HIGHER ORDER LAPLACIANS ON P.C.F. FRACTALS WITH THREE BOUNDARY POINTS AND DIHEDRAL SYMMETRY

SHIPING CAO AND HUA QIU*

Abstract. In this paper, we study higher order tangents and higher order Laplacians on fully symmetric p.c.f. self-similar sets with three boundary points. Firstly, we prove that for any function f defined near a vertex x, the higher order weak tangent of f at x, if exists, is the uniform limit of local multiharmonic functions that agree with f near x in some sense. Secondly, we prove that the higher order Laplacian on a fractal can be expressible as a renormalized uniform limit of higher order graph Laplacians. Some results can be extended to general p.c.f. self-similar sets. In the Appendix, we provide a recursive algorithm for the exact calculations of the boundary values of the monomials on $D3$ symmetric fractals, which is shorter and more direct than the previous work on the Sierpinski gasket.

1. Introduction

Laplacians on post critically finite(p.c.f.) self-similar sets are defined as renormalized limits of Laplacians on the graphs that approximate the fractals [Ki1,Ki2]. An important feature of Kigami's theory is its intrinsic approach. The derived analytical properties are only dependent on the fractal itself, not its embedding in Euclidean space. There are many works in exploring properties of these fractal Laplacians that are natural analogs of the usual Laplacian. See [BK, BST, DSV, Ki3-Ki6, KL, KSS, MT, T1, S1-S3] and the references therein.

Recently, there are several works in connection with the differential calculus on p.c.f. self-similar sets that involve derivatives, tangents, energy measures, multiharmonic functions, and higher order Laplacians, analogous to the theory of analysis on manifolds, see [BSSY, CQ, DRS, NSTY, P1, PT1, PT2, S3, SU,T2] and the references therein.

The study of "weak gradients" or "weak tangents" on p.c.f. self-similar sets, which is closely related to the theory of energy measures, goes back to the work of Kusuoka [Ku], under the assumption that the harmonic structures associated with the Dirichlet forms are nondegenerate. It is proved that the energy is equal to the integral of certain seminorm of the gradient. Based on Kusuoka's construction, Kigami [Ki6] introduced the notion of "measurable Riemannian structure" on the Sierpinski gasket $\mathcal{SG}$ (see Figure 1.1). He proved that $\mathcal{SG}$ can be embedded in $\mathbb{R}^2$ by a certain harmonic map, whose image is called the harmonic Sierpinski gasket. Then the notion of "weak gradients" in [Ku] can be transferred to be the gradients with respect to the "measurable Riemannian structure" inherited from $\mathbb{R}^2$ through this embedding. This work has rich further development, see [CFKR, Ka1, Ka2, Ke] for example, which are deeply related to the properties of intrinsic smooth functions on fractals.

Later, Kusuoka's study of "weak gradients" on p.c.f. self-similar sets was continued by Teplyaev [T2] and Strichartz [S3] to pointwise definition at generic points and vertices in fractals separately. For a generic point x, and a function f defined near x, one may regard the tangent $T_1(f)$ of f at x to be the local harmonic approximation of f at x, and the gradient of f at x to be the difference between the tangent and the value of f at x. This is a "global" definition since the associated

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* Corresponding author.

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harmonic function is defined on the entire fractal. It was proved in [T2] that for a function f in the domain of Laplacian, the gradient of f always exists and is continuous on the symbolic space associated with the fractal under some mild assumption on the Dirichlet form. However, in case that x is a vertex, things become quite different. The best harmonic approximation of the function f at x may not exist. In [S3], in a nondegenerate situation, assuming that every boundary point of the fractal is a fixed point of one of the contractive mappings that is associated with the fractal, Strichartz studied approximation of functions by "local tangents" at vertices and generic points. For a vertex x, there are a collection of derivatives at x for a function f defined near x, which makes up the gradient df of f at x. In [S3], a number of results about the existence and the rate of approximation by harmonic tangents, and by tangents of higher order are presented. See [CQ] for further investigation of the "weak" continuity and rate of approximation of derivatives at vertices.

In Teplyaev's work [T2] one can find a discussion on the relations between the different definitions and results of Kusuoka [Ku], Kigami [Ki3,Ki6], Teplyaev [T2] and Strichartz [S3] on this topic. There are also some other works concerning the gradients and tangents on fractals from different points of views, see [CGIS1-2, CS, H, IRT].

There are some other works developing theories about smooth functions, including a Borel theorem and bump functions, in connection with differential equations, see [ORS, RST] and the references therein. Also see [BCDEHKMST1-2, IPRRS, P2, RS1, RS2] for related works on differential equations, resolvent kernel and eigenfunctions. These works deeply explore properties of intrinsic smooth functions.

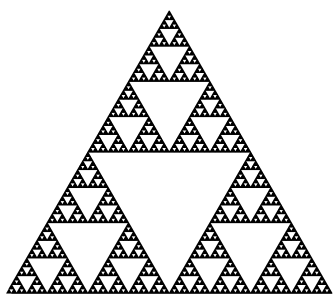


FIGURE 1.1. The Sierpinski gasket $\mathcal{SG}$.

In [S3], a theory of higher order tangents and local Taylor approximations of functions at vertices is also developed. Let f be a function defined near a vertex x. For the order n, denote $T_n(f)$ the n-harmonic function(solution of $\Delta_\mu^n h = 0$) with $\Delta_\mu^k f(x) = \Delta_\mu^k h(x)$ and $d\Delta_\mu^k f = d\Delta_\mu^k h(x)$ for all $k < n$. At the end of [S3], Strichartz posed several open problems that should be solved to complete the story of local Taylor approximations. Two of them are as follows.

Question 1. *For a smooth function f defined near a vertex x, can the higher order tangents $T_n(f)$ at x be expressible as limits of local multiharmonic functions that agree with f near x in a suitable sense?*

Question 2. *For a smooth function f, is it possible to express $\Delta_\mu^n f$ by a pointwise formula in terms of a uniform limit of linear combinations of values of f at graphs?*

The main goal of this paper is to answer these two questions.

We will mainly focus on the $D3$ symmetric fractals, i.e., those fractals whose boundary consists of 3 points and all structures possess full $D3$ symmetry. See [S3] for detailed discussions on $D3$ symmetric fractals, and see [KGMMOT] for a class of these fractals named $3N$-gaskets.

The paper is organized as follows. In Section 2, we collect some notations and facts about Laplacians and derivatives on general p.c.f. self-similar sets with regular harmonic structure, most of which can be found in [Ki5, S4]. First, in Section 3, we develop a lemma that will be useful in the later context, from which we also obtain a basic result on approximating Δ_μ^n on general p.c.f. fractals. Then, in the remainder of the paper, we turn to focus on the $D3$ symmetric fractals, since the fully symmetric structures provide certain advantages for the discussion. In Section 4, we introduce the theory of local monomials which form a basis of local multiharmonic functions near a vertex x, using which we give a positive answer to Question 1. This theory also plays a key role for solving Question 2. In Section 5, the main part of this paper, we answer Question 2 and prove a pointwise formula for the higher order Laplacians. Finally, in the Appendix, we provide a recursive algorithm for the exact calculations of the boundary values of the monomials, which play important roles in Section 4, for some typical $D3$ symmetric fractals. This algorithm is more direct and shorter than the one developed in [NSTY]. We remark that recently Strichartz et al[LJSS] utilized the algorithm to consider polynomials on the Sierpinski gasket with respect to a one-parameter family of symmetric and self-similar Laplacians.

2. Notations

We recall some standard notations and results on Kigami's Laplacians and Strichartz's derivatives on p.c.f. self-similar sets, which are the necessary background of this paper. Please refer to [Ki1-Ki2, Ki5, S4] for any unexplained notion.

Let $(K, N, \{F_l\}_{0\leq l<N})$ be a *p.c.f. self-similar structure*. That is, there is a finite set of contractive continuous injections $\{F_l\}_{0\leq l<N}$ on some metric space, with a compact invariant set K satisfying $K=\bigcup_{0\leq l<N} F_lK$. We define W_m as the space of *words* $w=w_1\cdots w_m$ of length $|w|=m$, taking values from the alphabet $\{0,\ldots,N-1\}$. For $w\in W_m$, we denote $F_w=F_{w_1}\circ\cdots\circ F_{w_m}$ and call F_wK a m-level *cell* of K. The term "p.c.f." means that K is connected, and there is a finite set $V_0\subseteq K$ called the *boundary* of K such that $F_wK\cap F_{w'}K\subset F_wV_0\cap F_{w'}V_0$ for different w and w' with the same length. We will require that each element in the boundary set V_0 is the fixed point of one of the mapping of $\{F_l\}_{0\leq l<N}$. Without loss of generality, we write $V_0=\{q_0,\cdots,q_{N_0-1}\}$ for $N_0\leq N$ and let $F_lq_l=q_l$ for $l<N_0$.

Let G_0 denote the complete graph on V_0. We approximate K by a sequence of *graphs* G_m with vertices V_m and edge relation $x\sim_m y$ defined by inductively applying the contractive mappings of $\{F_l\}$ to G_0. Let $V_*=\bigcup_{m\geq 0}V_m$ be the collection of all vertices of K.

We make the strong assumption that there is a regular *harmonic structure* on $(K, N, \{F_l\}_{0\leq l<N})$. Then by the standard theory(for example, see [Ki5]), there is a sequence of *renormalized graph energies* $\mathcal{E}_m$ on G_m with

$$\mathcal{E}_m(f,g)=\sum_{x\sim_m y}c_{xy}(f(x)-f(y))(g(x)-g(y))$$

for functions f,g defined on V_m, satisfying the self-similar identity

$$\mathcal{E}_m(f,g)=\sum_{l=0}^{N-1}r_l^{-1}\mathcal{E}_{m-1}(f\circ F_l,g\circ F_l),$$

where c_{xy} are the m-level *conductances* on graph G_m, and $\{r_l\}_{0\leq l<N}$ are the *renormalization factors* satisfying $0<r_l<1$. For $0\leq i,j<N_0$, we use c_{ij} to denote the 0-level conductances on graph G_0. Obviously, for $x\sim_m y$, we have

$$c_{xy}=r_w^{-1}c_{ij}, \tag{2.1}$$

where w is the word of length m such that $x=F_wq_i$, $y=F_wq_j$, with $r_w=r_{w_1}\cdots r_{w_m}$. Furthermore, if we denote $\mathcal{E}_m(f)=\mathcal{E}_m(f,f)$, then the trace of $\mathcal{E}_m$ to G_{m-1} equals $\mathcal{E}_{m-1}$, which means: if f is defined on G_{m-1}, then for all extension f' of f to G_m, the one $\tilde{f}$ that minimizes $\mathcal{E}_m$ satisfies

$\mathcal{E}_m(\tilde{f}) = \mathcal{E}_{m-1}(f)$. Hence the sequence $\{\mathcal{E}_m(f)\}$ is monotone increasing as m goes to infinity for any function f defined on K, and we can define

$$\mathcal{E}(f) = \lim_{m\to\infty} \mathcal{E}_m(f).$$

The domain $dom\mathcal{E}$ consists of the continuous functions f such that $\mathcal{E}(f) < \infty$. By polarization identity, for $f, g \in dom\mathcal{E}$, we define

$$\mathcal{E}(f,g) = \lim_{m\to\infty} \mathcal{E}_m(f,g).$$

The self-similar identity for the graph energy becomes

$$\mathcal{E}(f,g) = \sum_{w\in W_m} r_w^{-1}\mathcal{E}(f\circ F_w, g\circ F_w).$$

A function h is *harmonic* if it minimizes the energy from m-level to $(m+1)$-level for each m. Consequently the values on V_0 uniquely determine a harmonic function and the space $\mathcal{H}_0$ of harmonic functions has dimension N_0. In particular, for every $0 \le l < N_0$, there is a linear map $M_l : \mathcal{H}_0 \to \mathcal{H}_0$ defined by $M_l h = h \circ F_l$. We call M_l the l-th *harmonic extension matrix*.

Let μ be the *self-similar measure* with a set of probability weights $\{\mu_l\}$ on K, satisfying

$$\mu(A) = \sum_{0\le l<N} \mu_l \mu(F_l^{-1}A).$$

For $w \in W_m$, we denote $\mu_w = \mu_{w_1}\cdots\mu_{w_m}$ the measure of F_wK.

The *graph Laplacian* Δ_m on G_m is defined to be

$$\Delta_m f(x) = \sum_{y\sim_m x} c_{xy}(f(y) - f(x))$$

for $x \in V_m \setminus V_0$. The *Laplacian* with respect to μ on K is defined as the renormalized limit

$$\Delta_\mu f(x) = \lim_{m\to\infty} \tilde{\Delta}_m f(x), \tag{2.2}$$

where

$$\tilde{\Delta}_m f(x) = (\int_K \psi_x^m d\mu)^{-1}\Delta_m f(x). \tag{2.3}$$

(We avoid the notation $\tilde{\Delta}_{\mu,m}$ without causing any confusion.) Here ψ_x^m is a *tent function* which is harmonic on each m-level cell and has value 1 at x and 0 at other vertices in V_m. More precisely, $f \in dom\Delta_\mu$ and $\Delta_\mu f = g$ means f and g are continuous and the above limit converges to g uniformly on $V_*\setminus V_0$. There is an equivalent definition called the *weak formulation* of the Laplacian, which says that for $f \in dom\mathcal{E}$ and continuous function g, $f \in dom\Delta_\mu$ with $\Delta_\mu f = g$ if and only if

$$\mathcal{E}(f,v) = -\int_K gv d\mu$$

holds for all $v \in dom_0\mathcal{E}$, where $dom_0\mathcal{E}$ is the collection of functions in $dom\mathcal{E}$ that vanish on the boundary V_0.

There is a scaling identity

$$\Delta_\mu(f\circ F_w) = r_w\mu_w(\Delta_\mu f)\circ F_w$$

for the Laplacian Δ_μ, for any function $f \in dom\Delta_\mu$ and any word w.

The space of *multiharmonic functions* (solutions of $\Delta_\mu^n h = 0$ for some n) on fractals, which are analogous to *polynomials* on the unit interval, plays an important role in describing the approximation behavior of smooth functions, such as in the theory of Taylor approximations [S3], splines [SU], and power series expansions [NSTY]. Let $\mathcal{H}_n$ denote the collection of $(n+1)$-harmonic functions, the solutions of $\Delta_\mu^{n+1}h = 0$, which is of dimension $(n+1)N_0$ (see [SU] for more explanations).

There is a *Gauss-Green's formula*,

$$\mathcal{E}(f,g) = -\int_K \Delta_\mu f g d\mu + \sum_{q_l \in V_0} \partial_n f(q_l) g(q_l),$$

which connects the Laplacian Δ_μ with the important concept of normal derivatives (2.4).

We will not use the general theory of derivatives for general p.c.f. self-similar sets. In the rest of this section, we restrict our attention to the $D3$ *symmetric fractals*. This means we have $N_0 = 3$, and there exists a group $\mathcal{G}$ of homeomorphisms of K isomorphic to D_3 that acts as permutations on V_0. We further require that $\mathcal{G}$ preserves the harmonic structure and the self-similar measure, whose existence is always guaranteed by the symmetry conditions [L]. In this case, we can choose all $c_{ij} = 1$, and all the harmonic extension matrices M_l only differ by permutations, so that we must have $r_0 = r_1 = r_2$ and $\mu_0 = \mu_1 = \mu_2$. We denote r_l, μ_l's by r and μ for simplicity, and ρ the value of $r\mu$. In Section 4 and 5, we need the following assumptions.

Assumption 2.1. $r_l \mu_l = \rho$ *for all* $0 \leq l < N$.

Note that this assumption automatically holds when $N = N_0$. With this assumption, the scaling identity for the Laplacian becomes $\Delta_\mu(f \circ F_w) = \rho^{|w|} \Delta_\mu f \circ F_w$.

It is easy to verify that 1 is the largest eigenvalue and r is the second large eigenvalue of the matrix M_l, $l \in \{0,1,2\}$ (see [Ki5], Appendix A, for a proof). We denote the *third eigenvalue* by λ. The following assumption on λ is necessary for the definition of transverse derivatives (2.5).

Assumption 2.2. The matrices $M_l, l \in \{0,1,2\}$ are nondegenerate. In other words, $\lambda \neq 0$.

Below, we provide some examples satisfying Assumption 2.1 and 2.2. A typical example is the familiar *Sierpinski gasket* $\mathcal{SG}$, which is an invariant set generated by 3 contractive mappings with fixed points q_0, q_1, q_2 being the vertices of a triangle and with contraction ratio 1/2. For $\mathcal{SG}$, $r = 3/5$, $\mu = 1/3$, and $\rho = \lambda = 1/5$. Two more examples are the *level*-3 *Sierpinski gasket* $\mathcal{SG}_3$ and the *hexagasket* $\mathcal{HG}$. Here $\mathcal{SG}_3$ is an invariant set of six contractions of ratio 1/3 as shown in Fig. 2.1, which has $r = 7/15$, $\lambda = 1/15$, $\mu = 1/6$ and $\rho = 7/90$. While $\mathcal{HG}$, which is also named as Star of David, is generated by six mappings with simultaneously rotating and contracting by a ratio of 1/3 as shown in Fig. 2.1, having $r = 3/7, \lambda = 1/7$, $\mu = 1/6$ and $\rho = 1/14$. Please refer to [S4] for details.

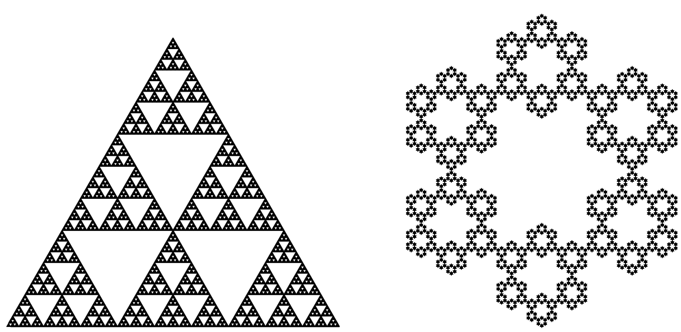


FIGURE 2.1. The level-3 Sierpinski gasket $\mathcal{SG}_3$(left) and the hexagasket $\mathcal{HG}$(right).

The *normal derivative* of a function f at the boundary point q_l is defined as

$$\partial_n f(q_l) = \lim_{m \to \infty} r^{-m}(2f(q_l) - f(F_l^m q_{l+1}) - f(F_l^m q_{l-1})) \tag{2.4}$$

(cyclic notation $q_{l+3} = q_l$), while the *transverse derivative* at q_l is defined as

$$\partial_T f(q_l) = \lim_{m \to \infty} \lambda^{-m}(f(F_l^m q_{l+1}) - f(F_l^m q_{l-1})), \tag{2.5}$$

providing the limits exist. For harmonic functions, these derivatives can be evaluated without taking the limit.

All the above notations and results are from a global viewpoint. Now we turn to the localized ones.

We localize the definition of derivatives as follows. Let x be a boundary point of cell F_wK, that is, there exists a q_l such that $x = F_w q_l$. We define the normal derivative at x with respect to F_wK by

$$\partial_n^w f(x) = \lim_{m\to\infty} r_w^{-1} r^{-m}(2f(x) - f(F_w F_l^m q_{l+1}) - f(F_w F_l^m q_{l-1}))$$

if the limit exists. We specify the superscript w since x may be a boundary point for more than one cell in same level. We will drop it when no confusion occurs. For $f \in dom\Delta_\mu$, the sum of all normal derivatives of f at x must vanish if x is not contained in V_0. This is called the *matching condition*. In general, the matching condition is necessary and sufficient for smoothly gluing functions with continuous Laplacians on neighboring cells, provided the functions agree at the common point of these cells (see [S4]).

Also at $x = F_w q_l$, there is a transverse derivative

$$\partial_T^w f(x) = \lim_{m\to\infty} r_w^{-1} \lambda^{-m}(f(F_w F_l^m q_{l+1}) - f(F_w F_l^m q_{l-1}))$$

if the limit exists. For $f \in dom\Delta_\mu$, the transverse derivatives at a point x with respect to different cells may be unrelated.

There are scaling identities for localized derivatives,

$$\partial_n^w f(F_w q_l) = r_w^{-1}\partial_n(f\circ F_w)(q_l), \text{ and } \partial_T^w f(F_w q_l) = r_w^{-1}\partial_T(f\circ F_w)(q_l).$$

Let $x \in V_* \setminus V_0$. Suppose m_0 is the minimal value for which $x \in V_{m_0}$. We say x is a *junction vertex* if there are at least two different m_0-cells containing x, i.e., x has at least two different representations $x = F_w q_l$ with $|w| = m_0$. Otherwise, we call x a *nonjunction vertex*, which has exactly one representation $x = F_w q_l$. For both types of vertices there is a canonical system of *neighborhoods* for each x. On each such neighborhood, there is a space of *local multiharmonic functions*.

Definition 2.3.

(a) For $x \in V_m \setminus V_0$, define the m-neighborhood of x as

$$U_m(x) = \bigcup\{F_wK | x \in F_wK, |w| = m\}.$$

Write $U(x) = U_{m_0}(x)$ for the sake of simplicity, which obviously is the largest one. The boundary of the m-neighborhood $U_m(x)$ is

$$\partial U_m(x) = \{y \in V_m | y \sim_m x\}.$$

(b) On each $U_m(x)$, define local $(n+1)$-harmonic functions to be those functions h on $U_m(x)$, with $h\circ F_w \in \mathcal{H}_n$ for each w, and $\Delta_\mu^i h$ satisfying the matching conditions at x for all $0 \le i \le n$.(If x is a nonjunction vertex, we say the matching conditions hold at x means $\partial_n \Delta_\mu^i h(x) = 0$ for all $0 \le i \le n$). Write the space of all such functions $\mathcal{H}_n(U_m(x))$.

We remark that our notations differ from that in [S3] when x is a nonjunction vertex. In our setting, we always view x as an interior point in $U_m(x)$.

Let $W(x)$ denote the set of words w of length m_0 such that there is a $q_l \in V_0$ with $x = F_w q_l$. Call $\#W(x)$ the *degree* of x. Obviously, $\#W(x) \ge 2$ when x is a junction vertex, while $\#W(x) = 1$ when x is a nonjunction vertex.

For convenience, we always sort the elements in $W(x)$ in lexicographical order. We use F_x to denote the contractive mapping on $U(x)$ with

$$F_x(y) = F_w F_l F_w^{-1}(y) \tag{2.6}$$

for $y \in F_wK$ and $w \in W(x)$, where for each such w the value l is such that $F_wq_l = x$. It is easy to see that $F_x(U_m(x)) = U_{m+1}(x)$.

3. Pointwise formula for Δ_μ^n on general p.c.f fractals

In this section, we provide some ideas considering the approximation of higher order Laplacians on general p.c.f. self-similar sets. The results will be applied in a more concrete setting in Section 5.

For the sake of applications, we consider the simple sets defined as follows.

Definition 3.1. *Call a finite union of cells $A = \bigcup_{w\in\Omega(A)} F_wK$ a simple set, where $\Omega(A)$ is a finite set of words. We define the boundary ∂A to be the collection of vertices y contained in A satisfying both*

(1) $y = F_wq_l$ with $w \in \Omega(A), l \in \{0, 1, 2\}$,

(2) $y \in V_0$ or $U_m(y)$ is not a subset of A for any m.

It is clear that all nonjunction vertices x are treated as interior points of $U_m(x)$ in the above definition, which is consistent with Definition 2.3. Also, there exists the possibility that y is a boundary point of A, while y belongs to more than one component cells of A simultaneously. See Fig. 3.1 for an example of such A in $\mathcal{SG}_3$.

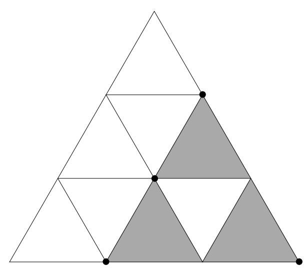


Figure 3.1. The shade area is a simple subset in $\mathcal{SG}_3$, whose boundary points are dotted. Note that the center point is a boundary point which belongs to two component cells.

Analogous to the global case, we define the local energy on A as

$$\mathcal{E}^A(f, g) = \sum_{w\in\Omega(A)} r_w^{-1}\mathcal{E}(f \circ F_w, g \circ F_w).$$

Let $dom(\mathcal{E}, A)$ be the space of continuous functions on A having finite energy, and denote $dom_0(\mathcal{E}, A)$ to be the subspace of such functions which vanish at ∂A. The Laplacian localized to A could be defined by the weak formulation in an analogous way. We denote by $dom(\Delta_\mu, A)$ the domain of Δ_μ on A. It is easy to check that if $f \in dom\Delta_\mu$, then $f|_A \in dom(\Delta_\mu, A)$. Additionally, the local multiharmonic function space on A is denoted $\mathcal{H}_n(A)$. Since one can assign data freely at the boundary ∂A, including those points belonging to more than one cell, we can easily check that $\mathcal{H}_n(A)$ has dimension $(n + 1)\#\partial A$.

The main result of this section is the following lemma, which extends the *mean value property* of harmonic functions.

Lemma 3.2. *Let A be a connected simple set, $l \in \mathbb{N}$, $\{y_j\}_{j=1}^l \subset A$ and $\{a_j\}_{j=1}^l \in \mathbb{R}$ with $\sum_{j=1}^l a_jh(y_j) = 0$ holding for any $h \in \mathcal{H}_{n-1}(A)$, then there exists a function $\phi_n \in L^1(A, \mu)$, such* *that*

$$\sum_{j=1}^{l} a_j f(y_j) = \int_A \phi_n \Delta_\mu^n f d\mu, \tag{3.1}$$

holds for any $f \in dom(\Delta_\mu^n, A)$.

Proof. There is a Green's operator G_A from $C(A)$ to $dom_0(\Delta_\mu, A)$ such that $-\Delta_\mu G_A v = v$ for any $v \in C(A)$. In fact, let G be the Green's operator on K, which is known to be bounded from $L^2(K,\mu)$ to $C(K)$ (see [Ki5,S4]), and then we can define $G_A f$ as

$$G_A f = (G\tilde{f})|_A - h,$$

where $\tilde{f}$ is defined by extending f with 0 on $K \setminus A$, and h is the unique harmonic function on A such that $G\tilde{f}|_{\partial A} = h|_{\partial A}$.

Clearly, G_A is bounded from $L^2(A,\mu)$ to $C(A)$, so $u \to \sum_{j=1}^{l} a_j(-G_A)^n u(y_j)$ is bounded from $L^2(A,\mu)$ to $\mathbb{R}$. Thus, by the Riesz representation theorem, there exists a function $\phi_n \in L^2(A,\mu)$ such that

$$\sum_{j=1}^{l} a_j(-G_A)^n u(y_j) = \int_A \phi_n u d\mu, \quad \forall u \in L^2(A,\mu).$$

Now, let $f \in dom(\Delta_\mu^n, A)$ and $u = \Delta_\mu^n f$, we have

$$\sum_{j=1}^{l} a_j f(y_j) = \sum_{j=1}^{l} a_j(-G_A)^n u(y_j) = \int_A \phi_n u d\mu,$$

where the first equality is due to the fact that $f - (-G_A)^n u \in \mathcal{H}_{n-1}(A)$ and the assumption that $\sum_{j=1}^{l} a_j h(y_j) = 0, \forall h \in \mathcal{H}_{n-1}(A)$. □

As an easy application of the above lemma, we have the following theorem.

Theorem 3.3. (Calculation of Δ_μ^n) *Let* $l \in \mathbb{N}$, $\{y_j\}_{j=1}^{l} \subset K$, *and* $\{a_j\}_{j=1}^{l} \subset \mathbb{R}$. *Assume*

$$\begin{cases} \sum_{j=1}^{l} a_j h(y_j) = 0, \forall h \in \mathcal{H}_{n-1}, \\ \sum_{j=1}^{l} a_j h'(y_j) = C, \forall h' \text{ with } \Delta_\mu^n h' = 1, \end{cases}$$

for some constant $C \neq 0$. *Then, for* $\omega \in \{0,1,2,\cdots,N-1\}^\infty$ *and* $x \in \bigcap_{m=0}^{\infty} F_{[\omega]_m} K$,

$$\Delta_\mu^n f(x) = \lim_{m\to\infty} C^{-1}(r_{[\omega]_m}\mu_{[\omega]_m})^{-n} \sum_{i=1}^{l} a_j f(F_{[\omega]_m} y_j).$$

The above limit is uniform on $\{0,1,\cdots,N-1\}^\infty$ *for any function* $f \in dom(\Delta_\mu^n)$.

Proof. By Lemma 3.2, taking $A = K$, there exists a function ϕ_n satisfying (3.1). Scaling the identity (3.1), for each $m \geq 0$, we have

$$\sum_{j=1}^{l} a_j f \circ F_{[\omega]_m}(y_j) = \int_K \phi_n \Delta_\mu^n (f \circ F_{[\omega]_m}) d\mu = (r_{[\omega]_m}\mu_{[\omega]_m})^n \int_K \phi_n (\Delta_\mu^n f) \circ F_{[\omega]_m} d\mu.$$

Taking the limit as $m \to \infty$, noticing that $\int_K \phi_n d\mu = C$ from using Lemma 3.2 again, we have proved the theorem. □

Remark. From the proof of Theorem 3.3, it is easy to find that the ratio of the uniform convergence depends only on the modulus of continuity of $\Delta_\mu^n f$. We will use the same idea in Section 5.

4. Local multiharmonic functions

From now on, from Section 4 to 6, we will focus on the $D3$ symmetric fractals. There are several bases of $\mathcal{H}_n$ for different purposes. In [NSTY], to develop a theory of the local behavior of functions at a single boundary point, a basis of $\mathcal{H}_n$, analogous to the monomials $x^j/j!$ on the unit interval, was described and studied on the Sierpinski gasket $\mathcal{SG}$. This could be easily extended to fractals having structures with full $D3$ symmetry, as follows. Throughout the following context, we drop the subscript μ of Δ_μ for simplicity.

Definition 4.1. *Fix a boundary point q_l. The monomials $Q_{jk}^{(l)}$ for $k \in \{1,2,3\}$ and $0 \le j \le n$ in $\mathcal{H}_n$ are the multiharmonic functions satisfying, for $0 \le i \le n$,*

$$\begin{aligned} \Delta^i Q_{jk}^{(l)}(q_l) &= \delta_{ij}\delta_{k1}, \\ \partial_n \Delta^i Q_{jk}^{(l)}(q_l) &= \delta_{ij}\delta_{k2}, \\ \partial_T \Delta^i Q_{jk}^{(l)}(q_l) &= \delta_{ij}\delta_{k3}. \end{aligned}$$

Remark 1. To see that the above definition makes sense, we refer to Lemma 6.1 of [S3], which states that a multiharmonic function $h \in \mathcal{H}_n$ is uniquely determined by the values of $\Delta^i h(q_l)$ and the derivatives $\partial_n \Delta^i h(q_l), \partial_T \Delta^i h(q_l)$, $0 \le i \le n$, and the values can be assigned freely.

Remark 2. Definition 4.1 relies on Assumption 2.2. If we do not assume Assumption 2.2, we may still define $Q_{j1}^{(l)}(q_l)$ and $Q_{j2}^{(l)}(q_l)$ as the multiharmonic functions in $\mathcal{H}_n$ that have mirror symmetry and satisfy $\Delta^i Q_{jk}^{(l)}(q_l) = \delta_{ij}\delta_{k1}, \partial_n \Delta^i Q_{jk}^{(l)}(q_l) = \delta_{ij}\delta_{k2}$. With this, we can apply some results in this section in Section 5. See the remark after Lemma 5.3.

It is easy to verify that these monomials form a basis of $\mathcal{H}_n$ with dimension $3(n+1)$. These monomials are related by the following identity,

$$\Delta Q_{jk}^{(l)} = Q_{(j-1)k}^{(l)}.$$

By the $D3$ symmetry, $Q_{jk}^{(l)}$ for different l's are same under simply rotations. $Q_{j1}^{(l)}$ and $Q_{j2}^{(l)}$ are symmetric while $Q_{j3}^{(l)}$ is skew-symmetric with respect to the reflection symmetry that fixes q_l and interchanges the other two boundary points. Moreover, the monomials satisfy the following self-similar identities that describe the decay ratios of these functions near q_l.

$$Q_{j1}^{(l)}(F_l^m x) = \rho^{jm} Q_{j1}^{(l)}(x), \tag{4.1}$$

$$Q_{j2}^{(l)}(F_l^m x) = r^m \rho^{jm} Q_{j2}^{(l)}(x), \tag{4.2}$$

$$Q_{j3}^{(l)}(F_l^m x) = \lambda^m \rho^{jm} Q_{j3}^{(l)}(x). \tag{4.3}$$

Denote

$$\alpha_j = Q_{j1}^{(0)}(q_1), \beta_j = Q_{j2}^{(0)}(q_1), \gamma_j = Q_{j3}^{(0)}(q_1), \tag{4.4}$$

for $j \ge 0$. In [NSTY], one can find an elaborate recursive algorithm for these numbers on the Sierpinski gasket $\mathcal{SG}$. An important observation is that all these numbers are non-zero. The calculation in [NSTY] is technical and hard to extend to the general $D3$ case. However, we still can verify that $\alpha_j, \beta_j, \gamma_j$ are never equal to 0 for some typical fractals with fully $D3$ symmetric structures, for example, the level-3 Sierpinski gasket $\mathcal{SG}_3$ and the hexagasket $\mathcal{HG}$.

Assumption 4.2. *All the numbers α_j, β_j and γ_j are non-zero.*

We will give the calculations of α_j, β_j and γ_j for $\mathcal{SG}$, $\mathcal{SG}_3$ and $\mathcal{HG}$ in the Appendix, by using a new algorithm modified from that in [NSTY], which is shorter and more direct. Please refer to [LJSS] for an application of this new algorithm.

We need to extend the above definitions and discussions to all vertices in V_*. Naturally, we have the following localized version of monomials, which will play an essential role in answering both the two questions posted in the Introduction.

Definition 4.3. *Fix a vertex $x \in V_* \setminus V_0$. The monomials P_{jk}^w in $\mathcal{H}_n(U(x))$ for $k \in \{1,2,3\}$, $0 \leq j \leq n$ and $w \in W(x)$ are the local multiharmonic functions satisfying*

$$\begin{aligned}
&\Delta^i P_{jk}^w(x) = \delta_{ij}\delta_{k1},\\
&\partial_n^{w''}\Delta^i P_{jk}^w(x) = \delta_{ij}\delta_{k2}\delta_{ww''} - \delta_{ij}\delta_{k2}\delta_{w'w''},\\
&\partial_T^{w''}\Delta^i P_{jk}^w(x) = \delta_{ij}\delta_{k3}\delta_{ww''},
\end{aligned}$$

where w' is the word in $W(x)$ that follows w in lexicographical order.

Remark 1. For $k=1$, the superscript w is unnecessary, and we may not add it when discuss P_{j1}^w separately. For nonjunction vertices, there are no monomials in the $k=2$ case because of the matching condition assumed in Definition 2.3 (b).

Remark 2. It is easy to check that $\{P_{jk}^w|_{U_m(x)}\}$ forms a basis of $\mathcal{H}_n(U_m(x))$.

Similar to (4.1)-(4.3), the following self-similar identities

$$\begin{aligned}
&P_{j1}(F_x^m y) = \rho^{jm}P_{j1}(y),\\
&P_{j2}^w(F_x^m y) = r^m\rho^{jm}P_{j2}^w(y),\\
&P_{j3}^w(F_x^m y) = \lambda^m\rho^{jm}P_{j3}^w(y),
\end{aligned}$$

describe the decay behaviors of these monomials near x (recall the definition of F_x in (2.6)). It is apparent that these monomials have symmetry properties analogous to the global case.

Denote R_i the rotations in $D3$ symmetric group, with $R_i(q_l) = q_{l+i}$ (cyclic notation).

Definition 4.4. *Fix a vertex $x \in V_* \setminus V_0$. For $k \in \{1,2,3\}$, let P_k be the linear projection from $\mathcal{H}_n(U(x))$ into itself satisfying*

$$\Delta^i P_k(h)(x) = \delta_{k1}\Delta^i h(x), \tag{4.5}$$

$$\partial_n^w\Delta^i P_k(h)(x) = \delta_{k2}\partial_n^w\Delta^i h(x), \tag{4.6}$$

$$\partial_T^w\Delta^i P_k(h)(x) = \delta_{k3}\partial_T^w\Delta^i h(x), \tag{4.7}$$

for any $h \in \mathcal{H}_n(U(x))$, $w \in W(x)$, $0 \leq i \leq n$, let R be a linear mapping on $\mathcal{H}_n(U(x))$, defined by

$$R(h)(y) = (\sum_{w' \in W(x)} r_{w'}^{-1})^{-1}\sum_{w' \in W(x)} r_{w'}^{-1} h \circ F_{w'} \circ R_{l'-l} \circ F_w^{-1}(y),$$

for $y \in F_w K, w \in W(x)$, where the values l and l' are such that $F_w q_l = F_{w'} q_{l'} = x$, for any h in $\mathcal{H}_n(U(x))$.

Clearly, $P_k(h)$ is a linear combination of the monomials P_{jk}^w, and it is easy to check that

$$P_1(h) + P_2(h) + P_3(h) = h.$$

As for R, roughly speaking, it is an operator on $\mathcal{H}_n(U(x))$ which first rotates variables around x, then takes mean values with weights proportional to $r_{w'}^{-1}$.

Let g_x be the local symmetry in $U(x)$ which fixes $x = F_w q_l$ and permutes the other two boundary points of $F_w K$ for each $w \in W(x)$.

Theorem 4.5. *Assume Assumption 2.1 and 2.2 hold. Let $x \in V_* \setminus V_0$ and $h \in \mathcal{H}_n(U(x))$, then the following identities hold,*

$$P_1(h) = \frac{1}{2}(R(h) \circ g_x + R(h)), \tag{4.8}$$

$$P_2(h) = \frac{1}{2}(h + h \circ g_x - R(h) \circ g_x - R(h)), \tag{4.9}$$

$$P_3(h) = \frac{1}{2}(h - h \circ g_x). \tag{4.10}$$

Proof. The following equalities are consequences of the symmetric definitions of Δ, ∂_n and ∂_T. For $w \in W(x)$, $0 \leq i \leq n$, and for any $h \in \mathcal{H}_n(U(x))$,

$$\Delta^i R(h)(x) = \Delta^i h(x),$$
$$\partial_n^w \Delta^i R(h)(x) = 0,$$

and

$$\Delta^i h \circ g_x(x) = \Delta^i h(x),$$
$$\partial_n^w \Delta^i h \circ g_x(x) = \partial_n^w \Delta^i h(x),$$
$$\partial_T^w \Delta^i h \circ g_x(x) = -\partial_T^w \Delta^i h(x).$$

These yield the result of the theorem. □

Corollary 4.6. *Assume Assumption 2.1 and 2.2 hold. Let $x \in V_* \setminus V_0$ and $h \in \mathcal{H}_n(U(x))$, then for each $m \geq m_0$, $h|_{\partial U_m(x)} = 0$ if and only if $P_k(h)|_{\partial U_m(x)} = 0$ for $k \in \{1, 2, 3\}$.*

In the rest of this section we give an application of the local monomials to show that the higher order weak tangents of smooth functions f at any fixed vertex are expressible as limits of local multiharmonic functions that agree with f at the boundary of $U_m(x)$. This answers Question 1. To be more precise, we need the following definition of *higher order weak tangents.*

Definition 4.7. *Let x be a vertex in $V_* \setminus V_0$ and f a function defined in a neighborhood of x. We say that an $(n+1)$-harmonic function h is a weak tangent of order $n+1$ of f at x if*

$$(f-h)|_{\partial U_m(x)} = o((\rho^n r)^m) \tag{4.11}$$

and

$$(f-h-(f-h)\circ g_x)|_{\partial U_m(x)} = o((\rho^n \lambda)^m). \tag{4.12}$$

The following Theorem 4.8 (a) establishes that the weak tangent of order $(n+1)$ is the best $(n+1)$-harmonic approximant to the function locally at x.

Theorem 4.8. *Assume Assumption 2.1, 2.2 and Assumption 4.2 hold. Let $x \in V_* \setminus V_0$. Then the following two conclusions hold.*

(a) For any $m \geq m_0$, an $(n+1)$-harmonic function h on $U(x)$ is uniquely determined by the values $h|_{\partial U_{m+i}}, 0 \leq i \leq n$, and any such values may be freely assigned.

(b)Let f be a continuous function defined in a neighborhood of x, and assume f has a weak tangent of order n+1 at x, denoted by h. Let h_m be the (n+1)-harmonic function defined in $U(x)$ that assumes the same values as f at the boundary points of $U_{m+i}(x)$ for all $0 \leq i \leq n$. Then h_m converges to h uniformly on $U(x)$.

Remark 1. This theorem extends the previous result in [CQ,S3] for the 1-order tangents and 1-order harmonic functions. For the nonjunction vertices, there is an implicit restriction that f, h and h_m should satisfy the equation $\partial_n \Delta^i u(x) = 0$ for all $0 \leq i \leq n$, since we always view x as an interior point in $U(x)$.

Remark 2. There are some sufficient conditions to ensure the existence of the weak tangents. One can find more detailed discussion on the weak tangents (and *tangents*, *strong tangents*) in [S3].

Proof of Theorem 4.8. (a) The map from $\mathcal{H}_n(U(x))$ to the values $h|_{\partial U_{m+i}(x)}, i = 0, 1, ..., n$ is obviously a linear map, and the dimension of $\mathcal{H}_n(U(x))$ is $2(n+1)\#W(x)$, which is exactly equal to $\#\bigcup_{0 \leq i \leq n} \partial U_{m+i}(x)$. Thus to prove (a), we only need to show that the map is injective.

Fix a word $w \in W(x)$ with $x = F_w q_l$. Let $h \in \mathcal{H}_n(U(x))$. For $k \in \{1,2,3\}$, writing $P_k(h)\circ F_w = \sum_{j=0}^n a^w_{jk} Q^{(l)}_{jk}$, we have the following equalities

$$\begin{aligned} P_k(h)(F_x^{m-|w|+i}F_w q_{l+1}) &= P_k(h)(F_w F_l^{m-|w|+i} q_{l+1}) \\ &= \sum_{j=0}^n a^w_{jk} Q^{(l)}_{jk}(F_l^{m-|w|+i} q_{l+1}) \\ &= \sum_{j=0}^n (A_k)_{ij} a^w_{jk}, \end{aligned} \tag{4.13}$$

for $(A_k)_{ij} = Q^{(l)}_{jk}(F_l^{m-|w|+i} q_{l+1}) = Q^{(l)}_{jk}(F_l^{m'+i} q_{l+1})$, where we denote $m' = m - |w|$ for convenience. Thus the $(n+1)\times(n+1)$ matrix A_k induces a linear map from $\{a^w_{jk}\}_j$ to the values $\{P_k(h)(F_x^{m-|w|+i}F_w q_{l+1})\}_i$.

We now show that the matrix A_k is invertible for $k \in \{1,2,3\}$. Denote by $\Gamma^{(n)}$ an $(n+1)\times(n+1)$ matrix with

$$\Gamma^{(n)} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \rho & \rho^2 & \dots & \rho^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \rho^n & \rho^{2n} & \dots & \rho^{n^2} \end{pmatrix},$$

which is obviously invertible since $\rho \in (0,1)$. Then by using the self-similar identities (4.1) – (4.3), we have

$$\begin{aligned} (A_1)_{ij} &= \rho^{m'j+ij}\alpha_j = (\Gamma^{(n)})_{ij}\rho^{m'j}\alpha_j, \\ (A_2)_{ij} &= r^{m'+i}\rho^{m'j+ij}\beta_j = r^{m'+i}(\Gamma^{(n)})_{ij}\rho^{m'j}\beta_j, \\ (A_3)_{ij} &= \lambda^{m'+i}\rho^{m'j+ij}\gamma_j = \lambda^{m'+i}(\Gamma^{(n)})_{ij}\rho^{m'j}\gamma_j, \end{aligned}$$

which can be rewritten in matrix notation,

$$\begin{aligned} A_1 &= \Gamma^{(n)} diag(\alpha_0, \rho^{m'}\alpha_1, ..., \rho^{m'n}\alpha_n), \\ A_2 &= diag(r^{m'}, ..., r^{m'+n})\Gamma^{(n)} diag(\beta_0, \rho^{m'}\beta_1, ..., \rho^{m'n}\beta_n), \\ A_3 &= diag(\lambda^{m'}, ..., \lambda^{m'+n})\Gamma^{(n)} diag(\gamma_0, \rho^{m'}\gamma_1, ..., \rho^{m'n}\gamma_n), \end{aligned}$$

from which it is obvious that all the matrices A_1, A_2, A_3 are invertible because of Assumption 4.2.

The above discussion shows that $P_k(h)$ vanishes at $\partial U_{m+i}(x)$ if and only if $P_k(h) = 0$. According to Corollary 4.6, h vanishes at $\partial U_{m+i}(x)$ if and only if all $P_k(h)$ vanishes at $\partial U_{m+i}(x)$. Thus we have proved (a).

(b) We need to study the $(n+1)$-harmonic functions $h - h_m$. Notice that formula (4.13) still holds for $h - h_m$.

For $k = 1$, we have $P_1(h_m - h)(F_x^{m'+i}F_w q_{l+1}) = \sum_{j=0}^n (A_1)_{ij} a_{j1}$. Thus

$$\begin{aligned} a_{j1} &= \sum_{i=0}^n (A_1^{-1})_{ji} P_1(h_m - h)(F_x^{m'+i}F_w q_{l+1}) \\ &= \alpha_j^{-1}\rho^{-jm'} \sum_{i=0}^n (\Gamma^{(n)})^{-1}_{ji} P_1(h_m - h)(F_x^{m'+i}F_w q_{l+1}). \end{aligned}$$

According to (4.11), by using Theorem 4.5, we have $P_1(h - h_m)|_{\partial U_{m+i}(x)} = o(r^m \rho^{mn})$, which gives that

$$a_{j1} = o(r^m \rho^{m(n-j)}).$$

For $k = 2$, a similar discussion shows that

$$\begin{aligned} a^w_{j2} &= \sum_{i=0}^{n} (A_2^{-1})_{ji} P_2(h_m - h)(F_x^{m'+i} F_w q_{l+1}) \\ &= \beta_j^{-1} \rho^{-jm'} \sum_{i=0}^{n} (\Gamma^{(n)})^{-1}_{ji} r^{-m'-i} P_2(h_m - h)(F_x^{m'+i} F_w q_{l+1}). \end{aligned}$$

According to (4.11), still using Theorem 4.5, we have $P_2(h - h_m)|_{\partial U_{m+i}(x)} = o(r^m \rho^{mn})$, and hence

$$a^w_{j2} = o(\rho^{m(n-j)}).$$

For $k = 3$, the same argument yields that

$$\begin{aligned} a^w_{j3} &= \sum_{i=0}^{n} (A_3^{-1})_{ji} P_3(h_m - h)(F_x^{m'+i} F_w q_{l+1}) \\ &= \gamma_j^{-1} \rho^{-jm'} \sum_{i=0}^{n} (\Gamma^{(n)})^{-1}_{ji} \lambda^{-m'-i} P_3(h_m - h)(F_x^{m'+i} F_w q_{l+1}). \end{aligned}$$

Using (4.12) and Theorem 4.5, we can get $P_3(h - h_m)|_{\partial U_{m+i}(x)} = o(\lambda^m \rho^{mn})$, so

$$a^w_{j3} = o(\rho^{m(n-j)}).$$

Thus we have proved that for $k \in \{1, 2, 3\}$, $P_k(h - h_m)$ converges uniformly to zero on each cell $F_w K$, which yields that h_m converges uniformly to h on $U(x)$. □

5. Pointwise formula for the higher order Laplacians

In this section, we will deal with Question 2, restricted to the $D3$ symmetric fractals.

5.1. **Definition of pointwise formula.** Analogous to the pointwise formula for the Laplacian, we will show that we can approach the n-order Laplacian by the n-fold iteration of the renormalized discrete Laplacian, which means

$$\Delta^n f(x) = \lim_{m\to\infty} \tilde{\Delta}_m^n f(x), \tag{5.1}$$

where $\tilde{\Delta}_m f$ is defined in (2.3). Notice that $\tilde{\Delta}_m^n$ may not be defined on all vertices in $V_m \setminus V_0$ for $n \geq 2$.

Definition 5.1. *For two vertices $x, y \in V_m$, the m-distance $d_m(x, y)$ is the minimal number of edges on a path from x to y in G_m.*

It is easy to check that any vertex satisfying $d_m(x, V_0) \geq n$ has a well-defined $\tilde{\Delta}_m^n f(x)$. Obviously,

$$V_m^n = \{x \in V_m : d_m(x, V_0) \geq n\}$$

is the domain of definition of $\tilde{\Delta}_m^n$. See Fig. 5.1 for V_2^2, the domain of definition of $\tilde{\Delta}_2^2$ for $\mathcal{SG}$.

For fixed $x \in V_m^n$, the calculation of $\tilde{\Delta}_m^n f(x)$ involves the values of f at those vertices with m-distance to x no more than n, which are collected as

$$L_m^n(x) = \{y \in V_m : d_m(x, y) \leq n\} = \bigcup \{L_m^1(y) : y \in L_m^{n-1}(x)\}. \tag{5.2}$$

The area bounded by these vertices is obviously a neighborhood of x, which may be written as $U_m^n(x)$ (see Fig. 5.2), which is

$$U_m^n(x) = \bigcup \{U_m(y) : y \in L_m^{n-1}(x)\}. \tag{5.3}$$

It is natural that the *boundary* of $U_m^n(x)$ is

$$\partial U_m^n(x) = L_m^n(x) \setminus \{y \in V_m \setminus V_0 : U_m(y) \subset U_m^n(x)\},$$

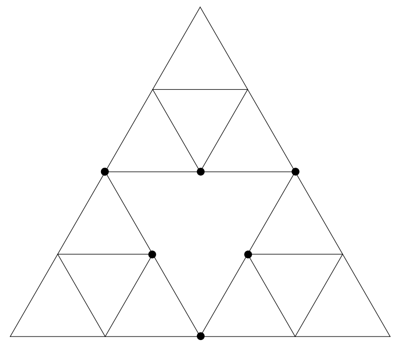


FIGURE 5.1. The domain of definition of $\tilde{\Delta}_2^2$ for $\mathcal{SG}$, denoted by dots.

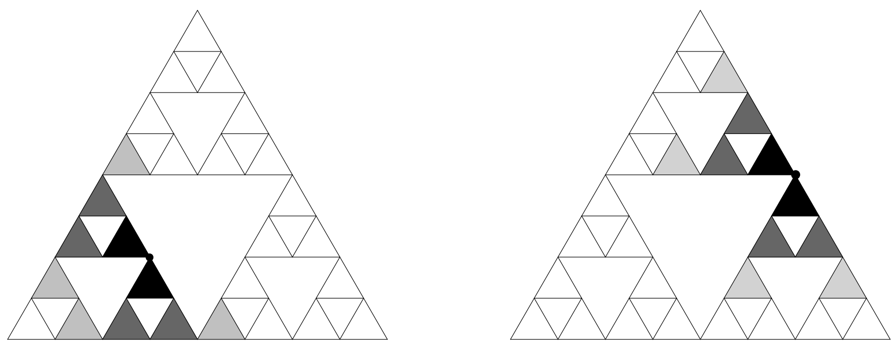


FIGURE 5.2. Some examples of $U_3^n(x)$ with $n \leq 3$ in $\mathcal{SG}$.

which is consistent with the boundary of $U_m(x)$ and the boundary of simple set A as introduced in Section 2 and 3. In our setting, nonjunction vertices always are viewed as interior points. It is easy to check that $\partial U_m^n(x) \subset L_m^n(x) \setminus L_m^{n-1}(x)$. There indeed may exist vertices that belong to $L_m^n(x) \setminus L_m^{n-1}(x)$, which are not boundary points of $U_m^n(x)$. For example, it is the case when we choose x to be the bottom dotted vertex in Fig 5.1 for $\mathcal{SG}$ for $n = m = 2$.

Remark. The shape of $U_m^n(x)$ varies for x in V_m^n and $m \geq 0$. We can give a classification of them. Let $x \in V_m^n$ and $y \in V_{m'}^n$. We say $U_m^n(x)$ and $U_{m'}^n(y)$ belong to a same *type* if there exists some mapping F which is a combination of rotations, reflections and scalings such that $FU_m^n(x) = U_{m'}^n(y)$.

We conclude that there are only *finitely many types* of $U_m^n(x)$ for any fixed n. In fact, the second equality of (5.2) shows that if there are finitely many types of $U_m^{n-1}(x)$, then the number of types of $U_m^n(x)$ is also finite. This observation will be useful in the proof of the uniform convergence of the pointwise formula. See Fig. 5.3 for the total types of $U_m^2(x)$ in $\mathcal{SG}$.

The following theorem is an answer to Question 2.

Theorem 5.2. *Assume Assumption 2.1 hold, then*

(a) For $f \in dom(\Delta^n)$, the pointwise formula (5.1) holds on $V_* \setminus V_0$. Moreover, we have the uniform control

$$\max_{x \in V_m^n} |\tilde{\Delta}_m^n f(x) - \Delta^n f(x)| \to 0, \text{ as } m \to \infty.$$

(b) Conversely, let $f \in C(K)$ and suppose there is $u \in C(K)$ such that

$$\max_{x \in V_m^n} |\tilde{\Delta}_m^n f(x) - u(x)| \to 0, \text{ as } m \to \infty.$$

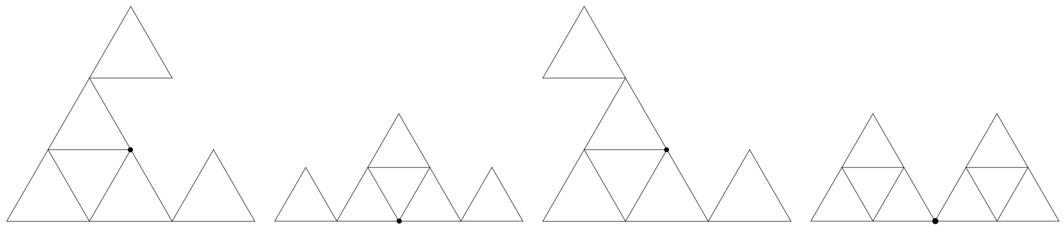


FIGURE 5.3. The total types of $U_m^2(x)$ in $\mathcal{SG}$.

Then $f \in dom(\Delta^n, K \setminus V_0)$ *with* $\Delta^n f = u$ *on* $K \setminus V_0$.

Before giving the proof, we remark that it looks that the (b) part of this theorem does not give complete information about the function f. This is to do with the existence of harmonic functions with singularities at boundary points (See more explanation on point singularities in [BSSY]). In fact, the conclusion is equivalent to that for any $g \in dom(\Delta^n)$ with $\Delta^n g = u$, we have $f - g$ is an n-harmonic function on $K \setminus V_0$ but maybe singular at V_0. However, if there is no multiharmonic function with singularity as occurs in the case of the unit interval, we obtain that $f \in dom(\Delta^n)$.

5.2. **Proof of Theorem 5.2(a).** We will take two steps to prove part (a) of Theorem 5.2. First, we deal with those functions which are local $(n+1)$-harmonic near x with $x \in V_m^n$, to get that (5.1) holds without taking the limit. Then, we prove the result for general functions in $dom(\Delta^n)$.

Lemma 5.3. *Let x be a vertex in $V_m \setminus V_0$, h be an $(n+1)$-harmonic function in $\mathcal{H}_n(U_m(x))$. Then*

$$\tilde{\Delta}_m h(x) = \sum_{j=1}^{n} \rho^{m(j-1)} \alpha_1^{-1} \alpha_j \Delta^j h(x). \tag{5.4}$$

In particular, $\alpha_1 = 1/6$.

Proof. Fix a word $w \in W(x)$ with $x = F_w q_l$. Note that $m \geq |w|$ (recall Definition 2.3) and $P_1(h \circ F_x^{m-|w|}) \circ F_w$ is a linear combination of monomials $Q_{j1}^{(l)}$. In fact, we have

$$P_1(h \circ F_x^{m-|w|}) \circ F_w = \sum_{j=0}^{n} \rho^{mj} \Delta^j h(x) Q_{j1}^{(l)},$$

by comparing the values at x when applying Δ^j on both sides. Thus we have

$$P_1(h \circ F_x^{m-|w|})(F_w q_{l+1}) = \sum_{j=0}^{n} \rho^{mj} \Delta^j h(x) Q_{j1}^{(l)}(q_{l+1}) = \sum_{j=0}^{n} \rho^{mj} \alpha_j \Delta^j h(x).$$

On the other hand, according to (4.8),

$$\begin{aligned} P_1(h \circ F_x^{m-|w|})(F_w q_{l+1}) &= (2 \sum_{w' \in W(x)} r_{w'}^{-1})^{-1} \sum_{y \in F_{w'}K, y \sim_m x} r_{w'}^{-1} h(y) \\ &= (\sum_{y \sim_m x} c_{xy})^{-1} \sum_{y \sim_m x} c_{xy} h(y), \end{aligned}$$

where we use the fact that $c_{xy} = r^{-m+|w'|}r_{w'}^{-1}$ by (2.1) in the second equality, noticing $c_{ij} \equiv 1$. Thus,

$$\begin{aligned}\tilde{\Delta}_m h(x) &= \frac{\sum_{y\sim_m x} c_{xy}}{\int \psi_x^m d\mu}\Big(P_1(h\circ F_x^{m-|w|})(F_w q_{l+1}) - h(x)\Big)\\ &= \frac{2\sum_{w\in W(x)} r_w^{-1} r^{-(m-|w|)}}{\int \psi_x^m d\mu}\sum_{j=1}^n \rho^{mj}\alpha_j \Delta^j h(x)\\ &= 6\rho^{-m}\sum_{j=1}^n \rho^{mj}\alpha_j\Delta^j h(x),\end{aligned}$$

where ψ_x^m is the same tent function in (2.3), the second equality comes from the fact that α_0 always equal to 1, and the third equality follows from Assumption 2.1.

From the arbitrariness of h, if we choose h to satisfy $\Delta h = 1$ in the above equality, this gives

$$\tilde{\Delta}_m h(x) = 6\alpha_1$$

for all m. By passing m to infinity, we get that $\alpha_1 = 1/6$. Thus we have proved the lemma. □

Remark. We use the decomposition of h based on the monomials in the above proof, and the definition of $Q_{jk}^{(l)}$ requires Assumption 2.2, which states that $\lambda \neq 0$. However, this requirement is not necessary. We can define $Q_{j1}^{(l)}, Q_{j2}^{(l)}$ as symmetric multiharmonic functions to avoid the occurence of transverse derivative as remarked below Definition 4.1.

It is interesting that the constant $\alpha_1 = 1/6$ is universal for all $D3$ symmetric fractals, which is an initial value for the calculations in the Appendix.

Lemma 5.4. *let x be a vertex in V_m^n and $h \in \mathcal{H}_n(U_m^n(x))$, then*

$$\tilde{\Delta}_m^n h(x) = \Delta^n h(x).$$

Proof. First, for $n = 1$, we have $\tilde{\Delta}_m h(x) = \Delta h(x), \forall x \in V_m \setminus V_0, h \in \mathcal{H}_1(U_m(x))$ as an immediate consequence of Lemma 5.3.

Next, we prove the lemma for $n > 1$, and we assume $\tilde{\Delta}_m^{n-1} h(y) = \Delta^{n-1}h(y), \forall y \in V_m^{n-1}, h \in \mathcal{H}_{n-1}(U_m^{n-1}(y))$ by induction. Let $h \in \mathcal{H}_n(U_m^n(x))$, it is obvious that $h|_{U_m(y)} \in \mathcal{H}_n(U_m(y))$ for any y with $d_m(x,y) \leq n-1$. So we can apply Lemma 5.3 to all points in $L_m^{n-1}(x)$. Thus we have

$$\begin{aligned}\tilde{\Delta}_m^n h(x) &= \tilde{\Delta}_m^{n-1}(\tilde{\Delta}_m h)(x)\\ &= \tilde{\Delta}_m^{n-1}\Big(\sum_{j=1}^n \rho^{m(j-1)}\alpha_1^{-1}\alpha_j\Delta^j h\Big)(x)\\ &= \sum_{j=1}^n \rho^{m(j-1)}\alpha_1^{-1}\alpha_j\tilde{\Delta}_m^{n-1}(\Delta^j h)(x).\end{aligned}$$

Since for each $j \geq 1$, $\Delta^j h$ belongs to $\mathcal{H}_{n-1}(U_m^{n-1}(x))$, by the inductive assumption, we then have

$$\tilde{\Delta}_m^n h(x) = \sum_{j=1}^n \rho^{m(j-1)}\alpha_1^{-1}\alpha_j\Delta^{n+j-1}h(x) = \Delta^n h(x).$$

Hence we have proved the lemma. □

Lemma 5.5. *For any m and any $x \in V_m^n$, there exists a function $\phi_{m,x}^{(n)} \in L^1(U_m^n(x),\mu)$ such that*

$$\tilde{\Delta}_m^n f(x) = \int_{U_m^n(x)} \phi_{m,x}^{(n)}\Delta^n f d\mu, \quad \forall f \in dom(\Delta^n, U_m^n(x)),$$

and it holds that $\int \phi_{m,x}^{(n)} d\mu = 1$. Furthermore, for any same type sets $U_m^n(x)$ and $U_{m'}^n(y)$, we have $\|\phi_{m,x}^{(n)}\|_1 = \|\phi_{m',y}^{(n)}\|_1$ if the m-level conductances on $U_m^n(x)$ are proportional to those on $U_{m'}^n(y)$.

Proof. By Lemma 5.4, we have $\tilde{\Delta}_m^n h(x) = 0$ for any $h \in \mathcal{H}_{n-1}(U_m^n(x))$, and $\tilde{\Delta}_m^n h(x) = 1$ if $\Delta^n h = 1$ on $U_m^n(x)$. The first half of the lemma follows from Lemma 3.2 by taking the simple set $A = U_m^n(x)$.

Let $U_m^n(x)$ and $U_{m'}^n(y)$ be in same type. It means there is a mapping F which is a combination of rotations, reflections and scalings, satisfying $FU_m^n(x) = U_{m'}^n(y)$. It is easy to find that

$$\phi_{m',y}^{(n)} = \frac{\mu(U_m^n(x))}{\mu(U_{m'}^n(y))}\phi_{m,x}^{(n)} \circ F^{-1},$$

by scaling. Hence $\|\phi_{m,x}^{(n)}\|_1 = \|\phi_{m',y}^{(n)}\|_1$. □

Since there are only finite types of $U_m^n(x)$, and for each type, there are only finite subtypes with proportional conductances, we have

Corollary 5.6. *Let $n \geq 2$ be fixed. For any m and any $x \in V_m^n$, $\|\phi_{m,x}^{(n)}\|_1$ is uniformly bounded.*

Proof of Theorem 5.2(a). Applying Lemma 5.5 and Corollary 5.6, we have

$$\begin{aligned}|\tilde{\Delta}_m^n f(x) - \Delta^n f(x)| &= |\int \phi_{m,x}^{(n)}(z)(\Delta^n f(z) - \Delta^n f(x))d\mu(z)| \\ &\leq \|\phi_{m,x}^{(n)}\|_1 \omega_{\Delta^n f}(U_m^n(x)) \leq C\omega_{\Delta^n f}(U_m^n(x))\end{aligned}$$

for some constant $C > 0$, where $\omega_{\Delta^n f}(U_m^n(x))$ is the oscillation of $\Delta^n f$ in $U_m^n(x)$. Since $\Delta^n f$ is continuous on K, $\omega_{\Delta^n f}(U_m^n(x))$ will go to zero uniformly as m goes to infinity. Thus we have (5.1) holds uniformly. □

Remark. The proof provides that the ratio of the convergence in (5.1) depends only on the *modulus of continuity* of $\Delta^n f$.

5.3. **Proof of Theorem 5.2(b).** In this subsection we will give the proof of the second part of Theorem 5.2.

Let A be a simple set. We denote by G_A the local Green's operator on A, i.e., for any continuous function u on A, $G_A u \in dom_0(\Delta, A)$, and satisfies

$$-\Delta G_A u = u.$$

There is a continuous Green's function $g_A \in C(A \times A)$ that satisfies

$$G_A u(x) = \int g_A(x,y)u(y)d\mu(y). \tag{5.5}$$

See Kigami's work [Ki7] for a detailed discussion on the Green's function on resistance spaces.

We use $S(\mathcal{H}_0, V_m, A)$ to denote the space of *harmonic splines*, which are harmonic in each m-level cell in A. For those harmonic splines vanishing at the boundary of A, we denote the collection of them by $S_0(\mathcal{H}_0, V_m, A)$. For any function $u_m \in l(V_m \cap (A \setminus \partial A))$, there is a unique solution $f_m \in S_0(\mathcal{H}_0, V_m, A)$ satisfying

$$-\tilde{\Delta}_m f_m(x) = u_m(x), \quad \forall x \in V_m \cap (A \setminus \partial A).$$

In fact, this may be realized using a Green's operator $G_{m,A}$ defined by

$$G_{m,A}u_m(x) = \sum_{y \in V_m \cap (A \setminus \partial A)} g_A(x,y)u_m(y)\int \psi_y^m d\mu, \tag{5.6}$$

where ψ_y^m is the tent function from (2.3). Indeed $f_m = G_{m,A}u_m$ satisfies $-\tilde{\Delta}_m f_m(x) = u_m(x)$ on $V_m \cap (A \setminus \partial A)$ as shown in [Ki7], Section 5. By comparing (5.5) and (5.6), we can easily see the following lemma.

Lemma 5.7. *For any simple set A in K, let $f \in dom_0(\Delta, A)$ and $f_m \in S_0(\mathcal{H}_0, V_m, A)$, $m \geq 1$. If $\tilde{\Delta}_m f_m \in l(V_m \cap (A \setminus \partial A))$ converges to Δf uniformly in the sense that*

$$\max_{x \in V_m \cap (A \setminus \partial A)} |\tilde{\Delta}_m f_m(x) - \Delta f(x)| \to 0, \text{ as } m \to \infty, \tag{5.7}$$

then f_m converges to f uniformly on A as m goes to infinity.

Proof. We restate this lemma as follows: suppose $u_m \in l(V_m \cap (A \setminus \partial A))$ converges uniformly to $u \in C(A)$ in a same sense as (5.7), then

$$\lim_{m\to\infty} G_{m,A}u_m = G_A u \tag{5.8}$$

holds uniformly on A.

Let $\tilde{u}_m \in l(V_m \cap (A \setminus \partial A))$ be defined as $\tilde{u}_m(x) = (\int \psi_x^m d\mu)^{-1} \cdot \int \psi_x^m u d\mu$. Clearly, $\tilde{u}_m$ converges to u uniformly in a same senese as (5.7). In addition, for any $x \in V_m \cap A$,

$$\begin{aligned} G_{m,A}\tilde{u}_m(x) - G_A u(x) &= \sum_{y\in V_m\cap(A\setminus\partial A)} g_A(x,y)\int_A \psi_y^m(z)1_A(z)u(z)d\mu(z) - \int_A g_A(x,z)u(z)d\mu(z) \\ &= \sum_{y\in V_m\cap A} g_A(x,y)\int_A \psi_y^m(z)1_A(z)u(z)d\mu(z) - \int_A g_A(x,z)u(z)d\mu(z) \\ &= \sum_{y\in V_m\cap A} \int_A \big(g_A(x,y) - g_A(x,z)\big)1_A(z)\psi_y^m(z)u(z)d\mu(z), \end{aligned}$$

where we use the facts that $g_A(x,y) = 0$ if $y \in \partial A$ in the second equality and $\sum_{y\in V_m\cap A} \psi_y^m = 1$ on A in the third equality. As a consequence, $\|G_{m,A}\tilde{u}_m - G_A u\|_{L^\infty(A)}$ converges to 0, since $g_A \in C(A\times A)$. Combining with the above the argument, we see that (5.8) holds uniformly by the estimate that

$$\begin{aligned} \|G_{m,A}u_m - G_A u\|_{L^\infty(A)} &\leq \|G_{m,A}u_m - G_{m,A}\tilde{u}_m\|_{L^\infty(A)} + \|G_{m,A}\tilde{u}_m - G_A u\|_{L^\infty(A)} \\ &\leq \mu(A)\|g_A\|_{L^\infty(A\times A)}\|u_m - \tilde{u}_m\|_{l^\infty(V_m\cap(A\setminus\partial A))} + \|G_{m,A}\tilde{u}_m - G_A u\|_{L^\infty(A)}. \end{aligned}$$

□

Proof of Theorem 5.2(b). Assume that we have $\lim_{m\to\infty}\tilde{\Delta}_m^n f(x) = u(x)$ uniformly on $V_* \setminus V_0$. Then by repeatedly using Lemma 5.7, on any A not intersecting the boundary V_0, we have

$$\lim_{m\to\infty}(-G_{m,A})^n\tilde{\Delta}_m^n f = (-G_A)^n u$$

converges uniformly. So we have $f - (-G_{m,A})^n\tilde{\Delta}_m^n f$ converges uniformly to the function $f - (-G_A)^n u$.

Now we prove $f - (-G_A)^n u \in \mathcal{H}_{n-1}(A)$.

Recall that in Lemma 5.3, we have shown that for any $(n+1)$-harmonic function $h \in \mathcal{H}_n(A)$, $\tilde{\Delta}_m h$ must be equal to some n-harmonic function on $A \cap V_m$, see (5.4). It is not hard to verify that any n-harmonic function can be written in the form on the right side of (5.4). Thus we have an inverse conclusion that for any n-harmonic function h' on A, there is a $(n+1)$-harmonic function $h \in \mathcal{H}_n(A)$ such that $\tilde{\Delta}_m h = h'$ on $V_m \cap (A \setminus \partial A)$.

Now we apply the above discussion in the proof. First we have

$$\tilde{\Delta}_m(\tilde{\Delta}_m^{n-1} f + G_{m,A}\tilde{\Delta}_m^n f) = 0,$$

and thus $\tilde{\Delta}_m^{n-1} f + G_{m,A}\tilde{\Delta}_m^n f$ equals to some harmonic function on A. Since for each $1 < i \leq n$,

$$\tilde{\Delta}_m(\tilde{\Delta}_m^{n-i} f - (-G_{m,A})^i\tilde{\Delta}_m^n f) = \tilde{\Delta}_m^{n-i+1} f - (-G_{m,A})^{i-1}\tilde{\Delta}_m^n f,$$

by repeatedly using the above discussion, we have that $f - (-G_{m,A})^n\tilde{\Delta}_m^n f$ equals to some n-harmonic function on $A \cap V_m$. Noticing that the space of n-harmonic functions on A is of finite dimension, the uniform limit $f - (-G_A)^n u$ of $f - (-G_{m,A})^n\tilde{\Delta}_m^n f$ is of course an n-harmonic function.

Thus we have $f = (-G_A)^n u + (f - (-G_A)^n u) \in dom(\Delta^n, A)$, and obviously $\Delta^n f = u$ on A. By the arbitrariness of A, we have proved $\Delta^n f = u$ on $K \setminus V_0$. □

6. Appendix

As an appendix of this paper, we focus on the calculation of α_j, β_j and γ_j, the boundary values of the monomials $\{Q^{(l)}_{jk}\}$. We will mainly discuss the $D3$ symmetric fractals. The most typical example $\mathcal{SG}$ has been well studied in [NSTY], where an iterated calculation of the values as well as the derivatives of $\{Q^{(l)}_{jk}\}$ at the boundary were given. However, their method is indirect, since it involves the boundary values and inner products of functions in what they call the "easy" basis, and they need to transform data from the "easy" basis to our monomial basis. Here we provide a new algorithm, which is more direct and shorter, using which we can calculate α_j, β_j and γ_j on some other examples, including $\mathcal{SG}_3$,$\mathcal{HG}$.

Our approach is based on the relationship between the Laplacian and the graph Laplacians of multiharmonic functions, established in Lemma 5.3. Taking $m = 1$ in (5.4), we get a recursive relation,

$$\tilde{\Delta}_1 Q^{(l)}_{jk}(x) = \sum_{i=1}^{j} \rho^{i-1}\alpha_1^{-1}\alpha_i \Delta^i Q^{(l)}_{jk}(x) = \sum_{i=1}^{j} \rho^{i-1}\alpha_1^{-1}\alpha_i Q^{(l)}_{(j-i)k}(x) \tag{6.1}$$

holding at all vertices $x \in V_1 \setminus V_0$ for all $j \geq 1$. Here in the second line of (6.1), we use the identity $\Delta^i Q^{(l)}_{jk} = Q^{(l)}_{(j-i)k}$.

Thus, assuming we already have the values $\alpha_j, j \geq 0$, (6.1) and the self-similar identities (4.1)-(4.3) form a system of equations to calculate $Q^{(l)}_{jk}|_{V_1}$ from the values $Q^{(l)}_{ik}|_{V_1}, 0 \leq i < j$. We use this idea to solve the $k \geq 2$ cases.

For the $k = 1$ case, it is a bit complicated, since we need to calculate all α_j simultaneously. We will give a theorem to show that α_j can be determined recursively by using (6.1). For convenience of the readers, we first introduce the new calculation on $\mathcal{SG}$ as an example, then give the proof for general $D3$ symmetric cases. First we introduce some observations as well as some notations, some of which are same as those in [NSTY].

Simplifing (6.1), we get

$$\Delta_1 Q^{(l)}_{jk}(x) = (\sum_{y\sim_1 x} c_{xy}) \sum_{i=1}^{j} \rho^i \alpha_i Q^{(l)}_{(j-i)k}(x).$$

Noticing that $\alpha_0 = 1$, we could rewrite the above identity into

$$\sum_{y\sim_1 x} c_{xy} Q^{(l)}_{jk}(y) = (\sum_{y\sim_1 x} c_{xy}) \sum_{i=0}^{j} \rho^i \alpha_i Q^{(l)}_{(j-i)k}(x) \tag{6.2}$$

for $j \geq 1$.

Also, we need a notation of *infinite dimensional semi-circulant matrices* α, β, γ. For example, $\alpha = \{\alpha_{ij}\}_{i,j=0,1,2\cdots}$, has $\alpha_{ij} = \alpha_{i-j}$ for $i \geq j$ and $\alpha_{ij} = 0$ for $i < j$. It is easy to check $(\alpha\beta)_{ij} = \sum_{l=i}^{j} \alpha_{il}\beta_{lj} = \sum_{l=0}^{i-j} \alpha_l \beta_{i-j-l}$ for $i \geq j$, and the multiplications among these matrices commute. We will need a linear operator τ on such matrices defined by

$$\tau \begin{pmatrix} d_0 & 0 & & & \\ d_1 & d_0 & 0 & & \\ d_2 & d_1 & d_0 & 0 & \\ d_3 & d_2 & d_1 & d_0 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} d_0 & 0 & & & \\ \rho^{-1}d_1 & d_0 & 0 & & \\ \rho^{-2}d_2 & \rho^{-1}d_1 & d_0 & 0 & \\ \rho^{-3}d_3 & \rho^{-2}d_2 & \rho^{-1}d_1 & d_0 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \tag{6.3}$$

where ρ is the scaling constant of the Laplacian defined before.

Example 6.1. The monomials have been well studied in [NSTY], with $\alpha_j, \beta_j, \gamma_j$ exactly calculated. The recursive relations are

$$\alpha_j = \frac{4}{5^j - 5} \sum_{i=1}^{j-1} \alpha_{j-i}\alpha_i, \forall j \geq 2,$$

$$\gamma_j = \frac{4}{5^{j+1} - 5} \sum_{i=0}^{j-1} \alpha_{j-i}\gamma_i, \forall j \geq 1,$$

$$\beta_j = \frac{1}{5^j - 1} \sum_{i=0}^{j-1} (\frac{2}{5} 5^{j-i}\alpha_{j-i}\beta_i - \frac{2}{3}\alpha_{j-i}5^i\beta_i + \frac{4}{5}\alpha_{j-i}\beta_i), \forall j \geq 1,$$

with initial data $\alpha_0 = 1, \alpha_1 = 1/6, \beta_0 = -1/2, \gamma_0 = 1/2$.

Now, we give a different calculation.

First, for $k = 1$, by considering the symmetry, (6.2) becomes

$$\begin{cases} \frac{a_j}{5^j} + \alpha_j + \frac{\alpha_j}{5^j} = 4\sum_{i=0}^{j} \frac{\alpha_i}{5^i}\frac{\alpha_{j-i}}{5^{j-i}}, \\ 2\alpha_j + \frac{2}{5^j}\alpha_j = 4\sum_{i=0}^{j} \frac{\alpha_i}{5^i}\frac{a_{j-i}}{5^{j-i}}, \end{cases}$$

for $j \geq 1$, where we denote $a_j = 5^j Q_{j1}^{(0)}(F_1 q_2)$. In addition, for $j = 0$, we have

$$\begin{cases} a_0 + 2\alpha_0 + 1 = 4\alpha_0, \\ 4\alpha_0 = 4a_0. \end{cases}$$

We could rewrite the above identities in matrix notation,

$$\begin{cases} a + \alpha + \tau(\alpha) + I = 4\alpha^2, \\ 2\alpha + 2\tau(\alpha) = 4\alpha a, \end{cases}$$

by multiplying them with 5^j on both sides, where a is the infinite matrix defined with $a_{ij} = a_{i-j}$ for $i \geq j$ and $a_{ij} = 0$ for $i < j$. Eliminating a, we get

$$8\alpha^3 - 2\alpha^2 - 3\alpha = 2\tau(\alpha)\alpha + \tau(\alpha),$$

which results that $\tau(\alpha) = 4\alpha^2 - 3\alpha$. So we get the recursion relation for α_j.

For $k = 2$, we can write (6.2) into

$$\begin{cases} \frac{b_j}{5^j} + \frac{3\beta_j}{5^{j+1}} + \beta_j = 4\sum_{i=0}^{j} \frac{\alpha_i}{5^i}\frac{3\beta_{j-i}}{5^{j-i+1}}, \\ 2\frac{3\beta_j}{5^{j+1}} + 2\beta_j = 4\sum_{i=0}^{j} \frac{\alpha_i}{5^i}\frac{b_{j-i}}{5^{j-i}}, \end{cases}$$

for $j \geq 0$, where we denote $b_j = 5^j Q_{j2}^{(0)}(F_1 q_2)$. Thus by multiplying both sides with 5^j, we have

$$\begin{cases} b + \frac{3}{5}\beta + \tau(\beta) = \frac{12}{5}\alpha\beta, \\ \frac{6}{5}\beta + 2\tau(\beta) = 4\alpha b. \end{cases}$$

With some calculation, we get

$$\frac{3}{5}\beta(2\alpha - I)(4\alpha + I) = \tau(\beta)(2\alpha + I),$$

which gives the recursion relation of β_j.

For $k = 3$, we have $Q_{j3}^{(0)}(F_1 q_2) = 0$ by symmetry, so only one system of equations need to be considered, which immediately yields the recursion relation of γ_j.

Now, we turn to the general cases.

Theorem 6.2. *Let $j \geq 2$. Then $Q_{j1}^{(l)}|_{V_1}$ is uniquely determined by the values of $Q_{i1}^{(l)}|_{V_1}$, $0 \leq i < j$, by the relations*

$$\begin{cases} \tilde{\Delta}_1 Q_{j1}^{(l)}(x) = \sum_{i=1}^{j} \rho^{i-1}\alpha_1^{-1}\alpha_i Q_{(j-i)1}^{(l)}(x), \forall x \in V_1 \setminus V_0, \\ Q_{j1}^{(l)}|_{F_l V_0} = \rho^j Q_{j1}^{(l)}|_{V_0}. \end{cases} \tag{6.4}$$

Proof. Obviously, $Q_{j1}^{(l)}|_{V_1}$ indeed satisfies the equations (6.4), which could be rewritten into an explicit form

$$\begin{cases} \tilde{\Delta}_1 Q_{j1}^{(l)}(x) - \rho^{j-1}\alpha_1^{-1}\alpha_j = \sum_{i=1}^{j-1} \rho^{i-1}\alpha_1^{-1}\alpha_i Q_{(j-i)1}^{(l)}(x), \forall x \in V_1 \setminus V_0, \\ Q_{j1}^{(l)}|_{F_l V_0} = \rho^j Q_{j1}^{(l)}|_{V_0}. \end{cases}$$

Thus, to prove that $Q_{j1}^{(l)}|_{V_1}$ is determined by (6.4) uniquely, we only need to prove the equations

$$\begin{cases} \tilde{\Delta}_1 f(x) - \rho^{j-1}\alpha_1^{-1} f(q_{l+1}) = 0, \forall x \in V_1 \setminus V_0, \\ f|_{F_l V_0} = \rho^j f|_{V_0}, \\ f \circ g_l = f \text{ on } V_1 \end{cases} \tag{6.5}$$

have a unique solution $f|_{V_1} = 0$, where g_l is the symmetry that fixes q_l and interchanges the other two vertices of V_0.

First we need to look at the equation

$$\tilde{\Delta}_1 h = 1, h|_{V_0} = 0. \tag{6.6}$$

It is not hard to check that $h = (Q_{11}^{(l)} - h')|_{V_1}$ is the unique solution of (6.6), where h' is the harmonic function with the same boundary values as those of $Q_{11}^{(l)}$, from which, one can find that

$$h(F_l q_{l+1}) = (\rho - r)\alpha_1. \tag{6.7}$$

Now suppose f is a solution of (6.5). Write $f = \rho^{j-1}\alpha_1^{-1} f(q_{l+1})h + \tilde{f}$. It is easy to check $\tilde{\Delta}_1 \tilde{f} = 0$, and $f|_{V_0} = \tilde{f}|_{V_0}$. Moreover, by using (6.7) and $h|_{V_0} = 0$, the relation $f|_{F_l V_0} = \rho^j f|_{V_0}$ implies

$$\tilde{f} \circ F_l(q_{l+1}) + \rho^{j-1} f(q_{l+1})(\rho - r) = \rho^j f(q_{l+1}).$$

This could be simplified into

$$r f(q_{l+1}) + \rho^{j-1} f(q_{l+1})(\rho - r) = \rho^j f(q_{l+1}),$$

since $f|_{V_0} = \tilde{f}|_{V_0}$ and $f \circ g_l = f$. Thus we have $(r - \rho^{j-1} r) f(q_{l+1}) = 0$, which implies that $f(q_{l+1}) = 0$, and thus $f = \tilde{f} = 0$ on V_1.

Hence we have proved the equations (6.5) only have a zero solution on V_1, which yields the result of the theorem. □

We give the recursive relations α, β and γ for $\mathcal{SG}_3$ and $\mathcal{HG}$ in the matrix form, which can be calculated in a similar way as in Example 6.1. Recall that the initial data are $\alpha_0 = 1, \alpha_1 = 1/6, \beta_0 = -1/2, \gamma_0 = 1/2$.

The $\mathcal{SG}_3$ case:

$$\begin{aligned} (1 + 6\alpha)\tau(\alpha) &= 1 + 12\alpha - 6\alpha^2 - 96\alpha^3 + 96\alpha^4, \\ (1 + 8\alpha + 12\alpha^2)\tau(\beta) &= (3 + 6\alpha - 60\alpha^2 - 96\alpha^3 + 192\alpha^4) r\beta, \\ \tau(\gamma) &= (-1 + 16\alpha^2)\lambda\gamma, \end{aligned}$$

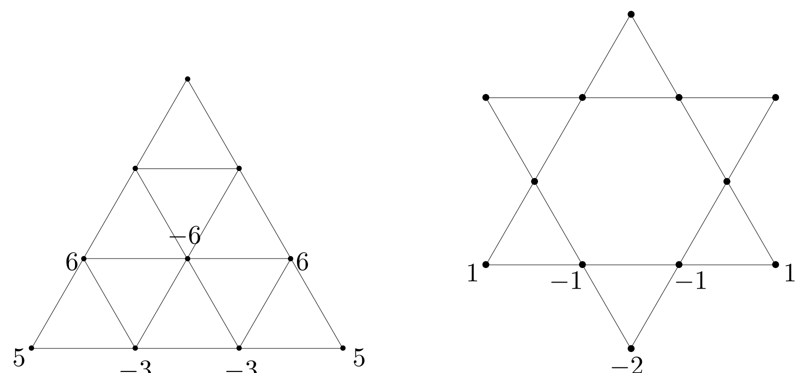
with $\rho = \frac{7}{90}$, $r = \frac{7}{15}$, $\lambda = \frac{1}{15}$.

The $\mathcal{HG}$ case:

$$(-1+2\alpha)\tau(\alpha) = -1+4\alpha+14\alpha^2-48\alpha^3+32\alpha^4,$$
$$(-1+4\alpha^2)\tau(\beta) = r(1+10\alpha-4\alpha^2-64\alpha^3+64\alpha^4)\beta,$$
$$\tau(\gamma) = \lambda(-1-8\alpha+16\alpha^2)\gamma,$$

with $\rho = \frac{1}{14}$, $r = \frac{3}{7}$ and $\lambda = \frac{1}{7}$.

Table 1-3 present numerical computations of α_j, β_j and γ_j for $\mathcal{SG}_3$ and $\mathcal{HG}$, respectively. (We are grateful to Mr. Wei Wei for providing an effective program.) For $\mathcal{SG}_3$ and $\mathcal{HG}$, it is easy to find that α_j, β_j behave like geometric progressions when j is large enough, with the reciprocal of common ratio $-124.68442107\cdots$ for $\mathcal{SG}_3$ and $-46.728917838\cdots$ for $\mathcal{HG}$. An explanation of this phenomenon comes from a slight generalization of Theorem 2.9 in [NSTY] (for $\mathcal{SG}$) involving a rather detailed knowledge of the description of eigenfunctions of $-\Delta$ by the spectral decimation. Recall that the natural of eigenvalues and eigenfunctions could be known explicitly via the method of spectral decimation for some fully symmetric p.c.f. fractals(See [FS, MT, Sh1-Sh2, ST,T1]). We refer the reader to the spectral decimation recipes for $\mathcal{SG}_3$, $\mathcal{HG}$ and $\mathcal{SG}^4$ in [DS], [BCDEHKMST2] and [FS] respectively, using which we could verify that $124.68442107\cdots$ is the eigenvalue of $-\Delta$ on $\mathcal{SG}_3$ with eigenfunction shown in Fig. 6.3(a), $46.728917838\cdots$ is the eigenvalue on $\mathcal{HG}$ with eigenfunction shown in Fig. 6.3(b).



(a). The $\mathcal{SG}_3$ case with $\lambda_1 = 14/3$. (b). The $\mathcal{HG}$ case with $\lambda_1 = 2$.

FIGURE 6.1. The values of the ultimate eigenfunctions on V_1. (We only show the non-zero values.)

TABLE 1. The data of $\alpha_j, \beta_j, \gamma_j$ for $\mathcal{SG}_3$.

j	α_j	β_j	γ_j
0	1	-0.5000000000	0.5000000000
1	0.1666666667	-0.04645247657	0.01499330656
2	$0.5332440874\times10^{-2}$	$-0.1029307014\times10^{-2}$	$0.1572291087\times10^{-2}$
3	$0.5981637501\times10^{-4}$	$-0.8969605760\times10^{-5}$	$0.7920041236\times10^{-6}$
4	$0.3116779311\times10^{-6}$	$-0.3931817387\times10^{-7}$	$0.2312923424\times10^{-8}$
5	$0.9411222994\times10^{-9}$	$-0.1015006336\times10^{-9}$	$0.4359699965\times10^{-11}$
6	$0.1768209338\times10^{-11}$	$-0.1801026965\times10^{-12}$	$0.5680708143\times10^{-14}$
7	$0.2703155181\times10^{-14}$	$-0.1686026983\times10^{-15}$	$0.5372013243\times10^{-17}$
8	$-0.9794948608\times10^{-19}$	$-0.5193342268\times10^{-18}$	$0.3827057814\times10^{-20}$
9	$0.2103310426\times10^{-19}$	$0.2555737442\times10^{-20}$	$0.2116632464\times10^{-23}$
10	$-0.1542910507\times10^{-21}$	$-0.2156048818\times10^{-22}$	$0.9317275066\times10^{-27}$
11	$0.1245430189\times10^{-23}$	$0.1723779132\times10^{-24}$	$0.3333118088\times10^{-30}$
12	$-0.9985135547\times10^{-26}$	$-0.1382754075\times10^{-26}$	$0.9861897696\times10^{-34}$
13	$0.8008453196\times10^{-28}$	$0.1108998433\times10^{-28}$	$0.2449517096\times10^{-37}$
14	$-0.6422974455\times10^{-30}$	$-0.8894451991\times10^{-31}$	$0.5172684514\times10^{-41}$
15	$0.5151385008\times10^{-32}$	$0.7133572856\times10^{-33}$	$0.9388504271\times10^{-45}$
16	$-0.4131538617\times10^{-34}$	$-0.5721302858\times10^{-35}$	$0.1478428756\times10^{-48}$
17	$0.3313596503\times10^{-36}$	$0.4588626969\times10^{-37}$	$0.2036817659\times10^{-52}$
18	$-0.2657586629\times10^{-38}$	$-0.3680192727\times10^{-39}$	$0.2471289873\times10^{-56}$
19	$0.2131450430\times10^{-40}$	$0.2951605903\times10^{-41}$	$0.2667519036\times10^{-60}$

TABLE 2. The data of $\alpha_j, \beta_j, \gamma_j$ for $\mathcal{HG}$.

j	α_j	β_j	γ_j
0	1	-0.5000000000	0.5000000000
1	0.1666666667	-0.04334554334	0.02197802198
2	0.00518925518	-0.0008741066739	0.0002728244486
3	$0.4189271589\times10^{-4}$	$-0.5294515600\times10^{-5}$	$0.14549879455\times10^{-5}$
4	$0.3320775837\times10^{-6}$	$-0.4109349118\times10^{-7}$	$0.41744883710\times10^{-8}$
5	$-0.1983647549\times10^{-8}$	$0.3457846603\times10^{-9}$	$0.73217281196\times10^{-11}$
6	$0.5477498983\times10^{-10}$	$-0.8500346754\times10^{-11}$	$0.85864350196\times10^{-14}$
7	$-0.1153381369\times10^{-11}$	$0.1804328531\times10^{-12}$	$0.71435664780\times10^{-17}$
8	$0.2470213442\times10^{-13}$	$-0.3862665725\times10^{-14}$	$0.44009750989\times10^{-20}$
9	$-0.5286113716\times10^{-15}$	$0.8266021437\times10^{-16}$	$0.20790960455\times10^{-23}$
10	$0.1131230500\times10^{-16}$	$-0.1768931223\times10^{-17}$	$0.75893835050\times10^{-27}$
11	$-0.2420836036\times10^{-18}$	$0.3785517208\times10^{-19}$	$0.24618203914\times10^{-30}$
12	$0.5180595119\times10^{-20}$	$-0.8101016218\times10^{-21}$	$0.19858084061\times10^{-34}$
13	$-0.1108648640\times10^{-21}$	$0.1733619479\times10^{-22}$	$0.76041321900\times10^{-37}$
14	$0.2372510837\times10^{-23}$	$-0.3709949982\times10^{-24}$	$-0.11425940714\times10^{-39}$
15	$-0.5077179071\times10^{-25}$	$0.7939302157\times10^{-26}$	$0.20348945233\times10^{-42}$
16	$0.1086517580\times10^{-26}$	$-0.1699012630\times10^{-27}$	$-0.34990746951\times10^{-45}$
17	$-0.2325150314\times10^{-28}$	$0.3635891238\times10^{-29}$	$0.59498904116\times10^{-48}$
18	$0.4975827434\times10^{-30}$	$-0.7780816262\times10^{-31}$	$-0.10013087692\times10^{-50}$
19	$-0.1064828304\times10^{-31}$	$0.1665096609\times10^{-32}$	$0.16709434786\times10^{-53}$

TABLE 3. The data of ratios of α_j, β_j.

j	$\alpha_{j-1}/\alpha_j(\mathcal{SG}_3)$	$\beta_{j-1}/\beta_j(\mathcal{SG}_3)$	$\alpha_{j-1}/\alpha_j(\mathcal{HG})$	$\beta_{j-1}/\beta_j(\mathcal{HG})$
0	/	/	/	/
1	6	10.76368876	6	11.53521127
2	31.25523013	45.12985529	32.11764706	49.58839080
3	89.14684104	114.7549894	123.8701068	165.0966282
4	191.9172615	228.1287475	126.1533989	128.8407348
5	331.1768633	387.3687528	-167.4075537	-118.8412787
6	532.2459729	563.5708715	-36.21447589	-40.67888879
7	654.1279429	1068.207675	-47.49078777	-47.11085928
8	-2759.744119	324.6516205	-46.69156721	-46.71200305
9	-4.656920103	-203.2032783	-46.73023651	-46.72944239
10	-136.3209607	-118.5380136	-46.72888255	-46.72890235
11	-123.8857481	-125.0768604	-46.72891855	-46.72891775
12	-124.7284209	-124.6627410	-46.72891783	-46.72891778
13	-124.6824487	-124.6849440	-46.72891784	-46.72891783
14	-124.6844940	-124.6842902	-46.72891784	-46.72891784
15	-124.6844188	-124.6843927	-46.72891784	-46.72891784
16	-124.6844211	-124.6844125	-46.72891784	-46.72891784
17	-124.6844211	-124.6844186	-46.72891784	-46.72891784
18	-124.6844211	-124.6844204	-46.72891784	-46.72891784
19	-124.6844211	-124.6844209	-46.72891784	-46.72891784

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Department of Mathematics, Cornell University, Ithaca, NY 14853, U.S.A.
Email address: `sc2873@cornell.edu`

Department of Mathematics, Nanjing University, Nanjing, 210093, P. R. China.
Email address: `huaqiu@nju.edu.cn`