SPECTRAL DECIMATION FOR A GRAPH-DIRECTED FRACTAL PAIR

SHIPING CAO, HUA QIU, HAORAN TIAN, AND LIJIAN YANG

Abstract. We introduce a graph-directed pair of planar self-similar sets that possess fully symmetric Laplacians. For these two fractals, due to Shima's celebrated criterion, we point out that one admits the spectral decimation and the other does not. For the second fractal, we adjust to choosing a new graph approximation guided by the directed graph, which still admits spectral decimation. Then we make a full description of the Dirichlet and Neumann eigenvalues and eigenfunctions of both of these two fractals.

1. Introduction

The Weyl's problem and the spectrum of the Laplacian on fractals have been widely studied since Kigami's construction of Laplacians on p.c.f. self-similar sets [11, 12] (see also the books [13] and [18]). The method of *spectral decimation*, first described for the Sierpinski gasket ($\mathcal{SG}$) by Fukushima and Shima [8, 15] and later extended to certain p.c.f. self-similar sets with strongly symmetric harmonic structure by Shima [16], describes a connection between eigenfunctions and eigenvalues for successive levels of graph Laplacians that approximate the fractal Laplacian. Using this connection, by extending graph eigenfunctions level by level with multiple decimation choices, and taking limits, one can recover all eigenfunctions of the fractal Laplacian. There is a large amount of literature dealing with the spectral decimation on distinct fractals, such as [2, 3] for D3 symmetric p.c.f. self-similar sets, [7] for a family of self-similar symmetric Laplacians on $\mathcal{SG}$, [4, 22] for the Vicsek sets, [6] for Hambly's homogeneous hierarchical gaskets, and [17, 20, 21] for certain fractalfolds based on $\mathcal{SG}$. Note that there are many extremely symmetric fractal Laplacians not satisfying spectral decimation, for example, see [1] for the case of pentagasket. Using spectral decimation, we are able to deal with many interesting problems about the spectrum of the fractal Laplacians, see [5, 10, 14, 19, 23] and the references therein.

In this paper, we will develop an exact spectral analysis on a pair of p.c.f. self-similar sets $\mathcal{T}$ and $\mathcal{S}$ (see Figure 1.1) which admit fully symmetric fractal Laplacians. We investigate $\mathcal{T}$ and $\mathcal{S}$ together since there is an obvious graph-directed construction relating them. See [9] for a delicate investigation by Hambly and Nyberg of Laplacians on finitely ramified graph-directed fractal families. Since $\mathcal{T}$ is D3 symmetric, there exists spectral decimation for the Laplacian on $\mathcal{T}$ related to its canonic graph approximation. But it fails for $\mathcal{S}$. Using the spectral decimation for $\mathcal{T}$, we are able to make a full description of the Dirichlet and Neumann spectra on $\mathcal{T}$. As for $\mathcal{S}$, we will adjust to choosing a new sequence of graphs that approximate $\mathcal{S}$ guided by the graph-directed construction. On this graph sequence, there still exists spectral decimation which is almost the same as that of $\mathcal{T}$. Basing on this, we can still make a full description of the Dirichlet and Neumann spectra on $\mathcal{S}$. Another interesting

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feature of the spectral decimation in this study is that there is a kind of graph eigenfunctions born at each level but never decimating to eigenfunctions of the fractal Laplacians. To the best of our knowledge, this does not happen on other well studied examples.

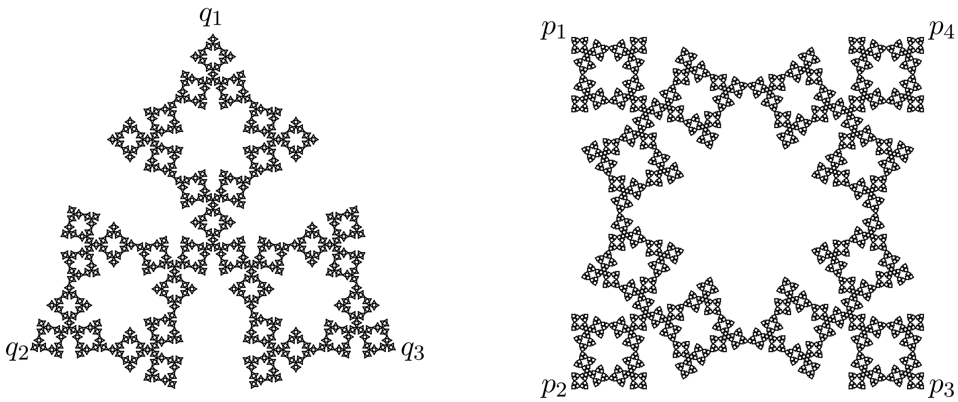


Figure 1.1. The fractal pair $\mathcal{T}$ and $\mathcal{S}$.

The paper is organized as follows. In Section 2, we will give details about the symmetric Laplacians on the fractals $\mathcal{T}$ and $\mathcal{S}$. In Section 3, we will describe the exact Dirichlet and Neumann eigenvalues and eigenfunctions on $\mathcal{T}$ by using the spectral decimation method. In Section 4, by using the graph-directed construction of $\mathcal{S}$ related to $\mathcal{T}$, we will adapt the techinique dealing with $\mathcal{T}$ to figure out the exact Dirichlet and Neumann eigenvalues and eigenfunctions on $\mathcal{S}$.

2. Laplacians on the fractal pair $\mathcal{T}$ and $\mathcal{S}$

2.1. **The fractal pair $\mathcal{T}$ and $\mathcal{S}$.**

The fractal pair $\mathcal{T}$ and $\mathcal{S}$ in Figure 1.1 are the invariant sets of a graph-directed iterated function system.

Let $R_\theta(x) = (\cos\theta x_1 - \sin\theta x_2, \sin\theta x_1 + \cos\theta x_2)$, which rotates $x = (x_1, x_2)$ in $\mathbb{R}^2$ counterclockwise by an angle θ. Let $\{q_k\}_{k=1}^3$ be the three vertices of a unit equilateral triangle, and $\{p_l\}_{l=1}^4$ be the four vertices of a unit square in $\mathbb{R}^2$. Define $\psi_k, k = 1,2,3$ and $\varphi_l, l = 1,2,3,4$ to be seven contractive mappings as

$$\psi_k(x) = \frac{\sqrt{6}}{6} R_{-\frac{\pi}{4}+\frac{2(k-1)}{3}\pi}(x - p_1) + q_k,$$
$$\varphi_l(x) = \frac{\sqrt{6}-\sqrt{2}}{2} R_{\frac{\pi}{4}+\frac{(l-1)}{2}\pi}(x - q_1) + p_l.$$

Then $\mathcal{T}, \mathcal{S}$ are the pair of *graph-directed self-similar sets* generated by ψ_k's and φ_l's,

$$\mathcal{T} = \bigcup_{k=1}^{3} \psi_k \mathcal{S}, \quad \mathcal{S} = \bigcup_{l=1}^{4} \varphi_l \mathcal{T}. \tag{2.1}$$

Iterating (2.1) twice, we get

$$\begin{aligned}\mathcal{T} &= \bigcup_{k=1}^{3}\bigcup_{l=1}^{4}\psi_{kl}\mathcal{T}, \quad \textit{with } \psi_{kl} := \psi_k \circ \varphi_l. \\ \mathcal{S} &= \bigcup_{l=1}^{4}\bigcup_{k=1}^{3}\varphi_{lk}\mathcal{S}, \quad \textit{with } \varphi_{lk} := \varphi_l \circ \psi_k.\end{aligned} \tag{2.2}$$

So both $\mathcal{T}$ and $\mathcal{S}$ are self-similar sets of iterated function systems consisting of 12 mappings. It is easy to check that they are all p.c.f. self-similar sets. We introduce two collections of *finite words* associated with $\mathcal{T}$ and $\mathcal{S}$, respectively. Let $m \in \mathbb{Z}^+$. Write

$$W^{\mathcal{T}}_{\frac{m}{2}} = \begin{cases} \big\{kl | k \in \{1,2,3\}, l \in \{1,2,3,4\}\big\}^{\frac{m}{2}}, & \text{if } m \text{ is even,} \\ \big\{kl | k \in \{1,2,3\}, l \in \{1,2,3,4\}\big\}^{\frac{m-1}{2}} \times \{1,2,3\}, & \text{if } m \text{ is odd,} \end{cases}$$

and

$$W^{\mathcal{S}}_{\frac{m}{2}} = \begin{cases} \big\{lk | l \in \{1,2,3,4\}, k \in \{1,2,3\}\big\}^{\frac{m}{2}}, & \text{if } m \text{ is even,} \\ \big\{lk | l \in \{1,2,3,4\}, k \in \{1,2,3\}\big\}^{\frac{m-1}{2}} \times \{1,2,3,4\}, & \text{if } m \text{ is odd.} \end{cases}$$

For a word $w = w_1w_2\cdots w_n \in \bigcup_{m\geq 0} W^{\mathcal{T}}_{\frac{m}{2}}$, we denote $\psi_w = \psi_{w_1}\varphi_{w_2}\psi_{w_3}\cdots\varphi_{w_n}$ for even n, and $\psi_w = \psi_{w_1}\varphi_{w_2}\psi_{w_3}\cdots\psi_{w_n}$ for odd n, and similarly φ_w for $w \in \bigcup_{m\geq 0} W^{\mathcal{S}}_{\frac{m}{2}}$.

Let

$$V_0 = \{q_1, q_2, q_3\} \text{ and } U_0 = \{p_1, p_2, p_3, p_4\}$$

be the *boundaries* of $\mathcal{T}$ and $\mathcal{S}$, respectively. For $m \in \mathbb{N}$, we iteratively define

$$V_{\frac{m}{2}} = \bigcup_{k=1}^{3}\psi_k U_{\frac{m-1}{2}} \text{ and } U_{\frac{m}{2}} = \bigcup_{l=1}^{4}\varphi_l V_{\frac{m-1}{2}}.$$

Clearly,

$$V_{\frac{m}{2}} = \begin{cases} \bigcup_{w\in W^{\mathcal{T}}_{\frac{m}{2}}} \psi_w V_0, & \text{if } m \text{ is even,} \\ \bigcup_{w\in W^{\mathcal{T}}_{\frac{m}{2}}} \psi_w U_0, & \text{if } m \text{ is odd,} \end{cases} \quad \textit{and} \quad U_{\frac{m}{2}} = \begin{cases} \bigcup_{w\in W^{\mathcal{S}}_{\frac{m}{2}}} \varphi_w U_0, & \text{if } m \text{ is even,} \\ \bigcup_{w\in W^{\mathcal{S}}_{\frac{m}{2}}} \varphi_w V_0, & \text{if } m \text{ is odd.} \end{cases}$$

Write $x \sim_{\frac{m}{2}} y$ for x, y in $V_{\frac{m}{2}}$ if there exists a word w in $W^{\mathcal{T}}_{\frac{m}{2}}$ such that $x, y \in \psi_w V_0$ when m is even, or $x, y \in \psi_w U_0$ when is m odd. Do the same for x, y in $U_{\frac{m}{2}}$. Obviously, $\{\Gamma_{\frac{m}{2}}\}_{m\geq 0} := \{(V_{\frac{m}{2}}, \sim_{\frac{m}{2}})\}_{m\geq 0}$ and $\{\Lambda_{\frac{m}{2}}\}_{m\geq 0} := \{(U_{\frac{m}{2}}, \sim_{\frac{m}{2}})\}_{m\geq 0}$ are two sequences of *graphs* that approximate $\mathcal{T}$ and $\mathcal{S}$, respectively. See Figure 2.1 for $\Gamma_0, \Gamma_{\frac{1}{2}}, \Gamma_1$, and Figure 2.2 for $\Lambda_0, \Lambda_{\frac{1}{2}}, \Lambda_1$.

Note that when we restrict to look at even integer m in $\mathbb{Z}^+$, we have

$$\begin{aligned} V_n &= \bigcup_{w\in W^{\mathcal{T}}_1} \psi_w V_{n-1} = \bigcup_{k=1}^{3}\bigcup_{l=1}^{4}\psi_{kl}V_{n-1}, \\ U_n &= \bigcup_{w\in W^{\mathcal{S}}_1} \varphi_w U_{n-1} = \bigcup_{l=1}^{4}\bigcup_{k=1}^{3}\varphi_{lk}U_{n-1}, \end{aligned}$$

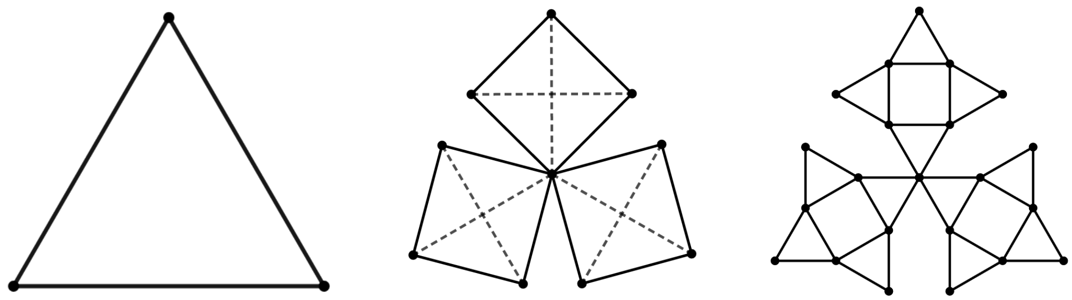


FIGURE 2.1. Γ_0, $\Gamma_{\frac{1}{2}}$, Γ_1.

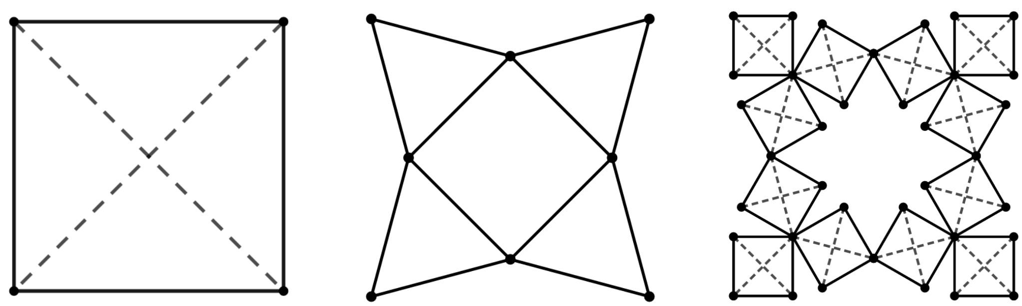


FIGURE 2.2. Λ_0, $\Lambda_{\frac{1}{2}}$, Λ_1.

for $n \in \mathbb{N}$, which are consistent with the ordinary notations of p.c.f. self-similar sets(for example, see[13, 18]). Write $V_* = \bigcup_{n\geq 0} V_n$ and $U_* = \bigcup_{n\geq 0} U_n$. Obviously, $V_* = \bigcup_{m\geq 0} V_{\frac{m}{2}}$, $U_* = \bigcup_{m\geq 0} U_{\frac{m}{2}}$, and they are dense in $\mathcal{T}$ and $\mathcal{S}$, respectively.

2.2. The Laplacians $\Delta^{\mathcal{T}}$, $\Delta^{\mathcal{S}}$ on $\mathcal{T}$, $\mathcal{S}$.

We begin with the (fully symmetric) *self-similar energy forms* on $\mathcal{T}$ and $\mathcal{S}$. For a finite set V, let $l(V)$ denote the space of all real-valued functions on V. We define the *graph energies* on Γ_0 and Λ_0 by

$$\begin{cases} \mathcal{E}_0^{\mathcal{T}}(u) = \sum_{k=1}^{3} \big(u(q_k) - u(q_{k+1})\big)^2, & \text{for } u \in l(V_0), \\ \mathcal{E}_0^{\mathcal{S}}(u) = \sum_{l=1}^{4} \big(u(p_l) - u(p_{l+1})\big)^2 + \frac{1}{2}\sum_{l=1}^{2} \big(u(p_l) - u(p_{l+2})\big)^2, & \text{for } u \in l(U_0), \end{cases}$$

where we use the cyclic index. Note that there are two kinds of conductances on Λ_0, the edges and diagonals, see Figure 2.3 for an illustration.

For $n \in \mathbb{Z}^+$, define the graph energies on Γ_n and Λ_n by

$$\begin{cases} \mathcal{E}_n^{\mathcal{T}}(u) = 2^{2n} \sum_{w\in W_n^{\mathcal{T}}} \mathcal{E}_0^{\mathcal{T}}(u \circ \psi_w), & \text{for } u \in l(V_n), \\ \mathcal{E}_n^{\mathcal{S}}(u) = 2^{2n} \sum_{w\in W_n^{\mathcal{S}}} \mathcal{E}_0^{\mathcal{S}}(u \circ \varphi_w), & \text{for } u \in l(U_n). \end{cases}$$

Clearly, we may rewrite the above energies in the following way,

$$\begin{cases} \mathcal{E}_n^{\mathcal{T}}(u) = \sum_{x\sim_n y} c_n^{\mathcal{T}}(x,y)\big(u(x) - u(y)\big)^2, \\ \mathcal{E}_n^{\mathcal{S}}(u) = \sum_{x\sim_n y} c_n^{\mathcal{S}}(x,y)\big(u(x) - u(y)\big)^2, \end{cases}$$

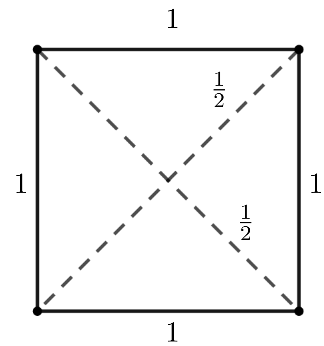


FIGURE 2.3. The conductances on Λ_0.

where

$$c_n^{\mathcal{T}}(x,y) = 2^{2n}, \quad c_n^{\mathcal{S}}(x,y) = \begin{cases} 2^{2n}, & \text{if } x=\varphi_w p_l, y=\varphi_w p_{l+1} \text{ for } 1 \le l \le 4, w \in W_n^{\mathcal{S}}, \\ 2^{2n-1}, & \text{if } x=\varphi_w p_l, y=\varphi_w p_{l+2} \text{ for } 1 \le l \le 4, w \in W_n^{\mathcal{S}} \end{cases}$$

are the level-n *conductances*. Using the polarization identity, defining

$$\mathcal{E}(u,v) = \frac{1}{4}\big(\mathcal{E}(u+v) - \mathcal{E}(u-v)\big),$$

we get the associated graph energy forms $\mathcal{E}_n^{\mathcal{T}}(u,v)$, $\mathcal{E}_n^{\mathcal{S}}(u,v)$ accordingly.

By using the standard electric network theory(for example, see [18]), one can easily verify that the sequences $\{\mathcal{E}_n^{\mathcal{T}}\}_{n\ge 0}$ and $\{\mathcal{E}_n^{\mathcal{S}}\}_{n\ge 0}$ are compatible, respectively. That is, $\forall n \in \mathbb{Z}^+$, we have

$$\mathcal{E}_n^{\mathcal{T}}(u) = \min\{\mathcal{E}_{n+1}^{\mathcal{T}}(v) : v|_{V_n} = u\}, \quad \mathcal{E}_n^{\mathcal{S}}(u) = \min\{\mathcal{E}_{n+1}^{\mathcal{S}}(v) : v|_{U_n} = u\}.$$

By the standard argument, we define the energy forms on $\mathcal{T}$ and $\mathcal{S}$ as

$$\begin{cases} \mathcal{E}^{\mathcal{T}}(u,v) = \lim_{n\to\infty} \mathcal{E}_n^{\mathcal{T}}(u,v), \\ \mathcal{E}^{\mathcal{S}}(u,v) = \lim_{n\to\infty} \mathcal{E}_n^{\mathcal{S}}(u,v). \end{cases}$$

A function h on $\mathcal{T}$ or $\mathcal{S}$ is called *harmonic* if it minimizes the energy $\mathcal{E}^{\mathcal{T}}(h)$ or $\mathcal{E}^{\mathcal{S}}(h)$, for given boundary value $h|_{V_0}$ or $h|_{U_0}$. Similarly, for $n \in \mathbb{N}$, a function is called *level-n piecewise harmonic* if it minimizes its energy for given initial value on V_n or U_n. In particular, for $x \in V_n \setminus V_0$ or $U_n \setminus U_0$, we denote $\Psi_n^{(x)}$ a *tent function*, which is level-n piecewise harmonic and takes value on V_n or U_n by $\Psi_n^{(x)}(y) = \delta_{xy}$.

Let μ and ν be the *normalized Hausdorff measures* on $\mathcal{T}$ and $\mathcal{S}$ respectively. An easy calculation yields that $\int_{\mathcal{T}} \Psi_n^{(x)} d\mu = \frac{1}{6}\frac{1}{12^n}\deg(x)$ for $x \in V_n \setminus V_0$, and $\int_{\mathcal{S}} \Psi_n^{(x)} d\nu = \frac{1}{12^{n+1}}\deg(x)$ for $x \in U_n \setminus U_0$, where $\deg(x)$ is the number of level-n edges attached to x.

As usual, the (symmetric) *Laplacians* on $\mathcal{T}$ or $\mathcal{S}$, denoted by $\Delta^{\mathcal{T}}$ or $\Delta^{\mathcal{S}}$, can be defined by the uniform limits of

$$\begin{cases} \widetilde{\Delta}_n^{\mathcal{T}} u(x) = \frac{1}{\int_{\mathcal{T}} \Psi_n^{(x)} d\mu} \sum_{y \sim_n x} c_n^{\mathcal{T}}(x,y)\big(u(y)-u(x)\big), & \text{for } x \in V_n \setminus V_0, \\ \widetilde{\Delta}_n^{\mathcal{S}} u(x) = \frac{1}{\int_{\mathcal{S}} \Psi_n^{(x)} d\nu} \sum_{y \sim_n x} c_n^{\mathcal{S}}(x,y)\big(u(y)-u(x)\big), & \text{for } x \in U_n \setminus U_0. \end{cases} \tag{2.3}$$

We can easily calculate that

$$\begin{cases} \widetilde{\Delta}_n^{\mathcal{T}} u(x) = 6\cdot 12^n 2^{2n}\cdot \Delta_n^{\mathcal{T}} u(x),\\ \widetilde{\Delta}_n^{\mathcal{S}} u(x) = 10\cdot 12^n 2^{2n}\cdot \Delta_n^{\mathcal{S}} u(x),\end{cases}$$

with

$$\begin{cases}\Delta_n^{\mathcal{T}} u(x) = \frac{1}{\deg(x)}\sum_{y\sim_n x} u(y) - u(x)\\ \Delta_n^{\mathcal{S}} u(x) = \frac{6}{5}\frac{1}{\deg(x)}\big(\sum_{y\sim_n x} u(y) + \frac{1}{2}\sum_{y\sim_n' x} u(y)\big) - u(x),\end{cases}$$

where for $\Delta_n^{\mathcal{S}}$, we use $y\sim_n' x$ to specify the case that $c_n^{\mathcal{S}}(x,y)=2^{2n-1}$ and use $y\sim_n x$ for $c_n^{\mathcal{S}}(x,y)=2^{2n}$.

2.3. **The Laplacians $\Delta^{\mathcal{T}}$, $\Delta^{\mathcal{S}}$ in graph-directed manner.**

The energy forms and the Laplacians introduced above can be alternatively constructed in a graph directed manner.

For the energy forms, we only need to fill up the energy sequence $\{\mathcal{E}_n^{\mathcal{T}}\}_{n\geq 0}$ or $\{\mathcal{E}_n^{\mathcal{S}}\}_{n\geq 0}$ to be $\{\mathcal{E}_{\frac{m}{2}}^{\mathcal{T}}\}_{m\geq 0}$ or $\{\mathcal{E}_{\frac{m}{2}}^{\mathcal{S}}\}_{n\geq 0}$ so that the new sequence is still compatible. For $n\in\mathbb{Z}^+$, let

$$\begin{cases}\mathcal{E}_{n+\frac12}^{\mathcal{T}}(u) = 2\sum_{k=1}^3 \mathcal{E}_n^{\mathcal{S}}(u\circ\psi_k) = 2^{2n+1}\sum_{w\in W_{n+\frac12}^{\mathcal{T}}}\mathcal{E}_0^{\mathcal{S}}(u\circ\psi_w), & \text{for } u\in l(V_{n+\frac12}),\\ \mathcal{E}_{n+\frac12}^{\mathcal{S}}(u) = 2\sum_{l=1}^4 \mathcal{E}_n^{\mathcal{T}}(u\circ\varphi_l) = 2^{2n+1}\sum_{w\in W_{n+\frac12}^{\mathcal{S}}}\mathcal{E}_0^{\mathcal{T}}(u\circ\varphi_w), & \text{for } u\in l(U_{n+\frac12}).\end{cases}$$

As before, we may rewrite them into

$$\begin{cases}\mathcal{E}_{n+\frac12}^{\mathcal{T}}(u) = \sum_{x\sim_{n+\frac12} y} c_{n+\frac12}^{\mathcal{T}}(x,y)\big(u(x)-u(y)\big)^2,\\ \mathcal{E}_{n+\frac12}^{\mathcal{S}}(u) = \sum_{x\sim_{n+\frac12} y} c_{n+\frac12}^{\mathcal{S}}(x,y)\big(u(x)-u(y)\big)^2,\end{cases}$$

where

$$c_{n+\frac12}^{\mathcal{S}}(x,y) = 2^{2n+1},\quad c_{n+\frac12}^{\mathcal{T}}(x,y) = \begin{cases}2^{2n+1}, & \text{if } x=\psi_w p_l, y=\psi_w p_{l+1} \text{ for } 1\leq l\leq 4, w\in W_{n+\frac12}^{\mathcal{T}},\\ 2^{2n}, & \text{if } x=\psi_w p_l, y=\psi_w p_{l+2} \text{ for } 1\leq l\leq 4, w\in W_{n+\frac12}^{\mathcal{T}}\end{cases}$$

are the level-$(n+\frac12)$ conductances.

It is easy to check that

$$\mathcal{E}_0^{\mathcal{T}}(u) = \min\{\mathcal{E}_{\frac12}^{\mathcal{T}}(v) : v|_{V_0}=u\},\quad \mathcal{E}_0^{\mathcal{S}}(u) = \min\{\mathcal{E}_{\frac12}^{\mathcal{S}}(v) : v|_{V_0}=u\},$$

so that the sequences $\{\mathcal{E}_{\frac{m}{2}}^{\mathcal{T}}\}_{m\geq 0}$ and $\{\mathcal{E}_{\frac{m}{2}}^{\mathcal{S}}\}_{m\geq 0}$ are compatible. In addition, we have

$$\mathcal{E}_{\frac{m+1}{2}}^{\mathcal{T}}(u) = 2\sum_{w\in W_{\frac12}^{\mathcal{T}}}\mathcal{E}_{\frac{m}{2}}^{\mathcal{S}}(u\circ\psi_w),\quad \mathcal{E}_{\frac{m+1}{2}}^{\mathcal{S}}(u) = 2\sum_{w\in W_{\frac12}^{\mathcal{S}}}\mathcal{E}_{\frac{m}{2}}^{\mathcal{T}}(u\circ\varphi_w),\quad \forall m\in\mathbb{Z}^+.$$

Similar to (2.3), by an easy calculation, the Laplacians $\Delta^{\mathcal{T}}$ and $\Delta^{\mathcal{S}}$ are the uniform limits of

$$\begin{cases}\widetilde{\Delta}_{n+\frac12}^{\mathcal{T}} u(x) = 3^{n+2}4^{n+1}2^{2n+1}\Big(\frac65\cdot\frac{1}{\deg(x)}\big(\sum_{y\sim_n x} u(y) \\ \qquad\qquad +\frac12\sum_{y\sim_n' x} u(y)\big) - u(x)\Big), \text{ for } x\in V_{n+\frac12}\setminus V_0,\\ \widetilde{\Delta}_{n+\frac12}^{\mathcal{S}} u(x) = 2\cdot 3^{n+1}4^{n+1}\cdot 2^{2n+1}\big(\frac{1}{\deg(x)}\sum_{y\sim_{n+\frac12} x} u(y) - u(x)\big), \text{ for } x\in U_{n+\frac12}\setminus U_0,\end{cases}$$

where for $\tilde{\Delta}^{\mathcal{T}}_{n+\frac{1}{2}}$, we use $y \sim'_{n+\frac{1}{2}} x$ to specify the case that $c^{\mathcal{T}}_{n+\frac{1}{2}}(x,y) = 2^{2n}$ and use $y \sim_{n+\frac{1}{2}} x$ for $c^{\mathcal{T}}_{n+\frac{1}{2}}(x,y) = 2^{2n+1}$.

3. Spectral decimation on the fractal $\mathcal{T}$

We have constructed a D3-symmetric Laplacian $\Delta^{\mathcal{T}}$ on $\mathcal{T}$. By Shima's criterion[16], there is a spectral decimation recipe on $\mathcal{T}$ which yields a complete description of Dirichlet and Neumann spectra of the minus Laplacian $-\Delta^{\mathcal{T}}$. Note that the Neumann spectrum may be defined in terms of normal derivatives, but it is more convenient to consider the double cover $\widetilde{\mathcal{T}}$, extending functions on $\mathcal{T}$ by even reflection and imposing the pointwise eigenfunction equation at the boundary points in V_0.

In this section, we will present the explicit details of eigenfunctions and eigenvalues of $-\Delta^{\mathcal{T}}$, which are limits of eigenfunctions and eigenvalues of $-\widetilde{\Delta}^{\mathcal{T}}_n$. For the sake of simplicity, we will omit the renormalization factor $6 \cdot 12^n 2^{2n}$ and instead to consider the unrenormalized Laplacian $\Delta^{\mathcal{T}}_n$, and will rescale the eigenvalues later.

3.1. **The spectral decimation recipe.**

Given an eigenfunction u_n on V_n of $-\Delta^{\mathcal{T}}_n$ with an eigenvalue λ_n, we want to extend it to an eigenfunction u_{n+1} on V_{n+1} of $\Delta^{\mathcal{T}}_{n+1}$ with an eigenvalue λ_{n+1}, that is:

$$-\Delta^{\mathcal{T}}_{n+1} u_{n+1}(x) = \lambda_{n+1} u_{n+1}(x), \quad \forall x \in V_{n+1} \setminus V_0, \tag{3.1}$$

with $u_{n+1}|_{V_n} = u_n$.

Consider any n-cell with boundary vertices s_1, s_2, s_3. We denote $u_n(s_k) = x_k$ respectively, and write the values of u_{n+1} at new points in $V_{n+1} \setminus V_n$ contained in this cell by $a_k, a'_k, b_k, b'_k, c_k, c'_k$ and d for $k = 1, 2, 3$, see Figure 3.1.

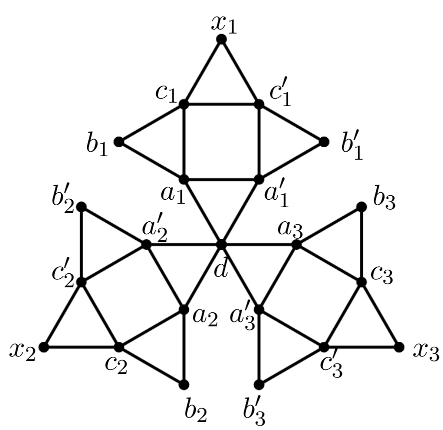


Figure 3.1. The values of u_{n+1} on an n-cell.

Evaluating (3.1) on points in $V_{n+1}\setminus V_n$ contained in this cell yields 19 equations, which will give values of $a_k,a'_k,b_k,b'_k,c_k,c'_k$ as functions of $x_1,x_2,x_3,\lambda_{n+1}$ by

$$\begin{cases} d=\frac{-(x_1+x_2+x_3)}{3(16\lambda_{n+1}^3-32\lambda_{n+1}^2+16\lambda_{n+1}-1)},\\ a_k=a'_k=\frac{-(5x_k+3d-14\lambda_{n+1}x_k-2\lambda_{n+1}d+8\lambda_{n+1}^2x_k)}{8(4\lambda_{n+1}^3-10\lambda_{n+1}^2+7\lambda_{n+1}-1)},\\ c_k=c'_k=\frac{-(5d+3x_k-14\lambda_{n+1}d-2\lambda_{n+1}x_k+8\lambda_{n+1}^2d)}{8(4\lambda_{n+1}^3-10\lambda_{n+1}^2+7\lambda_{n+1}-1)},\\ b_k=b'_k=\frac{a_k+c_k}{2(1-\lambda_{n+1})}. \end{cases} \tag{3.2}$$

More precisely, we have the following extension algorithm.

Lemma 3.1. *Let α_i's be the two positive roots of $4\lambda^2-6\lambda+1$, β_i's be the three positive roots of $16\lambda^3-32\lambda^2+16\lambda-1$. Then for $\lambda_{n+1}\notin\{\frac12,1,\frac32,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3\}$ and*

$$\lambda_n=-48\lambda_{n+1}(\lambda_{n+1}-1)^2(2\lambda_{n+1}-1)(4\lambda_{n+1}^2-6\lambda_{n+1}+1), \tag{3.3}$$

u_n is a λ_n-eigenfunction of $-\Delta_n^{\mathcal{T}}$ if and only if u_{n+1} is a λ_{n+1}-eigenfunction of $-\Delta_{n+1}^{\mathcal{T}}$ with $u_{n+1}|_{V_n}=u_n$ and values of u_{n+1} at points in $V_{n+1}\setminus V_n$ specified by formula (3.2).

Proof. First, denote by $a=\sum_{k=1}^3(a_k+a'_k), b=\sum_{k=1}^3(b_k+b'_k), c=\sum_{k=1}^3(c_k+c'_k), x=\sum_{k=1}^3 x_k$ for convenience. An easy combination of equations at points in $V_{n+1}\setminus V_n$ evaluated from (3.1) gives that

$$\begin{cases} (1-\lambda_{n+1})a=\frac14(a+b+c+6d),\\ (1-\lambda_{n+1})b=\frac12(a+c),\\ (1-\lambda_{n+1})c=\frac14(a+b+c+2x),\\ (1-\lambda_{n+1})d=\frac16 a. \end{cases} \tag{3.4}$$

Eliminating a,b,c, we get

$$(3-2\lambda_{n+1})(1-\lambda_{n+1})\big(3(16\lambda_{n+1}^3-32\lambda_{n+1}^2+16\lambda_{n+1}-1)d+x\big)=0.$$

This gives that if $\lambda_{n+1}\neq 1,\frac32,\beta_1,\beta_2,\beta_3$, then

$$d=\frac{-x}{3(16\lambda_{n+1}^3-32\lambda_{n+1}^2+16\lambda_{n+1}-1)}.$$

Next, by denoting $\widetilde{a_k}=a_k+a'_k, \widetilde{b_k}=b_k+b'_k, \widetilde{c_k}=c_k+c'_k$, another easy combination of equations at points in $V_{n+1}\setminus V_n$ gives that

$$\begin{cases} (1-\lambda_{n+1})\widetilde{a_k}=\frac14(\widetilde{a_k}+\widetilde{b_k}+\widetilde{c_k}+2d),\\ (1-\lambda_{n+1})\widetilde{b_k}=\frac12(\widetilde{a_k}+\widetilde{c_k}),\\ (1-\lambda_{n+1})\widetilde{c_k}=\frac14(\widetilde{a_k}+\widetilde{b_k}+\widetilde{c_k}+2x_k), \end{cases} \tag{3.5}$$

and

$$\begin{cases} (1-\lambda_{n+1})a_k=\frac14(\widetilde{a_k}-a_k+b_k+c_k+d),\\ (1-\lambda_{n+1})b_k=\frac12(a_k+c_k),\\ (1-\lambda_{n+1})c_k=\frac14(a_k+b_k+\widetilde{c_k}-c_k+x_k). \end{cases} \tag{3.6}$$

To ensure the above equations have a solution, we need that

$$\begin{vmatrix} (\frac{3}{4}-\lambda_{n+1}) & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & (1-\lambda_{n+1}) & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & (\frac{3}{4}-\lambda_{n+1}) \end{vmatrix} = \frac{1}{8}(4\lambda_{n+1}^2 - 6\lambda_{n+1}+1)(1-\lambda_{n+1}) \neq 0,$$

and

$$\begin{vmatrix} (\frac{5}{4}-\lambda_{n+1}) & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & (1-\lambda_{n+1}) & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & (\frac{5}{4}-\lambda_{n+1}) \end{vmatrix} = \frac{1}{8}(1-2\lambda_{n+1})(2\lambda_{n+1}-3)^2 \neq 0.$$

Then if $\lambda_{n+1} \notin \{\frac{1}{2}, 1, \frac{3}{2}, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3\}$, by solving (3.5) and (3.6), and substituting the value of d, we get the expressions in (3.2).

The formula (3.2) ensures that (3.1) holds for points in $V_{n+1} \setminus V_n$. We still need to look at the points in $V_n \setminus V_0$. Suppose s_k is a nonjunction vertex (the case that s_k is a junction vertex with $\deg(s_k) = 4$ or 6 is essentially the same), we then have

$$(1-\lambda_n)x_k = \frac{1}{2}(x_{k-1}+x_{k+1}),$$

which is an eigenvalue equation of $-\Delta_n^{\mathcal{T}}$ at s_k. By applying (3.1) at s_k, we also have

$$(1-\lambda_{n+1})x_k = \frac{1}{2}(c_k + c_k').$$

Using formula (3.2) and substituting $x_{k-1}+x_{k+1} = 2(1-\lambda_n)x_k$ into it, after an easy calculation, we get (3.3). □

Remark. *We call $\mathcal{F} := \{\frac{1}{2}, 1, \frac{3}{2}, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3\}$ forbidden eigenvalues, and the polynomial*

$$R(\lambda) := -48\lambda(\lambda-1)^2(2\lambda-1)(4\lambda^2-6\lambda+1)$$

appeared in (3.3) decimation function. See Figure 3.2 for the graph of $R(\lambda)$.

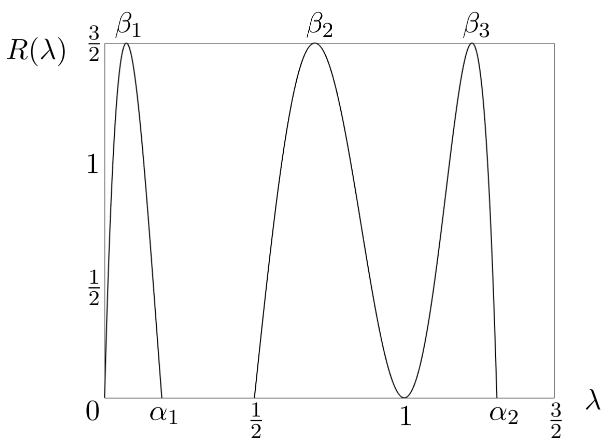


FIGURE 3.2. The graph of $R(\lambda)$.

Lemma 3.2. *If $\lambda_n \in (0, \frac{3}{2})$, there are 6 different values of λ_{n+1} satisfying (3.3), in addition, they all belong to $(0, \frac{3}{2}) \setminus \mathcal{F}$; if $\lambda_n = \frac{3}{2}$, the exact values λ_{n+1} satisfying (3.3) are $\beta_1, \beta_2, \beta_3$.*

Proof. It is straightforward by looking at Figure 3.2. □

3.2. **The Dirichlet and Neumann spectra.**

By Lemma 3.1, if $u_n \in l(V_n)$ is a λ_n-eigenfunction of $-\Delta_n^{\mathcal{T}}$, it will extend to a λ_{n+1}-eigenfunction $u_{n+1} \in l(V_{n+1})$ of $-\Delta_{n+1}^{\mathcal{T}}$, providing that $\lambda_n = R(\lambda_{n+1})$ and $\lambda_{n+1} \notin \mathcal{F}$. Every eigenfunction u on $\mathcal{T}$ has a *generation of birth* n_0, and a sequence $\{\lambda_n\}_{n\geq n_0}$ (related as above) such that $u|_{V_n}$ is a λ_n-eigenfunction of $-\Delta_n^{\mathcal{T}}$ for $n \geq n_0$, and $\lambda_n \notin \mathcal{F}$ for $n > n_0$. Call λ_{n_0} an *initial eigenvalue* and λ_n a *continued eigenvalue* for $n > n_0$. Let ϕ be the smallest inverse of R (it has 6 inverses). In order for the limit

$$\lambda = \lim_{n\to\infty} 6 \cdot 12^n 2^{2n} \lambda_n$$

to exist, which gives λ to be the eigenvalue of $-\Delta^{\mathcal{T}}$, it is necessary that $\lambda_n \to 0$, and thus $\lambda_{n+1} = \phi(\lambda_n)$ for all but a finite number of n's.

By the above discussion, to describe the explicit Dirichlet and Neumann spectra of $-\Delta^{\mathcal{T}}$, we only need to describe the discrete corresponding spectra of $-\Delta_n^{\mathcal{T}}$ for all n.

First, let's look at the Neumann case. It is easy to check that the Neumann eigenvalues of $-\Delta_0^{\mathcal{T}}$ are 0 and $\frac{3}{2}$ with multiplicity 1 and 2 respectively. For $\lambda_0 = 0$, its corresponding eigenfunction is a constant function on V_0; for $\lambda_0 = \frac{3}{2}$, we may choose two independent eigenfunctions u with $u(q_1) = 0$, $u(q_2) = 1$, $u(q_3) = -1$ or $u(q_1) = 0$, $u(q_2) = -1$, $u(q_3) = 1$. By Lemma 3.2, for $n \geq 1$, all the Neumann eigenvalues of $-\Delta_n^{\mathcal{T}}$ are contained in $[0, \frac{3}{2}]$, and all initial eigenvalues are contained in $\mathcal{F}$.

For $n \geq 1$, to describe the exact Neumann spectrum of $-\Delta_n^{\mathcal{T}}$, we only need to calculate the multiplicities of initial eigenvalues as well as their corresponding eigenfunctions. For this purpose, we introduce some notations. Call each copy of Γ_0 in Γ_n an *n-cell*, and a union of 4 n-cells as shown in Figure 3.3 an *n-star*. Call a circuit of n-stars around a "hole" in Γ_n an *n-starloop*, as shown in Figure 3.4. Denote

(a). $P_n = \#V_n = 3 + \frac{19}{11}(12^n - 1)$;

(b). $D_n^{(2)} = \#\{x \in V_n | \deg(x) = 2\} = 3 + \frac{6}{11}(12^n - 1)$;

(c). $D_n^{(4)} = \#\{x \in V_n | \deg(x) = 4\} = \frac{12}{11}(12^n - 1)$;

(d). $D_n^{(6)} = \#\{x \in V_n | \deg(x) = 6\} = \frac{1}{11}(12^n - 1)$;

(e). $C_n = 12^n$ the number of n-cells in Γ_n;

(f). $S_n = 3 \cdot 12^{n-1}$ the number of n-stars in Γ_n;

(g). $L_n = \frac{3}{11}(12^{n-1} - 1)$ the number of n-starloops in Γ_n.

For a Neumann λ_n-eigenvalue, we denote by $M^N_{\lambda_n,n}$ its multiplicity.

Theorem 3.3.(Neumann spectrum) *For $n \geq 1$, the initial Neumann eigenvalues of $-\Delta_n^{\mathcal{T}}$ are exactly the elements in $\mathcal{F}$, with the multiplicity:*

(a). $M^N_{\frac{1}{2},n} = 1 + S_n = 1 + 3 \cdot 12^{n-1}$;

(b). $M^N_{1,n} = 1 + L_n = 1 + \frac{3}{11}(12^{n-1} - 1)$;

(c). $M^N_{\frac{3}{2},n} = 2 \cdot S_n + D_n^{(6)} + D_{n-1}^{(4)} + D_{n-1}^{(2)} = 2 + \frac{8}{11}(12^n - 1)$;

(d). $M^N_{\alpha_i,n} = L_n = \frac{3}{11}(12^{n-1} - 1), i = 1, 2$;

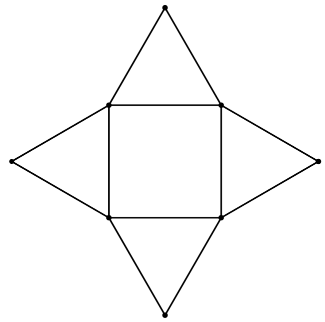


FIGURE 3.3. An n-star in Γ_n.

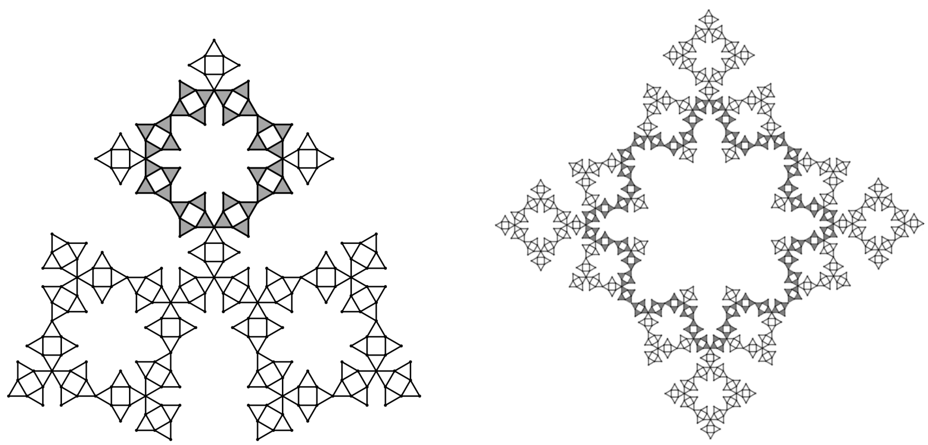


FIGURE 3.4. n-starloops in Γ_n.

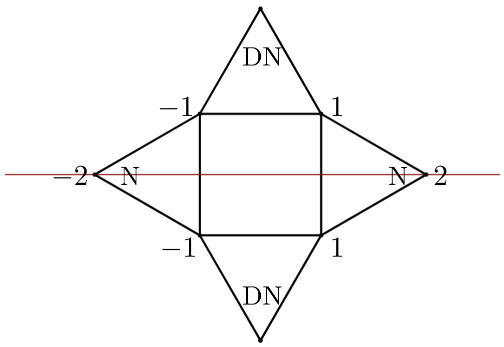
(e). $M^N_{\beta_i,n} = 10\cdot C_{n-2} - D^{(4)}_{n-2} - 2\cdot D^{(6)}_{n-2} = 2 + \frac{8}{11}(12^{n-1}-1), n \geq 2, i = 1,2,3$. *In addition,* $M^N_{\beta_i,1} = 2, i = 1,2,3$.

Their corresponding eigenfunctions can be described explicitly.

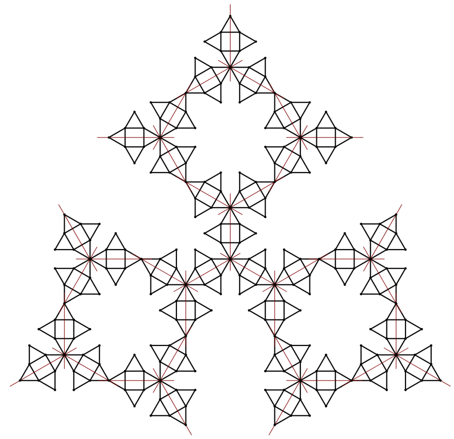
Proof. We prove the theorem by a dimension counting argument. That is, we need to count the numbers of initial eigenvalues and continued eigenvalues respectively, which sum up to P_n, the dimension of $l(V_n)$. Nevertheless, for each initial eigenvalue, we need to figure out its exact eigenspace.

First, let's look at the initial eigenvalues and eigenspaces. Since the only possible initial eigenvalues belong to $\mathcal{F}$, we make an analysis case by case. We omit the subscript n of λ_n for simplicity.

For $\lambda = \frac{1}{2}$, observe that Figure 3.5(a) provides a $\frac{1}{2}$-eigenfunction supported in an n-star, where we use a straight line to denote its symmetric axis. Note that it has two DN-boundary vertices (the up and down vertices) and two N-boundary vertices (the left and right vertices), where we use D or N to denote the Dirichlet or Neumann condition. Placing it into each of the n-stars in Γ_n with suitable rotation so that the DN vertices connecting with the outside, will give out S_n localized $\frac{1}{2}$-eigenfunctions. Furthermore, placing the copies of this function at the central 3 n-stars in Γ_n, and spreading it out continuously still by using copies will give a globally supported eigenfunction, see Figure 3.5(b) for an illustration on Γ_2. Obviously these functions are linearly independent. So we get $1 + S_n$ $\frac{1}{2}$-eigenfunctions in total.



(a)



(b)

FIGURE 3.5. The $\frac{1}{2}$-eigenfunctions with non-zero values marked.

For $\lambda = 1$, see Figure 3.6(a) for a 1-eigenfunction supported in an n-starloop in Γ_n. For each n-starloop, do similarly by a similar "chain of stars" arrangement. This will give out L_n localized eigenfunctions in Γ_n. Furthermore, there is still a globally supported eigenfunction by spreading continuously the copies of the function shown in Figure 3.6(b) in the way illustrated by Figure 3.5(b). So we get $1 + L_n$ independent 1-eigenfunctions in total.

For $\lambda = \frac{3}{2}$, we have 5 manners to get localized $\frac{3}{2}$-eigenfunctions. Figure 3.7 provides 5 distinct localized $\frac{3}{2}$-eigenfunctions with the DN or N boundary vertices marked. The (a), (b) and (e) functions are supported in a n-star; the (c) function is supported in a union of 3 n-stars; the (d) function is supported in a union of 2 n-stars. Placing each copy of one of these functions in suitable subgraphs of Γ_n by using DN vertices to connect with outside will provide 5 types of independent localized eigenfunctions. This will give S_n, S_n, $D_n^{(6)}$, $D_{n-1}^{(4)}$, $D_{n-1}^{(2)}$ $\frac{3}{2}$-eigenfunctions of each type.

For $\lambda = \alpha_i$, $i = 1, 2$. see Figure 3.8 for an α_i-eigenfunction supported in an n-starloop in Γ_n. Simiar as the $\lambda = 1$ case, there are L_n independent localized α_i-eigenfunctions associcated with the n-starloops in Γ_n. The cases for $i = 1, 2$ are same.

For $\lambda = \beta_i$, $i = 1, 2, 3$, things will be a little complicated, but the cases for distinct i are still same. First let's look at the eigenfunction as shown in Figure 3.9. Note that it has 3 N-boundary vertices and 1 D-boundary vertices as marked. We use a straight line to represent its symmetric axis. By linking copies of this function, we will figure out all the β_i-eigenfunctions in Γ_n. See Figure 3.10 for the manner of linking in two different cases.

We start by looking at Γ_2. See Figure 3.11 for 3 types of β_i-eigenfunctions where we use shadow to represent their supports. Note that the copies are linked in the manner as shown in Figure 3.10. By using obvious symmetries, we get 10 independent β_i-eigenfunctions in Γ_2. For higher level Γ_n, a straightforward way to figure out the eigenfunctions in Γ_n is to place each of the functions shown in Figure 3.11 into each of the copies of Γ_2 contained in Γ_n. This can be well done for the (a)-type and (c)-type functions in Figure 3.11. But for the (b)-type function we need to make a careful analysis on points in V_{n-2}. For $x \in V_{n-2}$ with

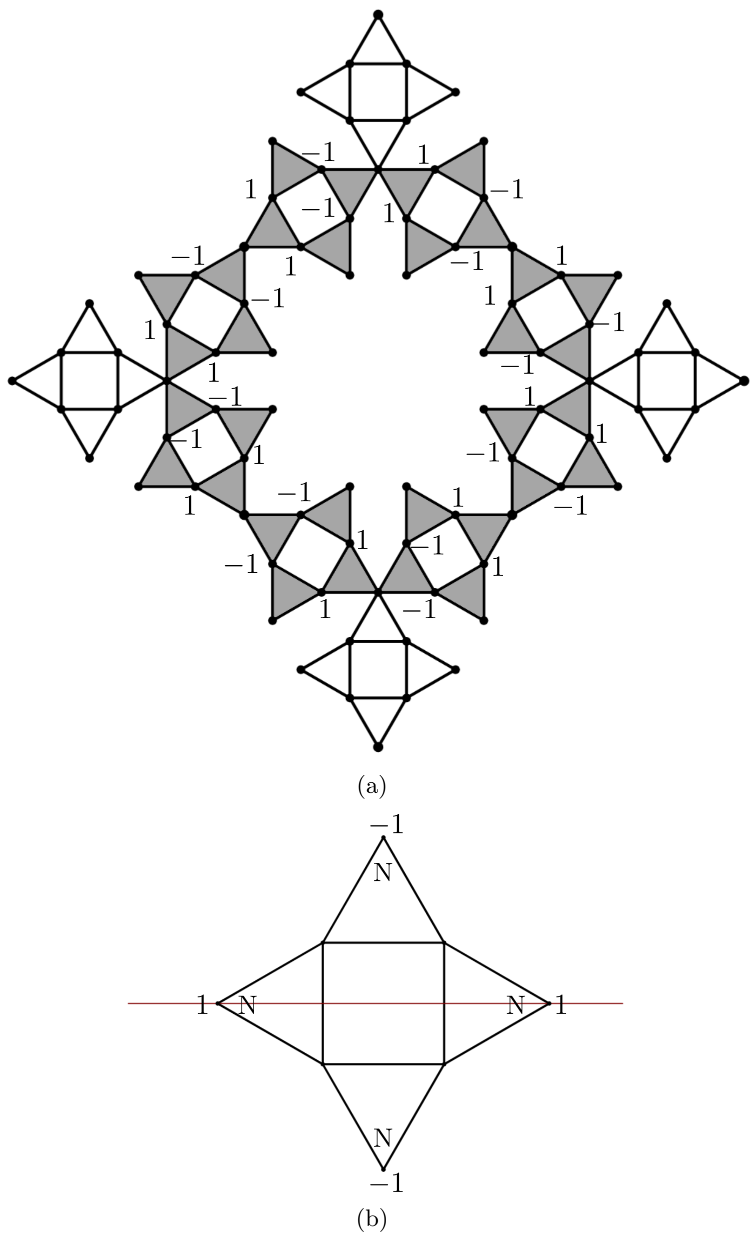


FIGURE 3.6. The 1-eigenfunctions.

$\deg(x) = 2$, the copies of (b)-type function related to x will still give 2 β_i-functions in Γ_n as in Γ_2; For $x \in V_{n-2}$ with $\deg(x) = 4$ or 6, since x is a junction point of 2 or 3 copies of Γ_2, the related eigenspace will be changed into what Figure 3.12 and 3.13 present, and thus each $x \in V_{n-2}$ with $\deg(x) = 4$ or 6 will give 3 or 4 eigenfunctions respectively. Basing on the above discussion, we have $10 \cdot C_{n-2} - D^{(4)}_{n-2} - 2 \cdot D^{(6)}_{n-2}$ β_i-eigenfunctions in Γ_n for $i = 1, 2, 3$.

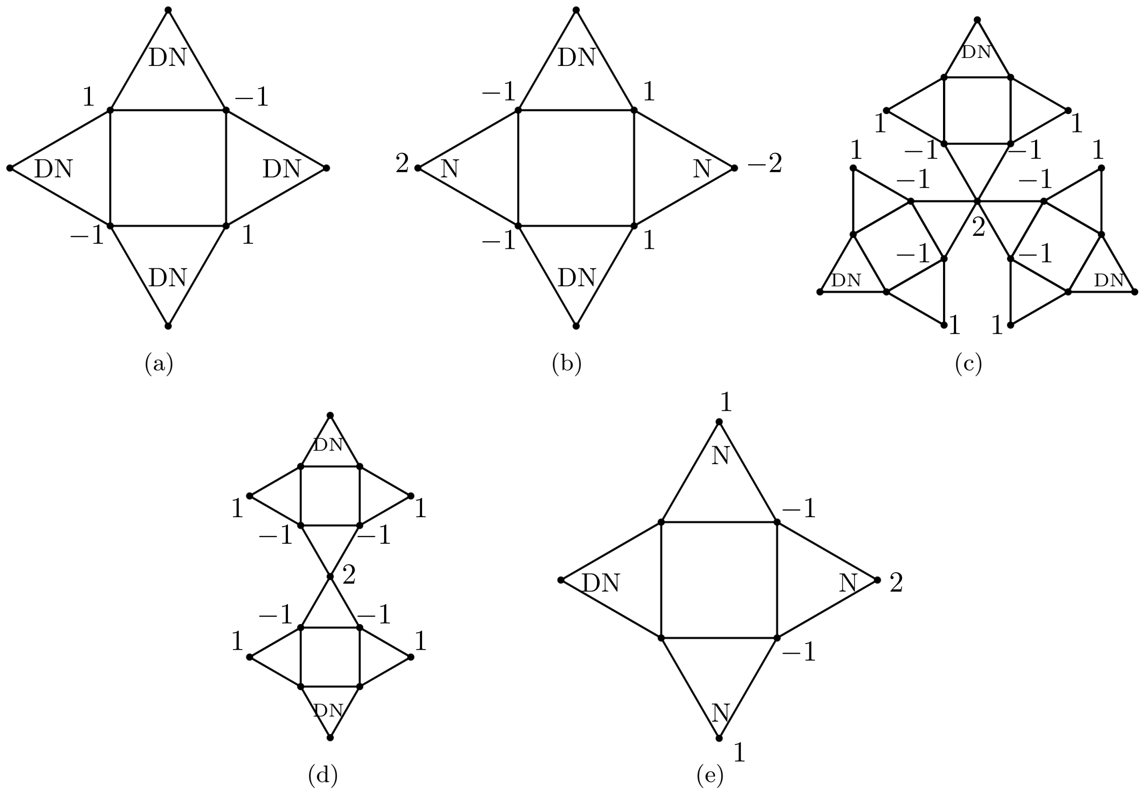


FIGURE 3.7. The $\frac{3}{2}$ eigenfunctions.

Till now, the total number of independent initial eigenfunctions we have figured out is

$$\begin{aligned} & (S_n+1)+(L_n+1)+(2\cdot S_n+D_n^{(6)}+D_{n-1}^{(4)}+D_{n-1}^{(2)}) \\ & \qquad\qquad +2\cdot L_n+3\cdot(10\cdot C_{n-2}-D_{n-2}^{(4)}-2\cdot D_{n-2}^{(6)}) \\ = \ & 21+\frac{162}{11}\cdot(12^{n-1}-1). \end{aligned}$$

Next, let's look at the continued eigenvalues and eigenfunctions. Using induction, the Neumann spectrum of $-\Delta_{n-1}^{\mathcal{T}}$ consists of a 0-eigenfunction(constant) with multiplicity 1; $\frac{3}{2}$-eigenfunctions with multiplicity $M^N_{\frac{3}{2},n-1}$; and $P_{n-1}-1-M^N_{\frac{3}{2},n-1}$ other eigenfunctions with eigenvalues in $(0,\frac{3}{2})$. By using Lemma 3.1 and Lemma 3.2, the total number of independent continued eigenfunctions of $-\Delta_n^{\mathcal{T}}$ will be

$$1+6\cdot(P_{n-1}-1-M^N_{\frac{3}{2},n-1})=1+6\cdot(12^{n-1}-1).$$

An easy calculation yields that $\big(21+\frac{162}{11}\cdot(12^{n-1}-1)\big)+\big(1+6\cdot(12^{n-1}-1)\big)=P_n$, which means that the eigenfunctions we found form a basis of $l(V_n)$. Thus each initial λ-eigenvalue with $\lambda\in\mathcal{F}$ has the exact dimension of its eigenspace as we have figured out. This finishes the proof of the theorem. $\square$

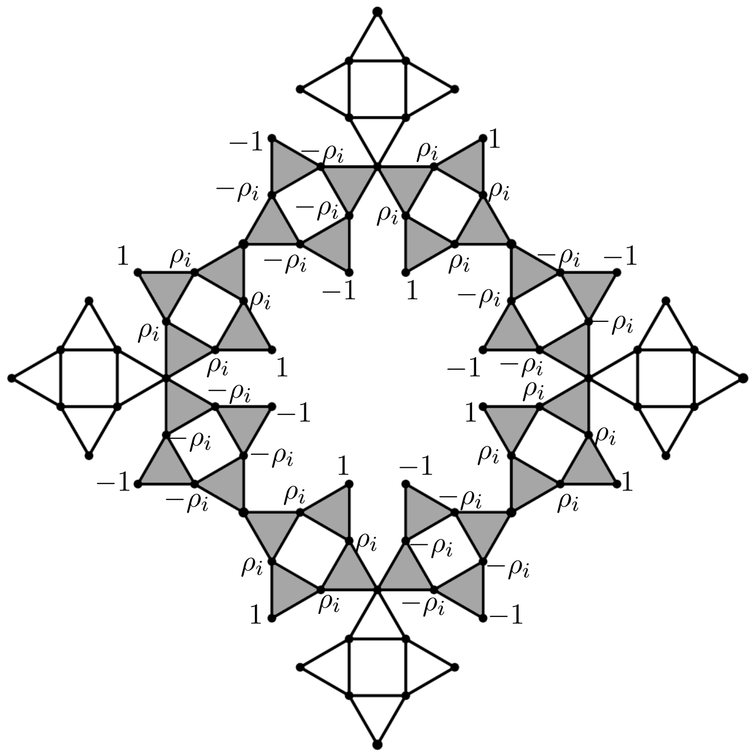
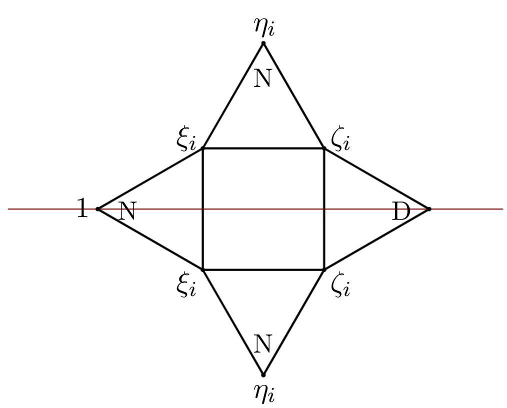


FIGURE 3.8. The α_i-eigenfunctions, with $\rho_i = 1 - \alpha_i$, i=1,2.

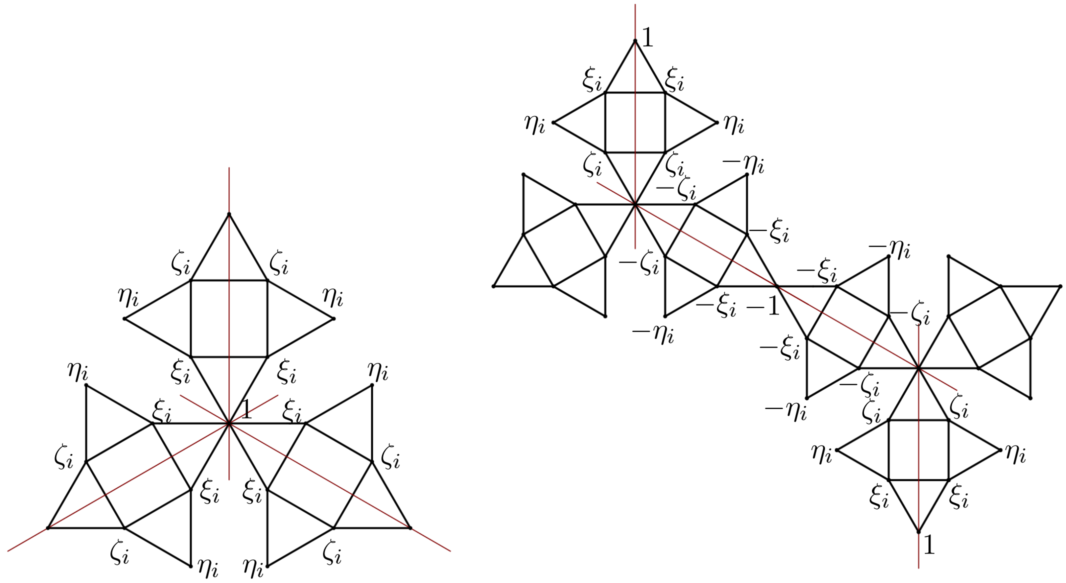


(a)

FIGURE 3.9. A β_i-eigenfunction, with $\xi_i = 1 - \beta_i$, $\eta_i = 1 - 2\beta_i$, $\zeta_i = (1 - \beta_i)(1 - 4\beta_i)$, $i = 1, 2, 3$.

Next, we turn to the Dirichlet case. For a Dirichlet λ_n-eigenvalue, we denote by $M^D_{\lambda_n,n}$ its multiplicity.

Theorem 3.4.(Dirichlet spectrum) *For $n \geq 1$, the initial Dirichlet eigenvalues of $-\Delta^{\mathcal{T}}_n$ are exactly the elements in $\mathcal{F}$, with the multiplicity:*



(a) (b)

FIGURE 3.10. The manner of linking copies of the function in Figure 3.9.

(a). $M^D_{\frac{1}{2},n} = S_n = 3 \cdot 12^{n-1}$*;*

(b). $M^D_{1,n} = 2 + L_n = 2 + \frac{3}{11}(12^{n-1} - 1)$*;*

(c). $M^D_{\frac{3}{2},n} = 2 \cdot S_n + D_n^{(6)} + D_{n-1}^{(4)} + D_{n-1}^{(2)} - 3 = -1 + \frac{8}{11}(12^n - 1)$*;*

(d). $M^D_{\alpha_1,n} = 2 + L_n = 2 + \frac{3}{11}(12^{n-1} - 1), i = 1, 2$*;*

(e). $M^D_{\beta_i,n} = 10 \cdot C_{n-2} - D_{n-2}^{(4)} - 2 \cdot D_{n-2}^{(6)} - 3 = -1 + \frac{8}{11}(12^{n-1} - 1), n \geq 2, i = 1, 2, 3.$ *In addition,* $M^D_{\beta_i,1} = 1, i = 1, 2, 3.$

Their corresponding eigenfunctions can be described explicitly.

Proof. The proof is very similar to the Neumann case, with only a few changes. $\square$

3.3. **Eigenvalue counting function and Weyl plot.**

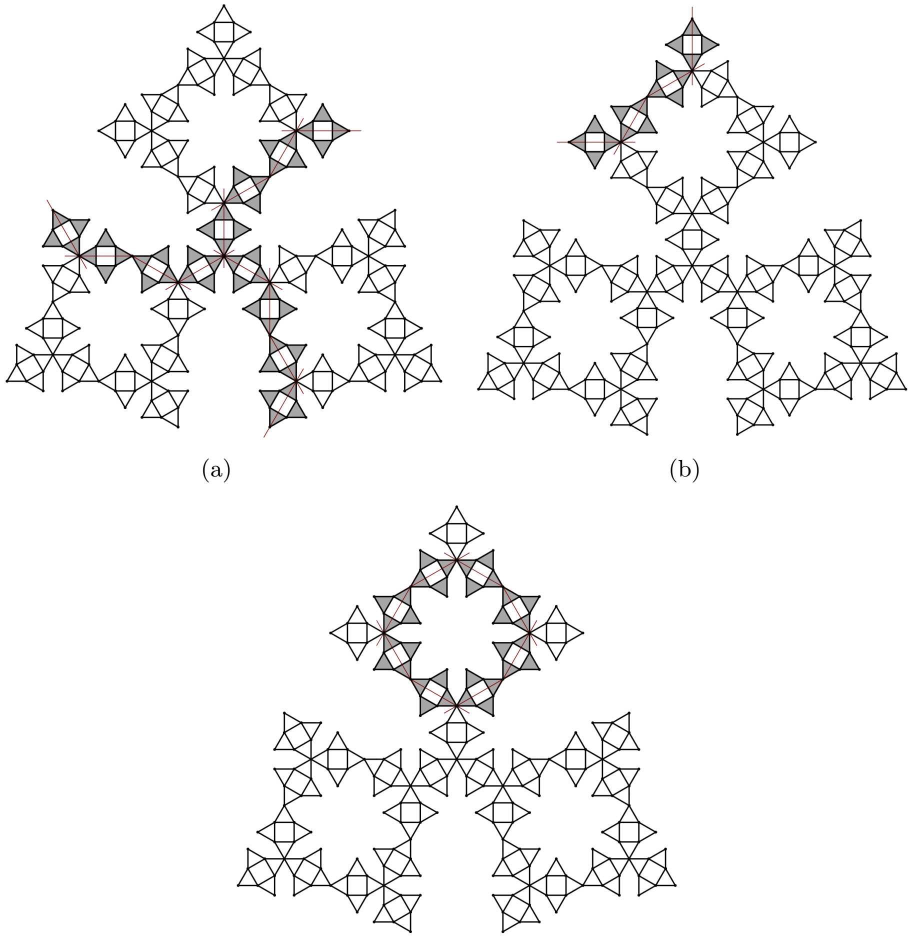
As a consequence of the previous discussion, we are able to estimate the asymptotic growth rate for the eigenvalue counting function

$$N(x) = \#\{\lambda \in S | \lambda \leq x\},$$

where S denotes the Dirichlet or Neumann spectrum (counting multiplicity). As at level-n, the Laplacian renormalization factor is 48^n, and the number of of cells in Γ_n is 12^n, we have $\alpha = \frac{ln 12}{ln 48}$, and thus the Weyl ratio

$$W(x) = \frac{N(x)}{x^\alpha},$$

which is bounded and bounded away from 0 by a routine discussion. We plot $N(x)$ at level-4, as well as $W(x)$ on a log scale, see Figure 3.14.



(a) (b) (c)

FIGURE 3.11. The 3 types of β_i-eigenfunctions in Γ_2, $i = 1, 2, 3$.

4. The Dirichlet and Neumann spectra on the fractal $\mathcal{S}$

In this section, we will turn to look at the fractal $\mathcal{S}$. It is easy to check that there is no spectral decimation recipe on $\mathcal{S}$ by using the graph approximation Λ_n by considering the equations

$$-\Delta_n^{\mathcal{S}} u_n(x) = \lambda_n u_n(x), \quad \forall x \in U_n \setminus U_0$$

for consecutive two level graphs. But if we instead to consider the $\{\Lambda_{n+\frac{1}{2}}\}_{n\geq 0}$ graph sequence, then luckly there is still a spectral decimation recipe which is essential the same as the case of the fractal $\mathcal{T}$.

So we need to look at the discrete spectra of $-\Delta^{\mathcal{S}}_{n+\frac{1}{2}}$ for $n \geq 0$. Below we list some useful notations.

(a). $\widetilde{P}_{n+\frac{1}{2}} = \#U_{n+\frac{1}{2}} = 8 + \frac{76}{11}(12^n - 1)$;

(b). $\widetilde{D}^{(2)}_{n+\frac{1}{2}} = \#\{x \in U_{n+\frac{1}{2}} \,|\, \deg(x) = 2\} = 4 + \frac{24}{11}(12^n - 1)$;

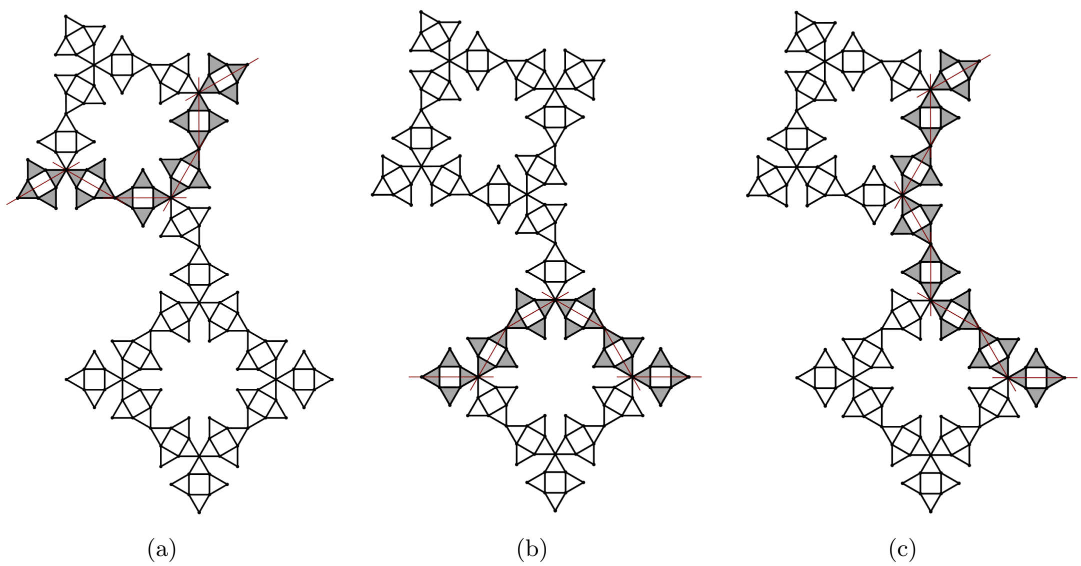


FIGURE 3.12. The case for $\deg(x) = 4$ with $x \in V_{n-2}$.

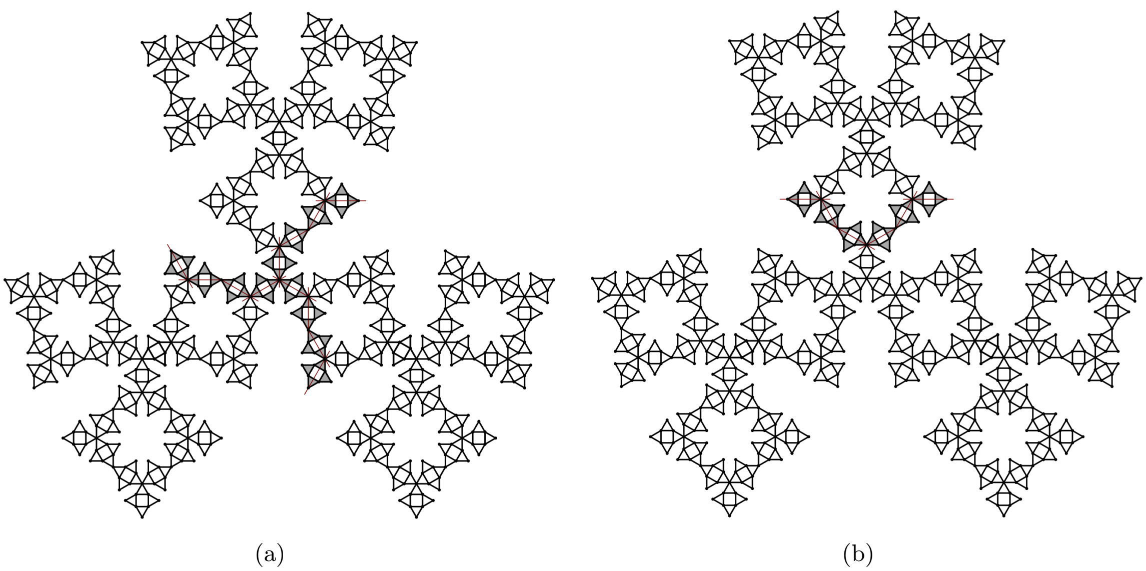


FIGURE 3.13. The case for $\deg(x) = 6$ with $x \in V_{n-2}$.

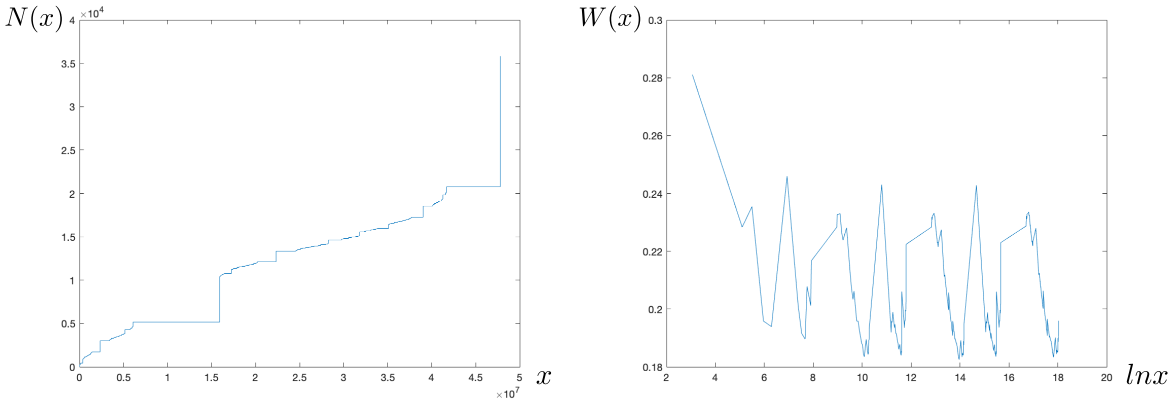
(c). $\widetilde{D}^{(4)}_{n+\frac{1}{2}} = \#\{x \in U_{n+\frac{1}{2}} \,|\, \deg(x) = 4\} = 4 + \frac{48}{11}(12^n - 1)$;

(d). $\widetilde{D}^{(6)}_{n+\frac{1}{2}} = \#\{x \in U_{n+\frac{1}{2}} \,|\, \deg(x) = 6\} = \frac{4}{11}(12^n - 1)$;

(e). $\widetilde{C}_{n+\frac{1}{2}} = 4 \cdot 12^n$, the number of $(n + \frac{1}{2})$-cells in $\Lambda_{n+\frac{1}{2}}$;

(f). $\widetilde{S}_{n+\frac{1}{2}} = 12^n$, the number of $(n + \frac{1}{2})$-stars in $\Lambda_{n+\frac{1}{2}}$;

(g). $\widetilde{L}_{n+\frac{1}{2}} = \frac{1}{11}(12^n - 1)$, the number of $(n + \frac{1}{2})$-starloops in $\Lambda_{n+\frac{1}{2}}$.



(a) $N(x)$ (b) $W(x)$

FIGURE 3.14. The graphs of $N(x)$ and $W(x)$ at level-4.

For a Dirichlet or Neumann eigenvalue λ, we denote by $\widetilde{M}^D_{\lambda,n+\frac{1}{2}}$ or $\widetilde{M}^N_{\lambda,n+\frac{1}{2}}$ its multiplicity.

Similar to Theorem 3.3 and 3.4, with almost a same proof, we have

Theorem 4.1. *For $n \geq 1$, the initial eigenvalues of $-\Delta^{\mathcal{S}}_n$ are exactly the elements in $\mathcal{F}$ with the multiplicity:*

(a). $\widetilde{M}^N_{\frac{1}{2},n+\frac{1}{2}} = \widetilde{S}_{n+\frac{1}{2}} + 1 = 1 + 12^n$;

(b). $\widetilde{M}^N_{1,n+\frac{1}{2}} = \widetilde{L}_{n+\frac{1}{2}} + 1 = 1 + \frac{1}{11}(12^n - 1)$;

(c). $\widetilde{M}^N_{\frac{3}{2},n+\frac{1}{2}} = 2\widetilde{S}_{n+\frac{1}{2}} + \widetilde{D}^{(6)}_{n+\frac{1}{2}} + \widetilde{D}^{(4)}_{n-\frac{1}{2}} + \widetilde{D}^{(2)}_{n-\frac{1}{2}} = 4 + \frac{32}{11}(12^n - 1)$;

(d). $\widetilde{M}^N_{\alpha_i,n+\frac{1}{2}} = \widetilde{L}_{n+\frac{1}{2}} = \frac{1}{11}(12^n - 1)$, $i = 1, 2$;

(e). $\widetilde{M}^N_{\beta_i,n+\frac{1}{2}} = 10\widetilde{C}_{n-\frac{3}{2}} - \widetilde{D}^{(4)}_{n-\frac{3}{2}} - 2\widetilde{D}^{(6)}_{n-\frac{3}{2}} = 4 + \frac{32}{11}(12^{n-1} - 1), n \geq 2, i = 1, 2, 3$;

(f). $\widetilde{M}^D_{\frac{1}{2},n+\frac{1}{2}} = \widetilde{S}_{n+\frac{1}{2}} = 12^n$;

(g). $\widetilde{M}^D_{1,n+\frac{1}{2}} = \widetilde{L}_{n+\frac{1}{2}} + 3 = 3 + \frac{1}{11}(12^n - 1)$;

(h). $\widetilde{M}^D_{\frac{3}{2},n+\frac{1}{2}} = 2\widetilde{S}_{n+\frac{1}{2}} + \widetilde{D}^{(6)}_{n+\frac{1}{2}} + \widetilde{D}^{(4)}_{n-\frac{1}{2}} + \widetilde{D}^{(2)}_{n-\frac{1}{2}} - 4 = \frac{32}{11}(12^n - 1)$;

(i). $\widetilde{M}^D_{\alpha_i,n+\frac{1}{2}} = \widetilde{L}_{n+\frac{1}{2}} + 3 = 3 + \frac{1}{11}(12^n - 1), i = 1, 2$;

(j). $\widetilde{M}^D_{\beta_i,n+\frac{1}{2}} = 10\widetilde{C}_{n-\frac{3}{2}} - \widetilde{D}^{(4)}_{n-\frac{3}{2}} - 2\widetilde{D}^{(6)}_{n-\frac{3}{2}} - 4 = \frac{32}{11}(12^{n-1} - 1), n \geq 2, i = 1, 2, 3.$

Before ending this section, we point out that there is another way to consider the spectra of $-\Delta^{\mathcal{S}}$ on $\mathcal{S}$ motivated by the result for the $\mathcal{T}$ fractal, by observing that a Dirichlet eigenfunction on $\mathcal{S}$ extends oddly to a Dirichlet eigenfunction on $\mathcal{T}$, and a Neumann eigenfunction on $\mathcal{S}$ extends evenly to a Neumann eigenfunction on $\mathcal{T}$.

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