HIGHER ORDER LAPLACIANS ON FULLY SYMMETRIC P.C.F. FRACTALS

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ABSTRACT. In this paper, we study higher order tangents and higher order Laplacians on fully symmetric p.c.f. self-similar sets. Firstly, we prove that for any function \(f\) defined near a vertex \(x\), the higher order weak tangent of \(f\) at \(x\), if exists, is the uniform limit of local multiharmonic functions that agree with \(f\) near \(x\) in some sense. Secondly, we prove that the higher order Laplacian on a fractal can be expressible as a renormalized uniform limit of higher order graph Laplacians. Some results can be extended to general p.c.f. self-similar sets. In the Appendix, we provide a recursive algorithm for the exact calculations of the boundary values of the monomials on \(D_3\) symmetric fractals, which is shorter and more direct than the previous work on the Sierpinski gasket.

1. Introduction

The Laplacians on post critically finite (p.c.f.) self-similar sets are defined as renormalized limits of Laplacians on the graphs that approximate the fractals [Ki1, Ki2]. An important feature of Kigami’s theory is its intrinsic approach. The derived analytical properties are only dependent on the fractal itself, not its embedding in Euclidean space. There are many works in exploring properties of these fractal Laplacians that are natural analogs of the usual Laplacian. See [BK, BST, DSV, Ki3-Ki6, KL, KSS, MT, T1, S1-S3] and the references therein.

Recently, there are several works in connection with the differential calculus on p.c.f. self-similar sets that involve derivatives, tangents, energy measures, multiharmonic functions, higher order Laplacians, analogous to the theory of analysis on manifolds, see [BSSY, CQ, DRS, NSTY, P1, PT1, PT2, S3, SU, T2] and the references therein.

The study of “weak gradients” or “weak tangents” on p.c.f. self-similar sets, which is closely related to the theory of energy measures, goes back to the work of Kusuoka [Ku], under the assumption that the harmonic structures associated with the Dirichlet forms are nondegenerate. It is proved that the energy is equal to the integral of certain seminorm of the gradient. Basing on Kusuoka’s construction, Kigami [Ki6] introduced the notion of “measurable Riemannian structure” on the Sierpinski gasket \(S_G\) (see Figure 1.1). He proved that \(S_G\) can be embedded in \(\mathbb{R}^2\) by a certain harmonic map, whose image is called the harmonic Sierpinski gasket. Then the notion of “weak gradients” in [Ku] can be transferred to be the gradients with respect to the “measurable Riemannian structure” inherited from \(\mathbb{R}^2\) through this embedding. This work has rich further development, see [CFKR, Ka1, Ka2, Ke] for example, which are deeply related to the properties of intrinsic smooth functions on fractals.

Later, Kusuoka’s study of “weak gradients” on p.c.f. self-similar sets was continued by Teplyaev [T2] and Strichartz [S3] to pointwise definition at generic points and vertices in fractals separately. For a generic point \(x\), and a function \(f\) defined near \(x\), one may think regard the tangent \(T_1(f)\) of \(f\) at \(x\) to be the harmonic approximation of \(f\) at \(x\), and the gradient of \(f\) at \(x\) to be the difference between the tangent and the value of \(f\) at \(x\). This is a “global” definition since the associated harmonic function is defined on the entire fractal. It was proved in [T2] that for a function \(f\) in

2000 Mathematics Subject Classification. Primary 28A80.

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The research of the second author was supported by the Nature Science Foundation of China, Grant 11471157.
the domain of Laplacian, the gradient of \( f \) always exists and is continuous on the symbolic space associated with the fractal under some mild assumption on the Dirichlet form. However, in case that \( x \) is a vertex, things become quite different. The best harmonic approximation of function \( f \) at \( x \) may not exist. In \([S3]\), in a nondegenerate situation, assuming that every boundary point of the fractal is a fixed point of one of the contractive mappings that associated with the fractal, Strichartz studied approximation of functions by “local tangents” at vertices and generic points. For a vertex \( x \), there are a collection of derivatives at \( x \) for a function \( f \) defined near \( x \), which makes up gradient \( df \) of \( f \) at \( x \). In \([S3]\), a number of results about the existence and the rate of approximation by harmonic tangents, and by tangents of higher order are presented. See \([CQ]\) for further investigation on the “weak” continuity and rate of approximation of derivatives at vertices.

In Teplyaev’s work \([T2]\) one could find a discussion on the relations between the different definitions and results of Kusuoka \([Ku]\), Kigami \([Ki3,Ki6]\), Teplyaev \([T2]\) and Strichartz \([S3]\) in this topic. There are also some other works concerning the gradients and tangents on fractals from different points of views, see \([CGIS1-2, CS, H, IRT]\).

There are some other works developing theories about smooth functions, including Borel theorem, bump functions, in connection with differential equations, see \([ORS, RST]\) and the references therein. Also see \([BCDEHKMST1-2, IPRRS, P2, RS1, RS2]\) for related works on differential equations, resolvent kernel and eigenfunctions. These works deeply explore properties of intrinsic smooth functions.

In \([S3]\), a theory of higher order tangents and local Taylor approximations of functions on vertices is also developed. Let \( f \) be a function defined near a vertex \( x \). For the order \( n \), denote \( T_n(f) \) the \( n \)-harmonic function (solution of \( \Delta^n_p h = 0 \)) with \( \Delta^k_p f(x) = \Delta^k_p h(x) \) and \( d\Delta^k_p f = d\Delta^k_p h(x) \) for all \( k < n \). In the end of \([S3]\), Strichartz posted several open problems that should be solved to complete the story of local Taylor approximations. Two of them are as follows.

**Question 1.** For a smooth function \( f \) defined near a vertex \( x \), could the higher order tangents \( T_n(f) \) at \( x \) be expressible as limits of local multiharmonic functions that agree with \( f \) near \( x \) in a suitable sense?

**Question 2.** For a smooth function \( f \), is it possible to express \( \Delta^n_p f \) by a pointwise formula in terms of a uniform limit of linear combinations of values of \( f \) at graphs?

The main goal of this paper is to answer these two questions.

We will mainly focus on the \( D3 \) symmetric fractals, i.e., those fractals whose boundary consists of 3 points and all structures possess full \( D3 \) symmetry. The results can be extended to fully symmetric p.c.f. self-similar sets with suitable modification.

Some of the results are extended to general p.c.f. self-similar sets.

The paper is organized as follows. In Section 2, we collect some notations and facts about Laplacians and derivatives on general p.c.f. self-similar sets, most of which can be found in \([K5, S4]\).
In Section 3, we introduce the theory of local monomials which form a basis of local multiharmonic functions near a vertex \( x \), using which we give a positive answer of Question 1. This theory also plays a key role for solving Question 2. Then in Section 4, the main part of this paper, we focus on Question 2 and prove a pointwise formula for the higher order Laplacians. In both Section 3 and Section 4 we only consider those \( D3 \) symmetric fractals, since the fully symmetric structures could provide certain advantages for our discussion. Then in Section 5, we turn to consider what extent we can extend the previous results to general p.c.f. self-similar sets without any symmetric assumption. Finally, in Appendix, we provide a recursive algorithm for the exact calculations of the boundary values of the monomials, which play important roles in Section 3, for some typical \( D3 \) symmetric fractals. This algorithm is more direct and shorter compared with the one developed in [NSTY]. We remark that recently Strichartz et al.[LJSS] utilized the algorithm to consider polynomials on the Sierpinski gasket with respect to a one-parameter family of symmetric and self-similar Laplacians.

2. Notations

We recall some standard notations and results on Kigami’s Laplacians and Stirchartz’s derivatives on p.c.f. self-similar sets, which are the necessary background of this paper. Please refer to [Ki1-Ki2, Ki5, S4] for any unexplained notion.

Let \( (K, N, \{F_i\}_{0 \leq i < N}) \) be a p.c.f. self-similar structure. That is, there is a finite set of contractive continuous injections \( \{F_i\}_{0 \leq i < N} \) on some metric space, with a compact invariant set \( K \) satisfying \( K = \bigcup_{0 \leq i < N} F_i K \). We define \( W_m \) as the space of words \( w = w_1 \cdots w_m \) of length \( |w| = m \), taking values from the alphabet \( \{0, \ldots, N-1\} \). For \( w \in W_m \), we denote \( F_w = F_{w_1} \circ \cdots \circ F_{w_m} \) and call \( F_w K \) a \( m \) level cell of \( K \). The term “p.c.f.” means that \( K \) is connected, and there is a finite set \( V_0 \subseteq K \) called the boundary of \( K \) such that \( F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0 \) for different \( w \) and \( w' \) with the same length. Moreover, each element in the boundary set \( V_0 \) is required to be the fixed point of one of the mapping of \( \{F_i\}_{0 \leq i < N} \). Without loss of generality, we write \( V_0 = \{q_0, \cdots, q_{N_0-1}\} \) for \( N_0 \leq N \) and let \( F_i q_l = q_l \) for \( l < N_0 \).

Let \( G_0 \) denote the complete graph on \( V_0 \). We approximate \( K \) by a sequence of graphs \( G_m \) with vertices \( V_m \) and edge relation \( x \sim_m y \) defined by inductively applying the contractive mappings of \( \{F_i\} \) to \( G_0 \). Let \( V = \bigcup_{m \geq 0} V_m \) be the collection of all vertices of \( K \).

Suppose there is a regular harmonic structure on \( (K, N, \{F_i\}_{0 \leq i < N}) \). Then by the standard theory (for example, see [Ki5]), there is a sequence of renormalized graph energies \( \mathcal{E}_m \) on \( G_m \) with

\[
\mathcal{E}_m(f, g) = \sum_{x \sim_m y} c_{xy}(f(x) - f(y))(g(x) - g(y))
\]

for functions \( f, g \) defined on \( V_m \), satisfying the self-similar identity

\[
\mathcal{E}_m(f, g) = \sum_{l=0}^{N-1} r_l^{-1} \mathcal{E}_{m-1}(f \circ F_l, g \circ F_l),
\]

where \( c_{xy} \) are the \( m \)-level conductances on graph \( G_m \), and \( \{r_l\}_{0 \leq l < N} \) are the renormalization factors satisfying \( 0 < r_l < 1 \). For \( 0 \leq i, j < N_0 \), we use \( c_{ij} \) to denote the 0-level conductances on graph \( G_0 \). Obviously, for \( x \sim_m y \), we have \( c_{xy} = r_w^{-1} c_{ij} \), where \( w \) is the word of length \( m \) such that \( x = F_w q_i, y = F_w q_j \), with \( w = w_1 \cdots w_m \). Furthermore, if we denote \( \mathcal{E}_m(f) = \mathcal{E}_m(f, f) \), then the restriction of \( \mathcal{E}_m \) to \( G_{m-1} \) equals \( \mathcal{E}_{m-1} \), which means, if \( f \) is defined on \( G_{m-1} \), then for all extension \( f' \) of \( f \) to \( G_m \), the one \( \tilde{f} \) that minimize \( \mathcal{E}_m \) satisfies \( \mathcal{E}_m(\tilde{f}) = \mathcal{E}_{m-1}(f) \). Hence the sequence \( \{\mathcal{E}_m(f)\} \) is monotone increasing as \( m \) goes to infinity for any function \( f \) defined on \( K \), and thus we could define

\[
\mathcal{E}(f) = \lim_{m \to \infty} \mathcal{E}_m(f).
\]
The domain $\text{dom}E$ consists of continuous functions $f$ such that $E(f) < \infty$. By polarization identity, for $f, g \in \text{dom}E$, we define

$$E(f, g) = \lim_{m \to \infty} E_m(f, g).$$

The self-similar identity for graph energy becomes

$$E(f, g) = \sum_{w \in W_m} r_w^{-1} E(f \circ F_w, g \circ F_w).$$

A function $h$ is harmonic if it minimizes the energy from level $m$ to level $m + 1$ for each $m$. All the harmonic functions form a $N_0$-dimensional space, denoted by $H_0$, and hence any values on the boundary can uniquely determine a harmonic function on $K$. In fact, for every $0 \leq l < N_0$, there is a linear map $M_l : H_0 \to H_0$ defined by $M_l h = h \circ F_l$. We call $M_l$ the $l$-th harmonic extension matrix.

Let $\mu$ be the self-similar measure with a set of probability weights $\{\mu_l\}$ on $K$, satisfying

$$\mu(A) = \sum_{0 \leq l < N} \mu_l(F_l^{-1}A).$$

For $w \in W_m$, we denote $\mu_w = \mu_{w_1} \cdots \mu_{w_m}$ the measure of $F_w K$.

The graph Laplacian $\Delta_m$ on $G_m$ is defined to be

$$\Delta_m f(x) = \sum_{y \sim x} c_{xy} (f(y) - f(x))$$

for $x \in V_m \setminus V_0$. The Laplacian with respect to $\mu$ on $K$ is defined as the renormalized limit

$$\Delta_\mu f(x) = \lim_{m \to \infty} \Delta_m f(x),$$

where $\Delta_m f(x) = (\int_K \psi^m_x d\mu)^{-1} \Delta_m f(x).$ (We avoid the notation $\Delta_{\mu,m}$ without causing any confusion.) Here $\psi^m_x$ is a tent function which is harmonic on each $m$-level cell taking value 1 at $x$ and 0 at other vertices in $V_m$. More precisely, $f \in \text{dom} \Delta_\mu$ and $\Delta_\mu f = g$ means $f$ and $g$ are continuous and the above limit converges to $g$ uniformly on $V_\ast \setminus V_0$. There is an equivalent definition called the weak formulation of the Laplacian, which says that for $f \in \text{dom}E$ and continuous function $g$, $f \in \text{dom} \Delta_\mu$ with $\Delta_\mu f = g$ if and only if

$$E(f, v) = -\int_K gv d\mu$$

holds for all $v \in \text{dom}_0 E$, where $\text{dom}_0 E$ is the collection of functions in $\text{dom}E$ that vanish on the boundary $V_0$.

There is a scaling identity

$$\Delta_\mu (f \circ F_w) = r_w \mu_w (\Delta_\mu f) \circ F_w$$

for the Laplacian $\Delta_\mu$, for any function $f \in \text{dom} \Delta_\mu$ and any word $w$.

The space of multiharmonic functions (solutions of $\Delta_n^h h = 0$ for some $n$) on fractals, analogous to polynomials on the unit interval, plays an important role in describing the approximation behavior of smooth functions, such as in the theory of Taylor approximations [S3], splines [SU], and power series expansions [NSTY]. Let $H_n$ denote the collection of $(n + 1)$-harmonic functions, the solutions of $\Delta_n^{n+1} h = 0$, which is of $(n + 1)N_0$-dimension.

There is a Gauss-Green’s formula,

$$E(f, g) = -\int_K \Delta_\mu fg d\mu + \sum_{q_i \in V_0} \partial_n f(q_i) g(q_i),$$

which connects the Laplacian $\Delta_\mu$ with the important concept of normal derivatives.

We would not involve the general theory of derivatives for general p.c.f. self-similar sets. In the rest of this section, we restrict our attention to the D3 symmetric fractals. Here D3 symmetry
means that $N_0 = 3$ and all structures are invariant under any homeomorphism of $K$. In this case, we could choose all $a_{ij} = 1$, and all the harmonic extension matrices $M_l$ only differ by permutations, so that we must have $r_0 = r_1 = r_2$ and $\mu_0 = \mu_1 = \mu_2$. We denote $r_l, \mu_l$’s by $r$ and $\mu$ for simplicity, and $\rho$ the value of $r\mu$. In the next two sections, we need the following assumption.

**Assumption 2.1.** $r_l \mu_l = \rho$ for all $0 \leq l < N$.

Note that this assumption automatically holds when $N = N_0$. With this assumption, the scaling identity for Laplacian becomes $\Delta_\mu(f \circ F_w) = \rho^{[\omega]} \Delta_\mu f \circ F_w$.

It is easy to verify that $1$ is the largest eigenvalue and $r$ is the second large eigenvalue of the matrix $M_l$, $l = 0, 1, 2$. We denote the third eigenvalue by $\lambda$. Here we require that the matrix $M_l$ to be nondegenerate. A typical example is the familiar Sierpinski gasket $SG$, which is an invariant set generated by 3 contractive mappings with fixed points $q_0, q_1, q_2$ being the vertices of a triangle and with contraction ratio $1/2$. For $SG$, $r = 3/5, \mu = 1/3$, and $\rho = \lambda = 1/5$. Two more examples are the level 3 Sierpinski gasket $SG_3$ and the hexagasket $HG$. Here $SG_3$ is an invariant set of six contractions of ratio $1/3$ as shown in Fig. 2.1, which has $r = 7/15, \lambda = 1/15, \mu = 1/6$ and $\rho = 7/90$. While $HG$, which is also named as Star of David, is generated by six mappings with simultaneously rotating and contracting by a ratio of $1/3$ as shown in Fig. 2.1, having $r = 3/7, \lambda = 1/7, \mu = 1/6$ and $\rho = 1/14$. Please refer to [S4] for details.

![Figure 2.1](image_url)

**Figure 2.1.** The level 3 Sierpinski gasket $SG_3$ (left) and the hexagasket $HG$ (right).

The normal derivatives of a function $f$ at the boundary point $q_l$ is defined as

$$\partial_n f(q_l) = \lim_{m \to \infty} r^{-m}(2f(q_l) - f(F_l^m q_{l+1}) - f(F_l^m q_{l-1}))$$

(cyclic notation $q_{l+3} = q_l$), while the transverse derivatives at $q_l$ is defined as

$$\partial_T f(q_l) = \lim_{m \to \infty} \lambda^{-m}(f(F_l^m q_{l+1}) - f(F_l^m q_{l-1})),$$

providing the limits exist. For harmonic functions, these derivatives can be evaluated without taking the limit.

All the above notations and results are from global viewpoint. Now we turn to the localized ones.

We localize the definition of derivatives as follows. Let $x$ be a boundary point of cell $F_w K$, that is, there exists a $q_l$ such that $x = F_w q_l$. We define the normal derivative at $x$ with respect to $F_w K$ by

$$\partial_n^w f(x) = \lim_{m \to \infty} r^{-m}(2f(x) - f(F_w F_l^m q_{l+1}) - f(F_w F_l^m q_{l-1}))$$

if the limit exists. We specify the superscript $w$ since $x$ may be boundary points of different cells in same level. We will drop it when no confusion occurs. For $f \in dom \Delta_\mu$, the sum of all normal derivatives of $f$ at $x$ must vanish if $x$ is not contained in $V_0$. This is called the matching condition.
In general, the matching condition is necessary and sufficient for gluing together two functions whose Laplacian is defined on neighboring cells.

Also at \( x = F_w q_l \), there is a transverse derivative
\[
\partial^w_T f(x) = \lim_{m \to \infty} r_w^{-1} \lambda^{-m}(f(F_w F_l^m q_l + 1) - f(F_w F_l^m q_l - 1))
\]
if the limit exists. For \( f \in \text{dom} \Delta_\mu \), the transverse derivatives at a point \( x \) with respect to different cells may be unrelated.

There are scaling identities for localized derivatives,
\[
\partial^w_T f(F_w q_l) = r_w^{-1} \partial_n (f \circ F_w)(q_l), \quad \text{and} \quad \partial^w_T f(F_w q_l) = r_w^{-1} \partial_T (f \circ F_w)(q_l).
\]

Let \( x \in V_s \setminus V_0 \). Suppose \( m_0 \) is the minimal value for which \( x \in V_{m_0} \). We say \( x \) is a junction vertex if there is at least two different \( m_0 \)-cells containing \( x \), i.e., \( x \) has at least two different representations \( x = F_w q_l \) with \( |w| = m_0 \). Otherwise, we call \( x \) a nonjunction vertex, which has exactly one representation \( x = F_w q_l \). For both the two different types of vertices, there is a canonical system of neighborhoods for each \( x \). On each certain neighborhood, there is a space of local multiharmonic functions. Our definition is slightly different from the definition in [S3].

**Definition 2.2.**

(a) For \( x \in V_m \setminus V_0 \), define the \( m \)-neighborhood of \( x \) as
\[
U_m(x) = \bigcup \{F_w K \mid x \in F_w K, |w| = m\}.
\]
Write \( U(x) = U_{m_0}(x) \) for the sake of simplicity, which obviously is the largest one. The boundary of the \( m \)-neighborhood \( U_m(x) \) is
\[
\partial U_m(x) = \{y \in V_m \mid y \sim_m x\}.
\]

(b) On each \( U_m(x) \), define local \((n + 1)\)-harmonic functions to be those functions \( h \) on \( U_m(x) \), with \( h \circ F_w \in \mathcal{H}_n \) for each \( w \), and \( \Delta^i_h \) satisfying the matching conditions at \( x \) for all \( 0 \leq i \leq n \). (If \( x \) is a nonjunction vertex, we say the matching conditions hold at \( x \) means \( \partial_n \Delta^i_h(x) = 0 \) for all \( 0 \leq i \leq n \). Write the space of all such functions \( \mathcal{H}_n(U_m(x)) \).

We remark that our notations differ from that in [S3] when \( x \) is a nonjunction vertex. In our setting, we always view \( x \) as an interior point in \( U_m(x) \).

Let \( W(x) \) denote the set of words of length \( m_0 \) such that there is a \( q_l \) with \( x = F_w q_l \) and \( w \in W(x) \). Call \( \#W(x) \) the degree of \( x \). Obviously, \( \#W(x) \geq 2 \) when \( x \) is a junction vertex, while \( \#W(x) = 1 \) when \( x \) is a nonjunction vertex. It is easy to verify that the dimension of the space \( \mathcal{H}_n(U_m(x)) \) is exactly \((n + 1)(N_0 - 1)\#W(x)\) for any \( x \in V_m \setminus V_0 \), with \( m \geq m_0 \).

For convenience, we always sort the elements in \( W(x) \) in lexicographical order. We use \( F_x \) to denote the contractive mapping on \( U(x) \) with \( F_x(y) = F_w F_l^{-1}(y) \) for \( y \in F_w K \) and \( w \in W(x) \). It is easy to see that \( F_x(U_m(x)) = U_{m+1}(x) \).

**Definition 2.3.** Call a finite union of cells \( A = \bigcup_{w \in \Omega(A)} F_w K \) a simple set, where \( \Omega(A) \) is a finite set of words. We define the boundary \( \partial A \) to be the collection of vertices \( y \) contained in \( A \) satisfying

1. \( y = F_w q_l \) with \( w \in \Omega(A), l = 0, 1, 2 \),
2. \( y \in V_0 \) or \( U_m(y) \) is not a subset of \( A \) for any \( m \).

It is clear that all nonjunction vertices \( x \) are treated as interior points of \( U_m(x) \) in the above definition, which is consistent with Definition 2.2. Also, there exists the possibility that \( y \) is a boundary point of \( A \), while \( y \) belongs to more than one component cells of \( A \) simultaneously. See Fig. 2.2 for an example of such \( A \) in \( SG_3 \).

Analogous to the global case, we define the local energy on \( A \) as
\[
E^A(f, g) = \sum_{w \in \Omega(A)} r_w^{-1} E(f \circ F_w, g \circ F_w).
\]
Let $\text{dom}(\mathcal{E}, A)$ to be the space of continuous functions on $A$ having the finite energy, and denote $\text{dom}_0(\mathcal{E}, A)$ to be the subspace of such functions which vanish at $\partial A$. The Laplacian localized to $A$ could be defined by the weak formulation in an analogous way. We denote $\text{dom}(\Delta_{\mu}, A)$ the domain of $\Delta_{\mu}$ on $A$. It is easy to check that if $f \in \text{dom}(\Delta_{\mu}, A)$, then $f|_A \in \text{dom}(\Delta_{\mu}, A)$. Additionally, the local multiharmonic function space on $A$ is denoted as $H_n(A)$ with dimension $(n + 1)!\#\partial A$.

The following Gauss-Green’s formula holds.

\begin{equation}
\mathcal{E}^A(f, g) = -\int_A \Delta_{\mu} fg d\mu + \sum_{z \in \partial A} \partial_n \Delta_{\mu} f(z)g(z)
\end{equation}

for any $g \in \text{dom}(\mathcal{E}, A)$ and $f \in \text{dom}(\Delta_{\mu}, A)$. Actually, this version of Gauss-Green’s formula can be obtained by adding Gauss-Green’s formula on component cells of $A$ together. For an interior vertex, by using the matching condition, the summation of normal derivatives vanishes automatically. For a boundary vertex $z$ of $A$, we write $\partial_n f(z) = \sum_{w \in \Omega(A), z \in F_w} \partial^w_n f(z)$.

3. Local multiharmonic functions

There are several bases of $H_n$ for different purposes. In [NSTY], to develop a theory of the local behavior of functions at a single boundary point, a basis of $H_n$ analogous to the monomials $x^j/j!$ on the unit interval, was described and studied on the Sierpinski gasket $\mathcal{SG}$. This could be easily extended to fractals whose all structures possess full $D3$ symmetry, as follows. Throughout this section, we drop the subscript $\mu$ of $\Delta_{\mu}$ for simplicity.

**Definition 3.1.** Fix a boundary point $q_l$. The monomials $Q_{jk}^{(l)}$ for $k = 1, 2, 3$ and $0 \leq j \leq n$ in $H_n$ are the multiharmonic functions satisfying

$\Delta^i Q_{jk}^{(l)}(q_l) = \delta_{ij}\delta_{k1},$

$\partial_n \Delta^i Q_{jk}^{(l)}(q_l) = \delta_{ij}\delta_{k2},$

$\partial_T \Delta^i Q_{jk}^{(l)}(q_l) = \delta_{ij}\delta_{k3}.$

It is easy to verify that these monomials form a basis of $H_n$ with dimension $3(n + 1)$. These monomials are related by the following identity,

$\Delta Q_{jk}^{(l)} = Q_{(j-1)k}^{(l)}.$

By the $D3$ symmetry, $Q_{jk}^{(l)}$ for different $l$’s are same under simply rotations. $Q_{j1}^{(l)}$ and $Q_{j2}^{(l)}$ are symmetric while $Q_{j3}^{(l)}$ is skew-symmetric with respect to the reflection symmetry that fixes $q_l$. 

**Figure 2.2.** The shade area is a simple subset in $\mathcal{SG}_3$, whose boundary points are dotted. Note that the center point is a boundary point which belongs to two component cells.
and interchanges the other two boundary points. Moreover, the monomials satisfy the following self-similar identities that describe the decay ratios of these functions near \( q_1 \).

\[
Q^{(l)}_{ij}(F^m x) = \rho^{lm}Q^{(l)}_{ij}(x),
\]

\[
Q^{(m)}_{ij}(F^m x) = r^{jm}Q^{(m)}_{ij}(x),
\]

\[
Q^{(m)}_{ij}(F^m x) = \lambda^{jm}Q^{(m)}_{ij}(x).
\]

Denote

\[
\alpha_j = Q^{(0)}_{ij}(q_1), \beta_j = Q^{(0)}_{ij}(q_1), \gamma_j = Q^{(0)}_{ij}(q_1),
\]

for \( j \geq 0 \). In [NSTY], one can find an elaborate recursive algorithm of these numbers on the Sierpinski gasket \( SG \). An important observation is that all these numbers are not equal to 0. The calculation there is technical and is hard to be extended to the general \( D3 \) case. However, we still could verify that \( \alpha_j, \beta_j, \gamma_j \) are never equal to 0 for some typical fractals with fully \( D3 \) symmetric structures, for example, the level 3 Sierpinski gasket \( SG_3 \) and the hexagasket \( HG \). We thus make a following technical assumption.

**Assumption 3.2.** All the numbers \( \alpha_j, \beta_j, \gamma_j \) are not equal to 0.

We will give the calculations of \( \alpha_j, \beta_j, \gamma_j \) for \( SG, SG_3 \) and \( HG \) in the Appendix, by using a new algorithm modified from that in [NSTY], which is shorter and more direct. Please refer to [LJSS] for an application of this new algorithm.

We need to extend the above definitions and discussions to all vertices in \( V_* \). Naturally, we have the following localized version of monomials, which will play an essential role in answering both the two questions posted in the Introduction.

**Definition 3.3.** Fix a vertex \( x \in V_* \setminus V_0 \). The monomials \( P^w_{jk} \) in \( \mathcal{H}_n(U(x)) \) for \( k = 1, 2, 3, 0 \leq j \leq n \) and \( w \in W(x) \) are the local multiharmonic functions satisfying

\[
\Delta^i P^w_{jk}(x) = \delta_{ij} \delta_{k1},
\]

\[
\partial_{w''}^w \Delta^i P^w_{jk}(x) = \delta_{ij} \delta_{k2} \delta_{w''w''} - \delta_{ij} \delta_{k2} \delta_{w''w''},
\]

\[
\partial_{x''}^w \Delta^i P^w_{jk}(x) = \delta_{ij} \delta_{k3} \delta_{w''w''},
\]

where \( w'' \) is the next word to \( w \) in \( W(x) \) in lexicographical order.

**Remark 1.** For \( k = 1 \), the superscript \( w \) is unnecessary, and we may not add it when discuss \( P^w_{jk} \) separately. For nonjunction vertices, there are no monomials in \( k = 2 \) case.

**Remark 2.** It is easy to check that \( \{P^w_{jk}|U_m(x)\} \) forms a basis of \( \mathcal{H}_n(U_m(x)) \).

Similar to (3.1)-(3.3), the following self-similar identities

\[
P^{(m)}_{i1}(F^m x y) = \rho^{lm}P^{(m)}_{i1}(y),
\]

\[
P^{(m)}_{i2}(F^m x y) = r^{jm}P^{(m)}_{i2}(y),
\]

\[
P^{(m)}_{i3}(F^m x y) = \lambda^{jm}P^{(m)}_{i3}(y),
\]

describe the decay behaviors of these monomials near \( x \). It is apparent that these monomials have symmetric properties analogous to the global case.

Denote \( R_i \) the rotations in \( D3 \) symmetric group, with \( R_i(q_1) = q_{i+1} \) (cyclic notation).

**Definition 3.4.** Fix a vertex \( x \in V_* \setminus V_0 \). For \( k = 1, 2, 3 \). Let \( P_k \) be a linear projection from \( \mathcal{H}_n(U(x)) \) into itself satisfying

\[
\Delta^i P_k(h)(x) = \delta_{k1} \Delta^i h(x),
\]

\[
\partial_{w''}^w \Delta^i P_k(h)(x) = \delta_{k2} \partial_{w''}^w \Delta^i h(x),
\]

\[
\partial_{x''}^w \Delta^i P_k(h)(x) = \delta_{k3} \partial_{x''}^w \Delta^i h(x).
\]
We say that an

1. To be more precise, we need the following definition of

These yield the result of the theorem.

\( y \in F_w K, w \in W(x) \), for any \( h \) in \( H_n(U(x)) \).

Clearly, \( P_k(h) \) is a linear combination of the monomials \( P_{jk}^w \), and it is easy to check that

\[ P_1(h) + P_2(h) + P_3(h) = h. \]

As for \( R \), roughly speaking, it is an operator on \( H_n(U(x)) \) which first rotates variables around \( x \), then takes mean values with weights proportional to \( r_w^{-1} \).

Let \( g_x \) be the relative symmetry in \( U(x) \), which fixes \( F_w q_l \) and permutes the other two boundary points of \( F_w K \) for each \( w \in W(x) \).

**Theorem 3.5.** Assume Assumption 2.1 holds. Let \( x \in V_0 \setminus V_0 \) and \( h \in H_n(U(x)) \), then the following identities hold,

\[
\begin{align*}
(3.8) & \quad P_1(h) = \frac{1}{2}(R(h) \circ g_x + R(h)), \\
(3.9) & \quad P_2(h) = \frac{1}{2}(h + h \circ g_x - R(h) \circ g_x - R(h)), \\
(3.10) & \quad P_3(h) = \frac{1}{2}(h - h \circ g_x).
\end{align*}
\]

**Proof.** The following equalities are consequences of symmetric definitions of \( \Delta, \partial_n \) and \( \partial_T \). For \( w \in W(x), 0 \leq i \leq n \), and for any \( h \in H_n(U(x)) \),

\[
\begin{align*}
\Delta^i R(h)(x) &= \Delta^i h(x), \\
\partial_n^w \Delta^i R(h)(x) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\Delta^i h \circ g_x(x) &= \Delta^i h(x), \\
\partial_n^w \Delta^i h \circ g_x(x) &= \partial_n^w \Delta^i h(x), \\
\partial_T^w \Delta^i h \circ g_x(x) &= -\partial_T^w \Delta^i h(x).
\end{align*}
\]

These yield the result of the theorem. \( \square \)

**Corollary 3.6.** Assume Assumption 2.1 holds. Let \( x \in V_0 \setminus V_0 \) and \( h \in H_n(U(x)) \), then for each \( m \geq m_0, |h|_{\partial U_m(x)} = 0 \) if and only if \( P_k(h)|_{\partial U_m(x)} = 0 \) for \( k = 1, 2, 3 \).

In the rest of this section, we give an application of the local monomials, to show that the higher order weak tangents of smooth functions \( f \) at any fixed vertex, could be expressible as limits of local multiharmonic functions that agree with \( f \) at the boundary of \( U_m(x) \). This answers Question 1. To be more precise, we need the following definition of **higher order weak tangents**.

**Definition 3.7.** Let \( x \) be a vertex in \( V_0 \setminus V_0 \) and \( f \) a function defined in a neighborhood of \( x \). We say that an \( (n+1) \)-harmonic function \( h \) is a weak tangent of order \( n+1 \) of \( f \) at \( x \) if

\[
(3.11) \quad (f - h)|_{\partial U_m(x)} = o((\rho^nr)^m)
\]

and

\[
(3.12) \quad (f - h - (f - h) \circ g_x)|_{\partial U_m(x)} = o((\rho^nx)^m).
\]

**Theorem 3.8.** Assume Assumption 2.1 and Assumption 3.2 hold. Let \( x \in V_0 \setminus V_0 \). Then the following two conclusions hold.

(a) An \( (n+1) \)-harmonic function \( h \) on \( U(x) \) is uniquely determined by the values \( h|_{\partial U_{m+i}}, 0 \leq i \leq n \), and any such values may be freely assigned.
(b) Let \( f \) be a continuous function defined in a neighborhood of \( x \), and assume \( f \) has a weak tangent of order \( n+1 \) at \( x \), denoted by \( h \). Let \( h_n \) be the \((n+1)\)-harmonic function defined in \( U(x) \), assuming the same values as \( f \) at the boundary points of \( U_{m+i}(x) \) for all \( 0 \leq i \leq n \). Then \( h_m \) converges to \( h \) uniformly on \( U(x) \).

**Remark 1.** This theorem extends the previous result in [CQ,S3] for the 1-order tangents and 1-order harmonic functions. For the nonjunction vertices, there is an implicit restriction that \( f, h \) and \( h_m \) should satisfy the equation \( \partial_n \Delta^i u(x) = 0 \) for all \( 0 \leq i \leq n \), since we always view \( x \) as an interior point in \( U(x) \).

**Remark 2.** There are some sufficient conditions to ensure the existence of the weak tangents. One can find more detailed discussion on the weak tangents (and tangents, strong tangents) in [S3].

**Proof of Theorem 3.8.** (a) The map from \( \mathcal{H}_n(U(x)) \) to the values \( h|_{\partial U_{m+i}(x)} \), \( i = 0, 1, \ldots, n \) is obviously a linear map, and the dimension of \( \mathcal{H}_n(U(x)) \) is \( 2(n+1)\#W(x) \), which is exactly equal to \( \# \bigcup_{0 \leq i \leq n} \partial U_{m+i}(x) \). Thus to prove (a), we only need to show that the map is injective.

Fix a word \( w \in W(x) \) with \( x = F_w q_0 \). Let \( h \in \mathcal{H}_n(U(x)) \). For \( k = 1, 2, 3 \), noticing that \( P_k(h) \circ F_w \) is a linear combination of \( Q^{(l)}_{jk} \), we have the following equalities

\[
P_k(h)(F_x^{m-|w|+i}F_w q_{l+1}) = P_k(h)(F_w F_l^{m-|w|+i} q_{l+1}) = \sum_{j=0}^{n} a_{j,k}^{w} Q^{(l)}_{jk} F_l^{m-|w|+i} q_{l+1}
\]

\[
= \sum_{j=0}^{n} (A_k)_{ij} a_{j,k}^{w},
\]

for \( (A_k)_{ij} = Q^{(l)}_{jk} F_l^{m-|w|+i} q_{l+1} = Q^{(l)}_{jk} F_l^{m+i} q_{l+1} \), where we denote \( m' = m - |w| \) for convenience. Thus the \((n+1) \times (n+1)\) matrix \( A_k \) induces a linear map from \( \{a_{j,k}^{w}\} \) to the values \( \{P_k(h)(F_x^{m-|w|+i}F_w q_{l+1})\} \).

We now show that the matrix \( A_k \) is invertible for \( k = 1, 2, 3 \). Denote by \( \Gamma^{(n)} \) an \((n+1) \times (n+1)\) matrix with

\[
\Gamma^{(n)} = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \rho & \rho^2 & \ldots & \rho^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \rho^n & \rho^{2n} & \ldots & \rho^{n^2}
\end{bmatrix},
\]

which is obviously invertible. Then by using the self-similar identities (3.1) – (3.3), we have

\[
(A_1)_{ij} = \rho^{m'j+i} \alpha_j = (\Gamma^{(n)})_{ij} \rho^{m'j} \alpha_j, \\
(A_2)_{ij} = r^{m'+i} \rho^{m'j+i} \beta_j = r^{m'+i} (\Gamma^{(n)})_{ij} \rho^{m'j} \beta_j, \\
(A_3)_{ij} = \lambda^{m'+i} \rho^{m'j+i} \gamma_j = \lambda^{m'+i} (\Gamma^{(n)})_{ij} \rho^{m'j} \gamma_j,
\]

which can be rewritten in matrix notation,

\[
A_1 = \Gamma^{(n)} \text{diag}(\alpha_0, \rho^{m'1}, \ldots, \rho^{m'n} \alpha_n),
\]

\[
A_2 = \text{diag}(r^{m'}, \ldots, r^{m'+n}) \Gamma^{(n)} \text{diag}(\beta_0, \rho^{m'1} \beta_1, \ldots, \rho^{m'n} \beta_n),
\]

\[
A_3 = \text{diag}(\lambda^{m'}, \ldots, \lambda^{m'+n}) \Gamma^{(n)} \text{diag}(\gamma_0, \rho^{m'1} \gamma_1, \ldots, \rho^{m'n} \gamma_n),
\]

from which it is obviously that all the matrices \( A_1, A_2, A_3 \) are invertible.

The above discussion shows that \( P_k(h) \) vanishes at \( \partial U_{m+i}(x) \) if and only if \( P_k(h) = 0 \). According to Corollary 3.6, \( h \) vanishes at \( \partial U_{m+i}(x) \) if and only if all \( P_k(h) \) vanishes at \( \partial U_{m+i}(x) \). Thus we have proved (a).
(b) We need to study the \((n+1)\)-harmonic functions \(h - h_m\). Notice that formula (3.13) still holds for \(h - h_m\).

For \(k = 1\), we have \(P_1(h_m - h)(F_x^{m'} + F_w q_{t+1}) = \sum_{j=0}^{n}(A_1)_{ij}a_{j1}\). Thus

\[
a_{j1} = \sum_{i=0}^{n}(A_1^{-1})_{ji}P_1(h_m - h)(F_x^{m'} + F_w q_{t+1})
= \alpha_j^{-1} \rho^{-jm'} \sum_{i=0}^{n} (\Gamma^{(n)})^{-1}_{ji} P_1(h_m - h)(F_x^{m'} + F_w q_{t+1}).
\]

According to (3.11), by using Theorem 3.5, we have \(P_1(h_m - h) \mid_{\partial U_{m+1}} = o(\rho^{m(n-j)})\), which gives that

\[
a_{j1} = o(\rho^{m(n-j)}).
\]

For \(k = 2\), a similar discussion shows that

\[
a_{j2}^{m} = \sum_{i=0}^{n}(A_2^{-1})_{ji}P_2(h_m - h)(F_x^{m'} + F_w q_{t+1})
= \beta_j^{-1} \rho^{-jm'} \sum_{i=0}^{n} (\Gamma^{(n)})^{-1}_{ji} r^{-m'i} P_2(h_m - h)(F_x^{m'} + F_w q_{t+1}).
\]

According to (3.11), still using Theorem 3.5, we have \(P_2(h_m - h) \mid_{\partial U_{m+1}} = o(\rho^{m(n-j)})\), and hence

\[
a_{j2}^{m} = o(\rho^{m(n-j)}).
\]

For \(k = 3\), the same argument yields that

\[
a_{j3}^{m} = \sum_{i=0}^{n}(A_3^{-1})_{ji}P_3(h_m - h)(F_x^{m'} + F_w q_{t+1})
= \gamma_j^{-1} \rho^{-jm'} \sum_{i=0}^{n} (\Gamma^{(n)})^{-1}_{ji} \lambda^{-m'i} P_3(h_m - h)(F_x^{m'} + F_w q_{t+1}).
\]

Using (3.12) and Theorem 3.5, we can get \(P_3(h_m - h) \mid_{\partial U_{m+1}} = o(\lambda^{m(n-j)})\), so

\[
a_{j3}^{m} = o(\lambda^{m(n-j)}).
\]

Thus we have proved that for \(k = 1, 2, 3\), \(P_k(h_m - h)\) converges uniformly to zero on each cell \(F_w K\), which yields that \(h_m\) converges uniformly to \(h\) on \(U(x)\). \(\square\)

4. POINTWISE FORMULA FOR THE HIGHER ORDER LAPLACIANS

In this section, we will deal Question 2, restricted to the \(D3\) symmetric fractals. We still drop the subscript \(\mu\) of \(\Delta_\mu\) for simplicity.

4.1. Definition of pointwise formula. Analogous to the pointwise formula of the Laplacian, we will show that we can approach the \(n\)-order Laplacian by the \(n\) times iteration of the renormalized discrete Laplacian, which means

\[
\Delta^n f(x) = \lim_{m \to \infty} \tilde{\Delta}_m^n f(x).
\]

Notice that \(\tilde{\Delta}_m^n\) may not be defined on all vertices in \(V_m \setminus V_0\) for \(n \geq 2\). For example, when \(n = 2\), there are exactly those vertices which are not connected with the boundary \(V_0\) having the operation \(\tilde{\Delta}_m^2\) well-defined.

Definition 4.1. For two vertices \(x, y \in V_m\), the \(m\)-distance \(d_m(x, y)\) is the minimal number of edges which connect \(x\) to \(y\) in \(G_m\).
It is easy to check that any vertex satisfying $d_m(x, V_0) \geq n$ has a well-defined $\tilde{\Delta}^n_m f(x)$. Obviously, 
\[ V^n_m = \{ x \in V_m : d_m(x, V_0) \geq n \} \]
is the definition domain of $\tilde{\Delta}^n_m$. See Fig. 4.1 for $V^2_2$, the definition domain of $\tilde{\Delta}^2_2$ for $SG$.

**Figure 4.1.** The definition domain of $\tilde{\Delta}^2_2$ for $SG$, denoted by dots.

For fixed $x \in V^n_m$, the calculation of $\tilde{\Delta}^n_m f(x)$ involves the values of $f$ at those vertices with $m$-distance to $x$ no more than $n$, which are collected as
\begin{equation}
L^n_m(x) = \{ y \in V_m : d_m(x, y) \leq n \} = \bigcup \{ L^n_m(y) : y \in L^{n-1}_m(x) \}.
\end{equation}
The area bounded by these vertices is obviously a neighborhood of $x$, which may be written as $U^n_m(x)$ (see Fig. 4.2), which is
\begin{equation}
U^n_m(x) = \bigcup \{ U_m(y) : y \in L^{n-1}_m(x) \}.
\end{equation}

**Figure 4.2.** Some examples of $U^n_3(x)$ with $n \leq 3$ in $SG$.

It is natural that the boundary of $U^n_m(x)$ is
\[ \partial U^n_m(x) = L^n_m \setminus \{ y \in V_m \setminus V_0 : U_m(y) \subset U^n_m(x) \}, \]
which is consistent with the boundary of $U_m(x)$ and the boundary of simple set $A$ as introduced in Section 2. In our setting, nonjunction vertices always are viewed as interior points. It is easy to check that $\partial U^n_m(x) \subset L^n_m(x) \setminus L^{n-1}_m(x)$. There indeed exist vertices that belong to $L^n_m(x) \setminus L^{n-1}_m(x)$, which are not boundary points of $U^n_m(x)$. For example, it is the case when we choose $x$ to be the bottom dotted vertex in Fig 4.1 for $SG$ for $n = m = 2$. 
Remark. The shape of $U^m_n(x)$ varies for $x$ in $V^m_n$ and $m \geq 0$. We could give a classification of them. Let $x \in V^m_n$ and $y \in V^m_n$. We say $U^m_n(x)$ and $U^m_n(y)$ belong to a same type if there exists some mapping $F$ which is a combination of rotations, reflections and scalings such that $FU^m_n(x) = U^m_n(y)$.

We conclude that there are only finite types of $U^m_n(x)$ for any fixed $n$. In fact, the second equality of (4.2) shows that if there are finite types of $U^m_n^{-1}(x)$, then the types of $U^m_n(x)$ is also finite. This observation will be useful in the proof of the uniform convergence of the pointwise formula. See Fig. 4.3 for the total types of $U^m_n(x)$ in $SG$.

![Figure 4.3. The total types of $U^m_n(x)$ in $SG$.](image)

The following theorem is an answer to Question 2.

**Theorem 4.2.** Assume Assumption 2.1 holds, then

(a) For $f \in \text{dom}(\Delta^n)$, the pointwise formula (4.1) holds with the uniform limit on $V_n \setminus V_0$.

(b) Conversely, let $f \in C(K)$ and the right side of (4.1) converges uniformly to a continuous function $u$ on $V_n \setminus V_0$. Then $f \in \text{dom}(\Delta^n, K \setminus V_0)$ with $\Delta^n f = u$ on $K \setminus V_0$.

Before proving, we remark that it looks that the (b) part of this theorem does not involve the whole information of the function $f$, and it has something to do with the existence of harmonic functions with singularities at boundary points (See more explanation on point singularities in [BSSY]). In fact, the conclusion is equivalent to that for any $g \in \text{dom}(\Delta^n)$ with $\Delta^n g = u$, we have $f - g$ may be an $n$-harmonic function with singularity. However, if there is no multiharmonic function with singularity, for example, the unit interval case, we see that $f \in \text{dom}(\Delta^n)$ already.

4.2. **Proof of Theorem 4.2(a).** We will take two steps to prove part (a) of Theorem 4.2. First, we deal with those functions which are local $(n + 1)$-harmonic near $x$ with $x \in V^m_n$, to get that (4.1) holds without taking the limit. Then, we prove the result for general functions in $\text{dom}(\Delta^n)$.

**Lemma 4.3.** Let $x$ be a vertex in $V_n \setminus V_0$, $h$ be an $(n + 1)$-harmonic function in $\mathcal{H}_n(U_n(x))$. Then

$$\tilde{\Delta}_n h(x) = \sum_{j=1}^n \rho^{m(j-1)}\alpha^{-1}_j \Delta^j h(x).$$

In particular, $\alpha_1 = 1/6$.

*Proof.* Fix a word $w \in W(x)$ with $x = F_w q_l$. Note that $P_1(h \circ F^{m-|w|}_x) \circ F_w$ is a linear combination of monomials $Q_{j_1}^{(l)}$. In fact, we have

$$P_1(h \circ F^{m-|w|}_x) \circ F_w = \sum_{j=0}^n \rho^{m_j} \Delta^j h(x) Q_{j_1}^{(l)}.$$ 

by comparing the values at $x$ when applying $\Delta^j$ on both sides. Thus we have

$$P_1(h \circ F^{m-|w|}_x)(F_w q_{l+1}) = \sum_{j=0}^n \rho^{m_j} \Delta^j h(x) Q_{j_1}^{(l)}(q_{l+1}) = \sum_{j=0}^n \rho^{m_j} \alpha_j \Delta^j h(x).$$
On the other hand, according to (3.8),
\[ P_1(h \circ F_{x}^{m-|w|})(F_w q_{l+1}) = (2 \sum_{w' \in W(x)} r_{w'}^{-1} \sum_{y \in F_{w'} K, y \sim m x} r_{w'}^{-1} h(y) \]
\[ = (\sum_{y \sim m x} c_{xy})^{-1} \sum_{y \sim m x} c_{xy} h(y). \]

Thus,
\[ \tilde{\Delta}_m h(x) = \frac{\sum_{y \sim m x} c_{xy}}{\int \psi_x^m d\mu} \left( P_1(h \circ F_{x}^{m-|w|})(F_w q_{l+1}) - h(x) \right) \]
\[ = 2 \frac{\sum_{w' \in W(x)} r_{w'}^{-1} r_{-(m-|w|)} \sum_{y \sim m x} \rho^{m j} \alpha_j \Delta^j h(x) \]
\[ = 6 \rho^{-m} \sum_{j=1}^{n} \rho^{m j} \alpha_j \Delta^j h(x). \]

The second equality above comes from the fact that \( \alpha_0 \) always equal to 1. From the arbitrariness of \( h \), if we choose \( h \) to satisfy \( \Delta h = 1 \) in the above equality, this gives
\[ \tilde{\Delta}_m h(x) = 6 \alpha_1 \]
for all \( m \). By passing \( m \) to infinity, we get that \( \alpha_1 = 1/6 \). Thus we have proved the lemma. \( \Box \)

**Remark.** We use the decomposition of \( h \) based on the monomials in the above proof, which requires the harmonic extension matrices \( M_0, M_1, M_2 \) to be non-degenerate. However, this requirement is not necessarily essential. In fact, we could alternatively start from an “easy” basis, which extends the conclusion to the degenerate cases. We will give this method in Section 6 when discussing the D4 symmetric fractals.

It is interesting that the constant \( \alpha_1 = 1/6 \) is universal for all D3 symmetric fractals, which is an initial value for the calculations in the Appendix.

**Lemma 4.4.** Let \( x \) be a vertex in \( V_m^n \) and \( h \in \mathcal{H}_n(U_m^n(x)) \), then
\[ \tilde{\Delta}_m h(x) = \Delta^n h(x). \]

**Proof.** It is obvious that \( h \mid_{U_m^n(y)} \in \mathcal{H}_n(U_m^n(y)) \) for any \( y \) with \( d_m(x, y) \leq n - 1 \). So we can apply Lemma 4.3 to all points in \( U_{m-1}^{n-1}(x) \). Thus we have
\[ \tilde{\Delta}_m h(x) = \tilde{\Delta}_m^{n-1}(\tilde{\Delta}_m h(x)) \]
\[ = \tilde{\Delta}_m^{n-1} \left( \sum_{j=1}^{n} \rho^{m(j-1)} \alpha_1^{-1} \alpha_j \Delta^j h(x) \right) \]
\[ = \sum_{j=1}^{n} \rho^{m(j-1)} \alpha_1^{-1} \alpha_j \tilde{\Delta}_m^{n-1}(\Delta^j h(x)). \]

Since for each \( j \geq 1 \), \( \Delta^j h \) belongs to \( \mathcal{H}_{n-1}(U_{m-1}^{n-1}(x)) \), by a standard inductive argument, we then have
\[ \tilde{\Delta}_m h(x) = \sum_{j=1}^{n} \rho^{m(j-1)} \alpha_1^{-1} \alpha_j \Delta^{n+j-1} h(x) = \Delta^n h(x). \]

Hence we have proved the lemma. \( \Box \)

Now for each \( x \in V_m^n \), we will define a global function \( \phi_{m,x}^{(n)} \) supported in \( U_m^n(x) \), belonging to \( \text{dom}(\Delta^{n-1}, U_m^n(x)) \), called \( n \)-tent function, which is piecewise \( n \)-harmonic and for any \( f \in \text{dom}(\Delta^n) \), it holds that
\[ \tilde{\Delta}_m f(x) = -\mathcal{E}(\phi_{m,x}^{(n)}, \Delta^{n-1} f). \]

We will define this function by an inductive argument.
When $n = 1$, we just need to choose $\phi^{(1)}_{m,x} = (\int \psi_x^m d\mu)^{-1}\psi_x^m$, and we have

$$\tilde{\Delta}_m f(x) = -\mathcal{E}(\phi^{(1)}_{m,x}, f).$$

When $n = 2$, first we define

$$\tilde{\phi}^{(2)}_{m,x} = \sum_{y \sim m,x} (\int \psi_x^m d\mu)^{-1}c_{xy}(\phi^{(1)}_{m,y} - \phi^{(1)}_{m,x}).$$

It is easy to check that

$$\tilde{\Delta}^2_m f(x) = \sum_{y \sim m,x} (\int \psi_x^m d\mu)^{-1}c_{xy} \left( \tilde{\Delta}_m f(y) - \tilde{\Delta}_m f(x) \right)$$

$$= -\sum_{y \sim m,x} (\int \psi_x^m d\mu)^{-1}c_{xy} \left( \mathcal{E}(\phi^{(1)}_{m,y}, f) - \mathcal{E}(\phi^{(1)}_{m,x}, f) \right)$$

$$= -\mathcal{E}(\tilde{\phi}^{(2)}_{m,x}, f).$$

Now let

$$\phi^{(2)}_{m,x}(\cdot) = -\int_K G_{m,2}(\cdot, z)\tilde{\phi}^{(2)}_{m,x}(z)d\mu(z),$$

where $G_{m,2}(\cdot, \cdot)$ is the local Green’s function (See [KSS, S4]) on $U^2_m(x)$.

We will show that it satisfies both the Dirichlet and Neumann boundary conditions at the boundary of $U^2_m(x)$, and thus could be extended to the whole $K$ by zero extension. Then by Gauss-Green’s formula, we have

$$\tilde{\Delta}^2_m f(x) = -\mathcal{E}(\phi^{(2)}_{m,x}, \Delta f).$$

Inductively, assume that we have already constructed the $(n - 1)$-level tent function $\phi^{(n-1)}_{m,x}$ for vertices $x$ in $V^m_{n-1}$, satisfying the Dirichlet boundary condition at $\partial U^m_{n-1}(x)$, as well as $\tilde{\Delta}^m_{n-1} f(x) = -\mathcal{E}(\phi^{(n-1)}_{m,x}, \Delta^{n-2} f)$. We will first define

$$\tilde{\phi}^{(n)}_{m,x} = \sum_{y \sim m,x} (\int \psi_x^m d\mu)^{-1}c_{xy}(\phi^{(n-1)}_{m,y} - \phi^{(n-1)}_{m,x}),$$

which obviously satisfies that

$$\tilde{\Delta}^n_m f(x) = -\mathcal{E}(\tilde{\phi}^{(n)}_{m,x}, \Delta^{n-2} f),$$

then define

$$\phi^{(n)}_{m,x}(\cdot) = -\int_K G_{m,n}(\cdot, z)\tilde{\phi}^{(n)}_{m,x}(z)d\mu(z),$$

where $G_{m,n}(\cdot, \cdot)$ is the local Green’s function on $U^n_m(x)$.

We will show that $\phi^{(n)}_{m,x}$ satisfies both the Dirichlet and Neumann boundary conditions at $\partial U^n_m(x)$, and then extend it to the whole $K$ by zero extension. Then

$$\tilde{\Delta}^n_m f(x) = -\mathcal{E}(\phi^{(n)}_{m,x}, \Delta^{n-1} f).$$

Thus for $n \geq 2$, $\phi^{(n)}_{m,x}$ defined using the above recipe will satisfy both the Dirichlet and the Neumann boundary conditions at the boundary $\partial U^n_m(x)$. The following lemma remains to be proved.

**Lemma 4.5.** Let $n \geq 2$. Suppose we have defined $\phi^{(n-1)}_{m,x}$ for vertices in $V^m_{n-1}$, satisfying $\tilde{\Delta}^m_{n-1} f(\cdot) = -\mathcal{E}(\phi^{(n-1)}_{m,x}, \Delta^{n-2} f)$, with the Dirichlet boundary condition holding at $\partial U^m_{n-1}(\cdot)$. Then the function $\phi^{(n)}_{m,x}$ defined by (4.5) and (4.7) satisfies both the Dirichlet and Neumann boundary conditions at the boundary of $U^n_m(x)$. Moreover, the equality (4.8) holds.
Proof. Let \( h \) be a multiharmonic harmonic function in \( \mathcal{H}_{n-1}(U^n_m(x)) \). By the definition of \( \tilde{\phi}^{(n)}_{m,x} \), it is easy to check that
\[
\mathcal{E}(\tilde{\phi}^{(n)}_{m,x}, \Delta^{n-2}h) = \sum_{y \sim m} \left( \int \psi_{m}^n d\mu \right)^{-1} c_{xy} \left( \mathcal{E}(\tilde{\phi}^{(n-1)}_{m,y}, \Delta^{n-2}h) - \mathcal{E}(\phi^{(n-1)}_{m,x}, \Delta^{n-2}h) \right)
= - \sum_{y \sim m} \left( \int \psi_{m}^n d\mu \right)^{-1} c_{xy} \left( \tilde{\Delta}^{n-1}h(y) - \tilde{\Delta}^{n-1}h(x) \right)
= - \sum_{y \sim m} \left( \int \psi_{m}^n d\mu \right)^{-1} c_{xy} \left( \Delta^{n-1}h(y) - \Delta^{n-1}h(x) \right)
= 0,
\]
where the second equality comes from the assumption of \( \phi^{(n-1)}_{m,x} \), the third equality is a result of Lemma 4.4, and the forth equality follows from the fact that \( \Delta^{n-1}h \in \mathcal{H}_0(U^n_m(x)) \).

On the other hand, by using the Dirichlet boundary condition of \( \phi^{(n-1)}_{m,x} \) and \( \phi^{(n)}_{m,x} \) at the boundary of \( U^n_m(x) \), and repeatedly using Gauss-Green’s formula, we have
\[
\mathcal{E}(\tilde{\phi}^{(n)}_{m,x}, \Delta^{n-2}h) = - \int_{U^n_m(x)} \tilde{\phi}^{(n)}_{m,x} \Delta^{n-1}h d\mu + \sum_{z \in \partial U^n_m(x)} \tilde{\phi}^{(n)}_{m,x}(z) \partial_n \Delta^{n-1}h(z)
= \int_{U^n_m(x)} \phi^{(n)}_{m,x} \Delta^n h d\mu - \int_{U^n_m(x)} \Delta \phi^{(n)}_{m,x} \Delta^{n-1}h d\mu
= \sum_{z \in \partial U^n_m(x)} \phi^{(n)}_{m,x}(z) \partial_n \Delta^{n-1}h(z) - \sum_{z \in \partial U^n_m(x)} \partial_n \phi^{(n)}_{m,x}(z) \Delta^{n-1}h(z)
= - \sum_{z \in \partial U^n_m(x)} \partial_n \phi^{(n)}_{m,x}(z) \Delta^{n-1}h(z).
\]
Thus we have proved that \( \sum_{z \in \partial U^n_m(x)} \partial_n \phi^{(n)}_{m,x}(z) \Delta^{n-1}h(z) = 0 \) holds for any \( h \in \mathcal{H}_{n-1}(U^n_m(x)) \), which yields that
\[
\sum_{z \in \partial U^n_m(x)} \partial_n \phi^{(n)}_{m,x}(z) h(z) = 0
\]
holds for any \( h \in \mathcal{H}_0(U^n_m(x)) \). By the arbitrariness of \( h \), we have proved that \( \phi^{(n)}_{m,x} \) satisfies both the Dirichlet and Neumann boundary conditions at \( \partial U^n_m(x) \).

Now for general function \( f \in dom(\Delta^n) \), by (4.6), using the boundary conditions of \( \phi^{(n)}_{m,x} \) at \( \partial U^n_m(x) \), and by using Gauss-Green’s formula, we finally get that
\[
\tilde{\Delta}^n_m f(x) = -\mathcal{E}(\tilde{\phi}^{(n)}_{m,x}, \Delta^{n-2}f) = -\mathcal{E}(\Delta \phi^{(n)}_{m,x}, \Delta^{n-2}f) = -\mathcal{E}(\phi^{(n)}_{m,x}, \Delta^{n-1}f). \quad \Box
\]

**Lemma 4.6.** For any \( m, x \in V^n_m \), it holds that \( \int \phi^{(n)}_{m,x} d\mu = 1 \). Furthermore, for any same type sets \( U^n_m(x) \) and \( U^n_m'(y) \), we have \( \|\phi^{(n)}_{m,x}\|_1 = \|\phi^{(n)}_{m',y}\|_1 \) if the \( m \)-level conductances on \( U^n_m(x) \) are proportional to those on \( U^n_m'(y) \).

Proof. Applying Gauss-Green’s formula to (4.8), using the Dirichlet boundary condition of \( \phi^{(n)}_{m,x} \) at \( \partial U^n_m(x) \), we have
\[
\tilde{\Delta}^n_m f(x) = -\mathcal{E}(\phi^{(n)}_{m,x}, \Delta^{n-1}f) = \int \phi^{(n)}_{m,x} \Delta^n f d\mu
\]
for any \( f \in dom(\Delta^n) \). Choosing a multiharmonic function \( h \in \mathcal{H}_n(U^n_m(x)) \) with \( \Delta^n h = 1 \) and taking it into the above equality, using Lemma 4.4, we get
\[
\int \phi^{(n)}_{m,x} d\mu = 1.
\]
Let $U^n_m(x)$ and $U^n_{m'}(y)$ be in same type. It means there is a mapping $F$ which is a combination of rotations, reflections and scalings, satisfying $F U^n_m(x) = U^n_{m'}(y)$. It is easy to find that

$$
\phi_{m',y}^{(n)} = \frac{\mu(U^n_m(x))}{\mu(U^n_{m'}(y))} \phi_{m,x}^{(n)} \circ F^{-1},
$$

by scaling. Hence $\|\phi_{m,x}^{(n)}\|_1 = \|\phi_{m',y}^{(n)}\|_1$. □

Since there are only finite types of $U^n_m(x)$, and for each type, there are only finite subtypes with proportional conductances, we have

**Corollary 4.7.** Let $n \geq 2$ be fixed. For any $m$, any $x \in V^n_m$, $\|\phi_{m,x}^{(n)}\|_1$ is uniformly bounded.

**Proof of Theorem 4.2(a).** Applying (4.9), Lemma 4.6 and Corollary 4.7, we have

$$
|\hat{\Delta}^n_m f(x) - \Delta^n f(x)| = \left| \int \phi_{m,x}^{(n)}(z)(\Delta^n f(z) - \Delta^n f(x)) \, d\mu(z) \right|
$$

for some constant $C > 0$, where $\omega_{\Delta^n f}(U^n_m(x))$ is the oscillation of $\Delta^n f$ in $U^n_m(x)$. Since $\Delta^n f$ is continuous on $K$, $\omega_{\Delta^n f}(U^n_m(x))$ will go to zero uniformly as $m$ goes to infinity. Thus we have (4.1) holds uniformly. □

**Remark.** The proof provides that the ratio of the convergence in (4.1) depends only on the modulus of continuity of $\Delta^n f$.

### 4.3. Proof of Theorem 4.2(b).

In this subsection we will give the proof of the second part of Theorem 4.2.

For a simple set $A$ in $K$, we use $S(H_0, V_m, A)$ to denote the space of harmonic splines, which are harmonic in each $m$-level cell in $A$. For those harmonic splines vanishing at the boundary of $A$, we denote the collection of them by $S_0(H_0, V_m, A)$. For any function $u \in l(V_m \cap (A \setminus \partial A))$, there is a unique solution $\psi \in S_0(H_0, V_m, A)$ satisfying

$$
\hat{\Delta}^n_m \psi(x) = u(x), \quad \forall x \in V_m \cap (A \setminus \partial A).
$$

In fact, for $\psi_1 \neq \psi_2 \in S_0(H_0, V_m, A)$, we have $\hat{\Delta}^n_m \psi_1 \neq \hat{\Delta}^n_m \psi_2$, so that $\hat{\Delta}^n_m$ is an injection. Comparing the dimension, one can find that $\hat{\Delta}^n_m$ is reversible. For convenience, we define $G_{m,A}$ the inverse operator of $-\hat{\Delta}^n_m$. It means that for any $u \in l(V_m \cap (A \setminus \partial A))$, $G_{m,A}u \in S_0(H_0, V_m, A)$ and

$$
-\hat{\Delta}^n_m G_{m,A} u = u.
$$

We denote $G_A$ the local Green’s operator on $A$, i.e., for any continuous function $u$ on $A$, $G_A u \in dom_0(\Delta, A)$, and satisfies

$$
-\Delta G_A u = u.
$$

The following lemma shows that $G_{m,A}$ will go to $G_A$ as $m$ goes to infinity.

**Lemma 4.8.** For any simple set $A$ in $K$, let $f \in dom_0(\Delta, A)$ and $\psi_m \in S_0(H_0, V_m, A)$. If $\hat{\Delta}^n_m \psi_m$ converges to $f$ uniformly as $m$ goes to infinity, then $\psi_m$ converges to $f$ uniformly as $m$ goes to infinity.

**Proof.** First we will show $\{\psi_m\}$ are equicontinuous and uniformly bounded. Observe that for any function $g \in dom_0(E, A)$,

$$
\mathcal{E}^A(\psi_m, g) = - \sum_{x \in V_m \cap (A \setminus \partial A)} g(x) \Delta_m \psi_m(x)
$$

$$
= - \sum_{x \in V_m \cap (A \setminus \partial A)} \left( \int_K \psi_m^x \, d\mu \right) g(x) \Delta_m \psi_m(x)
$$

$$
= - \sum_{F_{x,K} \subseteq A} \sum_{x \in F_{x,K} \cap V_0} \frac{1}{3} \mu_x g(x) \Delta_m \psi_m(x).
$$
Noticing that $\psi_m$ satisfies the Dirichlet boundary condition at $\partial A$, combining with the estimate $|\psi_m(x) - \psi_m(y)|^2 \leq R(x, y)E^A(\psi_m)$, with $R(x, y)$ the effective resistance metric (See [Ki4, S4]) between $x$ and $y$ on $A$, we have

$$\|\psi_m\|_\infty^2 \leq c_1E^A(\psi_m) \leq c_2\|\psi_m\|_\infty \|\tilde{\Delta}_m\psi_m\|_\infty,$$

for some constants $c_1, c_2 > 0$, which results that

$$\|\psi_m\|_\infty^2 \leq c_1E^A(\psi_m) \leq c_2\|\tilde{\Delta}_m\psi_m\|_\infty^2.$$

Since we have $\tilde{\Delta}_m\psi_m \rightarrow \Delta f$ uniformly, $\|\tilde{\Delta}_m\psi_m\|_\infty$ is uniformly bounded. So we have $\{\psi_m\}$ are uniformly bounded and equicontinuous. Denote $f - \psi_m = g_m$, then $\{g_m\}$ are also uniformly bounded and equicontinuous. Moreover,

$$E^A(g_m) = -\int_A \mu_d\Delta f d\mu + \sum_{F_w K \subset A} \sum_{x \in F_w V_0} \frac{1}{3}\mu_w g_m(x)\tilde{\Delta}_m \psi_m(x)$$

$$= \sum_{F_w K \subset A} \left( -\int_{F_w K} \mu_d\Delta f d\mu + \sum_{x \in F_w V_0} \frac{1}{3}\mu_w g_m(x)\tilde{\Delta}_m \psi_m(x) \right)$$

$$= \sum_{F_w K \subset A} \mu_w \left( -\int_K g_m \circ F_w(\Delta f) \circ F_w d\mu + \sum_{x \in F_w V_0} \frac{1}{3}g_m(x)\tilde{\Delta}_m \psi_m(x) \right).$$

All terms in the sum would converge to 0 uniformly as $m$ goes to $\infty$. In fact,

$$\left| -\int_K g_m \circ F_w(\Delta f) \circ F_w d\mu + \sum_{x \in F_w V_0} \frac{1}{3}g_m(x)\tilde{\Delta}_m \psi_m(x) \right|$$

$$\leq \sum_{x \in F_w V_0} \frac{1}{3} \sup_{y \in F_w K} |g_m(y)\Delta f(y) - g_m(x)\tilde{\Delta}_m \psi_m(x)|.$$

Using the equicontinuous and uniformly boundedness of $\{g_m\}$, we then have $\lim_{m \rightarrow \infty} E^A(f - \psi_m) = 0$. Together with the fact that $f - \psi_m$ satisfies the Dirichlet boundary condition at $\partial A$, it yields the result of Lemma 4.8. □

**Remark.** We restate Lemma 4.8 as follows. Suppose $\phi_m \in S(H_0, V_m, A)$ converges uniformly to a continuous function $u$, then

$$\lim_{m \rightarrow \infty} G_{m,A}\phi_m = G_A u$$

holds uniformly.

**Proof of Theorem 4.2(b).** Assume that we have $\lim_{m \rightarrow \infty} \tilde{\Delta}_m^n f(x) = u(x)$ uniformly on $V_s \setminus V_0$. Then by repeatedly using Lemma 4.8, on any $A$ not intersecting the boundary $V_0$, we have

$$\lim_{m \rightarrow \infty} (-G_{m,A})^n \tilde{\Delta}_m^n f = (-G_A)^n u$$

converges uniformly. So we have $f - (-G_{m,A})^n \tilde{\Delta}_m^n f$ converges uniformly to the function $f - (-G_A)^n u$.

Now we prove $f - (-G_A)^n u \in H_{n-1}(A)$.

Recall that in Lemma 4.3, we have shown that for any $(n+1)$-harmonic function $h \in H_n(A)$, $\tilde{\Delta}_m h$ must be equal to some $n$-harmonic function on $A \cap V_m$, see (4.4). It is not hard to verify that the right side of (4.4) could go through all $n$-harmonic functions. Thus we have an inverse
conclusion that for any $n$-harmonic function $h'$ on $A$, there is a $(n+1)$-harmonic function $h \in \mathcal{H}_n(A)$ such that $\tilde{\Delta}_m h = h'$ on $V_m \cap (A \setminus \partial A)$.

Now we apply the above discussion in the proof. First we have

$$\tilde{\Delta}_m (\tilde{\Delta}_m^{n-1} f + G_{m,A} \tilde{\Delta}_m^n f) = 0,$$

and thus $\tilde{\Delta}_m^{n-1} f + G_{m,A} \tilde{\Delta}_m^n f$ equals to some harmonic function on $A$. Since for each $1 < i \leq n$,

$$\tilde{\Delta}_m (\tilde{\Delta}_m^{n-1} f - (-G_{m,A}) \tilde{\Delta}_m^n f) = \Delta_m^{n-1} f - (-G_{m,A}) \tilde{\Delta}_m^n f,$$

by repeatedly using the above discussion, we have that $f - (-G_{m,A}) \tilde{\Delta}_m^n f$ equals to some $n$-harmonic function on $A \cap V_m$. Noticing that the space of $n$-harmonic functions on $A$ is of finite dimension, the uniform limit $f - (-G_{m,A})^n u$ of $f - (-G_{m,A}) \tilde{\Delta}_m^n f$ is of course an $n$-harmonic function.

Thus we have $f = (-G_{m,A})^n u + (f - (-G_{m,A}) \tilde{\Delta}_m^n f) \in \text{dom}(\tilde{\Delta}_m^n, A)$, and obviously $\tilde{\Delta}_m^n f = u$ on $A$. By the arbitrariness of $A$, we finally have proved $\tilde{\Delta}_m^n f = u$ on $K \setminus V_0$. \qed

5. Pointwise formula of $\Delta^n_\mu$ on general p.c.f. fractals

We have no idea on how to extend the previous results to other p.c.f. fractals. However, we still have some pointwise calculations of the higher order Laplacian in general.

The following is an extension of the mean value property of harmonic functions.

**Lemma 5.1.** Let $l \in \mathbb{N}$, $\{y_j\}_{j=1}^l \subset V_\ast$, and $\{a_j\}_{j=1}^l \subset \mathbb{R}$ with $\sum_{j=1}^l a_j = 0$. Then there exists a function $\phi \in \text{dom}\mathcal{E}$ such that

$$\sum_{j=1}^l a_j f(y_j) = -\mathcal{E}(f, \phi) \quad (5.1)$$

holds for any $f \in \text{dom}\mathcal{E}$. Furthermore, if we additionally assume $\sum_{j=1}^l a_j h(y_j) = 0$ holds for any $h \in \mathcal{H}_0$, then there is a unique such $\phi$ satisfying the Dirichlet boundary condition $\phi|_{V_0} = 0$.

**Proof.** Assume $\{y_j\}_{j=1}^l \subset V_m$ for some $m$, we now find the function $\phi$ in $S(\mathcal{H}_0, V_m)$, the space of continuous functions which are harmonic in each $m$-level cell of $K$.

In fact, For any $\psi \in S(\mathcal{H}_0, V_m)$, we have

$$-\mathcal{E}(f, \psi) = \sum_{y \in V_m \setminus V_0} \Delta_m \psi(y) f(y) - \sum_{y \in V_0} \partial_n \psi(y) f(y), \forall f \in \text{dom}\mathcal{E}.$$ 

It is easy to check that the map $\psi \mapsto \{\partial_n \psi|_{V_0}, \Delta_m \psi|_{V_m \setminus V_0}\}$ is injective from the space $S(\mathcal{H}_0, V_m)$ modulo constants to $\mathbb{R}^{\#V_m}$, and Gauss-Green’s formula, it always holds

$$\sum_{y \in V_m \setminus V_0} \Delta_m \psi(y) + \sum_{y \in V_0} \partial_n \psi(y) = 0.$$ 

Thus the map is a bijection from $S(\mathcal{H}_0, V_m)$ modulo constants to a $(\#V_m - 1)$-dimensional subspace of $\mathbb{R}^{\#V_m}$ by a counting dimension argument. Thus, we have proved the existence of function $\phi$ satisfying (5.1).

If we additionally assume $\sum_{j=1}^l a_j h(y_j) = 0$, then we have $\mathcal{E}(h, \phi) = 0$ for any $h \in \mathcal{H}_0$. Thus $\mathcal{E}_0(h, \phi) = 0, \forall h \in \mathcal{H}_0$.

which yields that $\phi|_{V_0} = C$, since $\mathcal{E}_0(\cdot, \cdot)$ is an inner product on $\mathbb{R}^{\#V_0}$ modulo constants. So there exists a unique $\phi$ satisfying (5.1) with the Dirichlet boundary condition. \qed

**Theorem 5.2.** (Calculation of $\Delta^n_\mu$) Let $l \in \mathbb{N}$, $\{y_j\}_{j=1}^l \subset V_\ast$ and $\{a_j\}_{j=1}^l \subset \mathbb{R}$. Assume

$$\begin{cases} 
\sum_{j=1}^l a_j h(y_j) = 0, \forall h \in \mathcal{H}_0, \\
\sum_{j=1}^l a_j h'(y_j) = A, \forall h' \text{ with } \Delta^n_\mu h' = 1,
\end{cases}$$

then there exists a unique $\phi \in \mathcal{H}_n(A)$ satisfying

$$\sum_{j=1}^l a_j f(y_j) = \mathcal{E}(f, \phi) = -\mathcal{E}(\phi, f) = \sum_{j=1}^l a_j f(y_j) - \sum_{y \in V_0} \partial_n \phi(y) f(y), \forall f \in \text{dom}\mathcal{E}.$$
for some constant \( A \neq 0 \). Then

\[
\Delta_n f(x) = \lim_{m \to \infty} A^{-1}(r_{[w]_m} \mu_{[w]_m})^{-1} \sum_{j=1}^l a_j f(F_{[w]_m} y_j)
\]

uniformly on \( K \) for any function \( f \in \text{dom} \Delta_n \). Here \( w \) is an infinite word corresponding to \( x \) and \( F_{[w]_m} K \) denotes the according \( m \)-cell containing \( x \) for each \( m \geq 0 \).

**Proof.** According to Lemma 5.1, there exists a piecewise harmonic spline \( \phi \) satisfying (5.1) with the Dirichlet boundary condition. Using Gauss-Green’s formula, for each \( f \in \text{dom} \Delta_n \),

\[
(5.2) \quad \sum_{j=1}^l a_j f(y_j) = -\mathcal{E}(f, \phi) = \int_K \phi \Delta_n f d\mu.
\]

Consider a \( h' \) with \( \Delta_n h' = 1 \), then \( \sum_{j=1}^l a_j h'(y_j) = A \), and so \( \int_K \phi d\mu = A \). By using a scaling of the identity (5.2), for each \( m \geq 0 \), we get

\[
\sum_{j=1}^l a_j f \circ F_{[w]_m}(y_j) = \int_K \phi \Delta_n (f \circ F_{[w]_m}) d\mu = r_{[w]_m} \mu_{[w]_m} \int_K \phi(\Delta_n f) \circ F_{[w]_m} d\mu.
\]

Taking the limit as \( m \to \infty \), we have proved the theorem. \( \Box \)

Now we turn to the higher order case.

**Lemma 5.3.** Let \( l \in \mathbb{N} \), \( \{y_j\}_{j=1}^l \subset V_1 \) and \( \{a_j\}_{j=1}^l \subset \mathbb{R} \) with \( \sum_{j=0}^l a_j h(y_j) = 0 \) holding for any \( h \in \mathcal{H}_{n-1} \), then there exists a function \( \phi_n \) with the Dirichlet boundary condition, such that

\[
(5.3) \quad \sum_{j=0}^l a_j f(y_j) = -\mathcal{E}(\Delta_n^{n-1} f, \phi_n)
\]

holds for any \( f \in \text{dom}(\Delta_n^{n-1}) \).

**Proof.** For \( n = 1 \), it is just what Lemma 5.1 says, so we get the initial function \( \phi_1 \).

Now, we assert that we could choose

\[
\phi_n = (-1)^{n-1} \int G(x, z_1) \cdots G(z_{n-2}, z_{n-1}) \phi_1(z_{n-1}) d\mu(z_{n-1}) \cdots d\mu(z_1)
\]

where \( G(\cdot, \cdot) \) is the Green’s function solving the Dirichlet problem of the Poisson equation on \( K \).

In fact, assume Lemma 5.3 holds for \( n - 1 \) case, then \( \forall f \in \text{dom}(\Delta_n^{n-1}) \),

\[
\sum_{j=0}^l a_j f(y_j) = -\mathcal{E}(\Delta_n^{n-2} f, \phi_{n-1}).
\]

Using Gauss-Green’s formula, by the assumption of \( \{a_j\}_{j=1}^l \), and using the Dirichlet boundary condition of \( \phi_{n-1}, \phi_n \), we have

\[
0 = \sum_{j=0}^l a_j h(y_j) = -\mathcal{E}(\Delta_n^{n-2} h, \phi_{n-1})
\]

\[
= \int_K \phi_{n-1} \Delta_n^{n-1} h d\mu - \sum_{z \in V_0} \phi_{n-1}(z) \partial_n \Delta_n^{n-2} h(z)
\]

\[
= \sum_{z \in V_0} (\Delta_n^{n-1} h(z) \partial_n \phi_{n}(z) - \phi_n(z) \partial_n \Delta_n^{n-1} h(z))
\]

\[
= \sum_{z \in V_0} \Delta_n^{n-1} h(z) \partial_n \phi_{n}(z)
\]

...
holds for any $h \in \mathcal{H}_{n-1}$. Noticing that $\Delta^{n-1}_\mu h$ goes through the whole space $\mathcal{H}_0$, we get
\[ \partial_n \phi_n |_{V_0} = 0. \]

Thus by using Gauss-Green’s formula again, we have
\[
\sum_{j=0}^l a_j f(y_j) = -\mathcal{E}(\Delta^{n-2}_\mu f, \phi_{n-1}) = \int_K \phi_{n-1} \Delta^{n-1}_\mu f d\mu = -\mathcal{E}(\Delta^{n-1}_\mu f, \phi_n). \quad \Box
\]

**Theorem 5.4. (Calculation of $\Delta^n_\mu$)** Let $l \in \mathbb{N}$, $(y_j)_{j=1}^l \subset V_*$, and $(a_j)_{j=1}^l \subset \mathbb{R}$. Assume
\[
\begin{align*}
\sum_{j=1}^l a_j h(y_j) &= 0, \forall h \in \mathcal{H}_{n-1}, \\
\sum_{j=1}^l a_j h'(y_j) &= A, \forall h' \text{ with } \Delta^n h' = 1,
\end{align*}
\]
for some constant $A \neq 0$. Then
\[
\Delta^n_\mu f(x) = \lim_{m \to \infty} A^{-1}(r_{[w]_m} \mu_{[w]_m})^{-n} \sum_{i=1}^l a_j f(F_{[w]_m} y_j),
\]
uniformly on $K$ for any function $f \in \text{dom}(\Delta^n_\mu)$.

**Proof.** By Lemma 5.3, there exists a function $\phi_n$ satisfying (5.3) with the Dirichlet boundary condition. Combining it with Gauss-Green’s formula, we get
\[
\sum_{j=1}^l a_j f(y_j) = -\mathcal{E}(\Delta^{n-1}_\mu f, \phi_n) = \int_K \phi_n \Delta^n_\mu f d\mu.
\]

Let $h'$ be any function with $\Delta^n_\mu h' = 1$. Then $\sum_{j=1}^l a_j h'(y_j) = A$ and thus $\int_K \phi_n d\mu = A$. Scaling the identity (5.4), for each $m \geq 0$, we have
\[
\sum_{j=1}^l a_j f \circ F_{[w]_m} (y_j) = \int_K \phi_n \Delta^n_\mu(f \circ F_{[w]_m}) d\mu = (r_{[w]_m} \mu_{[w]_m})^n \int_K \phi_n (\Delta^n_\mu f) \circ F_{[w]_m} d\mu.
\]

Taking the limit as $m \to \infty$, we have proved the theorem. $\Box$

**Remark.** From the proof of Theorem 5.4, it is easy to find that the ratio of the uniform convergence depends only on the modulus of continuity of $\Delta^n_\mu f$ as stated in Section 4.

### 6. Appendix

As an appendix of this paper, we focus on the calculation of $\alpha_j$, $\beta_j$ and $\gamma_j$, the boundary values of the monomials $\{Q^{(l)}_{j,k}\}$. We will mainly discuss the $D_3$ symmetric fractals. The most typical example $\mathcal{SG}$ has been well studied in [NSTY], where an iterated calculation of the values as well as the derivatives of $\{Q^{(l)}_{j,k}\}$ at the boundary were given. However, their method is indirect, since it involves the boundary values and inner products of functions in the “easy” basis, and need to transform data from the “easy” basis to our “monomial” basis. Here we provide a new algorithm, which is more direct and shorter, using which, we could calculate $\alpha_j$, $\beta_j$ and $\gamma_j$ on some other examples, including $\mathcal{SG}_3, \mathcal{HG}$.

Our approach is based on the relationship between the Laplacian and the graph Laplacians of multiharmonic functions, established in Lemma 4.3. Taking $m = 1$ in (4.4), we get a recursive relation,
\[
(6.1) \quad \tilde{\Delta}_1 Q^{(l)}_{j,k}(x) = \sum_{i=1}^j \rho^i-1 \alpha_1^{-1} \alpha_i \Delta^i Q^{(l)}_{j,k}(x) = \sum_{i=1}^j \rho^i-1 \alpha_1^{-1} \alpha_i Q^{(l)}_{(j-i)k}(x)
\]
holding at all vertices $x \in V_1 \setminus V_0$ for all $j \geq 1$. Here in the second line of (6.1), we use the identity $\Delta^i Q^{(l)}_{j,k} = Q^{(l)}_{(j-i)k}$. 

Thus, assuming we already have the values \( \alpha_j, j \geq 0 \), (6.1) as well as the self-similar identities (3.1)- (3.3) form a system of equations to calculate \( Q^{(i)}_{jk}(v) \) from the values \( Q^{(i)}_{1k}(v), 0 \leq i < j \). We do use this idea to solve the \( k \geq 2 \) cases.

For \( k = 1 \) case, it is a bit complicated, since we need to calculate all \( \alpha_j \) simultaneously. We will give a theorem to show that \( \alpha_j \) can be determined recursively by using (6.1).

For convenience of the readers, we first introduce the new calculation on \( S\mathcal{G} \) as an example, then give the proof for general \( D3 \) symmetric cases.

First we introduce some observations as well as some notations, some of which are same as those in [NSTY].

Simplifying (6.1), we get

\[
\Delta_1 Q^{(i)}_{jk}(x) = \left( \sum_{y \sim x} c_{xy} \right) \sum_{i=1}^j \rho^1 \alpha_i Q^{(i)}_{(j-i)k}(x).
\]

Noticing that \( \alpha_0 = 1 \), we could rewrite the above identity into

\[
(6.2) \quad \sum_{y \sim x} c_{xy} Q^{(i)}_{jk}(y) = \left( \sum_{y \sim x} c_{xy} \right) \sum_{i=0}^j \rho^1 \alpha_i Q^{(i)}_{(j-i)k}(x)
\]

for \( j \geq 1 \).

Also, we need a notation of infinite dimensional semi-circulant matrices \( \alpha, \beta, \gamma \). For example, \( \alpha = \{ \alpha_{ij} \}_{i,j=0,1,2,...} \), has \( \alpha_{ij} = \alpha_{i-j} \) for \( i \geq j \) and \( \alpha_{ij} = 0 \) for \( i < j \). It is easy to check \( (\alpha\beta)_{ij} = \sum_{l=0}^j \alpha_{ij} \beta_{jl} = \sum_{l=0}^j \alpha_{i-j} \beta_{l} \) for \( i \geq j \), and the multiplications among these matrices are commutable. We will need a linear operator \( \tau \) on such matrices defined by

\[
\tau \begin{pmatrix}
  d_0 & 0 & 0 \\
  d_1 & d_0 & 0 \\
  d_2 & d_1 & d_0 \\
  \vdots & \ddots & \ddots \\
\end{pmatrix} = \begin{pmatrix}
  d_0 & 0 & 0 \\
  \rho^{-1}d_1 & d_0 & 0 \\
  \rho^{-2}d_2 & \rho^{-1}d_1 & d_0 \\
  \vdots & \ddots & \ddots \\
\end{pmatrix},
\]

where \( \rho \) is the scaling constant of the Laplacian defined before.

**Example 6.1.** The monomials have been well studied in [NSTY], with \( \alpha_j, \beta_j, \gamma_j \) exactly calculated. The recursive relations are

\[
\alpha_j = \frac{4}{5j - 5} \sum_{i=1}^{j-1} \alpha_{j-i} \alpha_i, \forall j \geq 2,
\]

\[
\beta_j = \frac{1}{5j - 1} \sum_{i=0}^{j-1} \frac{2}{5} \alpha_{j-i} \beta_i - \frac{2}{3} \alpha_{j-i} \beta_i + \frac{4}{5} \alpha_{j-i} \beta_i, \forall j \geq 1,
\]

with initial data \( \alpha_0 = 1, \alpha_1 = 1/6, \beta_0 = -1/2, \gamma_0 = 1/2 \).

Now, we give a different calculation.

First, for \( k = 1 \), by considering the symmetry, (6.2) becomes

\[
\begin{cases}
  \alpha_j + \frac{\alpha_j}{\alpha_j^5} = 4 \sum_{i=0}^{j-1} \frac{\alpha_j}{\alpha_j^5}, \\
  2\alpha_j + \frac{2}{\alpha_j^5} \alpha_j = 4 \sum_{i=0}^{j-1} \frac{\alpha_j}{\alpha_j^5},
\end{cases}
\]
for \( j \geq 1 \), we denote \( a_j = 5^j Q^{(0)}_{j+1}(F_1 q_2) \). In addition, for \( j = 0 \), we have
\[
\begin{align*}
& \begin{cases} 
2a_0 + 2a_0 + 1 = 4a_0, \\
4a_0 = 4a_0.
\end{cases}
\end{align*}
\]

We could rewrite the above identities in matrix notation,
\[
\begin{align*}
& \begin{cases} 
\alpha + \alpha + \tau(\alpha) + I = 4\alpha^2, \\
2\alpha + 2\tau(\alpha) = 4\alpha a,
\end{cases}
\end{align*}
\]
by multiplying them with \( 5^j \) on both sides, where \( \alpha \) is the infinite matrix defined with \( a_{ij} = a_{i-j} \) for \( i \geq j \) and \( a_{ij} = 0 \) for \( i < j \). Eliminating \( a \), we get
\[
8\alpha^3 - 2\alpha^2 - 3\alpha = 2\tau(\alpha)\alpha + \tau(\alpha),
\]
which results that \( \tau(\alpha) = 4\alpha^2 - 3\alpha \). So we get the recursion relation for \( \alpha_j \).

For \( k = 2 \), we can write (6.2) into
\[
\begin{align*}
& \begin{cases} 
\frac{b_0}{5^0} + 3\frac{b_1}{5^1} + \beta_j = 4 \sum_{i=0}^{j} \frac{a_i 3\frac{b_{i-1}}{5^{i-1}}}{5^i}, \\
2\frac{3b_0 b_1}{5^{0+1}} + 2\beta_j = 4 \sum_{i=0}^{j} \frac{a_i b_{i-1}}{5^i},
\end{cases}
\end{align*}
\]
for \( j \geq 0 \), where we denote \( b_j = 5^j Q^{(0)}_{j+1}(F_1 q_2) \). Thus by multiplying both sides with \( 5^j \), we have
\[
\begin{align*}
& \begin{cases} 
\beta_0 + \frac{2}{5} \beta_1 + \tau(\beta) = \frac{12}{5} \alpha \beta, \\
\frac{6}{5} \beta + 2\tau(\beta) = 4\alpha \beta.
\end{cases}
\end{align*}
\]
With some calculation, we get
\[
\frac{3}{5} \beta(2\alpha - I)(4\alpha + I) = \tau(\beta)(2\alpha + I),
\]
which gives the recursion relation of \( \beta_j \).

For \( k = 3 \), we have \( Q^{(0)}_{j+1}(F_1 q_2) = 0 \) by symmetry, so only one system of equations need to be considered, which immediately yields the recursion relation of \( \gamma_j \).

Now, we turn to the general cases.

**Theorem 6.2.** Let \( j \geq 2 \). Then \( Q^{(l)}_{j+1} | V_1 \) is uniquely determined by the values of \( Q^{(l)}_{i+1} | V_1 \), \( 0 \leq i < j \), by the relations
\[
\begin{align}
& \begin{cases} 
\tilde{\Delta}_1 Q^{(l)}_{j+1}(x) = \sum_{i=1}^{j} \rho^{i-1} \alpha_1^{-1} \alpha_i Q^{(l)}_{i+1}(x), \forall x \in V_1 \setminus V_0; \\
Q^{(l)}_{j+1} | F_1 V_0 = \rho^j Q^{(l)}_{j+1} | V_0.
\end{cases}
\end{align}
\]

**Proof.** Obviously, \( Q^{(l)}_{i+1} | V_1 \) indeed satisfies the equations (6.4), which could be rewritten into an explicit form
\[
\begin{align*}
& \begin{cases} 
\tilde{\Delta}_1 Q^{(l)}_{j+1}(x) - \rho^{j-1} \alpha_1^{-1} \alpha_j = \sum_{i=1}^{j-1} \rho^{i-1} \alpha_1^{-1} \alpha_i Q^{(l)}_{i+1}(x), \forall x \in V_1 \setminus V_0; \\
Q^{(l)}_{j+1} | F_1 V_0 = \rho^j Q^{(l)}_{j+1} | V_0.
\end{cases}
\end{align*}
\]

Thus, to prove that \( Q^{(l)}_{j+1} | V_1 \) is determined by (6.4) uniquely, we only need to prove the equations
\[
\begin{align}
& \begin{cases} 
\tilde{\Delta}_1 f(x) - \rho^{j-1} \alpha_1^{-1} f(q_{l+1}) = 0, \forall x \in V_1 \setminus V_0, \\
f | F_1 V_0 = \rho^j f | V_0, \\
f \circ q_l = f \text{ on } V_1
\end{cases}
\end{align}
\]
have a unique solution \( f | V_1 = 0 \), where \( g_l \) is the symmetry that fixes \( q_l \) and interchanges the other two vertices of \( V_0 \).
First we need to look at the equation
\[(6.6) \quad \hat{\Delta}_1 h = 1, h|_{V_0} = 0.\]
It is not hard to check that \(h = (Q_1^{(l)} - h')|_{V_1}\) is the unique solution of (6.6), where \(h'\) is the harmonic function with the same boundary values as those of \(Q_1^{(l)}\), from which, one can find that
\[(6.7) \quad h(F(q_{l+1}) = (\rho - r)\alpha_1.\]

Now suppose \(f\) is a solution of (6.5). Write \(f = \rho^{j-1}\alpha_1^{-1}f(q_{l+1})h + \hat{f}\). It is easy to check \(\hat{\Delta}_1 \hat{f} = 0\), and \(f|_{V_0} = \hat{f}|_{V_0}\). Moreover, by using (6.7) and \(h|_{V_0} = 0\), the relation \(f|_{F(V_0)} = \rho^j f|_{V_0}\) implies
\[\hat{f} \circ F(q_{l+1}) + \rho^{j-1}f(q_{l+1})(\rho - r) = \rho^j f(q_{l+1}).\]
This could be simplified into
\[rf(q_{l+1}) + \rho^{j-1}f(q_{l+1})(\rho - r) = \rho^j f(q_{l+1}),\]

since \(f|_{V_0} = \hat{f}|_{V_0}\) and \(f \circ g_l = f\). Thus we have \((r - \rho^{j-1}r)f(q_{l+1}) = 0\), which implies that \(f(q_{l+1}) = 0\), and thus \(\hat{f} = 0\) on \(V_1\).

Hence we have proved the equations (6.5) only have a zero solution on \(V_1\), which yields the result of the theorem. \(\square\)

We give the recursive relations \(\alpha, \beta\) and \(\gamma\) for \(SG_3\) and \(HG\) in the matrix form, which can be calculated in a similar way as in Example 6.1. Recall that the initial data are \(\alpha_0 = 1, \alpha_1 = 1/6, \beta_0 = -1/2, \gamma_0 = 1/2\).

The \(SG_3\) case:
\[(1 + 6\alpha)\tau(\alpha) = 1 + 12\alpha - 6\alpha^2 - 96\alpha^3 + 96\alpha^4,
(1 + 8\alpha + 12\alpha^2)\tau(\beta) = (3 + 6\alpha - 60\alpha^2 - 96\alpha^3 + 192\alpha^4)\tau(\beta),
\tau(\gamma) = (-1 + 16\alpha^2)\lambda\lambda,\]
with \(\rho = \frac{7}{50}, r = \frac{7}{15}, \lambda = \frac{1}{15}\).

The \(HG\) case:
\[(-1 + 2\alpha)\tau(\alpha) = -1 + 4\alpha + 14\alpha^2 - 48\alpha^3 + 32\alpha^4,
(-1 + 4\alpha^2)\tau(\beta) = r(1 + 10\alpha - 4\alpha^2 - 64\alpha^3 + 64\alpha^4)\beta,
\tau(\gamma) = \lambda(-1 - 8\alpha + 16\alpha^2)\gamma,\]
with \(\rho = \frac{1}{14}, r = \frac{2}{3}\) and \(\lambda = \frac{1}{2}\).

Table 1-3 present numerical computations of \(\alpha_j, \beta_j\) and \(\gamma_j\) for \(SG_3\) and \(HG\), respectively. (We are grateful to Mr. Wei Wei for providing an effective program.) For \(SG_3\) and \(HG\), it is easy to find that \(\alpha_j, \beta_j\) behave similar to geometric progressions when \(j\) is large enough, with the reciprocal of common ratio \(-124.68442107 \cdots\) for \(SG_3\) and \(-46.728917838 \cdots\) for \(HG\). An explanation of this phenomenon comes from a slight generalization of Theorem 2.9 in [NSTY] (for \(SG\)) involving a rather detailed knowledge of the description of eigenfunctions of \(-\Delta\) by the spectral decimation. Recall that the natural of eigenvalues and eigenfunctions could be known explicitly via the method of spectral decimation for some fully symmetric p.c.f. fractals (See [FS, MT, Sh1-Sh2, ST,T1]).

We refer to the reader to find the spectral decimation recipes for \(SG_3\), \(HG\) and \(SG^4\) in [DS], [BCDEHKMST2] and [FS] respectively, using which, we could verify that \(124.68442107 \cdots\) is the eigenvalue of \(-\Delta\) on \(SG_3\) with eigenfunction shown in Fig. 6.3(a), \(46.728917838 \cdots\) is the eigenvalue on \(HG\) with eigenfunction shown in Fig. 6.3(b).
### Table 1. The data of $\alpha_j, \beta_j, \gamma_j$ for $\mathcal{S}_{G_3}$.

<table>
<thead>
<tr>
<th>j</th>
<th>$\alpha_j$</th>
<th>$\beta_j$</th>
<th>$\gamma_j$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>1</td>
<td>-0.5000000000</td>
<td>0.5000000000</td>
</tr>
<tr>
<td>1</td>
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<tr>
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</tr>
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</tr>
<tr>
<td>6</td>
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### Table 2. The data of $\alpha_j, \beta_j, \gamma_j$ for $\mathcal{H}_G$.

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<th>$\alpha_j$</th>
<th>$\beta_j$</th>
<th>$\gamma_j$</th>
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The $SG_3$ case with $\lambda_1 = 14/3$.

The $HG$ case with $\lambda_1 = 2$.

Figure 6.1. The values of the ultimate eigenfunctions on $V_1$. (We only denote the non-zero values.)

Table 3. The data of ratios of $\alpha_j, \beta_j$.

<table>
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<th>$j$</th>
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<th>$\beta_{j-1}/\beta_j(SG_3)$</th>
<th>$\alpha_{j-1}/\alpha_j(HG)$</th>
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HIGHER ORDER LAPLACIANS ON FULLY SYMMETRIC P.C.F. FRACTALS


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