_{\boldsymbol{e_1}}$, $K_{\boldsymbol{e_2}}$ with the same type and same size, we require $r_{\boldsymbol{e_1}} = r_{\boldsymbol{e_2}}$. We call $(\boldsymbol{D}, \boldsymbol{r})$ a *homogeneous regular harmonic structure* if the above condition is satisfied. Then it is easy to verify that we need to require $r_3 = r_4 = r_5 = r_6$, and we use the symbol s to denote them hereafter.

It is convenient to use the $\Delta - Y$ transformation for resistance networks here, see Figure 14 for an illustration of the $\Delta - Y$ transformation. See Figure 15 for the transformations between the first two level resistance networks.

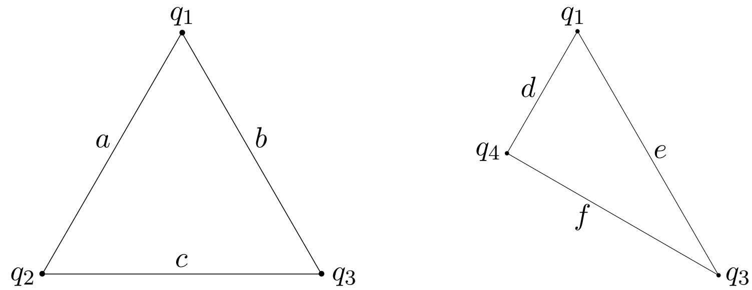


FIGURE 13. The resistance networks $(V_i, D_i), i = 1, 2$.

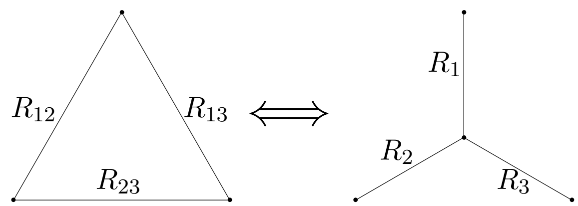


FIGURE 14. An illustration of the $\Delta - Y$ transformation, with $R_i = \frac{R_{ij}R_{ik}}{R_{12}+R_{23}+R_{13}}$, $\{i,j,k\} = \{1,2,3\}$. All R_i's and R_{ij}'s are resistances.

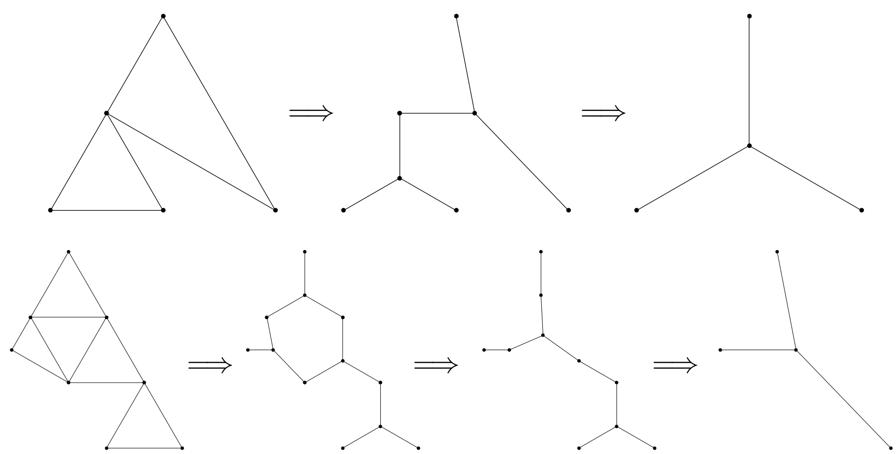


FIGURE 15. Transformations between the first two level resistance networks.

In view of Figure 15, it is more convenient to use the Y-shaped networks for the calculation, then solve the Δ-shaped networks by doing the inverse $\Delta - Y$ transformation. See Figure 16 for the Y-shaped networks with resistances a', b', e', d', f' marked there.

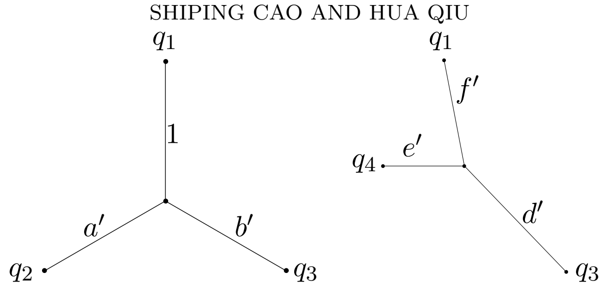


FIGURE 16. The Y-shaped resistance network of (V_i, D_i), i=1,2.

Put the resistances and renormalization factors into the transformations shown in Figure 15. We get

$$f' = 1, d' = b', e' + r_1 + r_1 a' = a', \tag{3.9}$$

by the transformation on $V_1^{(1)}$, and

$$\begin{cases} s + \frac{s(1+a')(1+b')}{2(1+a'+b')} = 1, \\ se' + \frac{s(1+a')(a'+b')}{2(1+a'+b')} = e', \\ 2sb' + s + \frac{s(1+b')(a'+b')}{2(1+a'+b')} = b', \end{cases}$$

by the transformation on $V_2^{(1)}$, using equations in (3.9). Solving these equations, we get

$$\begin{cases} b' = d' = \frac{2+3a'}{a'-2}, e' = \frac{1}{2a'} + \frac{1}{4} + \frac{a'}{4}, f' = 1, \\ s = \frac{2+a'}{4+3a'}, r_1 = \frac{-2-a'+3(a')^2}{4a'+4(a')^2}. \end{cases} \tag{3.10}$$

The solution depends on the parameter a' and gives us the homogeneous regular harmonic structures when $a' > 2$.

3.4. **Vicsek windmill.** Let $\{q_i\}_{i=1}^4$ be the vertices of a square in $\mathbb{R}^2$, say $\{(0,0),(1,0),(1,1),(0,1)\}$ for convenience. The *Vicsek windmill*, denoted by $\mathcal{VW}$, is the invariant set of the i.f.s. $\{F_i\}_{i=1}^8$,

$$F_i(x) = \frac{1}{4}x + \frac{3}{4}q_i, \ i = 1,2,3,4,$$
$$F_5(x) = \frac{1}{4}x + (\frac{1}{4}, 0), F_6(x) = \frac{1}{4}x + (\frac{1}{2}, \frac{1}{4}),$$
$$F_7(x) = \frac{1}{4}x + (\frac{1}{4}, \frac{1}{2}), F_8(x) = \frac{1}{4}x + (\frac{1}{2}, \frac{3}{4}),$$

see Figure 17 for $\mathcal{VW}$.

There is an f.r.g.d. construction of $\mathcal{VW}$ which has three states $S = \{1,2,3\}$, with

$$K_1 = \mathcal{VW}, \quad K_2 = F_1\mathcal{VW} \cup F_5\mathcal{VW}, \quad K_3 = F_1F_8\mathcal{VW} \cup F_1F_3\mathcal{VW} \cup F_5F_4\mathcal{VW}.$$

As the f.r.g.d. construction of $\{K_1, K_2, K_3\}$ involves long equations, we omit the exact expressions. But readers can get all the information from Figure 17.

The set $\mathcal{VW}$ possesses an obvious rotational symmetry. It is reasonable to require the harmonic structures to possess the same symmetry as well as to be homogeneous and regular. At the first glance, to determine a homogeneous regular harmonic structure $(\boldsymbol{D}, \boldsymbol{r})$, we need 8 parameters for the renormalization factors. For $i, j \in \{1,2,3\}$, we use r_{ij} to denote the

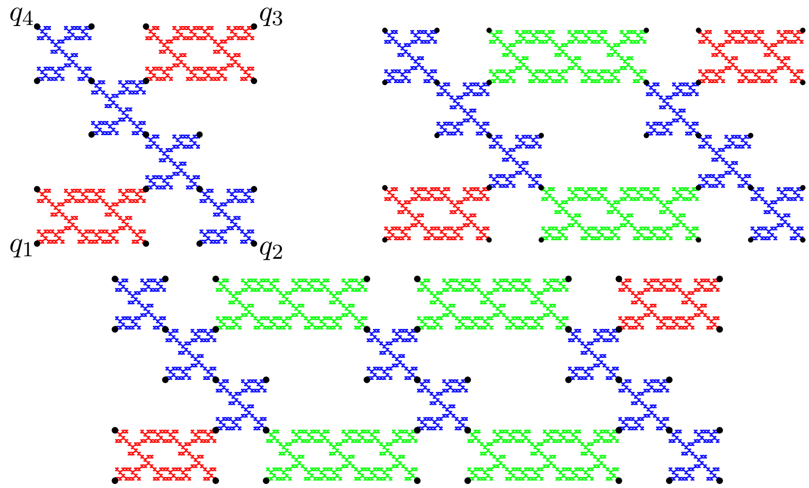


FIGURE 17. The Vicsek windmill $\mathcal{VW}$ (up left), K_2 (up right) and K_3 (down).

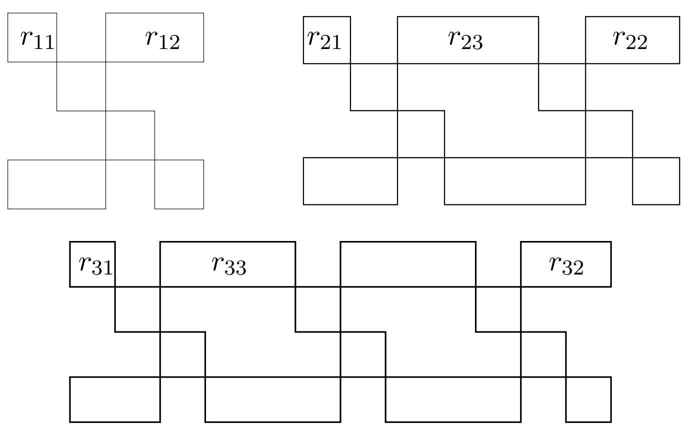


FIGURE 18. The f.r.g.d. construction and the renormalization factors for $\mathcal{VW}$.

renormalization factor associated with the edge in E from i to j, see Figure 18. Note that there is no r_{13} since no such edge exists in E.

Noticing that the homogeneity requirement of $(\boldsymbol{D}, \boldsymbol{r})$ implies that

$$\prod_{k=1}^{n} r_{i_{k-1}i_k} = r_{11}^n,$$

for any finite head-to-tail sequences of factors $r_{i_0i_1}, r_{i_1i_2}, \cdots, r_{i_{n-1}i_n}$ with $i_0 = i_n = 1$. We can see that $r_{11} = r_{22} = r_{33}$ by the equations $r_{12}r_{21}r_{11} = r_{12}r_{22}r_{21}$ and $r_{12}r_{23}r_{31}r_{11} = r_{12}r_{23}r_{33}r_{31}$. In a similar way, we can get that $r_{12}r_{21} = r_{11}^2, r_{12}r_{23}r_{31} = r_{11}^3$ and $r_{23}r_{32} = r_{22}^2 = r_{11}^2$. So there are only three free parameters r_{11}, r_{21}, r_{31}. Furthermore, if $(\boldsymbol{D}, \boldsymbol{r})$ is a homogeneous regular harmonic structure with $\boldsymbol{r} = \{r_{ij}\}$ as above, then by letting $D_1' = D_1$, $D_2' = \frac{r_{11}}{r_{21}} D_2$, $D_3' = \frac{r_{11}}{r_{31}} D_3$, and $\boldsymbol{r'}$ the vector of constant r_{11}, it is easy to check that $(\boldsymbol{D'}, \boldsymbol{r'})$ is also a homogeneous regular harmonic structure, which yields the same resistance form induced by $(\boldsymbol{D}, \boldsymbol{r})$.

Basing on the above discussion, we only need to consider the homogeneous regular harmonic structures $(\boldsymbol{D}, \boldsymbol{r})$ with $\boldsymbol{r}$ being a constant vector. To simplify the notations, we denote the common factor by r.

On the other hand, unlike the previous examples, it is easy to observe that for each cell, its boundary is not fully involved when intersecting with other cells. Thus, regarding the rotational symmetry, we only need to consider certain restricted resistance networks of (V_i, D_i)'s. Firstly, we choose a Y-shaped restricted network on $\{q_2, q_3, q_4\}$ of (V_1, D_1), and denote the resistances to be $a, b, c,$ see Figure 19 (a). In addition, the restricted network on $\{q_1, q_2, q_4\}$ is given by symmetry, see Figure 19 (b). By simple series connection, the effective resistance between q_2 and q_4 is always $a + c$. Secondly, for (V_2, D_2), we restrict the network onto $\{q_1, F_5 q_3\}$, two of the diagonal vertices in V_2, and set the resistance between them to be d, see Figure 19 (c). Lastly, for (V_3, D_3), we restrict the network onto $\{F_1F_8q_1, F_5F_4q_2\}$ or $\{F_1F_8q_4, F_5F_4q_3\}$, two of the four vertices in V_3 lying on a long side, and denote the resistance by e, which are same by the rotational symmetry, see Figure 19 (d).

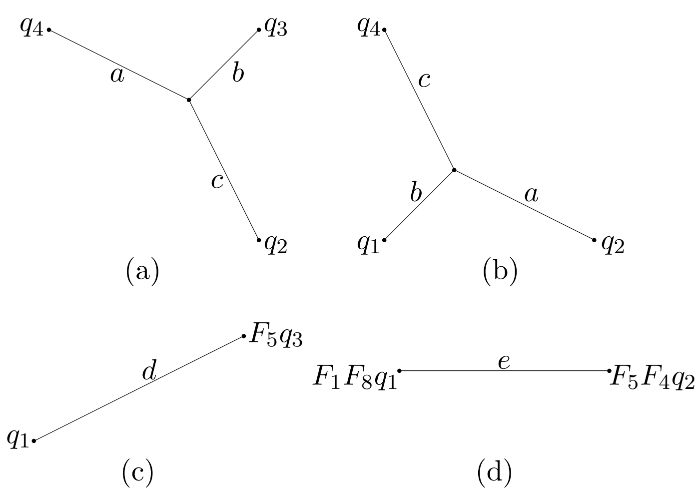


FIGURE 19. The three types of restricted resistance networks in (V_i, D_i)'s.

Now we come to the calculations.

First, look at the level-1 resistance network corresponding to (V_1, D_1), generated by the above level-0 restricted networks, see Figure 20. By comparing the effective resistances between q_i, q_j with that of (V_1, D_1) for distinct (i, j)'s, using series connection, we have

$$\begin{cases} r(2a + b + c + d) = a + b, \\ r(4a + 4c) = a + c, \\ r(2a + b + 3c + d) = b + c. \end{cases}$$

Solving the equations, we get

$$r = \frac{1}{4}, \quad c = 2a, \quad d = 3b. \tag{3.11}$$

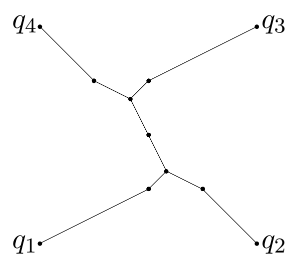


FIGURE 20. The Level-1 resistance network corresponding to (V_1, D_1).

Next, we look at the level-1 resistance network corresponding to (V_2, D_2), shown in Figure 21. We just need to compare the effective resistance between q_1 and F_5q_3 with that of (V_2, D_2). Using series and parallel connection of resistors, we get

$$d = r\big(2b + 2d + \frac{1}{2}(a + b + 2c + e)\big).$$

Substituting (3.11) into the above equation, we get

$$e = 7b - 5a. \tag{3.12}$$

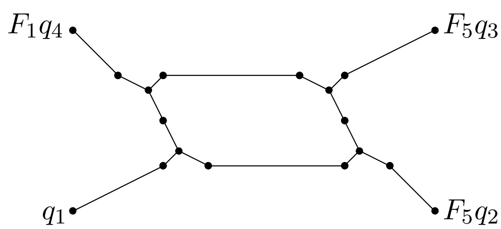


FIGURE 21. The Level-1 resistance network corresponding to (V_2, D_2).

Finally, we look at the level-1 resistance network corresponding to (V_3, D_3), shown in Figure 22. By using series connection operation, we simplify the network into what Figure 23 presents.

Then by using the symmetry of the above network, we can easily calculate the effective resistance between $F_1F_8q_1$ and $F_5F_4q_2$, which gives that

$$r(2a + b + c + d + \frac{2}{(a + b + 2c + e)^{-1} + (a + b + e)^{-1}}) = e.$$

Substituting (3.11) and (3.12) into the above equation, we get

$$b = \frac{13 + \sqrt{73}}{16}a, \quad c = 2a, \quad d = \frac{3(13 + \sqrt{73})}{16}a, \quad e = \frac{11 + 7\sqrt{73}}{16}a, \quad r = \frac{1}{4}. \tag{3.13}$$

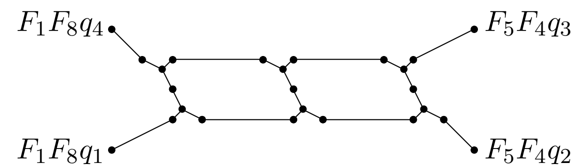


FIGURE 22. The Level-1 resistance network corresponding to (V_3, D_3).

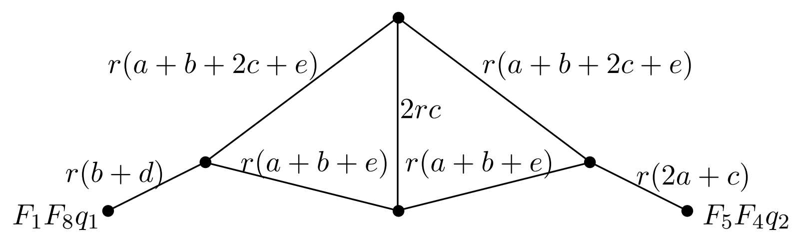


FIGURE 23. A simplification of the resistance network in Figure 22.

Thus we get a unique rotational symmetric homogeneous regular harmonic structure up to scalar constants.

4. F.R.F.T. STRUCTURES

It looks like both (**F1**) and (**F2**) are convenient to be formulated and checked for a self-similar set K. But unfortunately they are not necessary for K to possess an f.r.g.d. construction. For example, let $r \in (0,1)$ be an irrational number, let's consider the unit segment $I = [0,1]$, which can be regarded as an invariant set of the i.f.s.,

$$F_1 : x \to rx, \quad F_2 : x \to (1-r)x + r. \tag{4.1}$$

It is easy to see that (**F1**) can not hold by looking at the small copies of I around the point $x = r$. But of cause there is a canonical f.r.g.d. construction of I associated with the i.f.s., whose state set is a singleton. In this section, we will introduce a more general setting, called *finitely ramified of finite type* (f.r.f.t. for short) cell structure for a self-similar set K, which is necessary and sufficient for K to be an f.r.g.d. fractal.

Definition 4.1. *Let K be a connected self-similar set and $\{K_\alpha\}_{\alpha\in\Lambda}$ be a countable collection of distinct compact connected subsets in K, containing no singleton, satisfying that*

1. there is an index $\vartheta \in \Lambda$, called the root of Λ, such that $K = K_\vartheta$;

2. for any $\alpha \in \Lambda$, there is a finite set $\Lambda_\alpha \subset \Lambda$ with $\#\Lambda_\alpha \geq 2$, such that $K_\alpha = \bigcup_{\beta\in\Lambda_\alpha} K_\beta$, call α the parent of β, and β the child of α;

3. any $\alpha \in \Lambda \setminus \{\vartheta\}$ is an offspring of ϑ.

Call $\{K_\alpha, \Lambda_\alpha\}_{\alpha\in\Lambda}$ a cell structure of K, and Λ its index set.

Definition 4.2. *(a). For $\alpha, \beta \in \Lambda$, write $\alpha \sim \beta$ if there exists a similitude $\phi_{\alpha,\beta}$ such that $\phi_{\alpha,\beta}(K_\alpha) = K_\beta$. Denote $\Lambda/\sim$ the collection of equivalent classes in Λ with respect to "$\sim$".*

(b). Fix the similitude $\phi_{\alpha,\beta}$ so that $\phi_{\alpha,\alpha} = id$ and $\phi_{\gamma,\beta} \circ \phi_{\alpha,\gamma} = \phi_{\alpha,\beta}, \forall \alpha \sim \beta, \beta \sim \gamma$.

Remark. To achieve Definition 4.2 (b), we choose one α for each equivalent class and fix a $\phi_{\alpha,\beta}$ for each $\beta \sim \alpha$. Then we define $\phi_{\beta,\alpha} = \phi_{\alpha,\beta}^{-1}$, and define $\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}\phi_{\beta,\alpha}$ for any $\beta \sim \alpha, \gamma \sim \alpha$. □

Now, we introduce the following conditions for a cell structure $\{K_\alpha, \Lambda_\alpha\}_{\alpha\in\Lambda}$.

(**A1**). *Assume* $\#(\Lambda/\sim) < \infty$.

(**A2**). *For* $\alpha \sim \alpha'$, *there is a one to one correspondence between* Λ_α *and* $\Lambda_{\alpha'}$ *such that* $\forall\beta \in \Lambda_\alpha$, *there exists a unique* $\beta' \in \Lambda_{\alpha'}$ *satisfying* $\beta \sim \beta'$ *and* $\phi_{\beta,\beta'} = \phi_{\alpha,\alpha'}$.

(**A3**). *For each* α, *there exists a finite set* $V_\alpha \subset K_\alpha$ *such that* $\forall\alpha,\beta \in \Lambda$, *if* K_α, K_β *are not contained in each other, then it holds that* $K_\alpha \cap K_\beta = V_\alpha \cap V_\beta$; *and if in addition* $\alpha \sim \beta$, *then* $\phi_{\alpha,\beta}V_\alpha = V_\beta$.

Definition 4.3. *We say a cell structure* $\{K_\alpha, \Lambda_\alpha\}_{\alpha\in\Lambda}$ *is finitely ramified of finite type (f.r.f.t. for short) if (**A1**), (**A2**) and (**A3**) are satisfied, and call* K *an f.r.f.t. self-similar set.*

We can construct an f.r.g.d. construction from a given f.r.f.t. cell structure. We write $\Lambda/\sim := \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_M\}$. For each $\alpha \in \Lambda$, write $t(\alpha)$ so that $\alpha \in \mathcal{T}_{t(\alpha)}$, and call $\mathcal{T}_{t(\alpha)}$ the *type* of α. Choose an element α_s in $\mathcal{T}_s$ for $1 \le s \le M$ (for convenience, we require K_{α_s} has the largest diameter in cells of type $\mathcal{T}_s$, and obviously $\alpha_1 = \vartheta$). Then we have

$$K_{\alpha_s} = \bigcup_{\beta\in\Lambda_{\alpha_s}} K_\beta = \bigcup_{\beta\in\Lambda_{\alpha_s}} \phi_{\alpha_{t(\beta)},\beta}(K_{\alpha_{t(\beta)}}), \quad \forall 1 \le s \le M. \tag{4.2}$$

Thus we have a directed-graph $G = (S, E)$ with state set S and edge set E defined by

$$S = \{1, 2, \cdots, M\}, \quad E = \{(s, t(\beta)) : 1 \le s \le M, \beta \in \Lambda_{\alpha_s}\}, \tag{4.3}$$

and K is an f.r.g.d. fractal as a member of the f.r.g.d. family $\{K_{\alpha_s}\}_{s=1}^M$.

Let $\mathcal{G} = (S, E, \{\psi_e\}_{e\in E})$ be the associated f.r.g.d. construction. There is a one to one correspondence between Λ and walks in E_* starting from state 1. We write $\boldsymbol{e}(\vartheta, \alpha)$ for the unique walk associated with $\alpha \in \Lambda$, then clearly $\psi_{\boldsymbol{e}} = \phi_{\alpha_{t(\alpha)},\alpha}$.

Conversely, if we have an f.r.g.d. construction $(S, E, \{\psi_e\}_{e\in E})$ with $K = K_s$ for some $s \in S$, then we can construct an f.r.f.t. cell structure $\{K_{\boldsymbol{e}}, \Lambda_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\Lambda}$, with $\Lambda = \{\boldsymbol{e} \in E_* : i(\boldsymbol{e}) = s\}$ and $\Lambda_{\boldsymbol{e}} = \{\boldsymbol{e}e : i(e) = f(\boldsymbol{e}), e \in E\}$.

We conclude the above discussion with the following theorem.

Theorem 4.4. *A connected self-similar set possesses an f.r.f.t. cell structure if and only if it is an f.r.g.d. fractal.*

See [35] for a more general discussion on finitely ramified cell structures.

5. HOMOGENEOUS REGULAR HARMONIC STRUCTURES

Throughout this section, we assume K to be an f.r.f.t. self-similar set. Let $\{K_\alpha, \Lambda_\alpha\}_{\alpha\in\Lambda}$ be an f.r.f.t. cell structure of K and $\mathcal{G} = (S, E, \{\psi_e\}_{e\in E})$ be the associated f.r.g.d. construction. We say $(\boldsymbol{D}, \boldsymbol{r})$ a harmonic structure of $\{K_\alpha, \Lambda_\alpha\}_{\alpha\in\Lambda}$, if it is a harmonic structure of $\mathcal{G} = (S, E, \{\psi_e\}_{e\in E})$. We will focus on homogeneous regular harmonic structures. As indicated in previous examples, there may exist multiple f.r.g.d. constructions, or equivalently, multiple f.r.f.t. cell structures of K. It is of interest to ask whether they lead to same resistance forms.

Definition 5.1. *Let $(\boldsymbol{D},\boldsymbol{r})$ be a harmonic structure of an f.r.f.t. cell structure $\{K_\alpha,\Lambda_\alpha\}_{\alpha\in\Lambda}$. We say $(\boldsymbol{D},\boldsymbol{r})$ is homogeneous if for any two cells K_α, K_β with the same type and same size, $r_{\boldsymbol{e}(\vartheta,\alpha)}=r_{\boldsymbol{e}(\vartheta,\beta)}$.*

In the case that the associated directed-graph $G=(S,E)$ is *strongly connected*, i.e., for any two states s,t in S, there is a walk $\boldsymbol{e}$ such that $i(\boldsymbol{e})=s$, $f(\boldsymbol{e})=t$, we have the following proposition. Note that the examples in Section 3 are all in this case.

Proposition 5.2. *Let $(\boldsymbol{D},\boldsymbol{r})$ be a homogeneous harmonic structure of an f.r.f.t. cell structure $\{K_\alpha,\Lambda_\alpha\}_{\alpha\in\Lambda}$. Suppose the associated directed-graph $G=(S,E)$ is strongly connected. Then for any $1\leq s\leq M$, for any $\alpha,\beta\in\mathcal{T}_s$, the ratio $r_{\boldsymbol{e}(\vartheta,\alpha)}/r_{\boldsymbol{e}(\vartheta,\beta)}$ depends only on the ratio $diam(K_\alpha)/diam(K_\beta)$, i.e., there exists a function $c(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ such that $r_{\boldsymbol{e}(\vartheta,\alpha)}/r_{\boldsymbol{e}(\vartheta,\beta)}=c\big(diam(K_\alpha)/diam(K_\beta)\big)$.*

Proof. Let $\tilde{\alpha},\tilde{\beta}\in\mathcal{T}_t$ be another pair of indices such that $\frac{diam(K_\alpha)}{diam(K_\beta)}=\frac{diam(K_{\tilde{\alpha}})}{diam(K_{\tilde{\beta}})}$. Since G is strongly connected, we can always find a walk $\boldsymbol{e}$ such that $i(\boldsymbol{e})=s$ and $f(\boldsymbol{e})=1$. Connecting the walks $\boldsymbol{e}(\vartheta,\alpha)$ and $\boldsymbol{e}(\vartheta,\tilde{\beta})$ by $\boldsymbol{e}$, we get a walk $\boldsymbol{e}_1=\boldsymbol{e}(\vartheta,\alpha)\boldsymbol{e}\boldsymbol{e}(\vartheta,\tilde{\beta})$, and similarly $\boldsymbol{e}_2=\boldsymbol{e}(\vartheta,\beta)\boldsymbol{e}\boldsymbol{e}(\vartheta,\tilde{\alpha})$. Note that both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are walks from 1 to t. Obviously, $\psi_{\boldsymbol{e}_1}$ and $\psi_{\boldsymbol{e}_2}$ have the same similarity ratio, which gives that

$$r_{\boldsymbol{e}_1}=r_{\boldsymbol{e}_2},$$

since $(\boldsymbol{D},\boldsymbol{r})$ is homogeneous. As a result, $r_{\boldsymbol{e}(\vartheta,\alpha)}r_{\boldsymbol{e}(\vartheta,\tilde{\beta})}=r_{\boldsymbol{e}(\vartheta,\beta)}r_{\boldsymbol{e}(\vartheta,\tilde{\alpha})}$. Thus the ratio $r_{\boldsymbol{e}(\vartheta,\alpha)}/r_{\boldsymbol{e}(\vartheta,\beta)}$ only depends on $diam(K_\alpha)/diam(K_\beta)$. $\square$

Now for the f.r.f.t. self-similar set K, let $\mathcal{S}:=\{K_\alpha,\Lambda_\alpha\}_{\alpha\in\Lambda}$, $\mathcal{S}':=\{K_{\alpha'},\Lambda'_{\alpha'}\}_{\alpha'\in\Lambda'}$ be its two distinct f.r.f.t. cell structures. We use $G=(S,E)$ and $G'=(S',E')$ to denote their associated directed-graphs respectively.

For a cell $K_{\alpha'},\alpha'\in\Lambda'$, we can always find an at most countable set of indices $L_{\alpha'}\subset\Lambda$ such that

$$K_{\alpha'}\setminus V_{\alpha'}=\bigcup_{\alpha\in L_{\alpha'}}K_\alpha,\tag{5.1}$$

with

$$\#K_\alpha\cap K_\beta<\infty,\forall\alpha,\beta\in L_{\alpha'}.\tag{5.2}$$

Definition 5.3. *We say $\mathcal{S}'$ can be tiled by $\mathcal{S}$, and write $\mathcal{S}'\blacktriangleleft\mathcal{S}$, if for any $\alpha',\beta'\in\Lambda'$ with $\alpha'\sim\beta'$, there exist $L_{\alpha'},L_{\beta'}\subset\Lambda$ satisfying equations (5.1) and (5.2) such that there is a one to one correspondence $p_{\alpha',\beta'}:L_{\alpha'}\to L_{\beta'}$ satisfying*

$$\alpha\sim p_{\alpha',\beta'}(\alpha)\text{ and }\phi_{\alpha,p_{\alpha',\beta'}(\alpha)}=\phi_{\alpha',\beta'},\quad\forall\alpha\in L_{\alpha'}.$$

Recall the two f.r.f.t. cell structures $\mathcal{S},\mathcal{S}'$ of $\mathcal{OV}$, see Figure 7 and 10. This provides an example that $\mathcal{S}\blacktriangleleft\mathcal{S}'$ and $\mathcal{S}'\blacktriangleleft\mathcal{S}$.

The following theorem is the main result in this section.

Theorem 5.4. *Let K be a self-similar set, with two distinct f.r.f.t. cell structures $\mathcal{S}=\{K_\alpha,\Lambda_\alpha\}_{\alpha\in\Lambda}$ and $\mathcal{S}'=\{K_{\alpha'},\Lambda'_{\alpha'}\}_{\alpha'\in\Lambda'}$. Suppose $(\boldsymbol{D},\boldsymbol{r})$ is a homogeneous regular harmonic structure of $\mathcal{S}$, and $(\mathcal{E},\mathcal{F})$ is its induced resistance form. Assume $S'\blacktriangleleft S$ and G is strongly* *connected, where $G = (S, E)$ is the associated directed-graph of $\mathcal{S}$. Then there is a homogeneous regular harmonic structure $(\boldsymbol{D}', \boldsymbol{r}')$ of $\mathcal{S}'$ inducing the same resistance form $(\mathcal{E}, \mathcal{F})$.*

Proof. For each function $u \in \mathcal{F}$, we denote its associated *energy measure* by $\mu_{\mathcal{E},u}$, then for each $\alpha \in \mathcal{T}_s$ with $1 \leq s \leq M$,

$$\mu_{\mathcal{E},u}(K_\alpha) = r^{-1}_{\boldsymbol{e}(\vartheta,\alpha)} \mathcal{E}_{\alpha_s}(u|_{K_\alpha} \circ \phi_{\alpha_s,\alpha}).$$

The energy measure $\mu_{\mathcal{E},u}$ has no atom by a routine discussion, see [12] for example. So for each $\alpha' \in \Lambda'$, we have

$$\mu_{\mathcal{E},u}(K_{\alpha'}) = \sum_{\alpha \in L_{\alpha'}} \mu_{\mathcal{E},u}(K_\alpha),$$

where $L_{\alpha'} \subset \Lambda$ is a countable set of indices satisfying (5.1) and (5.2).

For each $\alpha' \in \Lambda'$, let $\mathcal{F}_{\alpha'} := \{u|_{K_{\alpha'}} : u \in \mathcal{F}\}$, and denote $\mathcal{E}_{\alpha'}(f) := \mu_{\mathcal{E},u}(K_{\alpha'})$ for each $f \in \mathcal{F}_{\alpha'}$ with $f = u|_{K_{\alpha'}}$. It is easy to check that the value of $\mathcal{E}_{\alpha'}(f)$ is independent of the choice of u. By using the polarization identity, we can get a bilinear form $(\mathcal{E}_{\alpha'}, \mathcal{F}_{\alpha'})$ defined by

$$\mathcal{E}_{\alpha'}(f,g) := \frac{1}{4}\big(\mathcal{E}_{\alpha'}(f+g) - \mathcal{E}_{\alpha'}(f-g)\big), \quad \forall f, g \in \mathcal{F}_{\alpha'},$$

and $(\mathcal{E}_{\alpha'}, \mathcal{F}_{\alpha'})$ turns out to be a resistance form on $K_{\alpha'}$. In fact, we can easily see that $\mathcal{F}_{\alpha'} = \mathcal{F}|_{K_{\alpha'}}$ and there is a constant C such that for any $f \in \mathcal{F}_{\alpha'}$, it holds that $C\mathcal{E}|_{K_{\alpha'}}(f) \leq \mathcal{E}_{\alpha'}(f) \leq \mathcal{E}|_{K_{\alpha'}}(f)$. The Markov property easily follows from the definition of $\mathcal{E}_{\alpha'}$.

We have the following claims.

Claim 1. $\mathcal{E}_{\alpha'}(f) = \sum_{\beta' \in \Lambda'_{\alpha'}} \mathcal{E}_{\beta'}(f|_{K_{\beta'}}), \quad \forall f \in \mathcal{F}_{\alpha'}$.

By definition, there exists $u \in \mathcal{F}$ so that $f = u|_{K_{\alpha'}}$. Thus

$$\mathcal{E}_{\alpha'}(f) = \mu_{\mathcal{E},u}(K_{\alpha'}) = \sum_{\beta' \in \Lambda'_{\alpha'}} \mu_{\mathcal{E},u}(K_{\beta'}) = \sum_{\beta' \in \Lambda'_{\alpha'}} \mathcal{E}_{\beta'}(f|_{K_{\beta'}}). \qquad \square$$

Claim 2. Let $\tilde{\mathcal{F}}_{\alpha'} = \{f \in \mathcal{F}_{\alpha'} : f|_{K_\alpha} \neq const.$ for at most finitely many $\alpha \in L_{\alpha'}\}$. Then $\tilde{\mathcal{F}}_{\alpha'}$ is dense in $\mathcal{F}_{\alpha'}$ with respect to the norm $\|\cdot\|_{\mathcal{F}_{\alpha'}} := \mathcal{E}^{1/2}_{\alpha'}(\cdot) + \|\cdot\|_{L^\infty(K_{\alpha'})}$.

Denote by A the set of accumulation points of $\bigcup_{\alpha \in L_{\alpha'}} V_\alpha$. Let $f \in \mathcal{F}_{\alpha'}$ and $q \in A$. For any $\epsilon > 0$, there is a function $f_q \in \mathcal{F}_{\alpha'}$ which is constant in a neighborhood of q, such that $\|f - f_q\|_{\mathcal{F}_{\alpha'}} < \epsilon$. In fact, from the definition of $\mathcal{F}_{\alpha'}$, there is a function $u \in \mathcal{F}$ such that $f = u|_{K_{\alpha'}}$. Choose a neighborhood U_q of q, which is a finite union of cells K_β with $\beta \in \Lambda$, such that $\mu^{1/2}_{\mathcal{E},u}(U_q) + \|u - u(q)\|_{L^\infty(U_q)} < \frac{\epsilon}{2}$. In a routine way, it is easy to construct $u_q \in \mathcal{F}_{\alpha'}$, such that $u_q = u$ on $K \setminus U_q$ and u_q is constant in a smaller neighborhood of q, with $\mu^{1/2}_{\mathcal{E},u_q}(U_q) + \|u_q - u(q)\|_{L^\infty(U_q)} < \frac{\epsilon}{2}$. Then $f_q = u_q|_{K_{\alpha'}}$ is the desired function.

Noticing that A consists of at most countably many points, we write $A = \{q_1, q_2, \cdots\}$. Using the above argument, for $f \in \mathcal{F}_{\alpha'}$ and $\epsilon > 0$, we could inductively construct a Cauchy sequence of functions $\{f_n\}_{n \geq 1}$ in $\mathcal{F}_{\alpha'}$ with $f_0 = f$ and $\|f_n - f_{n-1}\|_{\mathcal{F}_{\alpha'}} < \epsilon/2^n$, such that for $n \geq 1$, f_n is constant in a neighborhood of q_k, $\forall 1 \leq k \leq n$. The limit function $\tilde{f}$ takes constant on a neighborhood of q for each $q \in A$, and clearly is in $\tilde{\mathcal{F}}_{\alpha'}$. Thus we have proved Claim 2. $\square$

Claim 3. Suppose $\alpha' \in \mathcal{T}'_s$, $1 \leq s \leq M'$, then $f \in \mathcal{F}_{\alpha'}$ if and only if $f \circ \phi_{\alpha'_s,\alpha'} \in \mathcal{F}_{\alpha'_s}$. In addition, for any $f \in \mathcal{F}_{\alpha'}$,

$$\mathcal{E}_{\alpha'}(f) = r^{-1}_{\boldsymbol{e}(\vartheta',\alpha')}\mathcal{E}_{\alpha'_s}(f \circ \phi_{\alpha'_s,\alpha'}),$$

where $r_{\boldsymbol{e}(\vartheta',\alpha')} := c\big(\frac{diam(K_{\alpha'_s})}{diam(K_{\alpha'})}\big)^{-1}$ with $c(\cdot)$ being the function introduced in Proposition 5.2.

The difficulty is to show the one to one correspondence between $\mathcal{F}_{\alpha'}$ and $\mathcal{F}_{\alpha'_s}$. This can be overcome by Claim 2. In fact, it is easy to see that $f \to f \circ \phi_{\alpha'_s,\alpha'}$ is bijective from $\tilde{\mathcal{F}}_{\alpha'}$ to $\tilde{\mathcal{F}}_{\alpha'_s}$. In addition, for any $f \in \tilde{\mathcal{F}}_{\alpha'}$, by using Proposition 5.2 and the fact $S' \blacktriangleleft S$, we have

$$\begin{aligned}
\mathcal{E}_{\alpha'}(f) &= \sum_{t=1}^{M} \sum_{\alpha \in L_{\alpha'}, \alpha \in \mathcal{T}_t} r^{-1}_{\boldsymbol{e}(\vartheta,\alpha)}\mathcal{E}_{\alpha_t}(f|_{K_\alpha} \circ \phi_{\alpha_t,\alpha}) \\
&= \sum_{t=1}^{M} \sum_{\alpha \in L_{\alpha'}, \alpha \in \mathcal{T}_t} r^{-1}_{\boldsymbol{e}(\vartheta,\alpha)}\mathcal{E}_{\alpha_t}\big(f|_{K_\alpha} \circ \phi_{p(\alpha),\alpha} \circ \phi_{\alpha_t,p(\alpha)}\big) \\
&= c\big(\frac{diam(K_{\alpha'_s})}{diam(K_{\alpha'})}\big) \sum_{t=1}^{M} \sum_{\alpha \in L_{\alpha'}, \alpha \in \mathcal{T}_t} r^{-1}_{\boldsymbol{e}\left(\vartheta,p(\alpha)\right)}\mathcal{E}_{\alpha_t}\big((f \circ \phi_{\alpha'_s,\alpha'})|_{K_{p(\alpha)}} \circ \phi_{\alpha_t,p(\alpha)}\big) \\
&= r^{-1}_{\boldsymbol{e}(\vartheta',\alpha')}\mathcal{E}_{\alpha'_s}(f \circ \phi_{\alpha'_s,\alpha'}),
\end{aligned}$$

where $p : L_{\alpha'} \to L_{\alpha'_s}$ is a one to one correspondence such that $\phi_{\alpha,p(\alpha)} = \phi_{\alpha',\alpha'_s}$, $\forall \alpha \in L_{\alpha'}$. Since $\tilde{\mathcal{F}}_{\alpha'}$ and $\tilde{\mathcal{F}}_{\alpha'_s}$ are dense in $\mathcal{F}_{\alpha'}$ and $\mathcal{F}_{\alpha'_s}$ respectively by Claim 2, we get Claim 3. $\square$

For $1 \leq s \leq M'$, let $D_{\alpha'_s}$ be the Laplacian induced by the trace of $\mathcal{E}_{\alpha'_s}$ on $V_{\alpha'_s}$. Then, in view of Claim 1 and Claim 3, $(\{D_{\alpha'_s}\}_{s=1}^{M'}, \boldsymbol{r}')$ is a homogenous regular harmonic structure of the f.r.f.t. cell structure $\mathcal{S}' = \{K_{\alpha'}, \Lambda'_{\alpha'}\}_{\alpha' \in \Lambda'}$ with $r_{\boldsymbol{e}(\vartheta',\alpha')} = c\big(\frac{diam(K_{\alpha'_s})}{diam(K_{\alpha'})}\big)^{-1}$ for any $\alpha' \in \mathcal{T}'_s, 1 \leq s \leq M'$. $\square$

Remark. In the case that $\mathcal{S}' \not\blacktriangleleft \mathcal{S}$, the conclusion of Theorem 5.4 may fail to hold. Recall the example we have mentioned at the beginning of Section 4. For $r \in (0,1)$, let $\mathcal{S}_r$ be the canonical f.r.f.t. cell structure of the unit segment $I = [0,1]$ associated with the i.f.s.(4.1). Suppose r is not an algebraic number, then any harmonic structure of $\mathcal{S}_r$ is homogeneous, but there is only one of them inducing the same resistance form as that of the homogeneous regular harmonic structure of $\mathcal{S}_{1/2}$.

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