BROWNIAN MOTION ON THE GOLDEN RATIO SIERPINSKI GASKET

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Abstract. We construct a strongly local regular Dirichlet form on the golden ratio Sierpinski gasket, which is a self-similar set without any finitely ramified cell structure, via a study on the trace of electrical networks on an infinite graph. The Dirichlet form is self-similar in the sense of an infinite iterated function system, and is decimation invariant with respect to a graph-directed construction. A theorem of uniqueness is also provided. Lastly, the associated process satisfies the two-sided sub-Gaussian heat kernel estimate.

1. introduction

The golden ratio Sierpinski gasket $\mathcal{G}$ is a typical example of self-similar sets satisfying the finite type property [2], which arises in the study of the Hausdorff dimension of self-similar sets with overlaps [25, 30, 31]. Let $q_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), q_1 = (0,0), q_2 = (1,0)$ be the three vertices of an equilateral triangle in $\mathbb{R}^2$, and

$$F_0(x) = \rho^2(x - q_0) + q_0,$$
$$F_1(x) = \rho(x - q_1) + q_1, \quad F_2(x) = \rho(x - q_2) + q_2,$$

with $\rho = \frac{\sqrt{5}-1}{2}$ being the golden ratio. The gasket $\mathcal{G}$ is the invariant set associated with the iterated function system (i.f.s. for short) $\{F_0, F_1, F_2\}$. See Figure 1.

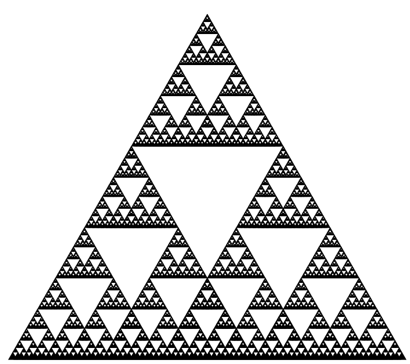

Figure 1. The golden ratio Sierpinski gasket $\mathcal{G}$.

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The large overlap $F_1\mathcal{G}\cap F_2\mathcal{G}$ makes $\mathcal{G}$ different from the existing examples of self-similar sets on which Brownian motions are constructed.

First, any effort to disconnect the bottom line of $\mathcal{G}$ requires the removal of infinitely many points, so there is not a finitely ramified cell structure [34] on $\mathcal{G}$. Well-known classes of fractals with finitely ramified cell structures include Lindstrøm's nested fractals [26], Kigami's post-critically finite (p.c.f.) self-similar sets [19, 20], finitely ramified graph-directed fractals [9, 18], and some Julia sets of polynomials [1, 13, 32] or rational functions [10]. See [8, 15, 23] for pioneering works on the Sierpinski gasket, and also books [3, 21] for systematic discussions.

Second, although there is a graph-directed construction related with $\mathcal{G}$ (see Section 2), by dividing $\mathcal{G}$ into blocks of nearly the same size, the graph will be much complicated. The deep and famous constructions on the Sierpinski carpet [4, 5, 6] by Barlow and Bass, and on certain symmetric fractals [24] by Kusuoka and Zhou will be extremely difficult here. See also [7] for a theorem of uniqueness on the Sierpinski carpet.

Instead, thanks to the golden ratio, there is an 'infinite cell structure' on $\mathcal{G}$. For the first level, we consider the cell $F_0\mathcal{G}$ and its images of combinations of F_1, F_2. The union of these cells covers $\mathcal{G}$ except the bottom line. For each such cell, we can find a finite word w, and a contraction map $F_w = F_{w_1}\circ F_{w_2}\circ\cdots\circ F_{w_m}$, so that the cell can be written as $F_w\mathcal{G}$. We name the collection of all such words W_1, and construct a resistance form [21] on $\mathcal{G}$, that is self-similar in the sense of the infinite i.f.s. $\{F_w\}_{w\in W_1}$. Roughly speaking, we have the following theorem, see Theorem 6.5, 6.6 and 6.8 for detailed and formal results.

Theorem 1. *There exists a unique strongly local regular resistance form $(\mathcal{E},\mathcal{F})$ on $\mathcal{G}$ such that $f\in\mathcal{F}$ if and only if $f\circ F_w\in\mathcal{F}$ for all $w\in W_1$ and $\sum_{w\in W_1}\rho_w^{-\theta}\mathcal{E}(f\circ F_w, f\circ F_w)<\infty$, where ρ_w is the similarity ratio of F_w and $0<\theta<1$ is a constant. In addition,*

$$\mathcal{E}(f,f)=\sum_{w\in W_1}\rho_w^{-\theta}\mathcal{E}(f\circ F_w, f\circ F_w).$$

Moreover, the form is decimation invariant with respect to the graph-directed construction of $\mathcal{G}$.

The form $(\mathcal{E},\mathcal{F})$ is then a strongly local regular Dirichlet form on $L^2(\mathcal{G},\mu_H)$, where μ_H is the normalized Hausdorff measure on $\mathcal{G}$. There is a diffusion process associated by the well-known theorem [14]. Although, our construction is based on an infinite i.f.s., the behavior of the process is same on each cell before hitting the boundary, up to a time scaling, since any cell can be decomposed in a same manner.

In addition, by following the well-established method of Hambly and Kumagai [17], which is organized in Barlow's book [3], we can obtain a sub-Gaussian heat kernel estimate (see Section 7). We refer to [8, 12, 22] for earlier results on transition density estimates on fractals.

Theorem 2. *There is a symmetric transition density $p(t,x,y)$ associated with the form $(\mathcal{E},\mathcal{F})$ on $\mathcal{G}$. In addition, there are constants c_1,c_2,c_3,c_4 so that*

$$c_1t^{-d_H/\beta}\exp\big(-c_2(\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big)\leq p(t,x,y)\leq c_3t^{-d_H/\beta}\exp\big(-c_4(\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big),$$

for $0<t\leq 1$, with $\beta=\theta+d_H$, where $d_H\approx 1.6824$ is the Hausdorff dimension of $\mathcal{G}$.

The main effort is to get an estimate of the resistance metric R on $\mathcal{G}$ as $c_1d(x,y)^\theta\leq R(x,y)\leq c_2d(x,y)^\theta$ for some constant $c_1,c_2>0$.

We organize the structure of the paper as follows. In Section 2, we will briefly introduce some facts about the geometry of $\mathcal{G}$. From Section 3 to 5, we study the trace of forms on an infinite graph. In Section 3, we establish the resistance forms on the graph. In Section 4, we study the trace map and a related renormalization map. We will show the jointly continuity of the renormalization map. In Section 5, we show that there is a unique solution to a renormalization problem. With all these preparations, we construct the resistance form on $\mathcal{G}$ in Section 6, and at the same time we get an upper bound estimate of the resistance metric. Lastly, we obtain the transition density estimate through a lower bound estimate of the resistance metric in Section 7.

Before ending this section, we remark that the result in this paper has a natural extension, by replacing $0<\rho<1$ to be a real root of x^n-2x+1 with $n\geq 4$, and taking the contraction ratios corresponding to F_0,F_1,F_2 to be $1-\rho,\rho,\rho,\rho$. Indeed, we will obtain a class of gaskets that possess a similar overlapping structure of $\mathcal{G}$, see Figure 2.

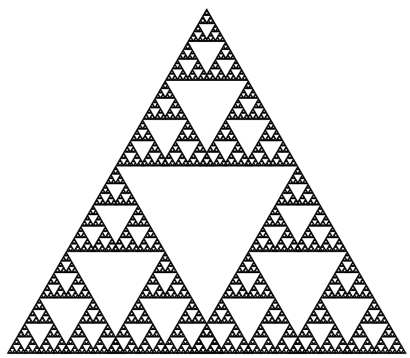


FIGURE 2. A gasket with $0<\rho<1$ being a root of $x^4-2x+1=0$.

2. Preliminary

The golden ratio Sierpinski gasket $\mathcal{G}$ is one of the typical examples of self-similar sets with overlaps but satisfying the finite type property.

Let K be a general self-similar set associated with an i.f.s. $\{F_i\}_{i=0}^{N-1}$ with contraction ratios $\{\rho_i\}_{i=0}^{N-1}$. For $m\geq 1$, we call $w=w_1w_2\cdots w_m$ with $0\leq w_i<N$, a *word* of length m(denoted by $|w|$), and call $\emptyset$ the empty word. We denote the set of all words by $\tilde{W}_*$. For any word $w\in\tilde{W}_*$, we write $F_w=F_{w_1}\circ F_{w_2}\cdots\circ F_{w_{|w|}}$, and let $F_\emptyset$ be the identity map for consistency. Let $\rho_*=\min\{\rho_i:0\leq i<N\}$.

Definition 2.1 (finite type property)**.** *A self-similar set K is of* finite type *if there are only finite many maps $h=F_w^{-1}F_v$ with $w,v\in\tilde{W}_*$ and $F_wK\cap F_vK\neq\emptyset$, and with similarity ratio $\rho_h\in(\rho_*,1/\rho_*)$.*

The finite type property of K, formulated in algebraic terms, was introduced in [2] by Bandt and Rao. It guarantees the existence of an 'almost non-overlapping' graph-directed construction (see [2, 30] for details) of K, which is quite useful for calculating the Hausdorff dimension of K. See [25, 31] for more flexible variants of the finite type property.

It is easy to verify that $\mathcal{G}$ satisfies the finite type property, noticing that $F_{122}\mathcal{G} = F_{211}\mathcal{G}$. In particular, it has the following *graph-directed construction* [27].

Definition 2.2 (a graph-directed construction of $\mathcal{G}$)**.** *(a). Let $K_1 = \mathcal{G}$ and $K_2 = \overline{\mathcal{G} \setminus F_{22}\mathcal{G}}$.*

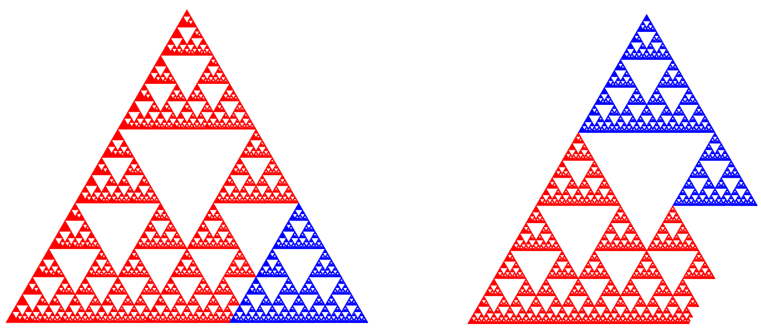
(b). Let $\Gamma(S, E)$ be a directed graph with the vertex set $S = \{1, 2\}$, and the edge set $E = \{e_i\}_{i=1}^{6}$, where $e_1 = (1, 2), e_2 = (1, 1), e_3 = (2, 1), e_4 = (2, 2), e_5 = (2, 2), e_6 = (2, 1)$.

(c). Define $\psi_{e_1} = Id$, $\psi_{e_2} = F_{22}$, $\psi_{e_3} = F_0$, $\psi_{e_4} = F_1$, $\psi_{e_5} = F_{21}$, $\psi_{e_6} = F_{20}$.

Clearly, we have

$$K_1 = \bigcup_{i=1}^{2} \psi_{e_i} K_{e_{i,2}} \text{ and } K_2 = \bigcup_{i=3}^{6} \psi_{e_i} K_{e_{i,2}},$$

where we use the notation $e_i = (e_{i,1}, e_{i,2})$ for a directed edge. In addition, there exist bounded open sets O_1 and O_2 such that $\bigcup_{i=1}^{2} \psi_{e_i} O_{e_{i,2}} \subset O_1$ and $\bigcup_{i=3}^{6} \psi_{e_i} O_{e_{i,2}} \subset O_2$, where the unions are disjoint. See Figure 3 for an illustration.



K_1 K_2

FIGURE 3. A graph-directed construction of $\mathcal{G}$.

Then similar to the open set condition situation, one can calculate the exact value of the Hausdorff dimension of $\mathcal{G}$ to be

$$d_H = \frac{\log \eta}{-2\log \rho} \approx 1.6824$$

with η being the largest root of $x^3 - 6x^2 + 5x - 1$. In addition, the associated Hausdorff measure of $\mathcal{G}$ is positive and finite. See details in [30] by Ngai and Wang.

In this paper, we take μ_H to be the normalized Hausdorff measure on $\mathcal{G}$, i.e. $\mu_H(\mathcal{G}) = 1$. It is not hard to verify that μ_H is volume doubling.

Lemma 2.3. *Let $B_s(p) = \{q \in \mathcal{G} : d(p, q) < s\}$. There are constants $c_1, c_2 > 0$ such that*

$$c_1 s^{d_H} \leq \mu_H(B_s(p)) \leq c_2 s^{d_H}, \quad \forall p \in \mathcal{G}, 0 < s \leq 1.$$

There is a *geodesic metric* d_g on $\mathcal{G}$ equivalent to the Euclidean metric d.

Lemma 2.4. *For $p, q \in \mathcal{G}$, let $d_g(p, q) = \inf\{|\gamma| : \gamma$ is a path connecting p, q, and $\gamma \subset \mathcal{G}\}$. Then there exists a constant $c \geq 1$ such that*

$$d(p, q) \leq d_g(p, q) \leq c\,d(p, q), \quad \forall p, q \in \mathcal{G}.$$

The proof relies on the finite type property. The rough idea is to link p, q with a bounded number of cells of diameter approximating to $d(p, q)$.

Clearly, there is always a path admitting the infimum length between p, q. So the metric space $(\mathcal{G}, d_g)$ satisfies the so-called *midpoint property*, i.e. for any $p, q \in \mathcal{G}$, there exists p' so that $d_g(p, p') = d_g(p', q) = \frac{1}{2} d_g(p, q)$. The space $(\mathcal{G}, d_g, \mu_H)$ is then a *fractional metric space*, see [3], Definition 3.2.

We will return to look at the geometric properties of $\mathcal{G}$ listed in this section. But first, from Section 3 to 5, we will instead consider an infinite i.f.s. and the associated infinite graph.

3. Resistance forms on the infinite graph V_1

The golden ratio Sierpinski gasket $\mathcal{G}$ can be realized as an invariant set of an infinite i.f.s. For convenience, we introduce some notations. For any word $w, w' \in \tilde{W}_*$, we write ww' for the concatenation of w, w'. For $w = w_1 w_2 \cdots w_m$ and $0 \leq l \leq m$, we write $[w]_l = w_1 w_2 \cdots w_l$. The following notations will be a little different from the standard ones.

Notation. Choose a set of finite words $W_1 \subset \bigcup_{n=0}^{\infty} \{1,2\}^n \times \{0\}$ so that
1. for any $w \in \bigcup_{n=0}^{\infty} \{1,2\}^n \times \{0\}$, there exists $w' \in W_1$ such that $F_w = F_{w'}$;
2. for different words $w, w' \in W_1$, we have $F_w \neq F_{w'}$.

In addition, based on W_1, we introduce some more notations.
(a). For $n \geq 1$, define $W_{1,n} = \{w \in W_1 : |w| = n\}$;
(b). For $m \geq 2$, define $W_m := W_1^m = \{w_1 w_2 \cdots w_m : w_i \in W_1, 1 \leq i \leq m\}$;
(c). Write $V_0 = \{q_i\}_{i=0}^2$ and for $m \geq 1$, $V_m = \bigcup_{w \in W_m} F_w V_0$. Denote $\bar{V}_m$ the closure of V_m;
(d). For distinct $p, q \in V_1$, we denote $p \sim q$ if and only if $p, q \in F_w V_0$ for some $w \in W_1$, which induce an infinite graph $(V_1, \sim)$. See Figure 4 for an illustration.

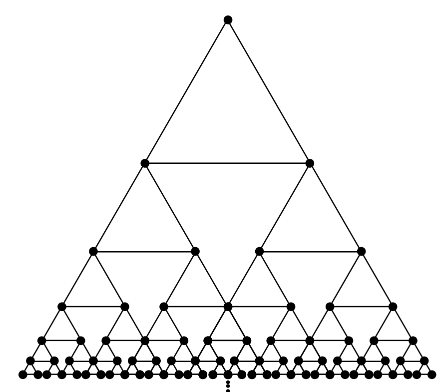


Figure 4. The infinite graph $(V_1, \sim)$. (The bottom line equals to $\bar{V}_1 \setminus V_1$.)

Obviously, we have

$$\mathcal{G} = \overline{\bigcup_{w \in W_1} F_w \mathcal{G}},$$

and thus $\{F_w\}_{w\in W_1}$ is an infinite i.f.s associated with $\mathcal{G}$. See [29] for more details about infinite i.f.s. The advantage of this i.f.s. lies in the fact that

$$F_w\mathcal{G}\cap F_{w'}\mathcal{G}=F_wV_0\cap F_{w'}V_0,\quad \forall w\neq w'\in W_1.$$

In the rest of this section, we consider a class of resistance forms generated by decimation. For convenience of readers, we recall the general definition of resistance forms in the following. See [21] for more details.

Definition 3.1. *Let X be a set, and $l(X)$ be the space of all real-valued functions on X. A pair $(\mathcal{E},\mathcal{F})$ is called a (non-degenerate)* resistance form *on X if it satisfies the following conditions:*
(RF1). $\mathcal{F}$ is a linear subspace of $l(X)$ containing constants and $\mathcal{E}$ is a nonnegative symmetric quadratic form on $\mathcal{F}$; $\mathcal{E}(f):=\mathcal{E}(f,f)=0$ if and only if f is constant on X.
(RF2). Let '$\sim$' be an equivalent relation on $\mathcal{F}$ defined by $f\sim g$ if and only if $f-g$ is constant on X. Then $(\mathcal{F}/\sim,\mathcal{E})$ is a Hilbert space.
(RF3). For any finite subset $V\subset X$ and any $u\in l(V)$, there exists a function $f\in\mathcal{F}$ such that $f|_V=u$.
(RF4). For any distinct $p,q\in X$, $R(p,q):=\sup\{\frac{|f(p)-f(q)|^2}{\mathcal{E}(f)}:f\in\mathcal{F},\mathcal{E}(f)>0\}$ is finite.
(RF5). If $f\in\mathcal{F}$, then $\bar{f}=\min\{\max\{f,0\},1\}\in\mathcal{F}$ and $\mathcal{E}(\bar{f})\leq\mathcal{E}(f)$.

Sometimes, we write $\mathcal{F}=Dom(\mathcal{E})$, and abbreviate $(\mathcal{E},\mathcal{F})$ to $\mathcal{E}$ when no confusion occurs. It is well-known that $R(p,q)$ defined in (RF3) is a metric on X, named the *effective resistance metric*.

On the finite set V_0, a resistance form $\mathcal{D}$ always has the form

$$\mathcal{D}(f,g)=\frac{1}{2}\sum_{i,j}a_{i,j}\big(f(q_i)-f(q_j)\big)\big(g(q_i)-g(q_j)\big),\quad \forall f,g\in l(V_0),\tag{3.1}$$

where $a_{i,i}=0$ and the 3×3 matrix $(a_{i,j})$ is positive, symmetric and irreducible. For convenience, we write $\mathcal{M}$ for the collection of all resistance forms on V_0. We view $\mathcal{M}$ as a subset of $\mathbb{R}^3$, which is not closed with induced topology.

Given a resistance form $\mathcal{D}$, we define a resistance form on V_1 associated with $\mathcal{D}$ in a self-similar manner, respecting the infinite i.f.s. $\{F_w\}_{w\in W_1}$.

Definition 3.2. *For $r>0$, $\mathcal{D}\in\mathcal{M}$, we define $\Psi_r\mathcal{D}$ as*

$$\Psi_r\mathcal{D}(f,g)=\sum_{w\in W_1}r^{-|w|+1}\mathcal{D}(f\circ F_w,g\circ F_w),$$

with $Dom(\Psi_r\mathcal{D})=\{f\in l(V_1):\Psi_r\mathcal{D}(f)<\infty\}$.

It is not hard to show that $\Big(\Psi_\lambda\mathcal{D},Dom(\Psi_r\mathcal{D})\Big)$ is a resistance form on V_1. However, to get a good resistance form, we need to restrict the range of r.

Proposition 3.3. *Let $\mathcal{D}\in\mathcal{M}$ and $r<1$, then $Dom(\Psi_r\mathcal{D})\subset C(\bar{V}_1)$ by a natural identification. In addition, if $\rho<r<1$, then $\Big(\Psi_r\mathcal{D},Dom(\Psi_r\mathcal{D})\Big)$ is a resistance form on $\bar{V}_1$, with the associated resistance metric $R(p,q)$ satisfying the estimate*

$$R(p,q)\leq\frac{4}{r^2(1-r)}R_0^*(\mathcal{D})d(p,q)^{\frac{\log r}{\log\rho}},\quad \forall p,q\in\bar{V}_1,\tag{3.2}$$

where $R_0^*(\mathcal{D}) = \max_{p,q\in V_0} R_0(p,q)$ *with* R_0 *being the resistance metric on* V_0 *associated with* $\mathcal{D}$.

Proof. Obviously, $R(p,q) \leq R_0^*(\mathcal{D})r^{n-1}$ for any distinct $p,q \in F_wV_0$ with $w \in W_{1,n}$ and $n \geq 1$. For $w \in \{1,2\}^n$, write $[w]_l = w_1w_2\cdots w_l$ and $p_l = F_{[w]_l}(q_0)$, with $0 \leq l \leq n$, then

$$R(p_i,p_j) \leq \sum_{l=i}^{j-1} R(p_l,p_{l+1}) \leq R_0^*(\mathcal{D})\sum_{l=i}^{j-1} r^l < R_0^*(\mathcal{D})\frac{r^i}{1-r}, \quad \forall 0 \leq i < j \leq n.$$

In particular, this implies that $R(p,q) < \frac{2r^n}{1-r}R_0^*(\mathcal{D})$ for any $p,q \in F_wV_1$ and $w \in \{1,2\}^n$. Now, if $p,q \in V_1$ and $d(p,q) < \rho^{n+1}$, then there exist $w,w' \in \{1,2\}^n$ such that $p \in F_wV_1, q \in F_{w'}V_1$, and $F_wV_1 \cap F_{w'}V_1 \neq \emptyset$, which implies that $R(p,q) \leq \frac{4r^n}{1-r}R_0^*(\mathcal{D})$. As a consequence, we have

$$R(p,q) \leq \frac{4}{r^2(1-r)}R_0^*(\mathcal{D})d(p,q)^{\frac{\log r}{\log \rho}}, \quad \forall p,q \in V_1. \tag{3.3}$$

On the other hand, for any $f \in Dom(\Psi_r\mathcal{D})$, we immediately have

$$|f(p) - f(q)| \leq \big(R(p,q)\Psi_r\mathcal{D}(f)\big)^{1/2}. \tag{3.4}$$

Combining (3.3) and (3.4), we then get $Dom(\Psi_r\mathcal{D}) \subset C(\bar{V}_1)$.

To show the second assertion, we let (X,R) be the completion of (V_1,R), and recall Theorem 2.3.10 in [21] to get that $\Big(\Psi_\lambda\mathcal{D}, Dom(\Psi_r\mathcal{D})\Big)$ extends to be a resistance form on X. It suffices to show that the identity map $Id : V_1 \to V_1$ extends to an homeomorphism from $(\bar{V}_1,d)$ to (X,R) under the assumption $\rho < r < 1$. First, by (3.3), Id is continuous from (V_1,d) to (V_1,R). Next, let $f \in l(V_1)$ be a restriction of a linear function on $\mathbb{R}^2$. We have

$$\begin{aligned}\Psi_r\mathcal{D}(f) &= \sum_{w\in W_1} r^{-|w|+1}\mathcal{D}(f\circ F_w) = \sum_{n=1}^{\infty}\sum_{w\in W_{1,n}} r^{-n+1}\mathcal{D}(f\circ F_w)\\ &= \sum_{n=1}^{\infty}\#W_{1,n}r^{-n+1}\rho^{2(n+1)}\mathcal{D}(f|_{V_0}),\end{aligned}$$

where the last equality follows from the fact that f is linear. Since $\#W_{1,n} \asymp \rho^{-n}$, we have $\Psi_r\mathcal{D}(f) < \infty$ when $r > \rho$, so that $f \in Dom(\Psi_r\mathcal{D})$. Noticing that for any points $p \neq q \in \bar{V}_1$, we can find a linear function f such that $f(p) \neq f(q)$, we have Id is injective. Finally, due to the fact that $(\bar{V}_1,d)$ is a compact Hausdorff space and (X,R) is the completion of (V_1,R), we then have that Id is an homeomorphism from $(\bar{V}_1,d)$ to (X,R). This implies that $\Big(\Psi_r\mathcal{D}, Dom(\Psi_r\mathcal{D})\Big)$ is a resistance form on $\bar{V}_1$, and (3.2) follows immediately from (3.3). $\square$

Remark. The restriction $\rho < r < 1$ is sharp. If $r \leq \rho$, there is no $f \in Dom(\Psi_r\mathcal{D})$ such that $f(q_1) = 0$ and $f(q_2) = 1$. In fact, for any $f \in C(\bar{V}_1)$ with $f(q_1) = 0$ and $f(q_2) = 1$, the total energy of f on the union of the cells $F_wV_0, w \in W_{1,n} \cup W_{1,n+1}$ (noticing that this union will induce a connected subgraph in $(V_1,\sim)$) will be bounded away from 0 as $n \to \infty$.

4. A renormalization map

In Proposition 3.3, we have shown that $\Psi_r\mathcal{D}$ extends to be a resistance form on $\bar{V}_1$ when $\rho<r<1$. It is natural to trace it back to V_0, noticing that $V_0\subset\bar{V}_1$.

Definition 4.1. *Let $(\mathcal{D}_1,\mathcal{F}_1)$ be a resistance form on $\bar{V}_1$, we write*

$$[\mathcal{D}_1]_{V_0}(u)=\inf\{\mathcal{D}_1(f):f|_{V_0}=u,f\in\mathcal{F}_1\},\quad \forall u\in l(V_0).$$

Note that $[\mathcal{D}_1]_{V_0}$ is a resistance form on V_0 by the polarization identity. For $\rho<r<1$ and $\mathcal{D}\in\mathcal{M}$, we define $\mathcal{R}_r\mathcal{D}=[\Psi_r\mathcal{D}]_{V_0}$, and call $\mathcal{R}_r$ the renormalization map. *Sometimes, we also write $\mathcal{R}(r,\mathcal{D}):=\mathcal{R}_r(\mathcal{D})$.*

The main purpose of this section is to show the continuity of the map $\mathcal{R}(r,\mathcal{D})$.

Theorem 4.2. *The map $\mathcal{R}(r,\mathcal{D})$ is jointly continuous from $(\rho,1)\times\mathcal{M}$ to $\mathcal{M}$.*

To prove Theorem 4.2, we need a study on the regularity of the resistance form $\Psi_r\mathcal{D}$.

Proposition 4.3. *Let $\mathcal{D}\in\mathcal{M}$ and $\rho<r_1<r_2<1$. Then*

(a). $Dom(\Psi_{r_1}\mathcal{D})$ depends only on r_1, and we have $Dom(\Psi_{r_1}\mathcal{D})\subset Dom(\Psi_{r_2}\mathcal{D})$.

(b). $Dom(\Psi_{r_1}\mathcal{D})$ is dense in $Dom(\Psi_{r_2}\mathcal{D})$ in the sense that for any $f\in Dom(\Psi_{r_2}\mathcal{D})$ and $\varepsilon>0$, there exists $g\in Dom(\Psi_{r_1}\mathcal{D})$ such that

$$\Psi_{r_2}\mathcal{D}(f-g)<\varepsilon,\ \text{ and } f|_{V_0}=g|_{V_0}.$$

Moreover, $Dom(\Psi_{r_1}\mathcal{D})$ is dense in $C(V_1)$ so that the resistance form is regular.

Proof. (a) is obvious, we only need to prove (b). Let $f\in Dom(\Psi_{r_2}\mathcal{D})$, and choose n large enough so that

$$\sum_{l=n}^{\infty}\sum_{w\in W_{1,l}} r_2^{-l+1}\mathcal{D}(f\circ F_w)<\varepsilon. \tag{4.1}$$

For convenience, we rename the vertices $\{F_wq_0\}_{w\in W_{1,n}}$ to be $\{p_i\}_{i=1}^N$ with $N=\#W_{1,n}$, so that for each i, p_i is on the left of p_{i+1}. Then, it is not hard to see

$$\begin{aligned}&r_2^{-n}\Big(\sum_{i=1}^{N-1}\big(f(p_i)-f(p_{i+1})\big)^2+\big(f(q_1)-f(p_1)\big)^2+\big(f(q_2)-f(p_N)\big)^2\Big)\\ \le &c_1\Big(\sum_{l=n}^{\infty}\sum_{w\in W_{1,l}} r_2^{-l+1}\mathcal{D}(f\circ F_w)\Big)<c_1\varepsilon,\end{aligned}$$

where c_1 is a constant depending on $\mathcal{D}$ and r_2, but not on n.

Write x_i for the x-coordinate of p_i, so we have $0<x_1<x_2<\cdots<x_N<1$. We introduce a piecewise linear function u on $\mathbb{R}^2$ such that

1. $u(x,y)$ depends only on x;
2. $u(q_1)=f(q_1)$, $u(q_2)=f(q_2)$, and $u(p_i)=f(p_i)$, $1\le i\le N$;
3. $u(x,0)$ is linear on each interval $(0,x_1)$, $(x_N,1)$ and (x_i,x_{i+1}), $1\le i\le N-1$.

We define $g\in l(V_1)$ as

$$g(p)=\begin{cases}f(p), & \text{if } p\in\bigcup_{l=1}^{n-1}\bigcup_{w\in W_{1,l}}\{F_wq_0\},\\ u(p), & \text{if } p\in\bigcup_{l=n}^{\infty}\bigcup_{w\in W_{1,l}}\{F_wq_0\}.\end{cases}$$

By a similar estimate as in Proposition 3.3, one can check that $g \in Dom(\Psi_{r_1}\mathcal{D})$, and

$$\sum_{l=n}^{\infty} \sum_{w\in W_{1,l}} r_2^{-l+1}\mathcal{D}(g\circ F_w) \leq c_2 r_2^{-n}\Big(\sum_{i=1}^{N-1} \big(f(p_i)-f(p_{i+1})\big)^2+\big(f(q_1)-f(p_1)\big)^2+\big(f(q_2)-f(p_N)\big)^2\Big),$$

where c_2 depends only on $\mathcal{D}$ and r_2. So we have $\Psi_{r_2}\mathcal{D}(f-g) \leq c_3\varepsilon$ for some constant c_3. Since ε can be arbitrarily small, we have that $Dom(\Psi_{r_1}\mathcal{D})$ is dense in $Dom(\Psi_{r_2}\mathcal{D})$. Finally, the claim that $Dom(\Psi_{r_1}\mathcal{D})$ is dense in $C(\bar{V}_1)$ follows from a same argument. $\square$

Proof of Theorem 4.2. Let $r_n \to r \in (\rho, 1)$ and $\mathcal{D}_n \to \mathcal{D} \in \mathcal{M}$. Also, let $u \in l(V_0)$.

First, we show that

$$\limsup_{n\to\infty} \mathcal{R}(r_n, \mathcal{D}_n)(u) \leq \mathcal{R}(r, \mathcal{D})(u). \tag{4.2}$$

We define f to be the unique function in $Dom(\Psi_r\mathcal{D})$ such that $f|_{V_0} = u$ and

$$\mathcal{R}(r, \mathcal{D})(u) = \Psi_r\mathcal{D}(f).$$

By Proposition 4.3, for any $\varepsilon > 0$, there is f_ε such that $f_\varepsilon|_{V_0} = u$, $f_\varepsilon \in Dom(\Psi_{r_n}\mathcal{D}_n)$ for any $n \geq 1$, and

$$\Psi_r\mathcal{D}(f_\varepsilon) \leq \Psi_r\mathcal{D}(f) + \varepsilon.$$

As a consequence, we have

$$\limsup_{n\to\infty} \mathcal{R}(r_n, \mathcal{D}_n)(u) \leq \lim_{n\to\infty} \Psi_{r_n}\mathcal{D}_n(f_\varepsilon) = \Psi_r\mathcal{D}(f_\varepsilon) \leq \mathcal{R}(r, \mathcal{D})(u) + \varepsilon,$$

where the equality is due to the dominated convergence theorem. Since ε can be arbitrarily chosen, we get (4.2).

Next, for each n, let f_n be the unique function in $Dom(\Psi_{r_n}\mathcal{D}_n)$ such that $f_n|_{V_0} = u$ and

$$\mathcal{R}(r_n, \mathcal{D}_n)(u) = \Psi_{r_n}\mathcal{D}_n(f_n).$$

Then $\{f_n\}_{n\geq 1}$ is uniformly bounded by the Markov property (RF5). In addition, $\Psi_{r_n}\mathcal{D}_n(f_n) \leq \mathcal{R}(r_*, \mathcal{D}_n)(u)$ with $r_* = \inf_{n\geq 1} r_n$, so $\{\Psi_{r_n}\mathcal{D}_n(f_n)\}_{n\geq 1}$ is a bounded sequence. By the estimates (3.2) and (3.4), we have

$$|f_n(p) - f_n(q)| \leq c\Big(d(p,q)^{\frac{\log r^*}{\log \rho}} \sup_{n\geq 1} \Psi_{r_n}\mathcal{D}_n(f_n)\Big)^{1/2}, \quad \forall n \geq 1, \forall p, q \in \bar{V}_1,$$

where $r^* = \sup_{n\geq 1} r_n$ and $c^2 = \sup_{n\geq 1}\{\frac{4}{r_n^2(1-r_n)} R_0^*(\mathcal{D}_n)\}$, and so $\{f_n\}_{n\geq 1}$ is also equicontinuous. Thus, there is a subsequence $\{f_{n_k}\}_{k\geq 1}$ such that f_{n_k} converges uniformly to a function $f \in C(\bar{V}_1)$. Clearly, f is an extension of u. By Fatou's lemma,

$$\mathcal{R}(r, \mathcal{D})(u) \leq \Psi_r\mathcal{D}(f) \leq \liminf_{k\to\infty} \Psi_{r_{n_k}}\mathcal{D}_{n_k}(f_{n_k}) = \liminf_{k\to\infty} \mathcal{R}(r_{n_k}, \mathcal{D}_{n_k})(u).$$

Combining this with (4.2), we see that

$$\mathcal{R}(r, \mathcal{D})(u) = \lim_{k\to\infty} \mathcal{R}(r_{n_k}, \mathcal{D}_{n_k})(u).$$

Since the argument works for any sequence $(r'_n, \mathcal{D}'_n) \to (r, \mathcal{D})$, we actually have

$$\mathcal{R}(r, \mathcal{D})(u) = \lim_{n\to\infty} \mathcal{R}(r_n, \mathcal{D}_n)(u).$$

The theorem follows immediately since u can be any function in $l(V_0)$. $\square$

5. A fixed point problem

In this section, analogous to the case of p.c.f. self-similar sets (see[21, 33]), we consider the renormalization equation

$$\mathcal{R}_r\mathcal{D} = \lambda\mathcal{D}, \tag{5.1}$$

with $\lambda > 0$. We will prove that for each given $\rho < r < 1$, there always exists a positive λ such that (5.1) has a solution $\mathcal{D}$ in $\mathcal{M}$. Nevertheless, this is not enough for the construction of a satisfying resistance form on $\mathcal{G}$ for later purpose. In order that cells of same size will be assigned with same renormalization factors, we will in addition require $\lambda = r^2$, i.e.

$$\mathcal{R}_r\mathcal{D} = r^2\mathcal{D}. \tag{5.2}$$

The existence and uniqueness of such solution is the main purpose of this section.

It is natural to look at the symmetric resistance forms on V_0, which means that $a_{0,1} = a_{0,2}$ in (3.1). We denote $\mathcal{M}_S$ for the set of all such resistance forms.

Theorem 5.1. *(a). For each $\rho < r < 1$, there exists a unique pair of $\lambda(r)$ and $\mathcal{D}(r) \in \mathcal{M}$ (up to constants) satisfying (5.1), where $\lambda(r)$ is decreasing and continuous in r, and $\mathcal{D}(r)$ is in $\mathcal{M}_S$.*
(b). There exists a unique $\rho < r < 1$ such that (5.2) has a unique (up to constants) solution $\mathcal{D} \in \mathcal{M}$.

We will first prove that for each r, there exist a unique $\lambda(r)$ such that (5.1) has a solution $\mathcal{D}(r)$ in $\mathcal{M}_S$, then prove that $\mathcal{D}(r)$ is indeed the unique solution (up to constants) in $\mathcal{M}$. The existence and uniqueness of a solution to (5.2) will follow from the properties of $\lambda(r)$. We divide these into two subsections.

5.1. The existence of a symmetric solution.

We begin with some simple observations.

Lemma 5.2. *Let $\rho < r < 1$ be fixed, and suppose that there is a solution to (5.1). Then the constant λ depends only on r.*

Proof. This follows from a standard argument like the finite graph case [28]. Suppose that $\mathcal{D}, \mathcal{D}'$ are two solutions to (5.1) with λ, λ' being the corresponding constant. Let $u \in l(V_0)$ so that $\frac{\mathcal{D}'(u)}{\mathcal{D}(u)} = \sup_{v\neq constants} \frac{\mathcal{D}'(v)}{\mathcal{D}(v)} := \theta$, and let f be the harmonic extension of u with respect to $\Psi_r\mathcal{D}$. Then

$$\lambda'\mathcal{D}'(u) = \mathcal{R}_r\mathcal{D}'(u) \leq \Psi_r\mathcal{D}'(f) \leq \theta\Psi_r\mathcal{D}(f) = \theta\mathcal{R}_r\mathcal{D}(u) = \theta\lambda\mathcal{D}(u).$$

This implies that $\lambda' \leq \lambda$. A same argument also shows that $\lambda \leq \lambda'$. $\square$

Inspired by Lemma 5.2, we can view the constant λ in (5.1) as a function of r. On the other hand, the problem of solvability of (5.1) can be transferred to a fixed point problem.

Definition 5.3. *(a). Define*

$$\tilde{\mathcal{M}}_S = \big\{\mathcal{D} \in \mathcal{M} : \mathcal{D}(f) = a\big(f(q_0)-f(q_1)\big)^2+a\big(f(q_0)-f(q_2)\big)^2+(1-a)\big(f(q_1)-f(q_2)\big)^2, 0 < a \leq 1\big\},$$

and for $0 < s \leq 1$,

$$\tilde{\mathcal{M}}_S^{[s,1]} = \big\{\mathcal{D} \in \mathcal{M} : \mathcal{D}(f) = a\big(f(q_0)-f(q_1)\big)^2+a\big(f(q_0)-f(q_2)\big)^2+(1-a)\big(f(q_1)-f(q_2)\big)^2, s \leq a \leq 1\big\}.$$

(b). For each $\mathcal{D} \in \mathcal{M}_S$, there is a unique constant c such that $c\mathcal{D} \in \tilde{\mathcal{M}}_S$, and we denote the resulted form $T\mathcal{D}$. We define $\tilde{\mathcal{R}}_r : \mathcal{M}_S \to \tilde{\mathcal{M}}_S$ as $\tilde{\mathcal{R}}_r = T \circ \mathcal{R}_r$. As before, we write $\tilde{R}(r, \mathcal{D}) = \tilde{R}_r(\mathcal{D})$.

The following lemma will play an essential role.

Lemma 5.4. *For $\rho < r_0 < r_1 < 1$, there exists $0 < s \leq 1$ such that $\tilde{\mathcal{R}} : [r_0, r_1] \times \mathcal{M}_S \to \tilde{\mathcal{M}}_S^{[s,1]}$.*

Proof. Let $\mathcal{D} \in \mathcal{M}_S$, $r_0 \leq r \leq r_1$ and R be the resistance metric on V_1 associated to $\Psi_r\mathcal{D}$. For convenience, we write $\mathcal{D}(f) = a\big(f(q_0) - f(q_1)\big)^2 + a\big(f(q_0) - f(q_2)\big)^2 + b\big(f(q_1) - f(q_2)\big)^2$, with $a > 0, b \geq 0$.

First, an immediate observation shows that for any $f \in l(V_1)$,

$$\Psi_r\mathcal{D}(f) \geq \sum_{n=0}^{\infty} ar^{-n}\big(f(F_1^n q_0) - f(F_1^{n+1} q_0)\big)^2 \geq a(1-r)\big(f(q_0) - f(q_1)\big)^2,$$

so we have $R(q_0, q_1) \leq \frac{1}{a(1-r)} \leq \frac{1}{a(1-r_1)}$.

Next, let f be the linear function on $\mathbb{R}^2$ such that $f(q_1) = 0, f(q_2) = 1$ and $f(q_0) = \frac{1}{2}$, so f only depends on the x-coordinate. We introduce an equivalence relation '$\sim_h$' on V_1,

$$p \sim_h q \text{ if there exists } w \in W_1 \text{ so that } p, q \in \{F_w q_1, F_w q_2\}.$$

Then we modify f on V_1 into a function $g \in l(V_1)$ as

$$g(p) = \frac{\sum_{q \sim_h p} f(q)}{\sum_{q \sim_h p} 1}, \quad \forall p \in V_1.$$

By doing this we have
1. $g(p) = g(q)$ if $p \sim_h q$;
2. $|g(p) - g(q)| \leq c_1\rho^n$ if $p, q \in F_w V_0$ with $w \in W_{1,n}$.

Thus we have

$$\begin{aligned}
\Psi_r\mathcal{D}(g) &= \sum_{l=1}^{\infty} \sum_{w \in W_{1,l}} r^{-l+1}\mathcal{D}(g \circ F_w) \\
&\leq 2c_1^2 a \sum_{l=1}^{\infty} r^{-l+1}\rho^{2l} \# W_{1,l} \\
&\leq c_2 a \sum_{l=1}^{\infty} r^{-l}\rho^l = \frac{c_2\rho}{r-\rho} a \leq \frac{c_2\rho}{r_0 - \rho} a,
\end{aligned}$$

where we use the estimate $\# W_{1,l} \asymp \rho^{-l}$. Thus, g extends to $g \in C(\bar{V}_1)$ by Proposition 3.3, and it is direct to check that $g|_{V_0} = f|_{V_0}$. As a consequence, we get $R(q_1, q_2) \geq \frac{r_0 - \rho}{c_2\rho} a^{-1}$.

Due to the above two estimates, there exists $c_3 > 0$ independent of $\mathcal{D}$ such that

$$\frac{R(q_0, q_1)}{R(q_1, q_2)} \leq c_3.$$

The lemma follows immediately. □

By using Lemma 5.2, Lemma 5.4 and Theorem 4.2, we can easily prove the following proposition.

Proposition 5.5. *Let $\rho < r < 1$, there always exists a solution to (5.1) in $\mathcal{M}_S$, with λ uniquely determined by r. In addition, regarding λ as a function of r, $\lambda(r)$ is decreasing and continuous in r.*

Proof. First, we have $\tilde{\mathcal{R}}_r : \tilde{\mathcal{M}}_S^{[s,1]} \to \tilde{\mathcal{M}}_S^{[s,1]}$ for some $s > 0$ by Lemma 5.4. The existence of a fixed point of $\tilde{\mathcal{R}}_r$ is then an immediate consequence.

Next, let $r_1 < r_2$, and assume that $\mathcal{R}_{r_1}\mathcal{D}_1 = \lambda(r_1)\mathcal{D}_1$ and $\mathcal{R}_{r_2}\mathcal{D}_2 = \lambda(r_2)\mathcal{D}_2$. Let $u \in l(V_0)$ so that $\frac{\mathcal{D}_2(u)}{\mathcal{D}_1(u)} = \sup_{v \neq constants} \frac{\mathcal{D}_2(v)}{\mathcal{D}_1(v)} := \theta$, and let f be the harmonic extension of u with respect to $\Psi_{r_1}\mathcal{D}_1$, then we have $\lambda(r_2)\mathcal{D}_2(u) \leq \Psi_{r_2}\mathcal{D}_2(f) \leq \theta\Psi_{r_1}\mathcal{D}_1(f) = \theta\lambda(r_1)\mathcal{D}_1(u)$. So we get $\lambda(r_2)\leq\lambda(r_1)$.

Finally, let $r_n \to r$, and let $\mathcal{D}_n \in \tilde{\mathcal{M}}_S$ be a sequence of solutions to $\mathcal{R}_{r_n}\mathcal{D}_n = \lambda(r_n)\mathcal{D}_n$. Clearly, we have $\rho < \inf_{n\geq 1} r_n < \sup_{n\geq 1} r_n < 1$, so $\{\mathcal{D}_n\}_{n\geq 1} \subset \tilde{\mathcal{M}}_S^{[s,1]}$ for some $s > 0$ by Lemma 5.4. Thus, there exists a subsequence $\{n_k\}_{k\geq 1}$ such that $\mathcal{D}_{n_k}$ converges to some $\mathcal{D} \in \tilde{\mathcal{M}}_S$ and $\lambda(r_{n_k})$ converges. By Theorem 4.2, we conclude that $\mathcal{R}_r\mathcal{D} = \big(\lim_{k\to\infty} \lambda(r_{n_k})\big)\mathcal{D}$. So $\lambda(r) = \lim_{k\to\infty} \lambda(r_{n_k})$. Since the argument works for any sequence $r_n \to r$, $\lambda(r)$ is continuous in r. □

We have an easy estimate of $\lambda(r)$.

Lemma 5.6. *For $\rho < r < 1$, we have $\big(\frac{1}{1-r} - \frac{r}{2+2r+2r^2}\big)^{-1} \leq \lambda(r) \leq \frac{2}{2+r}$.*

Proof. We consider a function $u \in l(V_0)$ with $u(q_0) = 0$ and $u(q_1) = u(q_2) = 1$. Without loss of generality, we assume the solution $\mathcal{D}$ has $\mathcal{D}(u) = 2$.

To get the upper bound of $\lambda(r)$, we construct an extension $f \in l(V_1)$ of u as

$$f(p) = \begin{cases} 0, & \text{if } p = q_0, \\ \frac{2}{2+r}, & \text{if } p \in \{F_1q_0, F_2q_0\}, \\ 1, & \text{if } p \in F_1V_1 \cup F_2V_1. \end{cases}$$

Then the upper bound follows easily from the following estimate,

$$\mathcal{R}_\lambda\mathcal{D}(u) \leq \Psi_r\mathcal{D}(f) = \frac{4}{2+r} = \frac{2}{2+r}\mathcal{D}(u).$$

To get the lower bound, we look at a subgraph in $(V_1, \sim)$, whose vertices are $\{F_i^l q_0\}_{i,l}$ with $i \in \{1, 2\}$ and $l \geq 0$, together with

$$p_{i,0} = F_iq_0, \quad p_{i,1} = F_iF_jq_0, \quad p_{i,2} = F_iF_jF_iq_0,$$
$$p_{i,3} = F_iF_jF_i^2q_0, \quad p_{i,4} = F_i^2F_jq_0, \quad p_{i,5} = F_i^2q_0,$$

with $i, j \in \{1, 2\}$ and $j \neq i$, and edges inherit from $(V_1, \sim)$. Let $f \in l(V_1)$ be the harmonic extension of u, denote $c_l = r^{-l-1}$ for $l \in \{0, 1, 2\}$, and $c_l = r^{l-6}$ for $l \in \{3, 4\}$, then the lower bound follows from the estimate that

$$\begin{aligned}\mathcal{R}_r\mathcal{D}(u) = \Psi_r\mathcal{D}(f) &\geq \sum_{i=1,2} \Big(\sum_{l=0}^{\infty} r^{-l}\big(f(F_i^l q_0) - f(F_i^{l+1}q_0)\big)^2 + \sum_{l=0}^{4} c_l\big(f(p_{i,l}) - f(p_{i,l+1})\big)^2 \Big) \\ &\geq 2\big(\frac{1}{1-r} - \frac{r}{2+2r+2r^2}\big)^{-1} = \big(\frac{1}{1-r} - \frac{r}{2+2r+2r^2}\big)^{-1}\mathcal{D}(u),\end{aligned}$$

where the last inequality can be done by an easy computation of the effective resistances on the subgraph. □

Using Proposition 5.5 and Lemma 5.6, we arrive the main result of this subsection, concerning the solvability of (5.2).

Theorem 5.7. *There exists a unique $\rho < r < 1$ such that (5.2) has a solution $\mathcal{D} \in \mathcal{M}_S$.*

Proof. By Proposition 5.5, we see that there is a continuous function $\lambda(r)$ so that $\mathcal{R}_\lambda(\mathcal{D}) = \lambda(r)\mathcal{D}$ has a solution. Noticing that

$$(\frac{1}{1-\rho} - \frac{\rho}{2+2\rho+2\rho^2})^{-1} > 1-\rho = \rho^2, \text{ and } \frac{2}{3} < 1,$$

there exists $\rho < r < 1$ such that $\lambda(r) = r^2$ by Lemma 5.6. The uniqueness follows from the fact that $\lambda(r)$ is decreasing in r, while r^2 is strictly increasing. □

Remark. We can see the uniqueness of r from another point of view. Let $\theta = \frac{\log r}{\log \rho}$, we will see in Section 7 that $\theta + d_H$ is the *walk dimension* of the resulting diffusion process on the metric measure space $(\mathcal{G}, d, \mu_H)$, whose uniqueness is shown in [16] under some weak conditions.

5.2. **The uniqueness.** In this subsection, we consider the uniqueness of the solution to (5.1) or (5.2). The proof is inspired by Sabot's work [33].

Theorem 5.8. *Let $\rho < r < 1$ and $\mathcal{D} \in \mathcal{M}_S$ be a symmetric solution to (5.1). Then $\mathcal{D}$ is the unique solution in $\mathcal{M}$ to (5.1).*

For fixed $\rho < r < 1$ and $\mathcal{D} \in \mathcal{M}_S$ satisfying (5.1), for convenience, we always write

1. h_s for the harmonic function with $h_s(q_0) = 0$, $h_s(q_1) = h_s(q_2) = 1$, and denote $E_s = \{f \in l(V_0) : f(q_0) = 0, f(q_1) = f(q_2) = c, c \in \mathbb{R}\}$;
2. h_a for the harmonic function with $h_a(q_0) = 0$, $h_a(q_1) = -h_a(q_2) = 1$, and denote $E_a = \{f \in l(V_0) : f(q_0) = 0, f(q_1) = -f(q_2) = c, c \in \mathbb{R}\}$.

Both h_s, h_a are harmonic with respect to $\Psi_r\mathcal{D}$ on $\bar{V}_1 \setminus V_0$, i.e. $\Psi_r\mathcal{D}(h_s, f) = \Psi_r\mathcal{D}(h_a, f) = 0$ for any $f \in Dom(\Psi_r\mathcal{D})$ such that $f|_{V_0} = 0$.

Lemma 5.9. *For r, $\mathcal{D}$ as above, we have*

$$h_s(F_1q_0) = h_s(F_2q_0) = \lambda(r), \quad h_a(F_1q_0) = -h_a(F_2q_0) = \eta,$$

for some $|\eta| < \lambda(r)$.

Proof. For convenience, we write $\mathcal{D}$ in the form $\mathcal{D}(f) = a\big(f(q_0)-f(q_1)\big)^2 + a\big(f(q_0)-f(q_2)\big)^2 + b\big(f(q_1)-f(q_2)\big)^2$, with $a > 0, b \geq 0$.

First, let $h = 1 - h_s$, we have $\mathcal{R}_r\mathcal{D}(h_s, h) = -2a\lambda(r)$. On the other hand, let $f \in l(V_1)$ be defined as $f(p) = \delta_{q_0,p}$, then clearly $f \in Dom(\Psi_r\mathcal{D})$, and $f|_{V_0} = h|_{V_0}$. Since h_s is harmonic,

$$\Psi_r\mathcal{D}(h_s, h) = \Psi_r\mathcal{D}(h_s, f) = -ah_s(F_1q_0) - ah_s(F_2q_0).$$

This shows the first assertion since $\mathcal{R}_r\mathcal{D}(h_s, h) = \Psi_r\mathcal{D}(h_s, h)$.

Next, by the symmetry of $\mathcal{D}$, there exists a number η such that $h_a(F_1q_0) = -h_a(F_2q_0) = \eta$. We need to show that $|\eta| < \lambda(r)$. We consider the matrix M such that

$$\big(h(F_1q_0), h(F_2q_0)\big)^t = M\big(h(q_1), h(q_2)\big)^t,$$

holds for any harmonic function h with $h(q_0) = 0$. Due to the Perron-Frobenius theorem, it suffices to show that each entry of M is positive. This can be deduced by proving the harmonic function h_1 with boundary value $h_1(q_1) = 1, h_1(q_0) = h_1(q_2) = 0$ is positive on $V_1 \setminus V_0$. To see this, we assume there exists $p \in V_1 \setminus V_0$ such that $h(p) = 0$. Let $\psi_p \in Dom(\Psi_r\mathcal{D})$ be defined as $\psi_p(q) = \delta_{p,q}$, then $\Psi_r\mathcal{D}(\psi_p, h_1) = 0$, so $h_1(p)$ is the weighted average of its neighbours. Thus h_1 is zero on the neighbours of p. Repeating the argument, we see that $h|_{V_1} = 0$. A contradiction. □

Proof of Theorem 5.8. Assume there is another solution $\mathcal{D}' \in \mathcal{M}$ to (5.1).

Firstly, we will show that $\mathcal{D}'$ is also symmetric. By diagonalizing $\mathcal{D}'$ with respect to $\mathcal{D}$, we have two 1-dimensional subspaces L_1, L_2 of $l(V_0)$ such that
1. L_1, L_2 are orthogonal with respect to both $\mathcal{D}$ and $\mathcal{D}'$;
2. $\mathcal{D}'|_{L_1} = \kappa_1\mathcal{D}|_{L_1}$ and $\mathcal{D}'|_{L_2} = \kappa_2\mathcal{D}|_{L_2}$, with $0 < \kappa_1 < \kappa_2$.

Let $u \in L_2$ and h_u be the harmonic extension of u with respect to $\Psi_r\mathcal{D}$. Then

$$\begin{aligned}\lambda(r)\mathcal{D}'(u) = \kappa_2\lambda(r)\mathcal{D}(u) = \kappa_2\Psi_r\mathcal{D}(h_u) &= \sum_{w\in W_1} r^{-|w|+1}\kappa_2\mathcal{D}(h_u\circ F_w)\\ &\geq \sum_{w\in W_1} r^{-|w|+1}\mathcal{D}'(h_u\circ F_w) = \Psi_r\mathcal{D}'(h_u) \geq \lambda(r)\mathcal{D}'(u).\end{aligned}$$

Clearly, this implies that $h_u \circ F_w \in L_2 + \mathit{constants}$ for each $w \in W_1$. In particular, we have $h_u \circ F_0 \in L_2 + \mathit{constants}$, which means $L_2 + \mathit{constants}$ is an invariant space under the mapping u to $h_u \circ F_0$. By Lemma 5.9, we see that $L_2 + \mathit{constants}$ is either $E_s + \mathit{constants}$ or $E_a + \mathit{constants}$. Thus, we have $\mathcal{D}' \in \mathcal{M}_S$.

Secondly, from the above argument, it is not hard to see that $h_s \circ F_w \in E_s + \mathit{constants}$ and $h_a \circ F_w \in E_a + \mathit{constants}$, for any $w \in W_1$.

Lastly, arbitrarily pick a $\tilde{\mathcal{D}} \in \mathcal{M}_S$, we will prove that $\tilde{\mathcal{D}}$ must also solve (5.1), which obviously contradicts Lemma 5.4. To achieve this purpose, let $\tilde{h}_s$ and $\tilde{h}_a$ be the harmonic functions with respect to $\Psi_r\tilde{\mathcal{D}}$, with the same boundary value on V_0 as h_s, h_a. By following a same argument as Lemma 5.9 in Sabot's paper [33], we can see that $\tilde{h}_s = h_s$ and $\tilde{h}_a = h_a$. For convenience of readers, we reproduce the proof here. Write $g = h_s - \tilde{h}_s$. Also, for each $w \in W_1$, let $g_{w,s} \in E_s + \mathit{constants}, g_{w,a} \in E_a + \mathit{constants}$ such that $g_w =: g \circ F_w = g_{w,s} + g_{w,a}$. Then, we can see that

$$\begin{aligned}\Psi_r\tilde{\mathcal{D}}(g) = \Psi_r\tilde{\mathcal{D}}(h_s, g) &= \sum_{w\in W_1} r^{-|w|+1}\tilde{\mathcal{D}}(h_{s,w}, g_w) = \sum_{w\in W_1} r^{-|w|+1}\tilde{\mathcal{D}}(h_{s,w}, g_{w,s})\\ &= c\sum_{w\in W_1} r^{-|w|+1}\mathcal{D}(h_{s,w}, g_{w,s}) = c\sum_{w\in W_1} r^{-|w|+1}\mathcal{D}(h_{s,w}, g_w) = c\Psi_r\mathcal{D}(h_s, g) = 0,\end{aligned}$$

for some constant c, with $h_{s,w} =: h_s \circ F_w$. Thus $g = 0$ as desired. As a consequence, we can easily see that, $\tilde{\mathcal{D}}$ is a solution to (5.1), so we arrive the desired contradiction. □

Finally, Theorem 5.1 immediately follows from Proposition 5.5, Theorem 5.7 and 5.8.

6. Construction of the Dirichlet form on $\mathcal{G}$

We will construct a resistance form on the golden ratio Sierpinski gasket $\mathcal{G}$ in this section. Let $\rho < r < 1$, $\mathcal{D}$ be the unique solution to (5.2), i.e. $\mathcal{R}_r\mathcal{D} = r^2\mathcal{D}$. We will focus on this standard form in most contents. For short, we write

$$\theta = \frac{\log r}{\log \rho}, \quad \rho_w = \prod_{n=1}^{|w|} \rho_{w_n}, \quad r_w = \rho_w^\theta,$$

with $\rho_0 = \rho^2$ and $\rho_1 = \rho_2 = \rho$. Obviously, ρ_w is the contraction ratio of F_w.

The following definition is similar to the construction in [21], though we use the infinite graphs at each level.

Definition 6.1. *(a). For $m \geq 0$ and $f \in C(\bar{V}_m)$, we write $\mathcal{D}^{(m)}(f) = \sum_{w \in W_m} r_w^{-1} \mathcal{D}(f \circ F_w)$, and $\mathcal{F}^{(m)} = \{f \in C(\bar{V}_m) : \mathcal{D}^{(m)}(f) < \infty\}$. In addition, for $f, g \in \mathcal{F}^{(m)}$, we define*

$$\mathcal{D}^{(m)}(f, g) = \sum_{w \in W_m} r_w^{-1} \mathcal{D}(f \circ F_w, g \circ F_w).$$

(b). Define $\mathcal{F} = \{f \in C(\mathcal{G}) : \lim_{m \to \infty} \mathcal{D}^{(m)}(f) < \infty\}$. For $f, g \in \mathcal{F}$, define

$$\mathcal{E}(f, g) = \lim_{m \to \infty} \mathcal{D}^{(m)}(f, g).$$

The limit in (b) always exists due to fact that

$$\mathcal{D}^{(m+1)}(f) = \sum_{w \in W_{m+1}} r_w^{-1} \mathcal{D}(f \circ F_w) = \sum_{w \in W_m} r_w^{-1} r^{-2} \Psi_r \mathcal{D}(f \circ F_w) \geq \sum_{w \in W_m} r_w^{-1} \mathcal{D}(f \circ F_w) = \mathcal{D}^{(m)}(f).$$

In the rest of this section, we will show that $(\mathcal{E}, \mathcal{F})$ is a good form.

Lemma 6.2. *For $m \geq 0$, $(\mathcal{D}^{(m)}, \mathcal{F}^{(m)})$ is a resistance form on $\bar{V}_m$. In addition, let*

$$R_m(p, q) = \sup_{f \in \mathcal{F}^{(m)}} \frac{|f(p) - f(q)|^2}{\mathcal{D}^{(m)}(f)},$$

then we have $R_n(p, q) = R_m(p, q)$ if $p, q \in \bar{V}_m$ and $n \geq m$.

Proof. (RF1) and (RF5) are trivial. We only need to verify (RF2)-(RF4). For convenience, we focus on $(\mathcal{D}^{(2)}, \mathcal{F}^{(2)})$ only, while for larger m, a same proof works inductively.

(RF2). Let $\{f_k\}_{k \geq 1}$ be a Cauchy sequence in $\mathcal{F}^{(2)}$. Then, $f_k|_{\bar{V}_1}$ converges in $\mathcal{F}^{(1)}$ to some $\tilde{f}$ in $\mathcal{F}^{(1)}$, since $(\mathcal{D}^{(1)}, \mathcal{F}^{(1)})$ is a resistance form. Also, for each $w \in W_1$, $f_k \circ F_w$ converges in $\mathcal{F}^{(1)}$ to a function $\tilde{f}_w$. Now, define $f \in l(\bar{V}_2)$ such that $f \circ F_w = \tilde{f}_w$ and $f|_{\bar{V}_1 \setminus V_1} = \tilde{f}$.

We show that $f \in C(\bar{V}_2)$. It suffices to prove that f is continuous at any point $p \in \bar{V}_1 \setminus V_1$. In fact, for any ε, there exists δ and N such that

1. for $q \in B_\delta(p) \cap \bar{V}_1$, we have $|f(p) - f(q)| < \varepsilon$;
2. for $w \in \bigcup_{n=N}^\infty W_{1,n}$ and $q, q' \in F_w \bar{V}_1$, we have $|f(q) - f(q')| < \varepsilon$. This follows from the fact that $\mathcal{D}^{(1)}(f \circ F_w) \leq r_w \sup_{k \geq 1} \mathcal{D}^{(2)}(f_k)$.

The continuity of f follows immediately. Lastly, by using Fatou's lemma, we can directly check that f_k converges to f in $\mathcal{F}^{(2)}$.

(RF3). First, we observe that the minimal energy extension of $f \in \mathcal{F}^{(1)}$ to $l(V_2)$ is continuous by a same argument as in (RF2). So we have enough functions in $\mathcal{F}^{(2)}$.

Let V be a finite set and $u \in l(V)$. First, we always have $f_1 \in \mathcal{F}^{(1)}$ such that $f_1|_{V \cap \bar{V}_1} = u|_{V \cap \bar{V}_1}$. Then we can extend f_1 to be a desired function in $\mathcal{F}^{(2)}$.

(RF4). Let $p,q\in \bar{V}_2$ and $f\in \mathcal{F}^{(2)}$. If $p\in \bar{V}_1$, we let $p'=p$; otherwise we choose $p'\in V_1$ so that $p,p'\in F_w\bar{V}_1$ for some $w\in W_1$, and thus

$$\mathcal{D}^{(2)}(f)\geq r_w^{-1}\mathcal{D}^{(1)}(f\circ F_w)\geq c_1\big(f(p)-f(p')\big)^2,$$

for some $c_1>0$. Also, we define q' in a same manner. It then follows that

$$\mathcal{D}^{(2)}(f)\geq c_2\Big(\big(f(p)-f(p')\big)^2+\big(f(p')-f(q')\big)^2+\big(f(q')-f(q)\big)^2\Big)\geq c_3\big(f(p)-f(q)\big)^2.$$

(RF4) follows immediately.

Thus, we have proved that $(\mathcal{D}^{(2)},\mathcal{F}^{(2)})$ is a resistance form on $\bar{V}_2$. The claim that $R_2(p,q)=R_1(p,q)$ for $p,q\in\bar{V}_1$ is obvious. The same arguments can be used inductively for $m\geq 3$. $\square$

In some situations, it is convenient to involve words in $\tilde{W}_*$.

Lemma 6.3. *Let $w\in\tilde{W}_*$ and m be the number of $0's$ in w. Then we have*

$$\mathcal{D}^{(1)}(f\circ F_w)\leq r_w\mathcal{D}^{(m+1)}(f),$$

for any $f\in\mathcal{F}^{(m+1)}$. As a consequence, there is a constant $c>0$ such that, for any $p,q\in F_w\bar{V}_1$, we have

$$R_{m+1}(p,q)\leq cd(p,q)^\theta.$$

Proof. Noticing that $\{w\tau:\tau\in W_1\}\subset W_{m+1}$, the lemma is obvious by the definition of $\mathcal{D}^{(m+1)}$ and Proposition 3.3. $\square$

Using Lemma 6.2 and 6.3, we have the following estimate of the resistance metric.

Lemma 6.4. *For $m\geq 0$ and $p,q\in\bar{V}_m$, define $\tilde{R}(p,q)=R_m(p,q)$. Then $\tilde{R}(p,q)$ is well defined on $(\bigcup_{m\geq0}\bar{V}_m)\times(\bigcup_{m\geq0}\bar{V}_m)$, and we have $\tilde{R}(p,q)\leq cd(p,q)^\theta$ for some $c>0$.*

Proof. First, we claim that there is a constant $c_1>0$ such that

$$\tilde{R}(p,q)\leq c_1\rho_w^\theta,\quad \forall w\in\tilde{W}_*,\forall p,q\in F_w\mathcal{G}\cap(\bigcup_{m\geq0}\bar{V}_m).$$

We first consider the case $q\in F_w\bar{V}_1$. Assume that $p\in F_w\bar{V}_n$ for some $n\geq1$, then we can find $\tau\in W_{n-1}$ such that $p\in F_wF_\tau\bar{V}_1$. We can then find a sequence

$$q=p_0,p_1,\cdots,p_{|\tau|+1}=p,$$

such that $p_i\in F_wF_{[\tau]_{i-1}}\bar{V}_1\cap F_wF_{[\tau]_i}\bar{V}_1$ for $1\leq i\leq|\tau|$. As a consequence, by using Lemma 6.3, we see that

$$\tilde{R}(p,q)\leq\sum_{i=0}^{|\tau|}c_2d(p_i,p_{i+1})^\theta\leq\sum_{i=0}^{|\tau|}c_2(\rho_w\rho^i)^\theta\leq\frac{c_2}{1-\rho^\theta}\rho_w^\theta,$$

where c_2 is the same constant in Lemma 6.3. For general q, we only need to set $c_1=\frac{2c_2}{1-\rho^\theta}\rho_w^\theta$.

Now, let $p,q\in\bigcup_{m=0}^\infty\bar{V}_m$. We choose $w,w'\in\tilde{W}_*$ such that $p\in F_w\mathcal{G}$, $q\in F_{w'}\mathcal{G}$ and

$$\rho d(p,q)\leq\rho_w,\rho_{w'}<\rho^{-1}d(p,q).$$

In addition, we can find a chain

$$w=w^{(0)},w^{(1)},\cdots,w^{(k)}=w'$$

such that $\min\{\rho_w,\rho_{w'}\}\leq\rho_{w^{(i)}}<\rho^{-2}\min\{\rho_w,\rho_{w'}\}$ of length at most c_3, where c_3 is a constant independent of p,q. By choosing a sequence $p=p_0,p_1,\cdots,p_{k+1}=q$ such that $p_i\in F_{w^{(i-1)}}\bar V_1\cap F_{w^{(i)}}\bar V_1$, $1\leq i\leq k$, we get the desired estimate as above. □

Now, we can show that $(\mathcal{E},\mathcal{F})$ is a good form.

Theorem 6.5. *$(\mathcal{E},\mathcal{F})$ defined in Definition 6.1 is a strongly local regular resistance form on $\mathcal{G}$.*

Proof. First, we claim that $(\mathcal{E},\mathcal{F})$ is a resistance form on $\bigcup_{m\geq 0}\bar V_m$. (RF1) and (RF5) are obvious. Observing that by keeping doing the minimal energy extension, we can extend any $f\in\mathcal{F}^{(m)}$ to $f\in\mathcal{F}$ thanks to the upper bound estimate of the resistance metric in Lemma 6.4. (RF2), (RF3) and (RF4) are then easy to shown with Lemma 6.2. In addition, we see that

$$\tilde R(p,q)=R(p,q):=\sup_{f\in\mathcal{F}}\frac{|f(p)-f(q)|^2}{\mathcal{E}(f)},\quad\forall p,q\in\bigcup_{m\geq 0}\bar V_m.$$

Next, to prove that $(\mathcal{E},\mathcal{F})$ is a resistance form on $\mathcal{G}$, we need to show that $\mathcal{F}$ separates points in $\mathcal{G}$, just like in Proposition 3.3. It suffices to prove that $\mathcal{F}$ is dense in $C(\mathcal{G})$. Let $u\in C(\mathcal{G})$, we fix N large enough so that $|u(x)-u(y)|<\varepsilon$ if $x,y\in F_wK$ and $|w|\geq N$. We can apply Proposition 4.3 to create $f\in\mathcal{F}$ such that $\|f-u\|_{L^\infty(\mathcal{G})}<2\varepsilon$. First we find $f_1\in\mathcal{F}^{(1)}$ such that

1. $\|f_1-u|_{\bar V_1}\|_{L^\infty}<\varepsilon$;
2. $f_1(p)=u(p)$ for any $p\in\bigcup_{n=1}^N\bigcup_{w\in W_{1,n}}F_wV_0$.

Then we apply harmonic extension to f_1 on $\bar V_2\setminus\bigcup_{n=1}^N\bigcup_{w\in W_{1,n}}F_w\bar V_1$. On the cells $F_w\bar V_1$ with $|w|<N$, we apply a same construction to get f_2, but with $N-2$ replacing N this time. After $k=[N/2]+1$ times, we get $f_k\in\mathcal{F}^{(k)}$ such that $\|f_k-u|_{\bar V_k}\|_{L^\infty}<2\varepsilon$. Since all cells have size smaller than ρ^N, by doing harmonic extension, we get $f\in\mathcal{F}$ such that $\|f-u\|_{L^\infty(\mathcal{G})}<2\varepsilon$. Thus, $(\mathcal{E},\mathcal{F})$ is regular resistance form on $\mathcal{G}$.

It remains to show that the form is strongly local. Let $f,g\in\mathcal{F}$ with $supp(f)\cap supp(g)=\emptyset$, then there exists $\varepsilon>0$ such that $d\big(supp(f),supp(g)\big)>\varepsilon$. Thus, we have $\mathcal{D}^{(n)}(f,g)=0$ for large n. By taking the limit, we see that $\mathcal{E}(f,g)=0$. Clearly $1\in\mathcal{F}$ with $\mathcal{E}(1)=0$, and it follows that the form is strongly local. □

In the remaining of this section, we would like to characterize $(\mathcal{E},\mathcal{F})$ as the unique self-similar form associated with the infinite i.f.s. $\{F_w\}_{w\in W_1}$.

Theorem 6.6. *The resistance form $(\mathcal{E},\mathcal{F})$ satisfies the following properties:*

(a). $\mathcal{F}\subset C(\mathcal{G})$.

(b). For each $f\in\mathcal{F}$, we have $f\circ F_w\in\mathcal{F}$ for all $w\in W_1$, and in addition,

$$\mathcal{E}(f)=\sum_{w\in W_1}\rho_w^{-\theta}\mathcal{E}(f\circ F_w).$$

(c). Reversely, let $f\in C(\mathcal{G})$, if $f\circ F_w\in\mathcal{F}$ for all $w\in W_1$, and $\sum_{w\in W_1}\rho_w^{-\theta}\mathcal{E}(f\circ F_w)<\infty$, then $f\in\mathcal{F}$.

Moreover, $(\mathcal{E},\mathcal{F})$ (up to constants) and θ are uniquely determined by the above properties.

Proof. The claimed properties of $(\mathcal{E},\mathcal{F})$ are immediate consequences of the construction.

The uniqueness follows by a well-known argument, but in the infinite graph version. Let $(\mathcal{E}',\mathcal{F}')$ be another form satisfying the above properties with θ' replacing θ. Define $\mathcal{D}'$ to be the trace of $\mathcal{E}'$ onto V_0, and write $r'_w=\rho_w^{\theta'}$, $r'=\rho^{\theta'}$. For any $u\in l(V_0)$, let h_u be the harmonic extension of u to $\mathcal{F}'$, then we can see that

$$\mathcal{D}'(u)=\mathcal{E}'(h_u)=\sum_{w\in W_1}{r'}_w^{-1}\mathcal{E}'(h_u\circ F_w)\geq\sum_{w\in W_1}{r'}_w^{-1}\mathcal{D}'\big((h_u\circ F_w)|_{V_0}\big)\geq {r'}^{-2}\mathcal{R}_{r'}\mathcal{D}'(u),$$

where $\mathcal{R}_{r'}$ is the renormalization map introduced in Definition 4.1, and we use properties (a) and (b) in the inequalities.

On the other hand, we can do the harmonic extension of u in two steps: first we extend u to $f_1\in C(\bar{V}_1)$ so that f_1 minimizes $\Psi_{r'}\mathcal{D}'$, then we take harmonic extension of f_1 on each cell $F_w\mathcal{G},w\in W_1$, to $f\in C(\mathcal{G})$, by using property (a) and the Markov property (RF5). In addition, $f\in\mathcal{F}'$ by the property (c). Then, by property (b),

$${r'}^{-2}\mathcal{R}_{r'}\mathcal{D}'(u)={r'}^{-2}\Psi_{r'}\mathcal{D}'(f_1)=\sum_{w\in W_1}{r'}_w^{-1}\mathcal{E}'(f\circ F_w)=\mathcal{E}'(f)\geq\mathcal{D}'(u).$$

Thus, we get $\mathcal{R}_{r'}\mathcal{D}'={r'}^2\mathcal{D}'$, which implies that $\mathcal{D}'=\mathcal{D}$ and $\theta'=\theta$ by Theorem 5.1. Finally, by a similar argument, one can easily find that the restriction of $\mathcal{E}'$ to $\bar{V}_m$ is $\mathcal{D}^{(m)}$, and the claim that $\mathcal{E}'=\mathcal{E}$ follows immediately by taking the limit. □

Finally, the form $(\mathcal{E},\mathcal{F})$ is decimation invariant with respect to the graph-directed construction in Definition 2.2.

Definition 6.7. *Take the same notations as in Definition 2.2. Let $(\mathcal{E}_1,\mathcal{F}_1)=(\mathcal{E},\mathcal{F})$, and define $(\mathcal{E}_2,\mathcal{F}_2)$ as follows,*

$$\begin{cases}\mathcal{E}_2(f,g)=\sum_{w\in W_1,F_w\mathcal{G}\subset K_2}\rho_w^{-\theta}\mathcal{E}(f\circ F_w,g\circ F_w),\\ \mathcal{F}_2=\{f\in C(K_2):\ f\circ F_w\in\mathcal{F},\forall w\in W_1\text{ such that }F_w\mathcal{G}\subset K_2,\ \mathcal{E}_2(f)<\infty\}.\end{cases}$$

It is not hard to verify that $(\mathcal{E}_2,\mathcal{F}_2)$ is a resistance form on K_2. Moreover, we have

Theorem 6.8. *Take the same notations as in Definition 2.2, and write ρ_{e_j} for the similarity ratio of ψ_{e_j}, $1\leq j\leq 6$. Let $(\mathcal{E}_i,\mathcal{F}_i),i=1,2$ be defined as in Definition 6.7. Then, for $f_i\in\mathcal{F}_i$, $i=1,2$, we have $f_{e_{j,1}}\circ\psi_{e_j}\in\mathcal{F}_{e_{j,2}}$ for $1\leq j\leq 6$ and*

$$\mathcal{E}_1(f_1)=\sum_{j=1}^{2}\rho_{e_j}^{-\theta}\mathcal{E}_{e_{j,2}}(f_1\circ\psi_{e_j}),\quad \mathcal{E}_2(f_2)=\sum_{j=3}^{6}\rho_{e_j}^{-\theta}\mathcal{E}_{e_{j,2}}(f_2\circ\psi_{e_j}).$$

Reversely, let $f_1\in C(K_1)$, if $f_1\circ\psi_{e_j}\in\mathcal{F}_{e_{j,2}}$ for $j=1,2$, then $f_1\in\mathcal{F}_1$. The same holds for $(\mathcal{E}_2,\mathcal{F}_2)$.

Remark. At the end of this section, we remark that a same construction can be applied to get some non-standard self-similar forms on $\mathcal{G}$ with respect to the infinite i.f.s. $\{F_w\}_{w\in W_1}$, by starting with any solution $R_{r'}\mathcal{D}'=\lambda(r')\mathcal{D}'$. Theorem 6.5 and 6.8 still hold for the forms, with slight changes of the renormalization factors. Nevertheless, the good heat kernel estimate (Theorem 7.4 below) will not hold, but it is possible to get a heat kernel estimate in the form of Hambly and Kumagai's on p.c.f. self-similar sets [17].

7. Transition density estimate

Let μ_H be the normalized Hausdorff measure on $\mathcal{G}$. $(\mathcal{E},\mathcal{F})$ becomes a local regular Dirichlet form on $L^2(\mathcal{G},\mu_H)$ ($L^2(\mathcal{G})$ for short) in a standard way. By the celebrated result [14], there is a Hunt process $X=(\mathbb{P}^x, x\in\mathcal{G}, X_t, t\geq 0)$ associated with $(\mathcal{E},\mathcal{F})$ such that

$$\mathbb{E}^x[f(X_t)]=P_tf(x),\qquad \text{a.e. } x\in\mathcal{G},$$

where P_t is the associated hear operator. In this last section, we will show that X is a fractional diffusion. We refer to Barlow's book [3], Section 3, for the definition of this fractional diffusion.

Definition 7.1. *A Markov process $X=(\mathbb{P}^x, x\in\mathcal{G}, X_t, t\geq 0)$ is a* fractional diffusion *on the fractional metric space $(\mathcal{G},d_g,\mu_H)$ (see Section 2) if*

(a). X is a conservative Feller diffusion with state space $\mathcal{G}$;

(b). X is μ_H-symmetric;

(c). X has a symmetric transition density $p(t,x,y)=p(t,y,x)$, $t>0$, $x,y\in\mathcal{G}$, which satisfies the Chapman-Kolmogorov equations and is jointly continuous for $t>0$;

(d). there exist a constant β and c_1-$c_4>0$, such that for $0<t\leq 1$,

$$c_1t^{-d_H/\beta}\exp\big(-c_2(\frac{d_g(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big)\leq p(t,x,y)\leq c_3t^{-d_H/\beta}\exp\big(-c_4(\frac{d_g(x,y)^\beta}{t})^{\frac{1}{\beta-1}}\big),$$

where d_H is the Hausdorff dimension of $\mathcal{G}$.

Since $d_g\asymp d$ by Lemma 2.4, it suffices to consider the Eculidean metric d in the following.

We will closely follow Barlow's book [3] and Hambly and Kumagai's paper [17]. We only provide some essential estimates, including a Nash inequality and an estimate of the resistance metric R.

For convenience, for $0<s<1$, we write $\tilde{W}_s=\{w\in\tilde{W}_*:\rho_w\leq s<\rho_{([w]_{|w|-1})}\}$, and by identifying words representing the same cells, we get a quotient class $\hat{W}_s$.

Proposition 7.2 (Nash inequality)**.** *Let $d_S=\frac{2d_H}{d_H+\theta}$ with $\theta=\frac{\log r}{\log\rho}$, and $f\in\mathcal{F}$, we have*

$$\|f\|_{L^2(\mathcal{G})}^{2+4/d_S}\leq c\big(\mathcal{E}(f)+\|f\|^2_{L^2(\mathcal{G})}\big)\|f\|^{4/d_S}_{L^1(\mathcal{G})},$$

for some constant $c>0$ independent of f.

Proof. The proof is essentially the same as that for the p.c.f. self-similar sets [17]. We reproduce it here for convenience of readers. Write $f_w=f\circ F_w$ for $w\in\tilde{W}_*$ for short. Then for $0<s<1$,

$$\begin{aligned}\|f\|^2_{L^2(\mathcal{G})}&\leq\sum_{w\in\hat{W}_s}\rho_w^{d_H}\|f_w\|^2_{L^2(\mathcal{G})}\leq c_1\sum_{w\in\hat{W}_s}\rho_w^{d_H}\big(\mathcal{E}(f_w)+\|f_w\|^2_{L^1(\mathcal{G})}\big)\\&\leq c_2s^{d_H+\theta}\sum_{w\in\hat{W}_s}\rho_w^{-\theta}\mathcal{E}(f_w)+c_3s^{-d_H}\sum_{w\in\hat{W}_s}(\rho_w^{d_H}\|f_w\|_{L^1(\mathcal{G})})^2\\&\leq c_4\big(s^{d_H+\theta}\mathcal{E}(f)+s^{-d_H}\|f\|^2_{L^1(\mathcal{G})}\big),\end{aligned}$$

where in the last inequality, we use the observation that $\sum_{w\in\hat{W}_s}\rho_w^{-\theta}\mathcal{E}(f_w)\leq c'\mathcal{E}(f)$ for some $c'\geq 1$. In the case that $\mathcal{E}(f)>\|f\|^2_{L^1(\mathcal{G})}$, we choose s such that $s^{2d_H+\theta}\mathcal{E}(f)=\|f\|^2_{L^1(\mathcal{G})}$, then the result follows immediately. In the case that $\mathcal{E}(f) \leq \|f\|^2_{L^1(\mathcal{G})}$, we have $\|f\|^2_{L^2(\mathcal{G})} \leq c_1(\mathcal{E}(f) + \|f\|^2_{L^1(\mathcal{G})}) \leq 2c_1\|f\|^2_{L^1(\mathcal{G})}$, and the result follows. $\square$

The Nash inequality provides an upper bound estimate $p(t,x,y) \leq c_1 t^{-d_S/2}$. In addition, $|p(t,x,y) - p(t,x,y')| \leq c_2 t^{-1-d_S/2} R(y,y'), \forall 0 < t \leq 1, x, y, y' \in \mathcal{G}$. See [11] for a proof.

Proposition 7.3. *Let $R(\cdot,\cdot)$ be the resistance metric associated with $(\mathcal{E},\mathcal{F})$ on $\mathcal{G}$. Then there exist $c_1, c_2 > 0$ such that*

$$c_1 d(p,q)^\theta \leq R(p,q) \leq c_2 d(p,q)^\theta, \quad \forall p,q \in \mathcal{G}.$$

In addition, for $p \in \mathcal{G}$ and $A \subset \mathcal{G}$, define $R(p,A) = \sup\{\mathcal{E}(f)^{-1} : f \in \mathcal{F}, f(p) = 1, f|_A = 0\}$. Then there exists $c_3, c_4 > 0$ such that

$$c_3 s^\theta \leq R\big(p, B_s^c(p)\big) \leq c_4 s^\theta,$$

where $B_s(p) = \{q \in \mathcal{G} : d(p,q) < s\}$ with $p \in \mathcal{G}$ and $0 < s < 1$, and $B_s^c(p)$ is the complement of $B_s(p)$ in $\mathcal{G}$.

Proof. We already have the estimate $R(p,q) \leq c_2 d(p,q)^\theta$ in Lemma 6.4 and Theorem 6.5. Now we show $R\big(p, B_s^c(p)\big) \geq c_3 s^\theta$ for $p \in \mathcal{G}$ and $0 < s < 1$.

Define

$$U_{p,s,0} = \bigcup_{w \in \hat{W}_{p,s,0}} F_w\mathcal{G} \text{ with } \hat{W}_{p,s,0} = \{w \in \hat{W}_{s\rho^2} : p \in F_w\mathcal{G}\},$$

$$U_{p,s,1} = \bigcup_{w \in \hat{W}_{p,s,1}} F_w\mathcal{G} \text{ with } \hat{W}_{p,s,1} = \{w \in \hat{W}_{s\rho^2} : F_w\mathcal{G} \cap U_{p,s,0} \neq \emptyset\}.$$

Clearly, we have $U_{p,s,0} \subset \mathring{U}_{p,s,1} \subset U_{p,s,1} \subset B_s(p)$. Since $(\mathcal{E},\mathcal{F})$ is regular, there exists $f_{p,s} \in \mathcal{F}$ so that $f_{p,s}|_{\mathring{U}^c_{p,s,1}} = 0$ and $f_{p,s}|_{U_{p,s,0}} = 1$.

As $\mathcal{G}$ satisfies the finite type property, there exists a finite class $\{(p_i,s_i)\}_{i=1}^N$ such that for any $p \in \mathcal{G}$ and $0 < s < 1$, there exists $1 \leq i \leq N$ and an affine map ψ such that $\psi : U_{p,s,l} \to U_{p_i,s_i,l}$ for $l = 0,1$, which maps cells corresponding to $\hat{W}_{p,s,l}$ to those corresponding to $\hat{W}_{p_i,s_i,l}$. In addition, we can require that ψ maps the boundary of $U_{p,s,l}$ to the boundary of $U_{p_i,s_i,l}$, which only depend on how the outside cells of approximately same size intersect $U_{p_i,s_i,1}$. Thus, we can assume that

$$f_{p,s}(q) = \begin{cases} f_{p_i,s_i} \circ \psi(q), & \text{if } q \in U_{p,s,1}, \\ 0, & \text{if } q \in U^c_{p,s,1}. \end{cases}$$

By a similar observation as in Lemma 6.4, there exists $m \in \mathbb{Z}$ such that

$$\mathcal{D}^{(n)}(f_{p_i,s_i}) \leq \rho_\psi^\theta \mathcal{D}^{(n+m)}(f_{p,s}),$$

where ρ_ψ is the similarity ratio of ψ. So we have $\mathcal{E}(f_{p,s}) = \rho_\psi^{-\theta}\mathcal{E}(f_{p_i,s_i}) \leq c_3^{-1}s^{-\theta}$ for some constant c_3 independent of p,s,i. Since $f_{p,s}|_{B_s^c(p)} = 0$ and $f_{p,s}(p) = 1$, we get the estimate $R(p, B_s^c(p)) \geq c_3 s^\theta$.

Finally, the estimates $R(p,q) \geq c_1 d(p,q)^\theta$ follows from the fact that $R(p, B_s^c(p)) \geq c_3 s^\theta$, and $R\big(p, B_s^c(p)\big) \leq c_4 s^\theta$ follows from the fact that $R(p,q) \leq c_2 d(p,q)^\theta$. $\square$

By the resistance metric estimate in Proposition 7.3, the Ahlfors regularity of the measure μ_H and the resulted estimates from the Nash inequality, there exist a lower bound estimate $p(t,x,y) \geq c_3 t^{-d_S/2}$ and an estimate of the hitting time $c_4 s^{\theta+d_H} \leq \mathbb{E}^x \tau(x,s) \leq c_5 s^{\theta+d_H}$, where $\tau(x,s) = \inf\{t \geq 0 : X_t \notin B_s(x)\}$. See [3] for a proof. Finally, by Theorem 3.1.1 of Barlow's book [3] or by following [17], we can finally find that our diffusion is a fractional diffusion.

Theorem 7.4. *The Hunt process $X = (\mathbb{P}^x, x \in \mathcal{G}, X_t, t \geq 0)$ associated with the form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{G}, d_H)$ is a fractional diffusion, with $\beta = \theta + d_H$, in the sense of Definition 7.1.*

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