TOPOLOGICAL ENTROPY ON SUBSETS FOR FIXED-POINT FREE FLOWS

DOU DOU, MENG FAN AND HUA QIU

Abstract. By considering all possible reparametrizations of the flows instead of the time-1 maps, we introduce Bowen topological entropy and local entropy on subsets for flows. Through handling techniques for reparametrization balls, we prove a covering lemma for fixed-point free flows and then prove a variational principle.

1. Introduction

Throughout this paper, we let X be a compact metric space with metric d. A *flow* over X is a pair (X, ϕ), where $\phi : X \times \mathbb{R} \to X$ is a continuous map satisfying $\phi_t \circ \phi_s = \phi_{s+t}$ for all $t, s \in \mathbb{R}$ and $\phi_t(\cdot) = \phi(\cdot, t)$ is a homeomorphism on X. A Borel probability measure μ on X is called $\phi-$*invariant* if for any Borel set B, it holds $\mu(\phi_t(B)) = \mu(B)$ for all $t \in \mathbb{R}$. It is called *ergodic* if any $\phi-$invariant Borel set has measure 0 or 1. The set of all Borel probability measures, all $\phi-$invariant Borel probability measures and all ergodic $\phi-$invariant Borel probability measures on X are denoted by $\mathcal{M}(X)$, $\mathcal{M}_\phi(X)$ and $\mathcal{E}_\phi(X)$ respectively.

Entropy is an important concept in dynamical system. There are several ways to define the entropy for flows. The traditional idea to define the entropy is to consider the time-1 map ϕ_1. Then the entropy for flows is defined by the usual entropy for the discrete system (X, ϕ_1). (We call such system a topological dynamical system, or a TDS for short.) In [9], Thomas introduced a definition of the entropy by considering reparametrizations of the flows. In this paper, we will study the entropy for flows via reparametrizations.

For a closed interval I which contains the origin, a continuous map $\alpha : I \to \mathbb{R}$ is called a *reparametrization* if it is a homeomorphism onto its image and $\alpha(0) = 0$. The set of all such reparametrizations on I is denoted by $Rep(I)$. For a flow ϕ on X, $x \in X$, $t \in \mathbb{R}^+$ and $\varepsilon > 0$, we set

$$B(x, t, \varepsilon, \phi) = \{y \in X : \text{ there exists } \alpha \in Rep[0, t] \text{ such that } d(\phi_{\alpha(s)}x, \phi_s y) < \varepsilon, \text{ for all } 0 \le s \le t\},$$

and call $B(x, t, \varepsilon, \phi)$ a $(t, \varepsilon, \phi)-$*ball* or a *reparametrization ball* in X. Clearly, all the reparametrization balls are open sets.

In the literature of entropy for flows via reparametrizations, $(t, \varepsilon, \phi)-$balls are used in place of the usual Bowen balls. Topological entropy for one parameter flows on compact

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metric spaces is defined by Bowen [1, 2]. To investigate the topological entropies of mutually conjugate expansive flows, Thomas [9] first defined the entropy for flows arised from allowing reparametrizations of orbits. Later on, he developed this study in [10] and showed that his definition of entropy is equivalent to Bowen's definition for any flow without fixed points on compact metric spaces. Sun and Vargas studied measure-theoretic aspect of this manner in [7, 8].

In 1973, Bowen [3] introduced the topological entropy for any subset in a way resembling the Hausdorff dimension for discrete dynamical systems, which is called Bowen topological entropy. In particular, Bowen topological entropy for the whole space coincides with the original topological entropy for compact discrete dynamical systems. This definition plays a key role in topological dynamics and dimension theory [6]. Since the variational principles are fundamental results in ergodic theory and dynamical systems, it is nature to find a variational principle for Bowen topological entropy. Inspired by a classical result in dimension theory, Feng and Huang[4] proved that for any non-empty compact subset K, Bowen topological entropy on K is the supremum of the measure theoretic local entropies, where the supremum is taken over all the Borel probability measures that concentrate on K. The proof is along the following steps:

(1) define the weighted entropy;
(2) give the relation between Bowen entropy and weighted entropy (actually they coincide);
(3) prove a dynamical Frostman's Lemma via weighted entropy;
(4) prove the result for compact subsets.

In this paper, we will introduce Bowen entropy on subsets for compact metric flows through reparametrization balls and then apply Feng and Huang's steps to prove a variational principle for compact metric flows without fixed points. We should emphasize here that the technical difficulties arising from allowing reparametrizations of orbits need to be overcome. The paper is organized as follows. In section 2, we introduce Bowen topological entropy and local measure theoretic entropy for flows. Some basic properties are also listed therein. In section 3, we give some lemmas related to the reparametrization balls and then prove a covering lemma. These lemmas will play a key role for proving the main theorem. Finally, in section 4, we follow Feng and Huang's technical line to prove the theorem.

2. Bowen topological entropy and local measure theoretic entropy

Let (X, ϕ) be a flow and Z a subset of X. For $s \geq 0$, $N \in \mathbb{N}$, and $\varepsilon > 0$, define

$$\mathcal{M}^s_{N,\varepsilon}(\phi, Z) = \inf \sum_i \exp(-st_i),$$

where the infimum is taken over all finite or countable families of reparametrization balls $\{B(x_i, t_i, \varepsilon, \phi)\}$, $x_i \in X$ and $t_i \geq N$ such that $\bigcup B(x_i, t_i, \varepsilon, \phi) \supset Z$.

The quantity $\mathcal{M}^s_{N,\varepsilon}$ dose not decrease as N increases and ε decreases, hence the following limits exist:

$$\mathcal{M}^s_{\varepsilon}(\phi, Z) = \lim_{N\to\infty} \mathcal{M}^s_{N,\varepsilon}(\phi, Z), \mathcal{M}^s(\phi, Z) = \lim_{\varepsilon\to 0} \mathcal{M}^s_{\varepsilon}(\phi, Z).$$

Proposition 2.1. *Let (X,ϕ) be a flow.*

(1) *For any $s \geq 0, N \in \mathbb{N}$ and $\varepsilon > 0$, $\mathcal{M}^s_{N,\varepsilon}(\phi,\cdot)$ is an outer measure on X.*
(2) *For any $s \geq 0$, $\mathcal{M}^s(\phi,\cdot)$ is a metric outer measure on X.*

Proof. (1) is a direct result from the definition of $\mathcal{M}^s_{N,\varepsilon}(\phi,\cdot)$ and we only need to prove (2).

Suppose $d = d(E,F) > 0$ and let $0 < \varepsilon < d/2$, $N \in \mathbb{N}$. For any $\delta > 0$, we choose a family of reparametrization balls $\{B(x_i,t_i,\varepsilon,\phi)\}$ with all $t_i \geq N$ that covers $E \cup F$ such that $\mathcal{M}^s_{N,\varepsilon}(\phi, E\cup F) > \sum_i \exp(-st_i) - \delta$. Then $\{B(x_i,t_i,\varepsilon,\phi)\}$ can be divided into two disjoint families $\{B(x_{i'},t_{i'},\varepsilon,\phi)\}$ and $\{B(x_{i''},t_{i''},\varepsilon,\phi)\}$ that cover E and F respectively. Thus

$$\begin{aligned}\mathcal{M}^s_{N,\varepsilon}(\phi, E\cup F) &> \sum_i \exp(-st_i) - \delta \\ &= \sum_{i'} \exp(-st_{i'}) + \sum_{i''} \exp(-st_{i''}) - \delta \\ &\geq \mathcal{M}^s_{N,\varepsilon}(\phi, E) + \mathcal{M}^s_{N,\varepsilon}(\phi, F) - \delta,\end{aligned}$$

which implies that $\mathcal{M}^s_{N,\varepsilon}(\phi, E\cup F) \geq \mathcal{M}^s_{N,\varepsilon}(\phi, E) + \mathcal{M}^s_{N,\varepsilon}(\phi, F)$. Hence we have $\mathcal{M}^s(\phi, E\cup F) = \mathcal{M}^s(\phi, E) + \mathcal{M}^s(\phi, F)$ and this means that $\mathcal{M}^s(\phi,\cdot)$ is a metric outer measure on X. □

The *Bowen topological entropy* $h^B_{top}(\phi, Z)$ is defined as a critical value of the parameter s, where $\mathcal{M}^s(\phi, Z)$ jumps from ∞ to 0, i.e.

$$\begin{aligned}h^B_{top}(\phi, Z) &= \inf\{s : \mathcal{M}^s(\phi, Z) = 0\} \\ &= \sup\{s : \mathcal{M}^s(\phi, Z) = \infty\}.\end{aligned}$$

Proposition 2.2. *Let (X,ϕ) be a flow. Then*

(1) *for $Z \subseteq Z' \subseteq X$, we have*
$$h^B_{top}(\phi, Z) \leq h^B_{top}(\phi, Z');$$
(2) *for $Z \subseteq \bigcup_{i=1}^{\infty} Z_i, s \geq 0$, we have*
$$h^B_{top}(\phi, Z) \leq \sup_{i\geq 1} h^B_{top}(\phi, Z_i).$$

Proof. (1) It is easy to prove that $\mathcal{M}^s_{N,\varepsilon}(Z,\phi) \leq \mathcal{M}^s_{N,\varepsilon}(Z',\phi)$ when $Z \subseteq Z'$. Then $\mathcal{M}^s(Z',\phi) = 0$ implies $\mathcal{M}^s(Z,\phi) = 0$, which deduce that

$$h^B_{top}(\phi, Z) \leq h^B_{top}(\phi, Z').$$

(2) Assume that

$$h^B_{top}(\phi, Z) > \sup_{i\geq 1} h^B_{top}(\phi, Z_i).$$

Then for some $\delta > 0$, $h^B_{top}(\phi, Z) > h^B_{top}(\phi, Z_i) + \delta$ for any $i \geq 1$. For $s = h^B_{top}(\phi, Z) - \delta$, we have $\mathcal{M}^s(\phi, Z) = \infty$, but $\mathcal{M}^s(\phi, Z_i) = 0$ for each i. Hence

$$\mathcal{M}^s(\phi, Z) > \sum_{i=1}^{\infty} \mathcal{M}^s(\phi, Z_i),$$

a contradiction. $\square$

For $x \in X$, $\varepsilon > 0$, $t \geq 0$ and $n \in \mathbb{N}$, we can define the following two classes of usual Bowen balls:

$$B_t(x, \varepsilon, \phi) = \{y \in X : d(\phi_s x, \phi_s y) < \varepsilon, \text{ for all } 0 \leq s \leq t\}$$

and

$$B_n(x, \varepsilon, \phi_1) = \{y \in X : d(\phi_i x, \phi_i y) < \varepsilon, \text{ for all } i = 0, 1, \ldots, n-1\}.$$

Replacing the reparametrization balls by the usual Bowen balls $B_t(x, \varepsilon, \phi)$, we can have the definition of the usual Bowen topological entropy on a subset Z for the flow (X, ϕ), denote it by $\tilde{h}^B_{top}(Z, \phi)$. If we replace the reparametrization balls by the Bowen balls $B_n(x, \varepsilon, \phi_1)$, we can have the definition of the usual Bowen topological entropy on a subset Z for the time-1 map, denote it by $h^B_{top}(Z, \phi_1)$.

Remark 2.3. For any $\varepsilon > 0$, since X is compact and ϕ is continuous, there exists $\delta > 0$ such that for any $0 \leq s \leq 1$ and $x, y \in X$, we have $d(\phi_s x, \phi_s y) < \varepsilon$ whenever $d(x, y) < \delta$. Then it is easy to see that

$$B_{\lceil t \rceil}(x, \delta, \phi_1) \subset B_t(x, \epsilon, \phi) \subset B_{\lceil t \rceil}(x, \varepsilon, \phi_1), \tag{2.1}$$

where $\lceil t \rceil$ is the largest integer which is not smaller than t. Hence from the definitions of the above Bowen topological entropies, $h^B_{top}(Z, \phi_1) = \tilde{h}^B_{top}(Z, \phi)$ for any subset Z of X. Moreover, since the reparametrization ball $B(x, t, \varepsilon, \phi)$ always contains the usual Bowen ball $B_t(x, \varepsilon, \phi)$, we have that $h^B_{top}(Z, \phi) \leq \tilde{h}^B_{top}(Z, \phi)$. Hence for any $Z \subset X$,

$$h^B_{top}(Z, \phi) \leq h^B_{top}(Z, \phi_1).$$

But it is not clear whether the equality holds for every $Z \subset X$.

Let $\mu \in \mathcal{M}(X)$. The *measure-theoretical lower and upper local entropies* of μ are defined respectively by

$$\underline{h}_\mu(\phi) = \int \underline{h}_\mu(\phi, x)\, d\mu, \text{ and } \overline{h}_\mu(\phi) = \int \overline{h}_\mu(\phi, x)\, d\mu$$

where

$$\underline{h}_\mu(\phi, x) = \lim_{\varepsilon \to 0} \liminf_{t \to +\infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi))$$

and

$$\overline{h}_\mu(\phi, x) = \lim_{\varepsilon \to 0} \limsup_{t \to +\infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi)).$$

Remark 2.4. Similar to Remark 2.3, it holds that

$$\underline{h}_\mu(\phi) \leq \underline{h}_\mu(\phi_1), \text{ and } \overline{h}_\mu(\phi) \leq \overline{h}_\mu(\phi_1).$$

It is also not clear whether the equalities hold for every $\mu \in \mathcal{M}(X)$.

Now we state the main theorem.

Theorem 2.5. *Let (X,ϕ) be a compact metric flow without fixed points. If K is a non-empty compact subset of X, then*

$$h^B_{top}(\phi, K) = sup\{\underline{h}_\mu(\phi) : \mu \in \mathcal{M}(X), \mu(K) = 1\}. \tag{2.2}$$

We suggest here that there are some further results related to Theorem 2.5 for flows without fixed points. Due to Feng and Huang [4], the compact subsets K's can be improved to analytic sets under the finite entropy or even zero mean dimension assumption. And one can also consider another kind of concept, named as packing entropy, then there will be a variational principle via the measure-theoretical upper local entropy. The proofs may involve more results in ergodic theory to flows and more techniques in geometric measure theory.

3. Properties about reparametrization balls and a covering lemma

In this section, we first will give some properties about reparametrization balls for flows without fixed points. Then we will apply these results to prove a related covering lemma(Theorem 3.5). This lemma is crucial in the proof of Theorem 2.5.

Lemma 3.1 (Lemma 1.2 of [9])**.** *Let (X,ϕ) be a compact metric flow without fixed points. For any $\eta > 0$, there exists $\theta > 0$ such that for any $x, y \in X$, any interval I containing the origin, and any reparametrization $\alpha \in Rep(I)$, if $d(\phi_{\alpha(s)}(x), \phi_s(y)) < \theta$ for all $s \in I$, then it holds that*

$$|\alpha(s) - s| < \begin{cases} \eta|s|, & \text{if } |s| > 1, \\ \eta, & \text{if } |s| \le 1. \end{cases}$$

Lemma 3.2. *Let (X,ϕ) be a compact metric flow without fixed points. Then for any $0 < \eta < 1$, there exists $\theta > 0$ such that for any $x \in X$, $\varepsilon > 0$, $0 < \varepsilon' < \theta$, $t > \frac{1}{1-\eta}$ and $y \in B(x,t,\varepsilon,\phi)$, it holds that*

$$B(y, \tilde{t}, \varepsilon', \phi) \subseteq B(x, \tilde{t}, \varepsilon + \varepsilon', \phi), \tag{3.1}$$

where $\tilde{t} = (1-\eta)t$.

Proof. Let θ be the same as in Lemma 3.1. By the definition of the reparametrization ball, for any $y \in B(x,t,\varepsilon,\phi)$, there exists an $\alpha_1 \in Rep[0,t]$, such that $d(\phi_{\alpha_1(s)}x, \phi_s y) < \varepsilon,\ \ \forall s \in [0,t]$. And for any $z \in B(y,\tilde{t},\varepsilon',\phi)$, there exists an $\alpha_2 \in Rep[0,\tilde{t}]$, such that $d(\phi_{\alpha_2(s)}y, \phi_s z) < \varepsilon', \forall s \in [0,\tilde{t}]$.

From Lemma 3.1 and the definition of $\tilde{t}$, we know that $|\alpha_2(\tilde{t}) - \tilde{t}| < \eta\tilde{t}$. This can deduce that $\alpha_2(\tilde{t}) < (1+\eta)\tilde{t} = (1-\eta^2)t < t$. Hence $\alpha_2(s) \in [0,t]$ whenever $s \in [0,\tilde{t}]$ and $\alpha_1 \circ \alpha_2 \in Rep[0,\tilde{t}]$. Let $\alpha_3 = \alpha_1 \circ \alpha_2$, then it holds that

$$\begin{aligned} d(\phi_{\alpha_3(s)}x, \phi_s z) &\le d(\phi_{\alpha_1\circ\alpha_2(s)}x, \phi_{\alpha_2(s)}y) + d(\phi_{\alpha_2(s)}y, \phi_s z) \\ &< \varepsilon + \varepsilon', \end{aligned}$$

for any $s \in [0,\tilde{t}]$. Thus $z \in B(x,\tilde{t},\varepsilon+\varepsilon',\phi)$. $\square$

Lemma 3.3. *Let (X,ϕ) be a compact metric flow without fixed points. Let $0<\eta<1$, $t>1$, and θ be as in Lemma 3.1. Write $\tilde{t}=(1-\eta)t$. Then for any $0<\varepsilon<\theta$,*

(1) *if $y\in B(x,t,\varepsilon,\phi)$, then $x\in B(y,\tilde{t},\varepsilon,\phi)$;*
(2) *if $y\in B(x,t,\frac{\varepsilon}{2},\phi)$, then $B(x,t,\frac{\varepsilon}{2},\phi)\subseteq B(y,\tilde{t},\varepsilon,\phi)$.*

Proof. (1). Let $y\in B(x,t,\varepsilon,\phi)$ and α_1 be a reparametrization on $[0,t]$ such that $d(\phi_{\alpha_1(s)}x,\phi_s y)<\varepsilon$, $\forall s\in[0,t]$. By Lemma 3.1, $\alpha_1(t)>t-\eta t=\tilde{t}$. Letting $\alpha_2=\alpha_1^{-1}|_{[0,\tilde{t}]}\in Rep[0,\tilde{t}]$, we have

$$d(\phi_{\alpha_2(s)}y,\phi_s x)=d(\phi_{\alpha_1\circ\alpha_1^{-1}(s)}x,\phi_{\alpha_1^{-1}(s)}y)<\varepsilon,\forall s\in[0,\tilde{t}].$$

This proves (1).

(2). For $y\in B(x,t,\frac{\varepsilon}{2},\phi)$, let $\alpha_1\in Rep[0,t]$ such that $d(\phi_{\alpha_1(s)}x,\phi_s y)<\frac{\varepsilon}{2}$ for all $s\in[0,t]$. $\forall z\in B(x,t,\frac{\varepsilon}{2},\phi)$, there exists $\alpha_2\in Rep[0,t]$ such that $d(\phi_{\alpha_2(s)}x,\phi_s z)<\frac{\varepsilon}{2}$ for all $s\in[0,t]$.

If $\alpha_1(t)\geq\alpha_2(t)$, then $\alpha_1^{-1}\circ\alpha_2$ is well-defined on $[0,t]$. Define $\alpha_3\in Rep[0,t]$ by $\alpha_3=\alpha_1^{-1}\circ\alpha_2$. Then it holds that for every $s\in[0,t]$,

$$\begin{aligned}d(\phi_{\alpha_3(s)}y,\phi_s z)&\leq d(\phi_{\alpha_2(s)}x,\phi_s z)+d(\phi_{\alpha_2(s)}x,\phi_{\alpha_3(s)}y)\\&<\frac{\varepsilon}{2}+d(\phi_{\alpha_1\circ\alpha_3(s)}x,\phi_{\alpha_3(s)}y)\text{ (note that }\alpha_3(s)\leq t)\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

i.e., $z\in B(y,t,\varepsilon,\phi)$.

If $\alpha_1(t)<\alpha_2(t)$, a similar argument shows that $y\in B(z,t,\varepsilon,\phi)$. By (1), we then have that $z\in B(y,\tilde{t},\varepsilon,\phi)$.

Combining the above two cases, we conclude that $B(x,t,\frac{\varepsilon}{2},\phi)\subseteq B(y,\tilde{t},\varepsilon,\phi)$. □

Lemma 3.4. *Let (X,ϕ) be a compact metric flow without fixed points. For any $\varepsilon>0$, there exists $\delta>0$ depending only on ε, such that*

$$B(x,t_1,\varepsilon,\phi)\subset B(x,t_2,2\varepsilon,\phi),\ \textit{for any } x\in X, \tag{3.2}$$

whenever $t_1,t_2>0$ and $|t_1-t_2|<\delta$.

Proof. It is obvious true for $t_1\geq t_2$. Now we assume $t_1<t_2$.

For any $y\in B(x,t_1,\varepsilon,\phi)$, let $\alpha_1\in Rep[0,t_1]$ such that $d(\phi_{\alpha_1(s)}x,\phi_s y)<\varepsilon,\forall s\in[0,t_1]$. We now define $\alpha_2\in Rep[0,t_2]$ by

$$\alpha_2(s)=\begin{cases}\alpha_1(s), & \text{if } 0\leq s\leq t_1\\ \alpha_1(t_1)+s-t_1, & \text{if } t_1<s\leq t_2.\end{cases}$$

Then for $0\leq s\leq t_1$, $d(\phi_{\alpha_2(s)}x,\phi_s y)=d(\phi_{\alpha_1(s)}x,\phi_s y)<\varepsilon$. Since X is a compact space and ϕ is continuous, there exists $\delta>0$ such that $d(\phi_t x,x)<\frac{\varepsilon}{2}$ for any $x\in X$ whenever

$0 < t < \delta$. And hence for $t_1 < s \le t_2$,

$$\begin{aligned} d(\phi_{\alpha_2(s)}x, \phi_s y) &= d(\phi_{s-t_1}\phi_{\alpha_1(t_1)}x, \phi_{s-t_1}\phi_{t_1}y) \\ &\le d(\phi_{s-t_1}\phi_{\alpha_1(t_1)}x, \phi_{\alpha_1(t_1)}x) + d(\phi_{\alpha_1(t_1)}x, \phi_{t_1}y) + d(\phi_{s-t_1}\phi_{t_1}y, \phi_{t_1}y) \\ &< \frac{\varepsilon}{2} + \varepsilon + \frac{\varepsilon}{2} = 2\varepsilon. \end{aligned}$$

□

Now we give our covering lemma which is a variation of the classical $5r$-coving Lemma in fractal geometry(see, for example, Theorem 2.1 of [5]).

Theorem 3.5 (**A covering lemma for reparametrization balls).** *Let (X, ϕ) be a compact metric flow without fixed points. For $0 < \eta < 1$, let $\theta > 0$ be as in Lemma 3.1. Let $\mathcal{B} = \{B(x, t, \varepsilon, \phi)\}_{(x,t)\in\mathcal{I}}$ be a family of reparametrization balls in X with $0 < \varepsilon < \frac{\theta}{2}$ and $t > \frac{1}{(1-\eta)^2}$. Then there exists a finite or countable subfamily $\mathcal{B}' = \{B(x, t, \varepsilon, \phi)\}_{(x,t)\in\mathcal{I}'}(\mathcal{I}' \subset \mathcal{I})$ of pairwise disjoint reparametrization balls in $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{(x,t)\in\mathcal{I}'} B(x, \hat{t}, 5\varepsilon, \phi)$$

where $\hat{t} = (1-\eta)^2 t$.

Proof. We denote by $A = \{x : (x, t) \in \mathcal{I}\}$, the collection of central points of the reparametrization balls in $\mathcal{B}$. Let

$$M = \inf\{t : (x, t) \in \mathcal{I}\}$$

and

$$A_1 = \{x \in A : (x, t) \in \mathcal{I} \text{ and } M \le t < M + \delta\},$$

where $\delta > 0$ is choosen as in Lemma 3.4.

Choose an arbitrary $x_1 \in A_1$ and then inductively choose

$$x_{k+1} \in A_1 \setminus \bigcup_{i=1}^{k} B(x_i, \tilde{t_i}, 3\varepsilon, \phi) \quad \text{(recall that } \tilde{t_i} = (1-\eta)t_i)$$

as long as $A_1 \setminus \bigcup_{i=1}^{k} B(x_i, \tilde{t_i}, 3\varepsilon, \phi) \neq \emptyset$, where each t_i satisfies $(x_i, t_i) \in \mathcal{I}$ and $M \le t_i < M + \delta$. For each x_i, we only choose one such t_i, noticing that there may exist different t_i's with $(x_i, t_i) \in \mathcal{I}$.

Firstly we show that $B(x_i, t_i, \varepsilon, \phi)$'s are mutually disjoint. Suppose we have chosen x_i and x_j, $i > j$ and there exists $y \in B(x_i, t_i, \varepsilon, \phi) \cap B(x_j, t_j, \varepsilon, \phi)$. As $|t_i - t_j| < \delta$, by Lemma 3.4,

$$y \in B(x_i, t_i, \varepsilon, \phi) \subseteq B(x_i, t_j, 2\varepsilon, \phi).$$

By (1) of Lemma 3.3, it holds that

$$x_i \in B(y, \tilde{t_j}, 2\varepsilon, \phi).$$

And thus by Lemma 3.2, $x_i \in B(x_j, \tilde{t_j}, 3\varepsilon, \phi)$. This contradicts the choice of x_i.

Secondly we claim that there exists a finite k_1 such that

$$A_1 \subset \bigcup_{i=1}^{k_1} B(x_i, \tilde{t_i}, 3\varepsilon, \phi).$$

To see this, we note that there exists $r > 0$ such that $d(\phi_s x, \phi_s y) < \varepsilon$ for any $s \in [0, M+\delta]$, whenever $d(x,y) < r$. So each reparametrization ball $B(x_i, t_i, \varepsilon, \phi)$ contains an ordinary ball $B(x_i, r)$. Since a compact metric space cannot contain infinite many mutually disjoint balls with the same radius r, we can conclude that k_1 is finite.

For any $x \in A_1$, we can choose an x_i from $\{x_1, \cdots, x_{k_1}\}$ such that $x \in B(x_i, \tilde{t_i}, 3\varepsilon, \phi)$. Then for $(x,t) \in \mathcal{I}$ with $M \le t < M+\delta$, by Lemma 3.2 and 3.4, we have that

$$B(x, t, \varepsilon, \phi) \subset B(x, t_i, 2\varepsilon, \phi) \subset B(x, \tilde{t_i}, 2\varepsilon, \phi) \subset B(x_i, \hat{t_i}, 5\varepsilon, \phi).$$

Hence

$$\begin{aligned} \bigcup_{x \in A_1, (x,t) \in \mathcal{I}} B(x, t, \varepsilon, \phi) &= \bigcup_{x \in A_1, (x,t) \in \mathcal{I}, M \le t < M+\delta} B(x, t, \varepsilon, \phi) \\ &\subset \bigcup_{i=1}^{k_1} B(x_i, \hat{t_i}, 5\varepsilon, \phi). \end{aligned}$$

Let

$$A_2 = \{x \in A : (x,t) \in \mathcal{I} \text{ and } M+\delta \le t < M+2\delta\}$$

and

$$\begin{aligned} A_2' = \{x \in A_2 : &\text{ there exists } t \text{ with } (x,t) \in \mathcal{I} \text{ and } M+\delta \le t < M+2\delta \\ &\text{such that } B(x, t, \varepsilon, \phi) \cap \bigcup_{i=1}^{k_1} B(x_i, t_i, \varepsilon, \phi) = \emptyset\}. \end{aligned}$$

For $x \in A_2 \setminus A_2'$, for each t with $(x,t) \in \mathcal{I}$ and $M+\delta \le t < M+2\delta$, there exists some $i \in \{1, 2, \cdots, k_1\}$, such that

$$B(x, t, \varepsilon, \phi) \cap B(x_i, t_i, \varepsilon, \phi) \ne \emptyset.$$

Choose one such t and any $y \in B(x, t, \varepsilon, \phi) \cap B(x_i, t_i, \varepsilon, \phi)$. By (1) of Lemma 3.3, we have

$$x \in B(y, \tilde{t}, \varepsilon, \phi).$$

By Lemma 3.4 and 3.2, we have

$$B(y, \tilde{t}, \varepsilon, \phi) \subseteq B(y, \tilde{t_i}, 2\varepsilon, \phi) \subseteq B(x_i, \tilde{t_i}, 3\varepsilon, \phi).$$

Thus

$$x \in B(x_i, \tilde{t_i}, 3\varepsilon, \phi).$$

This yields that

$$A_2 \setminus A_2' \subseteq \bigcup_{i=1}^{k_1} B(x_i, \tilde{t_i}, 3\varepsilon, \phi). \tag{3.3}$$

Choose $x_{k_1+1} \in A_2'$ arbitrarily and then inductively choose

$$x_{k+1} \in A_2' \setminus \bigcup_{i=k_1+1}^{k} B(x_i, \tilde{t}_i, 3\varepsilon, \phi).$$

As above there is a finite k_2 such that the reparametrization balls $B(x_i, t_i, \varepsilon, \phi), i = 1, 2, \cdots, k_2$, are pairwise disjoint and

$$A_2' \subseteq \bigcup_{i=k_1+1}^{k_2} B(x_i, \tilde{t}_i, 3\varepsilon, \phi).$$

Combining with (3.3), using the same argument as above, we get

$$\bigcup_{x \in A_2, (x,t) \in \mathcal{I}} B(x, t, \varepsilon, \phi) \subset \bigcup_{i=1}^{k_2} B(x_i, \hat{t}_i, 5\varepsilon, \phi).$$

Repeating the above process, we finish the proof.

□

4. Proof of Theorem 2.5

With the preparation in Section 3, we can now proceed Feng-Huang's steps to prove Theorem 2.5.

Step 1. Defining a weighted entropy for flows.

Let (X, ϕ) be a compact metric flow. For any bounded function $f : X \to \mathbb{R}$, $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$\mathcal{W}^s_{N,\varepsilon}(\phi, f) = \inf \sum_i c_i \exp(-st_i), \tag{4.1}$$

where the infimum is taken over all finite or countable families $\{(B(x_i, t_i, \varepsilon, \phi), c_i)\}$ such that $0 < c_i < \infty, x_i \in X, t_i \geq N$ for all i and

$$\sum_i c_i \chi_{B_i} \geq f,$$

where $B_i := B(x_i, t_i, \varepsilon, \phi)$ and χ_A denotes the characteristic function of set A.

For $Z \subseteq X$, we set $\mathcal{W}^s_{N,\varepsilon}(\phi, Z) = \mathcal{W}^s_{N,\varepsilon}(\phi, \chi_Z)$. The quantity $\mathcal{W}^s_{N,\varepsilon}(\phi, Z)$ does not decrease as N increases and ε decreases, hence the following limits exist:

$$\mathcal{W}^s_{\varepsilon}(\phi, Z) = \lim_{N\to\infty} \mathcal{W}^s_{N,\varepsilon}(\phi, Z), \qquad \mathcal{W}^s(\phi, Z) = \lim_{\varepsilon\to 0} \mathcal{W}^s_{\varepsilon}(\phi, Z).$$

Clearly, there exists a critical value of the parameter s, which will be denoted as $h^{WB}_{top}(\phi, Z)$, where $\mathcal{W}^s(\phi, Z)$ jumps from ∞ to 0, i.e.

$$\begin{aligned} h^{WB}_{top}(\phi, Z) &= \inf\{s : \mathcal{W}^s(\phi, Z) = 0\} \\ &= \sup\{s : \mathcal{W}^s(\phi, Z) = \infty\}. \end{aligned}$$

We call $h^{WB}_{top}(\phi, Z)$ the *weighted Bowen topological entropy* (or just *weighted entropy* for short) of the flow ϕ on Z.

Step 2. Relations between Bowen entropy and weighted entropy.

Proposition 4.1. *Let (X,ϕ) be a compact metric flow without fixed points, and $Z\subset X$. Let $1<\eta<1$ and θ be as in Lemma 3.1. Then for any $s\geq 0,\delta>0$ and $0<\varepsilon<\frac{\theta}{6}$,*

$$\mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi,Z)\leq \mathcal{W}_{N,\varepsilon}^{s}(\phi,Z)\leq \mathcal{M}_{N,\varepsilon}^{s}(\phi,Z) \tag{4.2}$$

when $N\in\mathbb{N}$ is large enough.

Proof. Taking $f=\chi_Z$ and $c_i\equiv 1$ in (4.1), it is clear that the second inequality holds for each $N\in\mathbb{N}$. In the following, we prove the first inequality when N is large enough.

Let $N>\max\{\frac{1}{(1-\eta)^3},2\}$ such that $n^2e^{-(n-1)\delta}\leq e^{-s}$ for all $n\geq N$. Let $\{B(x_i,t_i,\varepsilon,\phi),c_i\}_{i\in\mathcal{I}}$ be a family so that $\mathcal{I}\subset\mathbb{N},x_i\in X,0<c_i<\infty,t_i\geq N$ and

$$\sum_{i\in\mathcal{I}}c_i\chi_{B_i}\geq\chi_Z, \tag{4.3}$$

where as in Step 1, we denote $B_i:=B(x_i,t_i,\varepsilon,\phi)$. Then we will show that

$$\mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi,Z)\leq\sum_{i\in\mathcal{I}}c_ie^{-st_i}, \tag{4.4}$$

and hence $\mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi,Z)\leq\mathcal{W}_{N,\varepsilon}^{s}(\phi,Z)$.

For simplicity, in the rest of the proof, we denote $\tilde{B}_i:=B(x_i,\tilde{t}_i,\varepsilon,\phi)$ and $5\hat{B}_i:=B(x_i,\hat{t}_i,5\varepsilon,\phi)$, for $i\in\mathcal{I}$.

Now we decompose $\mathcal{I}$ into subsets $\mathcal{I}_n:=\{i\in\mathcal{I}:t_i\in(n-1,n]\}$ and decompose each $\mathcal{I}_n$ into finite subsets $\mathcal{I}_{n,k}:=\{i\in\mathcal{I}_n:i\leq k\}$ for $n\geq N$ and $k\in\mathbb{N}$. For $\tau>0$, set

$$Z_{n,\tau}=\{x\in Z:\sum_{i\in\mathcal{I}_n}c_i\chi_{B_i}(x)>\tau\}\ \text{ and}$$
$$Z_{n,k,\tau}=\{x\in Z:\sum_{i\in\mathcal{I}_{n,k}}c_i\chi_{B_i}(x)>\tau\}.$$

For each $n\geq N,k\in\mathbb{N}$ and $\tau>0$, let us consider the set $Z_{n,k,\tau}$. We may assume that each c_i is a positive integer. This could be done as follows. Since $\mathcal{I}_{n,k}$ is finite and by approximating the c_i's from above, we may first assume c_i's are positive rational numbers. Also notice that $Z_{n,k,d\tau}$ for dc_i's is equal to $Z_{n,k,\tau}$ for c_i's, so by multiplying with a common denominator d, we may then assume that each c_i is a positive integer. Let $m=\lceil\tau\rceil$, the smallest integer no less than τ. Denote $\mathcal{B}=\{B_i:i\in\mathcal{I}_{n,k}\}$ and define $u:\mathcal{B}\to\mathbb{Z}$ by $u(B_i)=c_i$. We now inductively define integer-valued functions $v_1,v_2,\cdots,v_m$ on $\mathcal{B}$ and subfamilies $\mathcal{B}_1,\mathcal{B}_2,\cdots,\mathcal{B}_m$ of $\mathcal{B}$ starting with $v_0=u$. Using Theorem 3.5, we find a pairwise disjoint subfamily $\mathcal{B}_1$ of $\mathcal{B}$ such that

$$\bigcup_{B\in\mathcal{B}}B\subseteq\bigcup_{B\in\mathcal{B}_1}5\hat{B}$$

and hence $Z_{n,k,\tau} \subseteq \bigcup_{B\in\mathcal{B}_1} 5\hat{B}$. Then by repeatedly using Theorem 3.5, for $j = 1, \ldots, m$, we can define inductively disjoint subfamilies $\mathcal{B}_j$ of $\mathcal{B}$ such that

$$\mathcal{B}_j \subseteq \{B \in \mathcal{B} : v_{j-1}(B) \geq 1\}, \quad Z_{n,k,\tau} \subseteq \bigcup_{B\in\mathcal{B}_j} 5\hat{B}$$

and the functions v_j's such that

$$v_j(B) = \begin{cases} v_{j-1}(B) - 1 & \text{for } B \in \mathcal{B}_j, \\ v_{j-1}(B) & \text{for } B \in \mathcal{B} \setminus \mathcal{B}_j. \end{cases}$$

This is possible since for $j < m$,

$$Z_{n,k,\tau} \subseteq \{x : \sum_{B\in\mathcal{B}: x\in B} v_j(B) \geq m - j\},$$

whence every $x \in Z_{n,k,\tau}$ belongs to some reparametrization ball $B \in \mathcal{B}$ with $v_j(B) \geq 1$. Thus

$$\begin{aligned} \sum_{j=1}^{m} \#(\mathcal{B}_j)e^{-sn} &= \sum_{j=1}^{m} \sum_{B\in\mathcal{B}_j} (v_{j-1}(B) - v_j(B))e^{-sn} \\ &\leq \sum_{j=1}^{m} \sum_{B\in\mathcal{B}} (v_{j-1}(B) - v_j(B))e^{-sn} \\ &\leq \sum_{B\in\mathcal{B}} u(B)e^{-sn} = \sum_{i\in\mathcal{I}_{n,k}} c_i e^{-sn}. \end{aligned}$$

Denote $\mathcal{J}_{n,k,\tau} := \{i \in \mathcal{I} : B_i \in \mathcal{B}_{j_0}\}$, where $j_0 \in \{1, \ldots, m\}$ is chosen such that $\#(\mathcal{B}_{j_0})$ is the smallest. Then

$$\#(\mathcal{J}_{n,k,\tau})e^{-sn} \leq \frac{1}{m} \sum_{i\in\mathcal{I}_{n,k}} c_i e^{-sn} \leq \frac{1}{\tau} \sum_{i\in\mathcal{I}_{n,k}} c_i e^{-sn}.$$

Moreover, due to the construction of $\mathcal{B}_{j_0}$, $Z_{n,k,\tau} \subseteq \bigcup_{i\in\mathcal{J}_{n,k,\tau}} 5\hat{B}_i$.

We next show that for each $n \geq N$ and $\tau > 0$, we have

$$\mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi, Z_{n,\tau}) \leq \frac{1}{n^2\tau} \sum_{i\in\mathcal{I}_n} c_i e^{-st_i}. \tag{4.5}$$

Assume $Z_{n,\tau} \neq \emptyset$. Since $Z_{n,k,\tau} \uparrow Z_{n,\tau}$, we have that $Z_{n,k,\tau} \neq \emptyset$ when k is large enough. Let $\mathcal{J}_{n,k,\tau}$ be the sets constructed in the previous discussion. Denote $E_{n,k,\tau} = \{x_i : i \in \mathcal{J}_{n,k,\tau}\}$. Note that the family of all non-empty compact subsets of X is compact under the Hausdorff metric. So there exists a subsequence $\{k_j\}$ of natural numbers and a non-empty compact set $E_{n,\tau} \subset X$ such that $E_{n,k_j,\tau}$ converges to $E_{n,\tau}$ in the Hausdorff metric as j goes to infinity.

Since any two points in $E_{n,k,\tau}$ can not be contained in the same B_i, any two points in $E_{n,\tau}$ also can not. Note that each B_i for $i \in \mathcal{J}_{n,k,\tau}$ contains a ball with radius $r > 0$ (for the choice of r, one may refer to the proof of Theorem 3.5). Thus $E_{n,\tau}$ is a finite set, moreover, $\#(E_{n,k_j,\tau}) = \#(E_{n,\tau})$ when j is sufficiently large. By choosing subsequence of $\{k_j\}$(still denoted by $\{k_j\}$), when $x_{i_j} \in E_{n,k_j,\tau}$ tends to $x \in E_{n,\tau}$, we can make the corresponding parameters t_{i_j} converges to a number denoted by t_x. We note that each $t_x(x \in E_{n,\tau})$ lies in the interval $[n-1, n]$.

By Lemma 3.4, $B(x_{i_j}, \hat{t}_{i_j}, 5\varepsilon, \phi) \subseteq B(x_{i_j}, \hat{t}_x, 6\varepsilon, \phi)$ when j is large enough. Since when j is large enough, $x_{i_j} \in B(x, \hat{t}_x, \varepsilon, \phi)$, by Lemma 3.2 (here we require that $\varepsilon < \frac{\theta}{6}$), $B(x_{i_j}, \bar{t}_x, 6\varepsilon, \phi) \subseteq B(x, \bar{t}_x, 7\varepsilon, \phi)$, where $\bar{t} = (1-\eta)^3 t$. Hence

$$Z_{n,k_j,\tau} \subseteq \bigcup_{i \in \mathcal{J}_{n,k_j,\tau}} 5\hat{B}_i \subseteq \bigcup_{x \in E_{n,\tau}} B(x, \bar{t}_x, 7\varepsilon, \phi),$$

when j is large enough. And thus

$$Z_{n,\tau} \subseteq \bigcup_{x \in E_{n,\tau}} B(x, \bar{t}_x, 8\varepsilon, \phi).$$

Since $\#(E_{n,k_j,\tau}) = \#(E_{n,\tau})$ when j is large enough, we then have

$$\#(E_{n,\tau})e^{-ns} \leq \frac{1}{\tau} \sum_{i \in \mathcal{I}_n} c_i e^{-sn}.$$

This forces

$$\begin{aligned} \mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi, Z_{n,\tau}) &\leq \sum_{x \in E_{n,\tau}} e^{-\frac{s+\delta}{(1-\eta)^3}\bar{t}_x} = \sum_{x \in E_{n,\tau}} e^{-(s+\delta)t_x} \\ &\leq \#(E_{n,\tau})e^{-(s+\delta)(n-1)} \leq \frac{1}{e^{n\delta-(s+\delta)}\tau} \sum_{i \in \mathcal{I}_n} c_i e^{-sn} \\ &\leq \frac{1}{n^2\tau} \sum_{i \in \mathcal{I}_n} c_i e^{-sn} \leq \frac{1}{n^2\tau} \sum_{i \in \mathcal{I}_n} c_i e^{-st_i}. \end{aligned}$$

Thus we have (4.5).

Fix an $\tau \in (0,1)$. Note that $\sum_{n=N}^{\infty} n^{-2} < 1$. It follows that $Z \subset \bigcup_{n=N}^{\infty} Z_{n,n^{-2}\tau}$ from (4.3). Hence by Proposition 2.1 and (4.5), we have

$$\mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi, Z) \leq \sum_{n=N}^{\infty} \mathcal{M}_{N,8\varepsilon}^{\frac{s+\delta}{(1-\eta)^3}}(\phi, Z_{n,n^{-2}\tau}) \leq \sum_{n=N}^{\infty} \frac{1}{\tau} \sum_{i \in \mathcal{I}_n} c_i e^{-st_i} = \frac{1}{\tau} \sum_{i \in \mathcal{I}} c_i e^{-st_i}.$$

Let τ tend to 1, we get the desired result. □

Corollary 4.2. *$\mathcal{M}^{s+\delta}(\phi, Z) \leq \mathcal{W}^s(\phi, Z) \leq \mathcal{M}^s(\phi, Z)$ and $h_{top}^B(\phi, Z) = h_{top}^{WB}(\phi, Z)$.*

Step 3. A Frostman lemma for fixed-point free flows.

Proposition 4.3. *Let (X, ϕ) be a compact metric flow without fixed points and K a non-empty compact subset of X. Let $s \geq 0, N \in \mathbb{N}$ and $\varepsilon > 0$. Suppose that $c := \mathcal{W}_{N,\varepsilon}^s(\phi, K) > 0$. Then there exists a Borel probability measure μ on X such that $\mu(K) = 1$ and*

$$\mu(B(x, t, \varepsilon, \phi)) \leq \frac{1}{c} e^{-st}, \quad \forall x \in X, t \geq N. \tag{4.6}$$

Proof. Since $c < \infty$, we can define a function p on $C(X)$ (the space of continuous real-valued functions on X) by

$$p(f) = (1/c)\mathcal{W}^s_{N,\varepsilon}(\phi, \chi_K \cdot f).$$

Let $\mathbf{1} \in C(X)$ denote the constant function $\mathbf{1}(x) \equiv 1$. It is easy to verify that

(1) $p(f + g) \leq p(f) + p(g)$ for any $f, g \in C(X)$;
(2) $p(\ell f) = \ell p(f)$ for any $\ell \geq 0$ and $f \in C(X)$;
(3) $p(\mathbf{1}) = 1, 0 \leq p(f) \leq \| f \|_\infty$ for any $f \in C(X)$, and $p(g) = 0$ for $g \in C(X)$ with $g \leq 0$.

By the Hahn-Banach theorem, we can extend the linear functional $l \mapsto \ell p(\mathbf{1}), \ell \in \mathbb{R}$, from the subspace of the constant functions to a linear functional $L : C(X) \to \mathbb{R}$ satisfying

$$L(\mathbf{1}) = p(\mathbf{1}) = 1 \text{ and } -p(-f) \leq L(f) \leq p(f) \text{ for any } f \in C(X).$$

If $f \in C(X)$ with $f \geq 0$, then $p(-f) = 0$ and so $L(f) \geq 0$. Hence combining the fact $L(\mathbf{1}) = 1$, we can use the Riesz representation theorem to find a Borel probability measure μ on X such that $L(f) = \int f d\mu$ for $f \in C(X)$.

For any compact set $E \subset X \setminus K$, by the Uryson lemma, there is $f \in C(X)$ such that $0 \leq f \leq 1, f(x) = 1$ for $x \in E$ and $f(x) = 0$ for $x \in K$. Then $f \cdot \chi_K \equiv 0$ and thus $p(f) = 0$. Hence $\mu(E) \leq L(f) \leq p(f) = 0$. This shows $\mu(X \setminus K) = 0$, i.e. $\mu(K) = 1$.

For any compact set $E \subset B(x, t, \varepsilon, \phi)$, by the Uryson lemma again, there exists $f \in C(X)$ such that $0 \leq f \leq 1, f(y) = 1$ for $y \in E$ and $f(y) = 0$ for $y \in X \setminus B(x, t, \varepsilon, \phi)$. Then $\mu(E) \leq L(f) \leq p(f)$. Since $f \cdot \chi_K \leq \chi_{B(x,t,\varepsilon,\phi)}$ and $t \geq N$, we have $\mathcal{W}^s_{N,\varepsilon}(\phi, \chi_K \cdot f) \leq e^{-st}$ and thus $p(f) \leq \frac{1}{c} e^{-ts}$. Therefore $\mu(E) \leq \frac{1}{c} e^{-st}$. It follows that

$$\mu(B(x, t, \varepsilon, \phi)) = \sup\{\mu(E) : E \text{ is a compact subset of } B(x, t, \varepsilon, \phi)\} \leq \frac{1}{c} e^{-st}.$$

□

Step 4. Proof of Theorem 2.5.

Proof. We first show that $h^B_{top}(\phi, K) \geq \underline{h}_\mu(\phi)$ for any $\mu \in \mathcal{M}(X)$ with $\mu(K) = 1$. Let μ be a such measure. For any $x \in X$ and $\varepsilon > 0$, we write

$$\underline{h}_\mu(\phi, x, \varepsilon) = \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi)).$$

Clearly $\underline{h}_\mu(\phi, x, \varepsilon)$ is nonnegative and increases as ε decreases. Hence by the monotone convergence theorem and the definition of the lower local entropy,

$$\lim_{\varepsilon \to 0} \int \underline{h}_\mu(\phi, x, \varepsilon)\, d\mu = \int \underline{h}_\mu(\phi, x)\, d\mu = \underline{h}_\mu(\phi).$$

Let $0 < \eta < 1$, we will show that $h^B_{top}(\phi, K) \geq (1 - \eta)\underline{h}_\mu(\phi)$. Clearly, it is sufficient to show $h^B_{top}(\phi, K) \geq (1 - \eta) \int \underline{h}_\mu(\phi, x, \varepsilon)\, d\mu$ for sufficiently small $\varepsilon > 0$.

Let $\theta > 0$ be as in Lemma 3.3. Fix $0 < \varepsilon < \theta$ and $\ell \in \mathbb{N}$. Denote $u_\ell = \min\{\ell, \int \underline{h}_\mu(\phi, x, \varepsilon)\, d\mu - \frac{1}{\ell}\}$. Then there exist a Borel set $A_\ell \subset X$ with $\mu(A_\ell) > 0$ and $2 < N \in \mathbb{N}$ such that

$$\mu(B(x,t,\varepsilon,\phi)) \leq e^{-u_\ell t},\ \forall x \in A_\ell, \quad t \geq N. \tag{4.7}$$

Let $\{B(x_i, t_i, \frac{1}{2}\varepsilon, \phi)\}$ be a countable or finite family so that $x_i \in X, \tilde{t}_i = (1-\eta)t_i \geq N$ and $\bigcup_i B(x_i, t_i, \frac{1}{2}\varepsilon, \phi) \supset (K \cap A_\ell)$. We may assume that for each i, $B(x_i, t_i, \frac{1}{2}\varepsilon, \phi) \cap (K \cap A_\ell) \neq \emptyset$, and choose $y_i \in B(x_i, t_i, \frac{1}{2}\varepsilon, \phi) \cap (K \cap A_\ell)$. Then by (4.7) and (2) of Lemma 3.3, we have

$$\begin{aligned}\sum_i e^{-t_i(1-\eta)u_\ell} &= \sum_i e^{-\tilde{t}_i u_\ell} \geq \sum_i \mu(B(y_i, \tilde{t}_i, \varepsilon, \phi)) \\ &\geq \sum_i \mu(B(x_i, t_i, \frac{1}{2}\varepsilon, \phi)) \geq \mu(K \cap A_\ell) = \mu(A_\ell) > 0.\end{aligned}$$

It follows that $\mathcal{M}^{(1-\eta)u_\ell}(\phi, K) \geq \mathcal{M}^{(1-\eta)u_\ell}_{\lceil \frac{N}{1-\eta} \rceil, \frac{\varepsilon}{2}}(\phi, K) \geq \mu(K \cap A_\ell)$. Therefore $h^B_{top}(\phi, K) \geq (1-\eta)\mu_\ell$. Let $\ell \to \infty$, we have $u_\ell \to \int \underline{h}_\mu(\phi, x, \varepsilon)\, d\mu$, and the inequality $h^B_{top}(\phi, K) \geq (1-\eta)\int \underline{h}_\mu(\phi, x, \varepsilon)\, d\mu$ holds. Hence $h^B_{top}(\phi, K) \geq (1-\eta)\underline{h}_\mu(\phi)$.

Let $\eta \to 0$, we then have the desired inequality.

We next show that $h^B_{top}(\phi, K) \leq \sup\{\underline{h}_\mu(\phi) : \mu \in \mathcal{M}(X),\ \mu(K) = 1\}$. We can assume that $h^B_{top}(\phi, K) > 0$. By Corollary 4.2, we have $h^{WB}_{top}(\phi, K) = h^B_{top}(\phi, K)$. Let $0 < s < h^B_{top}(\phi, K)$. Then there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $c := \mathcal{W}^s_{N,\varepsilon}(\phi, K) > 0$. By Proposition 4.3, there exists $\mu \in \mathcal{M}(X)$ with $\mu(K) = 1$ such that $\mu(B(x_i, t_i, \varepsilon, \phi)) \leq \frac{1}{c}e^{-ts}$ for any $x \in X$ and $t \geq N$. Clearly $\underline{h}(\phi, x) \geq \underline{h}_\mu(\phi, x, \varepsilon) \geq s$ for each $x \in X$ and hence $\underline{h}_\mu(\phi) \geq \int \underline{h}_\mu(\phi, x)\, d\mu \geq s$. This finishes the proof of Theorem 2.5. □

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Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, P.R. China

E-mail address: doumath@163.com

Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, P.R. China

E-mail address: fmgiveme5@163.com

Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, P.R. China

E-mail address: huaqiu@nju.edu.cn