

Open set condition and pseudo Hausdorff measure of self-affine IFSs

Xiaoye Fu¹, Jean-Pierre Gabardo², Hua Qiu³

1. School of Mathematics and Statistics

and Hubei Key Laboratory of Mathematical Sciences,

Central China Normal University, Wuhan 430079, P.R.China

2. Department of Mathematics and Statistics, McMaster University,

Hamilton, Ontario, L8S 4K1, Canada

3. Department of Mathematics, Nanjing University, Nanjing, 210093, P.R.China

Abstract

Let A be an $n \times n$ real expanding matrix and \mathcal{D} be a finite subset of \mathbb{R}^n with $0 \in \mathcal{D}$. The family of maps $\{f_d(x) = A^{-1}(x + d)\}_{d \in \mathcal{D}}$ is called a self-affine iterated function system (self-affine IFS). The self-affine set $K = K(A, \mathcal{D})$ is the unique compact set determined by (A, \mathcal{D}) satisfying the set-valued equation $K = \bigcup_{d \in \mathcal{D}} f_d(K)$. The

number $s = n \ln(\#\mathcal{D})/\ln(q)$ with $q = |\det(A)|$, is the so-called pseudo similarity dimension of K . As shown by He and Lau, one can associate with A and any number $s \geq 0$ a natural pseudo Hausdorff measure denoted by \mathcal{H}_w^s . In this paper, we show that, if s is chosen to be the pseudo similarity dimension of K , then the condition $\mathcal{H}_w^s(K) > 0$ holds if and only if the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the open set condition (OSC). This extends the well-known result for the self-similar case that the OSC is equivalent to K having positive Hausdorff measure \mathcal{H}^s for a suitable s . Furthermore, we relate the exact value of pseudo Hausdorff measure $\mathcal{H}_w^s(K)$ to a notion of upper s -density with respect to the pseudo norm $w(x)$ associated with A for the measure $\mu = \lim_{M \rightarrow \infty} \sum_{d_0, \dots, d_{M-1} \in \mathcal{D}} \delta_{d_0 + Ad_1 + \dots + A^{M-1}d_{M-1}}$ in the case that $\#\mathcal{D} \leq |\det A|$.

1 Introduction

Definition 1.1. Let $M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries. A matrix $A \in M_n(\mathbb{R})$ is called expanding if all its eigenvalues λ_i satisfy $|\lambda_i| > 1$. A self-affine set in \mathbb{R}^n is a compact set $K \subseteq \mathbb{R}^n$ satisfying the set-valued equation $AK = \bigcup_{d \in \mathcal{D}} (K + d)$, where

$A \in M_n(\mathbb{R})$ is an expanding matrix and $\mathcal{D} \subseteq \mathbb{R}^n$ is a finite set of distinct real vectors, which is called a digit set. K is called a self-similar set if A is a similarity matrix, i.e. $A = \rho R$, where $\rho > 1$ and R is an orthogonal matrix. To simplify the notations, we let $q = |\det(A)|$.

Math Subject Classifications. 28A80, 28A78.

Keywords. Self-affine sets, Iterated function system, Open set condition, Pseudo norm, Pseudo Hausdorff measure, Upper Beurling density.

The research of Fu was supported by NSFC grant 11922109.

The research of Qiu was supported by NSFC grant 11471157.

The research of Gabardo was supported by an NSERC grant.

Email: xiaoyefu@mail.ccnu.edu.cn, gabardo@mcmaster.ca, huaqiu@nju.edu.cn.

For an expanding matrix $A \in M_n(\mathbb{R})$ and a digit set $\mathcal{D} \subseteq \mathbb{R}^n$, it has been shown that the pair (A, \mathcal{D}) can uniquely determine a self-affine set $K := K(A, \mathcal{D})$ (see [12]). Given the pair (A, \mathcal{D}) , define

$$f_d(x) = A^{-1}(x + d), \quad d \in \mathcal{D}.$$

The family of maps $\{f_d\}_{d \in \mathcal{D}}$ is called a *self-affine iterated function system* (self-affine IFS). An important property of these maps is that they are contractive with respect to a suitable norm on \mathbb{R}^n (see [18]). It is clear that the self-affine set $K := K(A, \mathcal{D})$ determined by the pair (A, \mathcal{D}) satisfies $K = \bigcup_{d \in \mathcal{D}} f_d(K)$.

Definition 1.2. For the pair (A, \mathcal{D}) as above, we say that the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the *open set condition (OSC)* if there exists a non-empty bounded open set V such that

$$\bigcup_{d \in \mathcal{D}} f_d(V) \subset V \quad \text{and} \quad f_d(V) \cap f_{d'}(V) = \emptyset \quad \text{for } d \neq d' \in \mathcal{D}.$$

The OSC is the most important separation condition in the theory of IFS and it is thus very useful to find conditions equivalent to it. When the IFS is self-similar, it is well-known [29] that the OSC is equivalent to the self-similar set generated by the IFS having positive Hausdorff measure. For the self-affine case, He and Lau [10] showed that if the OSC is satisfied, then the corresponding self-affine set has positive pseudo Hausdorff measure. This last measure is defined by using a pseudo norm constructed from the matrix A instead of the classical Euclidean norm. In this paper, we prove that the OSC is indeed equivalent to the self-affine set generated by the IFS having positive pseudo Hausdorff measure by showing that the converse also holds.

In the following, we always assume, without loss of generality, that $0 \in \mathcal{D}$. For an integer $M \geq 1$, consider the sets

$$\mathcal{D}_M = \left\{ \sum_{j=0}^{M-1} A^j d_j : d_j \in \mathcal{D} \right\}, \quad \text{and} \quad \mathcal{D}_\infty = \bigcup_{M \geq 1} \mathcal{D}_M.$$

Then $\mathcal{D}_M \subset \mathcal{D}_{M+1}$ for any $M \geq 1$ since $0 \in \mathcal{D}$. Combining our results with those proved by He and Lau (Theorem 4.4 in [10]), we provide some conditions equivalent to the OSC for self-affine IFSs.

Theorem 1.1. *The following conditions are equivalent.*

- (i) *The IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC;*
- (ii) *$0 < \mathcal{H}_w^s(K) < \infty$, where $s = n \ln(\#\mathcal{D}) / \ln(q)$ and $\mathcal{H}_w^s(K)$ denotes the s -dimensional pseudo Hausdorff measure of K generated by the IFS $\{f_d\}_{d \in \mathcal{D}}$ (the detailed definition of $\mathcal{H}_w^s(K)$ is given in Section 2);*
- (iii) *$\#\mathcal{D}_M = (\#\mathcal{D})^M$ and \mathcal{D}_∞ is a uniformly discrete set, i.e. there exists $\delta > 0$ such that $\|x - y\| > \delta$ for any distinct elements x, y of \mathcal{D}_∞ .*

For the proof of Theorem 1.1, we utilize the connection between pseudo norm and Euclidean norm as well as the technique used by Schief [29], Bishop and Peres [3] for the

self-similar case. We also would like to mention that there have been several equivalent characterizations for the OSC under special cases given by Lagarias and Wang (Theorem 1.1 in [18]), by He and Lau (Theorem 4.4 in [10]) and by Fu and Gabardo (Theorem 3.2 in [6]).

In fractal geometry, one of the classical questions is to study the Hausdorff dimension and the corresponding Hausdorff measure of the self-affine set $K(A, \mathcal{D})$ determined by the pair (A, \mathcal{D}) .

In the case that $K(A, \mathcal{D})$ has positive Lebesgue measure and $\#\mathcal{D} = |\det A| \in \mathbb{Z}$, K is called a *self-affine tile* and the corresponding set \mathcal{D} is called a *tile digit set*, where $\#\mathcal{D}$ denotes the number of elements in \mathcal{D} . The Lebesgue measure and many aspects of the theory of self-affine tiles including the structure and tiling properties, the connection to wavelet theory, the fractal structure of the boundaries and the classification of tile digit sets have been investigated thoroughly (see e.g. [18, 19, 7, 8, 20, 9, 21, 16, 17]).

The situation becomes more complicate when $\#\mathcal{D} > q := |\det A|$ because the sets $K+d$, $d \in \mathcal{D}$, might overlap. He, Lau and Rao [11] considered the problem as to whether or not the Lebesgue measure of $K(A, \mathcal{D})$ is positive for this case. Qiu [28] provided an algorithm for calculating the Hausdorff measure of a special class of Cantor sets $K(A, \mathcal{D}) \subset \mathbb{R}$ with overlaps.

It is easy to see that the Lebesgue measure of $K(A, \mathcal{D})$ is 0 if $\#\mathcal{D} < q$, a situation which has motivated many researchers to study the Hausdorff dimension and Hausdorff measure of such sets $K(A, \mathcal{D})$. For self-similar sets satisfying certain separating conditions (e.g. open set condition [5], weak separation condition [23, 24], finite type condition [26]), there exist methods to calculate their Hausdorff dimensions [5, 11, 26, 30] and the corresponding Hausdorff measures [1, 6, 14, 13, 15, 31, 32, 33]. However, no many results are available in that direction for self-affine sets. The difficulty stems from the non-uniform contraction in different directions, in contrast to the self-similar case where the contraction is uniform in every direction. In [10], He and Lau defined a pseudo norm $w(x)$ associated with the matrix A and replaced the Euclidean norm by this pseudo norm to define the Hausdorff dimension and the Hausdorff measure for subsets in \mathbb{R}^n . They called these the *pseudo Hausdorff dimension* \dim_H^w and the *pseudo Hausdorff measure* \mathcal{H}_w^s , respectively. This setup gives a convenient estimation to the classical Hausdorff dimension of $K(A, \mathcal{D})$ and, furthermore, it makes $K(A, \mathcal{D})$ have a structure similar to that of a self-similar set since the pseudo norm defined in terms of A absorbs the non-uniform contractivity from A .

In this paper, we are interested in the computation of the pseudo Hausdorff measure of self-affine sets in the case that $\#\mathcal{D} \leq q$. This is motivated by the results in [6], which gave an exact expression for the Lebesgue measure of $K(A, \mathcal{D})$ with $\#\mathcal{D} = q$ and the Hausdorff measure of the self-similar set $K(A, \mathcal{D})$ associated with its similarity dimension in the case that $\#\mathcal{D} \leq q$. One of the main results of this paper is to relate the pseudo Hausdorff measure of $K(A, \mathcal{D})$, namely $\mathcal{H}_w^s(K(A, \mathcal{D}))$ where $s = n \ln(\#\mathcal{D})/\ln(q)$ is the pseudo similarity dimension of K , to a notion of upper density with respect to (w.r.t.) $w(x)$ for the measure μ which is defined by

$$\mu = \lim_{M \rightarrow \infty} \sum_{d_0, \dots, d_{M-1} \in \mathcal{D}} \delta_{d_0 + Ad_1 + \dots + A^{M-1}d_{M-1}}. \quad (1.1)$$

The measure μ defined in (1.1) is indeed a counting measure on \mathcal{D}_∞ which counts the number of repetitions. It is different from the invariant measure σ determined by the pair (A, \mathcal{D}) , which is defined in (3.1).

Theorem 1.2. *Let $K := K(A, \mathcal{D})$ be a self-affine set and let $s = n \ln(\#\mathcal{D})/\ln(q)$ be the pseudo similarity dimension of K . Then $\mathcal{H}_w^s(K) = (\mathcal{E}_{w,s}^+(\mu))^{-1}$, where μ is defined by (1.1) and $\mathcal{E}_{w,s}^+(\mu)$ is the upper s -density of μ w.r.t. $w(x)$ defined by*

$$\mathcal{E}_{w,s}^+(\mu) = \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\mu(U)}{(\text{diam}_w U)^s},$$

where the supremum is over all convex sets U with $\text{diam}_w U \geq r > 0$ w.r.t. $w(x)$ and $\text{diam}_w U$ is defined in Section 2 by using $w(x)$ instead of the classical Euclidean norm in the definition of $\text{diam}U$.

We will divide the proof of Theorem 1.2 into two cases, (i) and (ii), with the case (i) corresponding to the situation where the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC and the case (ii) where it does not.

It follows from Theorem 1.1 that if the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC, then $K := K(A, \mathcal{D})$ is an s -set w.r.t. $w(x)$. By analyzing the upper convex s -density w.r.t. $w(x)$ of points in K , we have the following expression of $\mathcal{H}_w^s(K)$.

Lemma 1.3. *Let $K := K(A, \mathcal{D})$ be the self-affine set associated with an IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfying the OSC. Let $s = n \ln(\#\mathcal{D})/\ln(q)$ and let σ be the invariant measure supported on K satisfying*

$$\int f \, d\sigma = \frac{1}{\#\mathcal{D}} \sum_{d \in \mathcal{D}} \int f \circ f_d \, d\sigma$$

for any compactly supported continuous function f on \mathbb{R}^n . Then, for any $r_0 > 0$,

$$(\mathcal{H}_w^s(K))^{-1} = \sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s},$$

where the supremum is taken over all convex sets U with $U \cap K \neq \emptyset$ and $0 < \text{diam}_w U \leq r_0$.

For case (i), Theorem 1.2 will follow from Lemma 1.3 after we prove that

$$\mathcal{E}_{w,s}^+(\mu) = \sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s}.$$

For case (ii), we show $\mathcal{E}_{w,s}^+(\mu) = \infty$ by using the third equivalent condition in Theorem 1.1.

The paper is organized as follows. In Section 2, we collect some definitions and some known results on pseudo norm, pseudo Hausdorff dimension and pseudo Hausdorff measures that we will use. In Section 3, we prove Theorem 1.1. Some properties of upper convex s -density w.r.t. $w(x)$ of points in $K(A, \mathcal{D})$ and the upper s -density of μ w.r.t. $w(x)$ are investigated respectively in Section 4 and in Section 5. In Section 6, Lemma 1.3 and Theorem 1.2 are proved.

2 Preliminaries

In this section, we recall the notions of pseudo norm and pseudo Hausdorff measure defined in [10]. and collect some known results about these that we will use later.

Let $A \in M_n(\mathbb{R})$ be expanding with $q := |\det A| \in \mathbb{R}$. We can assume without loss of generality that A has the property that $\|x\| \leq \|Ax\|$ and equality holds only for $x = 0$, where the norm $\|\cdot\|$ is the Euclidean norm, since $\|\cdot\|$ in \mathbb{R}^n can be renormed with an equivalent norm $\|\cdot\|'$ so that $\|x\|' < \|Ax\|'$ for all $0 \neq x \in \mathbb{R}^n$ [18]. He and Lau [10] introduced a *pseudo norm* $w(x)$ associated with A as follows:

- For $0 < \delta < 1/2$, choose a positive function $\phi_\delta(x) \in C^\infty(\mathbb{R}^n)$ with support in $B_\delta := B(0, \delta)$ (the closed ball centered at 0 with radius δ) such that $\phi_\delta(x) = \phi_\delta(-x)$ and $\int \phi_\delta(x) dx = 1$.
- Let $V = AB_1 \setminus B_1$ and $h(x) = \chi_V * \phi_\delta(x)$. Define

$$w(x) = \sum_{j=-\infty}^{\infty} q^{-\frac{j}{n}} h(A^j x), \quad x \in \mathbb{R}^n. \quad (2.1)$$

Note that V is an annular region by our convention that $\|x\| < \|Ax\|$ for $x \neq 0$. It is clear that $\mathbb{R}^n \setminus \{0\} = \bigcup_{k \in \mathbb{Z}} A^k V$, where the union is disjoint.

Proposition 2.1 ([10]). *The $w(x)$ defined in (2.1) is a C^∞ function on \mathbb{R}^n and satisfies*

- (i) $w(x) = w(-x)$, $w(x) = 0 \Leftrightarrow x = 0$;
- (ii) $w(Ax) = q^{1/n} w(x)$, $x \in \mathbb{R}^n$;
- (iii) *there exists an integer $p > 0$ such that for each $x \in \mathbb{R}^n$, the sum in (2.1) has at most p non-zero terms and $\alpha \leq w(x) \leq pq^{p/n}$, $x \in V$, where $\alpha = \inf_{x \in V} h(x) > 0$.*

He and Lau [10] showed that the pseudo norm $w(x)$ is comparable with the Euclidean norm $\|x\|$ through λ_{max} and λ_{min} , the maximal and minimal moduli of the eigenvalues of A . For more details about the properties of $w(x)$ and its relationship with the Euclidean norm, please refer to [10, 4, 25].

Proposition 2.2 ([10]). *Let $A \in M_n(\mathbb{R})$ be an expanding matrix with $|\det A| = q$ and let $w(x)$ be a pseudo norm associated with A . Then for any $0 < \epsilon < \lambda_{min} - 1$, there exists $C > 0$ (depending on ϵ) such that*

$$C^{-1} \|x\|^{\ln q / (n \ln(\lambda_{max} + \epsilon))} \leq w(x) \leq C \|x\|^{\ln q / (n \ln(\lambda_{min} - \epsilon))}, \quad \|x\| > 1,$$

$$C^{-1} \|x\|^{\ln q / (n \ln(\lambda_{min} - \epsilon))} \leq w(x) \leq C \|x\|^{\ln q / (n \ln(\lambda_{max} + \epsilon))}, \quad \|x\| \leq 1.$$

Unlike Euclidean norm, the triangle inequality is not satisfied for pseudo norm any more. However, we have the following inequality instead.

Lemma 2.3 ([10]). *There exists $\beta > 0$ such that for any $x, y \in \mathbb{R}^n$,*

$$w(x + y) \leq \beta \max\{w(x), w(y)\}.$$

Furthermore, we can modify Lemma 2.3 into the following lemma, which will be used in Section 5.

Lemma 2.4. *For any $\epsilon > 0$, there is a positive number $\lambda_\epsilon > 1$ such that for any $x_1, x_2 \in \mathbb{R}^n$ with $w(x_2) > \lambda_\epsilon w(x_1)$, $w(x_1 + x_2) < (1 + \epsilon)w(x_2)$ holds.*

Proof. Let $V = AB_1 \setminus B_1$. Denote $\theta = \max\{\|x\| : x \in V\}$ and $V_1 = \bigcup_{x \in V} B(x, 1)$. Obviously, $w \in C(\overline{V_1})$ since $w \in C^\infty(\mathbb{R}^n)$. So, for any $\epsilon > 0$, there exists a number δ with $0 < \delta < 1$ such that $w(z_1) - w(z_2) < \alpha \epsilon$ whenever $z_1, z_2 \in V_1$ with $\|z_1 - z_2\| \leq \delta$, where $\alpha = \inf_{x \in V} h(x)$ as introduced in Proposition 2.1. Choose $\lambda_\epsilon > 1$ large enough such that

$$\frac{n \ln(\lambda_\epsilon \alpha / (p q^{p/n}))}{\ln q} \geq \frac{-\ln(\delta/\theta)}{\ln \lambda_{\min}},$$

where p, q are the same as in Proposition 2.1. For any $x_1, x_2 \in \mathbb{R}^n$ with $w(x_2) > \lambda_\epsilon w(x_1)$, without loss of generality, assume $x_1 \neq 0$ and write $x_1 = A^{l_1} y_1$ and $x_2 = A^{l_2} y_2$ with $l_1, l_2 \in \mathbb{Z}$ and $y_1, y_2 \in V$. It is easy to check that $w(x_i) = q^{l_i/n} w(y_i)$ for $i = 1, 2$, and hence

$$q^{(l_2 - l_1)/n} > \lambda_\epsilon w(y_1)/w(y_2) \geq \lambda_\epsilon \alpha / (p q^{p/n}),$$

since $\alpha \leq w(y_i) \leq p q^{p/n}$ for $i = 1, 2$ by Proposition 2.1 (iii). This gives that

$$l_2 - l_1 > \frac{n \ln(\lambda_\epsilon \alpha / (p q^{p/n}))}{\ln q}$$

and thus $l_2 - l_1 > \frac{-\ln(\delta/\theta)}{\ln \lambda_{\min}} > 0$. Hence

$$\|A^{l_1 - l_2} y_1\| = \|(A^{-1})^{l_2 - l_1} y_1\| \leq \lambda_{\min}^{l_1 - l_2} \theta < \delta.$$

So we have

$$w(x_1 + x_2) = w(A^{l_2}(A^{l_1 - l_2} y_1 + y_2)) = q^{l_2/n} w(A^{l_1 - l_2} y_1 + y_2) < q^{l_2/n} (w(y_2) + \alpha \epsilon)$$

since $y_1, y_2 \in V$ and $\|A^{l_1 - l_2} y_1\| < \delta$, and thus

$$w(x_1 + x_2) < (1 + \epsilon) q^{l_2/n} w(y_2) = (1 + \epsilon) w(x_2).$$

□

Next, we come to the definition of pseudo Hausdorff measure and pseudo Hausdorff dimension. For a given set $E \subset \mathbb{R}^n$, the *diameter of E w.r.t. $w(x)$* is defined by

$$\text{diam}_w E = \sup\{w(x - y) : x, y \in E\}.$$

A collection of sets $\{U_i\}_{i=1}^\infty$ in \mathbb{R}^n is called a δ -cover of $E \subset \mathbb{R}^n$ w.r.t. $w(x)$ if $E \subseteq \bigcup_{i=1}^\infty U_i$ and $\text{diam}_w U_i \leq \delta$. Such a collection is called an open δ -cover of E if U_i is open for all $i \geq 1$. For $E \subset \mathbb{R}^n$ and $s \geq 0$, $\delta > 0$, define

$$\mathcal{H}_{w,\delta}^s(E) = \inf \left\{ \sum_{i=1}^\infty (\text{diam}_w U_i)^s : \{U_i\}_{i=1}^\infty \text{ is a } \delta\text{-cover of } E \text{ w.r.t. } w(x) \right\}.$$

Since $\mathcal{H}_{w,\delta}^s(E)$ is increasing when δ tends to 0, we can define the *s-dimensional Hausdorff measure of E w.r.t. w(x)* (the *s-dimensional pseudo Hausdorff measure of E*) by

$$\mathcal{H}_w^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_{w,\delta}^s(E) = \sup_{\delta > 0} \mathcal{H}_{w,\delta}^s(E).$$

It is direct to see that \mathcal{H}_w^s is a Borel measure on \mathbb{R}^n . By Proposition 2.1 (ii), it is easy to obtain that

$$\mathcal{H}_w^s(AE) = q^{s/n} \mathcal{H}_w^s(E). \quad (2.2)$$

As usual, we define the *Hausdorff dimension of E w.r.t. w(x)* (the *pseudo Hausdorff dimension of E*) to be the quantity

$$\dim_H^w E = \inf\{s : \mathcal{H}_w^s(E) = 0\} = \sup\{s : \mathcal{H}_w^s(E) = \infty\}.$$

This setup gives a convenient estimation of the classical Hausdorff dimension and makes a self-affine set have a structure as a self-similar set since the pseudo norm defined in terms of A absorbs the non-uniform contractivity from A .

Theorem 2.5 ([10]). *Let $A \in M_n(\mathbb{R})$ be an expanding matrix with $|\det A| = q \in \mathbb{R}$ and let $w(x)$ be a pseudo norm associated with A . Then for any subset $E \subset \mathbb{R}^n$,*

$$\frac{\ln q}{n \ln \lambda_{max}} \dim_H^w E \leq \dim_H E \leq \frac{\ln q}{n \ln \lambda_{min}} \dim_H^w E,$$

where $\lambda_{max}, \lambda_{min}$ denote the maximal and minimal moduli of the eigenvalues of A , and $\dim_H E$ is the classical Hausdorff dimension of E .

It follows immediately that $\dim_H^w E = \dim_H E$ when $\lambda_{max} = \lambda_{min}$. This includes the special case that A is a similarity matrix.

3 Proof of Theorem 1.1

In the following, let $A \in M_n(\mathbb{R})$ be expanding with $|\det A| = q$ and $0 \in \mathcal{D} \subset \mathbb{R}^n$ be a digit set. Let $K := K(A, \mathcal{D})$ be a self-affine set associated with (A, \mathcal{D}) . We always assume that $w(x)$ is a pseudo norm associated with A .

He and Lau [10] proved the direction “OSC $\Rightarrow 0 < \mathcal{H}_w^s(K) < \infty$ ” for the self-affine case.

Theorem 3.1 ([10]). *Suppose that the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC. Then $\dim_H^w K = s := n \ln(\#\mathcal{D})/\ln(q)$ and $0 < \mathcal{H}_w^s(K) < \infty$.*

In particular, if A is a similarity matrix with scaling factor $\rho > 1$, then $s := \ln(\#\mathcal{D})/\ln(\rho)$ is the similarity dimension of the self-similar set $K(A, \mathcal{D})$. For consistency, we call $s := n \ln(\#\mathcal{D})/\ln(q)$ the *pseudo similarity dimension* of the self-affine set $K(A, \mathcal{D})$.

To prove the other direction “ $0 < \mathcal{H}_w^s(K) < \infty \Rightarrow \text{OSC}$ ”, Lemma 3.2 and Lemma 3.6 below are needed. It is well-known ([12]) that the IFS $\{f_d\}_{d \in \mathcal{D}}$ determines a unique Borel probability measure σ supported on the set $K(A, \mathcal{D})$ satisfying

$$\int f \, d\sigma = \frac{1}{\#\mathcal{D}} \sum_{d \in \mathcal{D}} \int f \circ f_d \, d\sigma, \quad (3.1)$$

for any compactly supported continuous function f on \mathbb{R}^n . We say that σ has *no overlap* if $\sigma(f_d(K) \cap f_{d'}(K)) = 0$ for $d \neq d' \in \mathcal{D}$. Lemma 3.2 and its proof show that if the self-affine set K has positive pseudo Hausdorff measure associated with the dimension $s := n \ln(\#\mathcal{D})/\ln(q)$, then the invariant measure σ has no overlap.

Lemma 3.2. *Suppose that $0 < \mathcal{H}_w^s(K) < \infty$ with $s := n \ln(\#\mathcal{D})/\ln(q)$ and σ is a self-affine measure defined in (3.1). Then*

$$\sigma = (\mathcal{H}_w^s(K))^{-1} \mathcal{H}_w^s \upharpoonright K,$$

(i.e. σ is the restriction of \mathcal{H}_w^s to K normalized so as to give $\sigma(K) = 1$).

Proof. For any Borel subset $E \subset \mathbb{R}^n$ and $d \in \mathcal{D}$, we have

$$\mathcal{H}_w^s(f_d^{-1}(E)) = \mathcal{H}_w^s(AE - d) = \mathcal{H}_w^s(AE) = q^{s/n} \mathcal{H}_w^s(E) = (\#\mathcal{D}) \mathcal{H}_w^s(E).$$

Similarly, $\mathcal{H}_w^s(f_d(E)) = \frac{1}{\#\mathcal{D}} \mathcal{H}_w^s(E)$. Then, we have

$$\begin{aligned} \mathcal{H}_w^s(K) &= \mathcal{H}_w^s\left(\bigcup_{d \in \mathcal{D}} f_d(K)\right) \leq \sum_{d \in \mathcal{D}} \mathcal{H}_w^s(f_d(K)) \\ &= \#\mathcal{D} \cdot \frac{1}{\#\mathcal{D}} \mathcal{H}_w^s(K) = \mathcal{H}_w^s(K). \end{aligned}$$

This implies that $\mathcal{H}_w^s(f_d(K) \cap f_{d'}(K)) = 0$ for $d \neq d' \in \mathcal{D}$ since $0 < \mathcal{H}_w^s(K) < \infty$. Then for any Borel set E ,

$$\mathcal{H}_w^s(E \cap K) = \sum_{d \in \mathcal{D}} \mathcal{H}_w^s(E \cap f_d(K)) = \sum_{d \in \mathcal{D}} \frac{1}{\#\mathcal{D}} \mathcal{H}_w^s(f_d^{-1}(E) \cap K).$$

This proves that $\mathcal{H}_w^s \upharpoonright K$ is invariant for the IFS $\{f_d\}_{d \in \mathcal{D}}$ and thus the probability measure $(\mathcal{H}_w^s(K))^{-1} \mathcal{H}_w^s \upharpoonright K$ coincides with σ as this last measure is unique. \square

For $E, F \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, we let

$$\begin{aligned} D(E, F) &= \inf\{d(x, y) : x \in E, y \in F\} \\ D(z, E) &= D(\{z\}, E) \quad \text{and} \quad D(E, z) = D(E, \{z\}). \end{aligned}$$

where d denotes the distance induced by the Euclidean norm. The *Hausdorff distance* between compact sets $E, F \subset \mathbb{R}^n$ is denoted by $D_{\mathcal{H}}(E, F)$ and defined by

$$D_{\mathcal{H}}(E, F) = \max\left\{\sup_{x \in E} D(x, F), \sup_{y \in F} D(E, y)\right\}.$$

Denote $\text{Comp}(\mathbb{R}^n)$ the set of compact subsets in \mathbb{R}^n . Then Blaschke selection Theorem [3] implies that

Theorem 3.3 ([3]). *(Comp(\mathbb{R}^n), $D_{\mathcal{H}}$) is a compact metric space.*

We use the pseudo norm to replace the Euclidean norm and let

$$\begin{aligned} D_w(E, F) &= \inf\{d_w(x, y) := w(x - y) : x \in E, y \in F\}, \\ D_w(z, E) &= D_w(\{z\}, E) \quad \text{and} \quad D_w(E, z) = D_w(E, \{z\}). \end{aligned}$$

Define the *Hausdorff distance w.r.t. $w(x)$* between compact sets E and F in \mathbb{R}^n by

$$D_{\mathcal{H},w}(E, F) = \max\{\sup_{x \in E} D_w(x, F), \sup_{y \in F} D_w(E, y)\}.$$

Denote $U_w(x, \epsilon) := \{y \in \mathbb{R}^n : d_w(x, y) < \epsilon\}$ to be the open ϵ -neighborhood of $x \in \mathbb{R}^n$ w.r.t. $w(x)$ and $U_w(F, \epsilon) = \bigcup\{U_w(x, \epsilon) : x \in F\}$. Let f_1, f_2, \dots, f_N be the IFS associated with the expanding matrix $A \in M_n(\mathbb{R})$ and the digit set $\mathcal{D} = \{d_1, d_2, \dots, d_N\} \subset \mathbb{R}^n$. Let $\Sigma = \{1, 2, \dots, N\}$ and $\Sigma^m = \{(i_1 i_2 \dots i_m) : 1 \leq i_j \leq N\}$ for $m \geq 1$. Write $\Sigma^* = \bigcup_{m \geq 0} \Sigma^m$ with $\Sigma^0 := \emptyset$. For $\mathbf{i} = (i_1 i_2 \dots i_m)$ and $\mathbf{j} = (j_1 j_2 \dots j_k)$ in Σ^* , we use the notation \mathbf{ij} for the element $(i_1 i_2 \dots i_m j_1 j_2 \dots j_k) \in \Sigma^*$, and say that \mathbf{i} and \mathbf{j} are incomparable if there exists no \mathbf{k} such that $\mathbf{i} = \mathbf{jk}$ or $\mathbf{j} = \mathbf{ik}$. It follows from Proposition 2.1 (ii) that for any $i \in \Sigma$

$$w(f_i(x) - f_i(y)) = q^{-\frac{1}{n}} w(x - y). \quad (3.2)$$

Let $r = q^{-\frac{1}{n}}$. For $\mathbf{i} \in \Sigma^m$, $m \geq 1$, the length of \mathbf{i} is denoted by $|\mathbf{i}| = m$. Define

$$f_{\mathbf{i}} = f_{i_1} \circ f_{i_2} \cdots \circ f_{i_m}, \quad K_{\mathbf{i}} = f_{\mathbf{i}}(K) \quad \text{and} \quad r_{\mathbf{i}} = r^{|\mathbf{i}|} = q^{-\frac{m}{n}}.$$

It is obvious that, for any $m \geq 1$, $K = \bigcup_{\mathbf{i} \in \Sigma^m} K_{\mathbf{i}}$. Particularly, for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \Sigma^*$, it follows from (3.2) that we can get some elementary property on diam_w, D_w and $D_{\mathcal{H},w}$ related to the IFS maps $f_{\mathbf{k}}$. For later use, we collect them as follows.

Proposition 3.4. *Given $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \Sigma^*$, we have the following identities.*

$$\begin{aligned} \text{diam}_w(K_{\mathbf{i}}) &= r^{|\mathbf{i}|} \text{diam}_w K, \\ D_w(f_{\mathbf{k}}(K_{\mathbf{i}}), f_{\mathbf{k}}(K_{\mathbf{j}})) &= r^{|\mathbf{k}|} D_w(K_{\mathbf{i}}, K_{\mathbf{j}}), \\ D_{\mathcal{H},w}(f_{\mathbf{k}}(K_{\mathbf{i}}), f_{\mathbf{k}}(K_{\mathbf{j}})) &= r^{|\mathbf{k}|} D_{\mathcal{H},w}(K_{\mathbf{i}}, K_{\mathbf{j}}). \end{aligned}$$

According to Lemma 3.2, it is direct to get the following result.

Corollary 3.5. *Suppose that $0 < \mathcal{H}_w^s(K) < \infty$ with $s := n \ln(\#\mathcal{D}) / \ln(q)$. Then $\mathbf{i}, \mathbf{j} \in \Sigma^*$ are incomparable if and only if $\mathcal{H}_w^s(K_{\mathbf{i}} \cap K_{\mathbf{j}}) = \emptyset$.*

Also if we admit only open sets in the covers of E , then $\mathcal{H}_{w,\delta}^s(E)$ (also $\mathcal{H}_w^s(E)$) does not change.

Lemma 3.6. *For $E \subset \mathbb{R}^n$ and $s \geq 0$, $\delta > 0$, define*

$$\tilde{\mathcal{H}}_{w,\delta}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}_w U_i)^s : \{U_i\}_{i=1}^{\infty} \text{ is an open } \delta\text{-cover of } E \text{ w.r.t. } w(x) \right\}.$$

Then $\tilde{\mathcal{H}}_{w,\delta}^s(E) = \mathcal{H}_{w,\delta}^s(E)$.

Proof. It is obvious that $\mathcal{H}_{w,\delta}^s(E) \leq \tilde{\mathcal{H}}_{w,\delta}^s(E)$. For any $\epsilon > 0$, by the definition of $\mathcal{H}_{w,\delta}^s(E)$, there exists a δ -cover $\{U_i\}_{i=1}^\infty$ of E w.r.t. $w(x)$ such that

$$\mathcal{H}_{w,\delta}^s(E) \geq \sum_{i=1}^{\infty} (\text{diam}_w U_i)^s - \epsilon.$$

Denote $U(U_i, 1) = \{y \in \mathbb{R}^n : \|y - x\| < 1 \text{ for some } x \in U_i\}$ to be the open 1-neighborhood of U_i . For the above $\epsilon > 0$, by using $w(x) \in C(\overline{U(U_i, 1)})$, there exists $\delta_i > 0$ such that $|w(x) - w(y)| < \text{diam}_w(U_i)\epsilon$ whenever $\|x - y\| \leq \delta_i$ and $x, y \in \overline{U(U_i, 1)}$. Take $\delta'_i = \min\{\delta_i, 1\}$ and $V_i = U(U_i, \frac{\delta'_i}{2})$. Then $U_i \subset V_i \subset U(U_i, 1)$ and V_i is open. For any $z_1, z_2 \in V_i$, by the definition of V_i , there exist $x_1, x_2 \in U_i$ such that $\|x_j - z_j\| \leq \frac{\delta'_i}{2}, j = 1, 2$. This and $w(x) \in C(\overline{V_i})$ imply that

$$w(z_1 - z_2) \leq w(x_1 - x_2) + \text{diam}_w(U_i)\epsilon \leq \text{diam}_w(U_i) + \text{diam}_w(U_i)\epsilon < (1 + \epsilon)\delta. \quad (3.3)$$

It follows from (3.3) that $\text{diam}_w(V_i) \leq (1 + \epsilon)\text{diam}_w(U_i) < (1 + \epsilon)\delta$ since $z_1, z_2 \in V_i$ are arbitrary. Using the definition of $\tilde{\mathcal{H}}_{w,\delta}^s$,

$$\begin{aligned} \tilde{\mathcal{H}}_{w,(1+\epsilon)\delta}^s(E) &\leq \sum (\text{diam}_w V_i)^s \leq (1 + \epsilon)^s (\text{diam}_w U_i)^s \\ &\leq (1 + \epsilon)^s (\mathcal{H}_{w,\delta}^s(E) + \epsilon). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, one can get $\tilde{\mathcal{H}}_{w,\delta}^s(E) \leq \mathcal{H}_{w,\delta}^s(E)$. \square

To prove the direction “ $0 < \mathcal{H}_w^s(K) < \infty \Rightarrow \text{OSC}$ ”, we use the idea due to [29, 3] for the self-similar case and Lemma 2.3 on the connection between Pseudo norm and Euclidean norm.

Theorem 3.7. *If $0 < \mathcal{H}_w^s(K(A, \mathcal{D})) < \infty$ with $s := n \ln(\#\mathcal{D})/\ln(q)$, then the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC.*

Proof. Let $t > 0$. By the definition of $\mathcal{H}_w^s(K)$ and Lemma 3.6, there exists a sequence of open sets $\{U_i\}_{i \geq 1}$ such that

$$U := \bigcup_{i=1}^{\infty} U_i \supset K \text{ and } \sum_{i=1}^{\infty} (\text{diam}_w U_i)^s \leq (1 + t^s) \mathcal{H}_w^s(K).$$

Claim 1: Denote $\delta = D_w(K, U^c)$, where U^c denotes the complement of U . Then for all incomparable \mathbf{i}, \mathbf{j} with $r_{\mathbf{j}} > tr_{\mathbf{i}}$, we have $D_{\mathcal{H},w}(K_{\mathbf{i}}, K_{\mathbf{j}}) \geq \delta r_{\mathbf{i}}$.

Proof. Suppose that Claim 1 does not hold. Then there exist a pair \mathbf{i}, \mathbf{j} with $r_{\mathbf{j}} > tr_{\mathbf{i}}$ and $D_{\mathcal{H},w}(K_{\mathbf{i}}, K_{\mathbf{j}}) < \delta r_{\mathbf{i}}$. Since clearly $D_w(K_{\mathbf{i}}, (f_{\mathbf{i}}(U))^c) = \delta r_{\mathbf{i}}$, we get

$$K_{\mathbf{j}} \subset U_w(K_{\mathbf{i}}, \delta r_{\mathbf{i}}) \subset f_{\mathbf{i}}(U).$$

This implies that

$$\begin{aligned} \mathcal{H}_w^s(K) r_{\mathbf{i}}^s (1 + t^s) &< \mathcal{H}_w^s(K) (r_{\mathbf{i}}^s + r_{\mathbf{j}}^s) = \mathcal{H}_w^s(K_{\mathbf{i}}) + \mathcal{H}_w^s(K_{\mathbf{j}}) \\ &= \mathcal{H}_w^s(K_{\mathbf{i}} \cup K_{\mathbf{j}}) \leq \sum_{i=1}^{\infty} (\text{diam}_w f_{\mathbf{i}}(U_i))^s \\ &= \sum_{i=1}^{\infty} r_{\mathbf{i}}^s (\text{diam}_w U_i)^s \leq \mathcal{H}_w^s(K) r_{\mathbf{i}}^s (1 + t^s), \end{aligned}$$

which is a contradiction. (The second to the last inequality follows from the fact that $K_{\mathbf{i}} \cup K_{\mathbf{j}} \subset f_{\mathbf{i}}(U)$ and the second equality is obtained from Corollary 3.5). \square

For $0 < b < 1$, we set $I_b = \{\mathbf{i} \in \Sigma^* : r^{|\mathbf{i}|} \leq b < r^{|\mathbf{i}|-1}\}$. The elements of I_b are obviously incomparable and satisfy $K = \bigcup_{\mathbf{i} \in I_b} K_{\mathbf{i}}$.

Fix $0 < \varepsilon < \min\{\text{diam}_w K, \beta \text{diam}_w K, (\beta \text{diam}_w K)^2, \lambda_{\min} - 1\}$, where β satisfies the inequality in Lemma 2.3 and λ_{\min} is the minimal moduli of the eigenvalues of A . For $\mathbf{k} \in \Sigma^*$, denote $G_{\mathbf{k}} = U_w(K_{\mathbf{k}}, \varepsilon r_{\mathbf{k}})$. Note that for any $k \geq 1$, the pair $(A, A^{-k}\mathcal{D})$ can determine a self-affine set $A^{-k}K$ if K is determined by the pair (A, \mathcal{D}) and the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC if and only if $\{f_{A^{-k}d}\}_{d \in \mathcal{D}}$ satisfies the OSC. To simplify the notations, WLOG we can assume that $\text{diam}_w K$ is small enough such that $\text{diam}_w G_{\mathbf{k}} < 1$ for any $\mathbf{k} \in \Sigma^*$ since we can always use $A^{-k}K$ and $\{f_{A^{-k}d}\}_{d \in \mathcal{D}}$ instead of K and $\{f_d\}_{d \in \mathcal{D}}$ if $\text{diam}_w K$ is not small enough.

Claim 2: Denote $I(\mathbf{k}) = \{\mathbf{i} \in I_{\text{diam}_w G_{\mathbf{k}}} : K_{\mathbf{i}} \cap G_{\mathbf{k}} \neq \emptyset\}$, and $\gamma = \sup_{\mathbf{k}} \#I(\mathbf{k})$. Then $\gamma < \infty$.

Proof. For the given $\varepsilon > 0$, let C_i and α_i , $i = 1, 2$, be the number as in Proposition 2.2 satisfying the inequality that $\|x - y\| \leq (C_i d_w(x, y))^{\alpha_i}$ for $\|x - y\| > 1$ and $\|x - y\| \leq 1$ respectively. Take $C = C_1$ and $\alpha = \alpha_1$ if $(C_1 \beta^3 (\text{diam}_w K)^2)^{\alpha_1} \geq (C_2 \beta^3 (\text{diam}_w K)^2)^{\alpha_2}$ and if not, we take $C = C_2$ and $\alpha = \alpha_2$. Let B be the closed $(C \beta^3 (\text{diam}_w K)^2)^{\alpha}$ -neighborhood of K , i.e. $B = \{x \in \mathbb{R}^n : D(x, K) \leq (C \beta^3 (\text{diam}_w K)^2)^{\alpha}\}$. Then for any $\mathbf{k} \in \Sigma^*$, it holds that

$$f_{\mathbf{k}}^{-1}(K_{\mathbf{i}}) \subset B, \quad \forall \mathbf{i} \in I(\mathbf{k}). \quad (3.4)$$

In fact, noticing that $K_{\mathbf{i}} \cap G_{\mathbf{k}} \neq \emptyset$ if $\mathbf{i} \in I(\mathbf{k})$, for any $y \in K_{\mathbf{i}}$, it follows from the definition of d_w and Lemma 2.3 that,

$$D_w(y, K_{\mathbf{k}}) \leq \beta \max\{d_w(y, z), D_w(z, K_{\mathbf{k}})\} \leq \beta \max\{\text{diam}_w K_{\mathbf{i}}, \varepsilon r_{\mathbf{k}}\},$$

where z is any point in $K_{\mathbf{i}} \cap G_{\mathbf{k}}$. This gives that

$$D_w(f_{\mathbf{k}}^{-1}(y), K) \leq \beta \max\{r_{\mathbf{k}}^{-1} r_{\mathbf{i}} \text{diam}_w K, \varepsilon\}. \quad (3.5)$$

On the other hand, if $\mathbf{i} \in I(\mathbf{k})$, then $\mathbf{i} \in I_{\text{diam}_w G_{\mathbf{k}}}$ and thus we have $r_{\mathbf{i}} \leq \text{diam}_w G_{\mathbf{k}}$ by the definition of $I_{\text{diam}_w G_{\mathbf{k}}}$. Next, we will utilize Lemma 2.3 to give an estimation on $\text{diam}_w G_{\mathbf{k}}$. Let $z_1, z_2 \in G_{\mathbf{k}}$. Then there exist $x_1, x_2 \in K_{\mathbf{k}}$ satisfying that $d_w(z_i, x_i) < \varepsilon r_{\mathbf{k}}$ for $i = 1, 2$. By Lemma 2.3, we obtain

$$\begin{aligned} d_w(z_1, z_2) &= w(z_1 - x_1 + x_1 - x_2 + x_2 - z_2) \\ &\leq \beta \max\{w(z_1 - x_1), w(x_1 - x_2 + x_2 - z_2)\} \\ &\leq \beta \max\{w(z_1 - x_1), \beta \max\{w(x_1 - x_2), w(x_2 - z_2)\}\} \\ &\leq \beta \max\{\varepsilon r_{\mathbf{k}}, \beta \max\{r_{\mathbf{k}} \text{diam}_w K, \varepsilon r_{\mathbf{k}}\}\} \\ &\leq \beta^2 r_{\mathbf{k}} \text{diam}_w K. \end{aligned}$$

The last inequality is obtained by the restriction of ε . This and $r_{\mathbf{i}} \leq \text{diam}_w G_{\mathbf{k}}$ give $r_{\mathbf{i}} \leq \beta^2 r_{\mathbf{k}} \text{diam}_w K$. Substituting this into (3.5), one can get $D_w(f_{\mathbf{k}}^{-1}(y), K) \leq \beta^3 (\text{diam}_w K)^2$. Then by using Proposition 2.2, we have

$$\begin{aligned}
D(f_{\mathbf{k}}^{-1}(y), K) &\leq \begin{cases} (C_1 D_w(f_{\mathbf{k}}^{-1}(y), K))^{\alpha_1}, & \text{if } D(f_{\mathbf{k}}^{-1}(y), K) > 1, \\ (C_2 D_w(f_{\mathbf{k}}^{-1}(y), K))^{\alpha_2}, & \text{if } D(f_{\mathbf{k}}^{-1}(y), K) \leq 1 \end{cases} \\
&\leq (C\beta^3(\text{diam}_w K)^2)^\alpha,
\end{aligned}$$

which proves (3.4).

Since for any $\mathbf{i}, \mathbf{j} \in \Sigma^m$, $r_{\mathbf{i}} = r_{\mathbf{j}} = r^m$. Then $r_{\mathbf{j}} > r_{\mathbf{i}} r$ holds. We may apply Claim 1 for $t = r$ to get $\delta > 0$ such that

$$D_{\mathcal{H}, w}(K_{\mathbf{i}}, K_{\mathbf{j}}) \geq \delta r_{\mathbf{i}} \geq \delta r_{\mathbf{k}} r \text{diam}_w G$$

for any distinct $\mathbf{i}, \mathbf{j} \in I(\mathbf{k})$, where $G = U_w(K, \epsilon)$. Hence, by Proposition 3.4, we have

$$D_{\mathcal{H}, w}(f_{\mathbf{k}}^{-1}(K_{\mathbf{i}}), f_{\mathbf{k}}^{-1}(K_{\mathbf{j}})) \geq \delta r \text{diam}_w G$$

and

$$D_{\mathcal{H}}(f_{\mathbf{k}}^{-1}(K_{\mathbf{i}}), f_{\mathbf{k}}^{-1}(K_{\mathbf{j}})) \geq (C' \delta r \text{diam}_w G)^{\alpha'},$$

with some positive C', α' for all $\mathbf{i}, \mathbf{j} \in I(\mathbf{k})$ by Proposition 2.2. By Theorem 3.3, $\#I(\mathbf{k})$ is bounded by the maximal number of compact subsets of B which are $(C' \delta r \text{diam}_w G)^{\alpha'}$ -separated in the Hausdorff metric, which is obviously independent of $\mathbf{k} \in \Sigma^*$. \square

Claim 3: Choose \mathbf{k} such that $\gamma = \#I(\mathbf{k})$. Then for any $\mathbf{j} \in \Sigma^*$, $I(\mathbf{j}\mathbf{k}) = \{\mathbf{j}\mathbf{i} : \mathbf{i} \in I(\mathbf{k})\}$.

Proof. Notice that $\emptyset \neq K_{\mathbf{i}} \cap G_{\mathbf{k}}$ implies

$$\begin{aligned}
\emptyset &\neq f_{\mathbf{j}}(K_{\mathbf{i}} \cap G_{\mathbf{k}}) = f_{\mathbf{j}}(K_{\mathbf{i}}) \cap f_{\mathbf{j}}(G_{\mathbf{k}}) = K_{\mathbf{j}\mathbf{i}} \cap f_{\mathbf{j}}(U_w(K_{\mathbf{k}}, \epsilon r_{\mathbf{k}})) \\
&= K_{\mathbf{j}\mathbf{i}} \cap U_w(K_{\mathbf{j}\mathbf{k}}, \epsilon r_{\mathbf{j}\mathbf{k}}) = K_{\mathbf{j}\mathbf{i}} \cap G_{\mathbf{j}\mathbf{k}}.
\end{aligned}$$

This shows that $\{\mathbf{j}\mathbf{i} : \mathbf{i} \in I(\mathbf{k})\} \subseteq I(\mathbf{j}\mathbf{k})$. On the other hand, we further note that $\#\{\mathbf{j}\mathbf{i} : \mathbf{i} \in I(\mathbf{k})\} = \#I(\mathbf{k}) = \gamma$. By the selection of \mathbf{k} and the maximality of $\#I(\mathbf{k})$, Claim 3 follows. \square

Claim 4: $D_w(K_{i\mathbf{i}\mathbf{k}}, K_j) \geq \epsilon r_{i\mathbf{i}\mathbf{k}}$ for any $j \neq i$ and any $\mathbf{i} \in \Sigma^*$.

Proof. For any word $j\mathbf{l}$ with $j \neq i$, Claim 3 implies that $j\mathbf{l} \notin I(i\mathbf{i}\mathbf{k})$. By the definition of $I(i\mathbf{i}\mathbf{k})$, for $j\mathbf{l} \in I_{\text{diam}_w G_{i\mathbf{i}\mathbf{k}}}$, $K_{j\mathbf{l}} \cap G_{i\mathbf{i}\mathbf{k}} = \emptyset$. Hence, $D_w(K_{i\mathbf{i}\mathbf{k}}, K_{j\mathbf{l}}) \geq \epsilon r_{i\mathbf{i}\mathbf{k}}$. Noticing that

$$K_j \subseteq \bigcup \{K_{j\mathbf{l}} : j\mathbf{l} \in I_{\text{diam}_w G_{i\mathbf{i}\mathbf{k}}}\},$$

then Claim 4 follows. \square

Claim 5: For $\mathbf{i} \in \Sigma^*$, denote $G_{\mathbf{i}}^* = U_w(K_{\mathbf{i}}, \beta^{-1} \epsilon r_{\mathbf{i}})$. Then $U = \bigcup_{\mathbf{j} \in \Sigma^*} G_{\mathbf{j}\mathbf{k}}^*$ gives the OSC.

Proof. Clearly, U is open and $K_{\mathbf{k}} \subset G_{\mathbf{k}}^* \subset U$. For each i ,

$$f_i(U) = \bigcup_{\mathbf{j} \in \Sigma^*} f_i(G_{\mathbf{j}\mathbf{k}}^*) = \bigcup_{\mathbf{j} \in \Sigma^*} G_{\mathbf{i}\mathbf{j}\mathbf{k}}^* \subset U.$$

For $i \neq j$, $f_i(U) \cap f_j(U) = \emptyset$. Indeed, if not, there exist \mathbf{i}, \mathbf{j} such that $G_{\mathbf{i}\mathbf{k}}^* \cap G_{\mathbf{j}\mathbf{k}}^* \neq \emptyset$. Let $y \in G_{\mathbf{i}\mathbf{k}}^* \cap G_{\mathbf{j}\mathbf{k}}^*$. Then there exist $y_1 \in K_{\mathbf{i}\mathbf{k}}$ and $y_2 \in K_{\mathbf{j}\mathbf{k}}$ such that $w(y - y_1) < \beta^{-1}\varepsilon r_{\mathbf{i}\mathbf{k}}$ and $w(y - y_2) < \beta^{-1}\varepsilon r_{\mathbf{j}\mathbf{k}}$. Without loss of generality, we assume that $r_{\mathbf{i}\mathbf{k}} \geq r_{\mathbf{j}\mathbf{k}}$. Then we have $w(y_1 - y_2) < \varepsilon r_{\mathbf{i}\mathbf{k}}$. Hence, $D_w(K_{\mathbf{i}\mathbf{k}}, K_{\mathbf{j}\mathbf{k}}) < \varepsilon r_{\mathbf{i}\mathbf{k}}$, which contradicts Claim 4. \square

This completes the proof of Theorem 3.7. \square

There is another equivalent condition for the OSC provided by He and Lau in [10].

Theorem 3.8 ([10]). *Let $A \in M_n(\mathbb{R})$ be expanding and let $\mathcal{D} \subset \mathbb{R}^n$ be a digit set. Then the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC if and only if $\#\mathcal{D}_M = (\#\mathcal{D})^M$ and \mathcal{D}_∞ is a uniformly discrete set.*

Theorem 3.7 together with Theorem 3.8 and Theorem 3.1 imply Theorem 1.1.

4 The upper convex density w.r.t. $w(x)$

In this section, we introduce the notion of s -sets w.r.t. the pseudo norm $w(x)$, and study the upper convex density of an s -set w.r.t. $w(x)$ at certain points. These are definitions analogous to those corresponding to the Euclidean norm. (See, for example, Section 2 in [5].)

A subset $E \subset \mathbb{R}^n$ is called an s -set ($0 \leq s \leq n$) w.r.t. $w(x)$ if E is \mathcal{H}_w^s -measurable and $0 < \mathcal{H}_w^s(E) < \infty$. The *upper convex s -density* of an s -set E w.r.t. $w(x)$ at x is defined as

$$D_{w,c}^s(E, x) = \lim_{r \rightarrow 0} \sup_{0 < \text{diam}_w U \leq r, x \in U} \frac{\mathcal{H}_w^s(E \cap U)}{(\text{diam}_w U)^s},$$

where the supremum is over all convex sets U with $x \in U$ and $0 < \text{diam}_w U \leq r$, and the limit exists obviously. We have the following result similar to Theorem 2.2 and Theorem 2.3 in [5].

Theorem 4.1. *If E is an s -set w.r.t. $w(x)$ in \mathbb{R}^n , then $D_{w,c}^s(E, x) = 1$ at \mathcal{H}_w^s -almost all $x \in E$ and $D_{w,c}^s(E, x) = 0$ at \mathcal{H}_w^s -almost all $x \in E^c$.*

We will prove Theorem 4.1 by showing that $D_{w,c}^s(E, x) = 0$ at \mathcal{H}_w^s -almost all $x \in E^c$ (Lemma 4.4) and $D_{w,c}^s(E, x) = 1$ at \mathcal{H}_w^s -almost all $x \in E$ (Lemma 4.5) respectively. We need an analogue of Vitali covering theorem [5] and the following lemma. We should mention that the sets encountered in the following can always be represented in terms of known \mathcal{H}_w^s -measurable sets using combinations of $\overline{\lim}$, $\underline{\lim}$, countable unions and intersections. So without explicit mention in this section, we always assume that the sets involved are \mathcal{H}_w^s -measurable.

Lemma 4.2. *Let $E \subset \mathbb{R}^n$ be \mathcal{H}_w^s -measurable with $\mathcal{H}_w^s(E) < +\infty$ and let $\varepsilon > 0$. Then there exists $\rho > 0$, depending only on E and ε , such that for any collection of Borel sets $\{U_i\}_{i=1}^\infty$ with $0 < \text{diam}_w U_i \leq \rho$, we have*

$$\mathcal{H}_w^s(E \cap \bigcup_i U_i) < \sum_i (\text{diam}_w U_i)^s + \varepsilon.$$

Proof. By the definition that $\mathcal{H}_w^s = \lim_{\delta \rightarrow 0} \mathcal{H}_{w,\delta}^s$, we may choose $\rho > 0$ such that

$$\mathcal{H}_w^s(E) \leq \sum_i \text{diam}_w(W_i) + \varepsilon/2 \quad (4.1)$$

for any ρ -cover $\{W_i\}$ of E w.r.t. $w(x)$. Given Borel sets $\{U_i\}$ with $0 < \text{diam}_w(U_i) \leq \rho$, by the definition of \mathcal{H}_w^s , we can find a ρ -cover $\{V_i\}$ of $E \setminus \bigcup_i U_i$ w.r.t. $w(x)$ satisfying

$$\mathcal{H}_w^s(E \setminus \bigcup_i U_i) + \varepsilon/2 > \sum_i \text{diam}_w(V_i).$$

Then $\{U_i\} \cup \{V_i\}$ is a ρ -cover of E w.r.t. $w(x)$, and using (4.1), we have

$$\mathcal{H}_w^s(E) \leq \sum_i \text{diam}_w(U_i) + \sum_i \text{diam}_w(V_i) + \varepsilon/2.$$

Hence,

$$\begin{aligned} \mathcal{H}_w^s(E \cap \bigcup_i U_i) &= \mathcal{H}_w^s(E) - \mathcal{H}_w^s(E \setminus \bigcup_i U_i) \\ &< \sum_i \text{diam}_w(U_i) + \sum_i \text{diam}_w(V_i) + \varepsilon/2 - \sum_i \text{diam}_w(V_i) + \varepsilon/2 \\ &= \sum_i \text{diam}_w(U_i) + \varepsilon. \end{aligned}$$

□

A collection of sets \mathcal{V} is called a *Vitali class* for E w.r.t. $w(x)$ if for each $x \in E$ and $\delta > 0$, there exists $U \in \mathcal{V}$ with $x \in U$ and $0 < \text{diam}_w U \leq \delta$.

Theorem 4.3 (Vitali covering theorem).

(a) Let E be an \mathcal{H}_w^s -measurable subset of \mathbb{R}^n and let \mathcal{V} be a Vitali class of closed sets for E w.r.t. $w(x)$. Then we may select a (finite or countable) disjoint sequence U_i from \mathcal{V} such that either $\sum_i (\text{diam}_w U_i)^s = \infty$ or $\mathcal{H}_w^s(E \setminus \bigcup_i U_i) = 0$.

(b) If $\mathcal{H}_w^s(E) < +\infty$, then for any given $\varepsilon > 0$, we may also require that

$$\mathcal{H}_w^s(E) \leq \sum_i (\text{diam}_w U_i)^s + \varepsilon.$$

Proof. (a). Fix $\rho > 0$. We may assume that $\text{diam}_w U \leq \rho$ for all $U \in \mathcal{V}$. Let $U_1 \in \mathcal{V}$ and $U_1 \cap E \neq \emptyset$. We choose U_i , $i \geq 2$ inductively. Suppose that U_1, \dots, U_m have been chosen, and let

$$d_m = \sup\{\text{diam}_w U : U \in \mathcal{V} \text{ and } U \cap U_i = \emptyset, i = 1, 2, \dots, m\}.$$

Note that $\{d_m\}_{m \geq 1}$ is decreasing. If $d_m = 0$, then $E \subset \bigcup_{i=1}^m U_i$. Indeed, if there existed a

point $x \in E \setminus \bigcup_{i=1}^m U_i$, then, letting

$$\delta_x = \frac{1}{2} \inf\{w(x-y), y \in \bigcup_{i=1}^m U_i\} > 0,$$

we could find $U \in \mathcal{V}$ such that $x \in U$ and $0 < \text{diam}_w U < \delta_x$, contradicting the fact that $d_m = 0$. So (a) follows and the process terminates. Otherwise, let $U_{m+1} \in \mathcal{V}$ be a set satisfying $U_{m+1} \cap (\bigcup_{i=1}^m U_i) = \emptyset$ and $\text{diam}_w(U_{m+1}) \geq \frac{1}{2}d_m$.

Suppose that the process continues indefinitely and that $\sum (\text{diam}_w U_i)^s < \infty$. For each i , let B_i be a pseudo ball centered in U_i with radius $2\beta \text{diam}_w(U_i)$, where β is the constant in Lemma 2.3. We claim that for every $k \geq 1$,

$$E \setminus \bigcup_{i=1}^k U_i \subset \bigcup_{i=k+1}^{\infty} B_i. \quad (4.2)$$

In fact, for $x \in E \setminus \bigcup_{i=1}^k U_i$, there exists $U \in \mathcal{V}$ with $x \in U$ and $U \cap (\bigcup_{i=1}^k U_i) = \emptyset$. By the assumption that $\sum (\text{diam}_w U_i)^s < \infty$, we obtain that $\lim_{i \rightarrow \infty} \text{diam}_w U_i = 0$. Hence, we have $\text{diam}_w U > 2 \text{diam}_w U_\ell \geq d_{\ell-1}$ for some $\ell \geq k+2$. If $U \cap U_j = \emptyset$ for $k < j < \ell$ and thus for $1 \leq j \leq \ell-1$, it would follow that

$$\text{diam}_w U > 2 \text{diam}_w U_\ell \geq d_{\ell-1} \geq \text{diam}_w U,$$

a contradiction. Let thus i be the smallest integer j with $k < j < \ell$ such that $U \cap U_j \neq \emptyset$. Since $U \cap U_j = \emptyset$ for $1 \leq j \leq i-1$, we have

$$\text{diam}_w U \leq d_{i-1} \leq 2 \text{diam}_w U_i.$$

By elementary geometry, we have $U \subset B_i$ and (4.2) follows.

Thus, if $\delta > 0$,

$$\mathcal{H}_{w,\delta}^s(E \setminus \bigcup_{i=1}^{\infty} U_i) \leq \mathcal{H}_{w,\delta}^s(E \setminus \bigcup_{i=1}^k U_i) \leq \sum_{i=k+1}^{\infty} (\text{diam}_w B_i)^s \leq 2^s \beta^{2s} \sum_{i=k+1}^{\infty} (\text{diam}_w U_i)^s,$$

provided that k is large enough to ensure that $\text{diam}_w B_i \leq \delta$ for $i > k$. Hence, for all $\delta > 0$,

$$\mathcal{H}_{w,\delta}^s(E \setminus \bigcup_{i=1}^{\infty} U_i) = 0.$$

So $\mathcal{H}_w^s(E \setminus \bigcup_{i=1}^{\infty} U_i) = 0$ which proves (a).

(b). Suppose that ρ chosen at the beginning of the proof is the number corresponding to ε and E given in Lemma 4.2. If $\sum_i (\text{diam}_w U_i)^s = +\infty$, then (b) is obvious. Otherwise, by (a) and Lemma 4.2, we obtain

$$\begin{aligned} \mathcal{H}_w^s(E) &= \mathcal{H}_w^s(E \setminus \bigcup_{i=1}^{\infty} U_i) + \mathcal{H}_w^s(E \cap (\bigcup_{i=1}^{\infty} U_i)) \\ &= 0 + \mathcal{H}_w^s(E \cap \bigcup_{i=1}^{\infty} U_i) < \sum_{i=1}^{\infty} (\text{diam}_w U_i)^s + \varepsilon. \end{aligned}$$

□

Lemma 4.4. *If E is an s -set w.r.t. $w(x)$ in \mathbb{R}^n , then $D_{w,c}^s(E, x) = 0$ for \mathcal{H}_w^s -almost all $x \in E^c$.*

Proof. Fix $\alpha > 0$, we show that the measurable set $F = \{x \notin E : D_w^s(E, x) > \alpha\}$ has zero pseudo Hausdorff measure. By the regularity of \mathcal{H}_w^s , for any given $\delta > 0$, there exists a closed set $E_1 \subset E$ such that $\mathcal{H}_w^s(E \setminus E_1) < \delta$. For $\rho > 0$, let

$$\mathcal{V} = \{U \text{ closed \& convex} : 0 < \text{diam}_w U \leq \rho, U \cap E_1 = \emptyset, \mathcal{H}_w^s(E \cap U) > \alpha(\text{diam}_w U)^s\}.$$

Then \mathcal{V} is a Vitali class of closed sets for F w.r.t. $w(x)$. It follows from Theorem 4.3 (a) that we can find a disjoint sequence of sets $\{U_i\}$ in \mathcal{V} with either $\sum(\text{diam}_w U_i)^s = +\infty$ or $\mathcal{H}_w^s(F \setminus \bigcup_i U_i) = 0$. However, by the definition of \mathcal{V} ,

$$\begin{aligned} \sum_i (\text{diam}_w U_i)^s &< \frac{1}{\alpha} \sum_i \mathcal{H}_w^s(E \cap U_i) = \frac{1}{\alpha} \mathcal{H}_w^s(E \cap \bigcup_i U_i) \\ &\leq \frac{1}{\alpha} \mathcal{H}_w^s(E \setminus E_1) < \frac{\delta}{\alpha} < +\infty. \end{aligned}$$

This implies that $\mathcal{H}_w^s(F \setminus \bigcup_i U_i) = 0$, and thus we have

$$\begin{aligned} \mathcal{H}_{w,\rho}^s(F) &\leq \mathcal{H}_{w,\rho}^s(F \setminus \bigcup_i U_i) + \mathcal{H}_{w,\rho}^s(F \cap \bigcup_i U_i) \\ &\leq \mathcal{H}_w^s(F \setminus \bigcup_i U_i) + \sum_i (\text{diam}_w U_i)^s < \frac{\delta}{\alpha} + 0. \end{aligned}$$

This is true for any $\delta > 0$ and any $\rho > 0$. So $\mathcal{H}_w^s(F) = 0$. □

Lemma 4.5. *If E is an s -set w.r.t. $w(x)$ in \mathbb{R}^n , then $D_{w,c}^s(E, x) = 1$ at \mathcal{H}_w^s -almost all $x \in E$.*

Proof. Firstly, we use the definition of pseudo Hausdorff measure w.r.t. $w(x)$ to show that $D_{w,c}^s(E, x) \geq 1$ a.e. in E . Take $\alpha < 1$ and $\rho > 0$. Let

$$F = \{x \in E : \mathcal{H}_w^s(E \cap U) \leq \alpha(\text{diam}_w U)^s \text{ for all convex } U \text{ with } x \in U \text{ and } \text{diam}_w U \leq \rho\}.$$

For any $\varepsilon > 0$, we may find a ρ -cover of F by convex sets $\{U_i\}$ such that

$$\sum (\text{diam}_w U_i)^s < \mathcal{H}_w^s(F) + \varepsilon.$$

Hence, assuming that each U_i contains some points of F and using the definition of F , we obtain

$$\mathcal{H}_w^s(F) \leq \sum_i \mathcal{H}_w^s(F \cap U_i) \leq \sum_i \mathcal{H}_w^s(E \cap U_i) \leq \alpha \sum_i (\text{diam}_w U_i)^s \leq \alpha(\mathcal{H}_w^s(F) + \varepsilon).$$

Since $\alpha < 1$ and the outer inequality holds for all $\varepsilon > 0$, we conclude that $\mathcal{H}_w^s(F) = 0$. We may define such F for any $\rho > 0$. So $D_{w,c}^s(E, x) \geq \alpha$ for a.e. $x \in E$ by the definition. This is true for all $\alpha < 1$, so we conclude that $D_{w,c}^s(E, x) \geq 1$ a.e. in E .

Secondly, we use a Vitali method to show that $D_{w,c}^s(E, x) \leq 1$ a.e. in E . Given $\alpha > 1$, let $F := \{x \in E : D_{w,c}^s(E, x) > \alpha\}$ be a measurable subset of E and let

$$F_0 = \{x \in F : D_{w,c}^s(E \setminus F, x) = 0\}.$$

Then $\mathcal{H}_w^s(F \setminus F_0) = 0$ by Lemma 4.4. By the definition of upper convex s -density, for $x \in F_0$, we have

$$D_{w,c}^s(F, x) \geq D_{w,c}^s(E, x) - D_{w,c}^s(E \setminus F, x) > \alpha.$$

Thus,

$$\mathcal{V} = \{U \text{ closed \& convex} : \mathcal{H}_w^s(F \cap U) > \alpha (\text{diam}_w U)^s\} \quad (4.3)$$

is a Vitali class for F_0 . Then, by Theorem 4.3 (b), for any given $\varepsilon > 0$, we can find a disjoint sequence $\{U_i\}_i$ in \mathcal{V} such that $\mathcal{H}_w^s(F_0) \leq \sum_i (\text{diam}_w U_i)^s + \varepsilon$. By (4.3), we obtain that

$$\mathcal{H}_w^s(F) = \mathcal{H}_w^s(F_0) \leq \sum_i (\text{diam}_w U_i)^s + \varepsilon < \frac{1}{\alpha} \sum_i \mathcal{H}_w^s(F \cap U_i) + \varepsilon \leq \frac{1}{\alpha} \mathcal{H}_w^s(F) + \varepsilon.$$

This inequality holds for any $\varepsilon > 0$. Hence, we have $\mathcal{H}_w^s(F) = 0$ if $\alpha > 1$ as required. \square

Theorem 3.1 implies that if the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC, then the corresponding self-affine set $K := K(A, \mathcal{D})$ is an s -set w.r.t. $w(x)$, where $s = \dim_H^w K = n \ln(\#\mathcal{D}) / \ln(q)$ is the pseudo similarity dimension of K . Thus Theorem 4.1 can be applied to K directly.

5 The upper s -density of μ w.r.t. $w(x)$

In this section, let μ be a Borel measure on \mathbb{R}^n , we use the pseudo norm $w(x)$ instead of the Euclidean norm to define the *upper s -density of μ w.r.t. $w(x)$* . It will be used to find a different expression for the pseudo Hausdorff measure of $K(A, \mathcal{D})$. This is motivated by the connection between the upper s -density of μ in (1.1) which was first introduced in [6] and the Hausdorff measure of a self-similar set $K(A, \mathcal{D})$.

Definition 5.1. *Let μ be a Borel measure in \mathbb{R}^n . The upper s -density of μ w.r.t. $w(x)$ is defined by*

$$\mathcal{E}_{w,s}^+(\mu) = \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\mu(U)}{(\text{diam}_w U)^s},$$

where the supremum is over all compact convex sets $U \subseteq \mathbb{R}^n$ with $\text{diam}_w U \geq r > 0$.

Let μ be a Borel measure and let σ be a Borel probability measure. The convolution $\mu * \sigma$ is defined to be the measure so that

$$\int_{\mathbb{R}^n} \phi(z) d(\mu * \sigma)(z) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x + y) d\mu(x) d\sigma(y),$$

holds for any compactly supported continuous function ϕ on \mathbb{R}^n .

Lemma 5.1. *Let μ and σ be two Borel measures on \mathbb{R}^n with σ being a probability measure. Then $\mathcal{E}_{w,s}^+(\mu * \sigma) = \mathcal{E}_{w,s}^+(\mu)$.*

Proof. By the definition of $\mathcal{E}_{w,s}^+(\mu)$ and the convolution of $\mu * \sigma$, we get

$$\begin{aligned}
\mathcal{E}_{w,s}^+(\mu * \sigma) &= \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\mu * \sigma(U)}{(\text{diam}_w U)^s} \\
&= \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_U(x+y) d\mu(x) d\sigma(y)}{(\text{diam}_w U)^s} \\
&= \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\int_{\mathbb{R}^n} \mu(U-y) d\sigma(y)}{(\text{diam}_w U)^s}, \\
&\leq \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \sup_{y \in \mathbb{R}^n} \frac{\mu(U-y)}{(\text{diam}_w U)^s} \quad (\text{Since } \sigma \text{ is a Borel probability measure}) \\
&= \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\mu(U)}{(\text{diam}_w U)^s} \\
&= \mathcal{E}_{w,s}^+(\mu), \tag{5.1}
\end{aligned}$$

where the supremum is over all convex sets $U \subset \mathbb{R}^n$ with $\text{diam}_w U \geq r > 0$. Thus, $\mathcal{E}_{w,s}^+(\mu * \sigma) \leq \mathcal{E}_{w,s}^+(\mu)$ by (5.1).

For the reverse inequality, fix a real number $R > 0$. Let $\epsilon > 0$ and $r \geq \lambda_\epsilon \beta^2 R$ where λ_ϵ is the same as in Lemma 2.4 and β is defined in Lemma 2.3. For any set $U \subset \mathbb{R}^n$ with $\text{diam}_w U \geq r$, choose a set $\tilde{U} = \bigcup_{y \in B_w(0,R)} (U + y)$. Obviously $U \subset \tilde{U} - y$ for any $y \in B_w(0,R)$, the closed ball centered at 0 with radius R w.r.t. $w(x)$. Moreover, we claim that $\text{diam}_w \tilde{U} \leq (1+\epsilon) \text{diam}_w U$. In fact, for any two points $x_1, x_2 \in \tilde{U}$, we write $x_i = z_i + y_i$ with $z_i \in U$ and $y_i \in B_w(0,R)$ for $i = 1, 2$. Then $w(y_1 - y_2) \leq \beta R$. If $w(z_1 - z_2) > \lambda_\epsilon \beta R$, then we have $w(z_1 - z_2) > \lambda_\epsilon w(y_1 - y_2)$, and this gives

$$w(x_1 - x_2) = w((z_1 - z_2) + (y_1 - y_2)) < (1 + \epsilon) w(z_1 - z_2)$$

by Lemma 2.4. Otherwise if $w(z_1 - z_2) \leq \lambda_\epsilon \beta R$, then we have

$$w(x_1 - x_2) \leq \beta \max\{w(z_1 - z_2), w(y_1 - y_2)\} \leq \beta \max\{\lambda_\epsilon \beta R, \beta R\} = \lambda_\epsilon \beta^2 R \leq r.$$

Thus we have $w(x_1 - x_2) \leq (1 + \epsilon) \text{diam}_w U$ in both cases, which yields the claim since x_1, x_2 are arbitrary points in \tilde{U} . Then we have

$$\begin{aligned}
\frac{\int_{B_w(0,R)} \mu(U) d\sigma(y)}{(\text{diam}_w U)^s} &\leq \frac{\int_{B_w(0,R)} \mu(\tilde{U} - y) d\sigma(y)}{(\text{diam}_w \tilde{U})^s} \cdot \frac{(\text{diam}_w \tilde{U})^s}{(\text{diam}_w U)^s} \\
&\leq \frac{\int_{B_w(0,R)} \mu(\tilde{U} - y) d\sigma(y)}{(\text{diam}_w \tilde{U})^s} \cdot (1 + \epsilon)^s.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\int_{B_w(0,R)} \mu(U) d\sigma(y)}{(\text{diam}_w U)^s} \\
& \leq \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U' \geq r > 0} \frac{\int_{B_w(0,R)} \mu(U' - y) d\sigma(y)}{(\text{diam}_w U')^s} \cdot (1 + \epsilon)^s \\
& \leq \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U' \geq r > 0} \frac{\int_{\mathbb{R}^d} \mu(U' - y) d\sigma(y)}{(\text{diam}_w U')^s} \cdot (1 + \epsilon)^s \\
& = \mathcal{E}_{w,s}^+(\mu * \sigma) \cdot (1 + \epsilon)^s.
\end{aligned}$$

By letting $\epsilon \rightarrow 0$ and $R \rightarrow \infty$, we obtain that $\mathcal{E}_{w,s}^+(\mu) \leq \mathcal{E}_{w,s}^+(\mu * \sigma)$. \square

Lemma 5.2. *Let σ be the Borel probability measure supported on $K(A, \mathcal{D})$ which satisfies (3.1). For $M \geq 1$, define $\mu_M = \sum_{x \in \mathcal{D}_M} \delta_x$, then for any Borel measurable set $W \subset \mathbb{R}^n$, we have $\sigma(A^{-M}W) = \frac{1}{(\#\mathcal{D})^M} (\mu_M * \sigma)(W)$.*

Proof. For any Borel measurable set $W \subset \mathbb{R}^n$, we deduce from the identity (3.1) that

$$\begin{aligned}
\sigma(A^{-M}W) &= \int_{\mathbb{R}^n} \chi_{A^{-M}W}(x) d\sigma(x) \\
&= \frac{1}{(\#\mathcal{D})^M} \sum_{d_1, d_2, \dots, d_M \in \mathcal{D}} \int_{\mathbb{R}^n} \chi_{A^{-M}W}(A^{-M}x + A^{-1}d_1 + \dots + A^{-M}d_M) d\sigma(x) \\
&= \frac{1}{(\#\mathcal{D})^M} \sum_{d_1, d_2, \dots, d_M \in \mathcal{D}} \int_{\mathbb{R}^n} \chi_W(x + A^{M-1}d_1 + \dots + d_M) d\sigma(x) \\
&= \frac{1}{(\#\mathcal{D})^M} \int_{\mathbb{R}^n} \chi_W(x) d(\sigma * \mu_M)(x) \\
&= \frac{1}{(\#\mathcal{D})^M} \sigma * \mu_M(W).
\end{aligned}$$

\square

6 Pseudo Hausdorff measure of self-affine sets

This section is devoted to proving Theorem 1.2 by considering the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies and does not satisfy the OSC separately. The following technical lemma is needed. We borrow the technique of its proof from [27] for the self-similar case.

Lemma 6.1. *Let the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfy the OSC. Then $\mathcal{H}_w^s(K \cap U) \leq (\text{diam}_w U)^s$ for any subset U in \mathbb{R}^n .*

Proof. We will prove the statement by a contradiction. Assume that there exists a subset $U \subset \mathbb{R}^n$ such that $\mathcal{H}_w^s(K \cap U) > (\text{diam}_w U)^s$. Then we can find some $0 < \kappa < 1$ such that

$(1 - \kappa)\mathcal{H}_w^s(K \cap U) > (\text{diam}_w U)^s$. Fix $\delta > 0$ and choose a positive integer m such that $\text{diam}_w f_{\mathbf{i}}(U) \leq \delta$ for all words $\mathbf{i} \in \Sigma^m$, where Σ^m is defined in Section 3. Note that

$$\bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(K \cap U) \subset K \cap \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U), \quad (6.1)$$

since $f_{\mathbf{i}}(K) \subset \bigcup_{\mathbf{j} \in \Sigma^m} f_{\mathbf{j}}(K) = K$ for each $\mathbf{i} \in \Sigma^m$. By the assumption that the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC, then by using Theorem 3.1 and Lemma 3.2, we have $\mathcal{H}_w^s(f_{\mathbf{i}}(K \cap U)) \cap f_{\mathbf{j}}(K \cap U) = 0$ for distinct $\mathbf{i}, \mathbf{j} \in \Sigma^m$. Therefore, by (6.1), we obtain

$$\begin{aligned} \mathcal{H}_w^s(K \cap \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) &\geq \mathcal{H}_w^s(\bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(K \cap U)) \\ &= \sum_{\mathbf{i} \in \Sigma^m} \mathcal{H}_w^s(f_{\mathbf{i}}(K \cap U)) = \mathcal{H}_w^s(K \cap U). \end{aligned} \quad (6.2)$$

Defining $\eta = \frac{1}{2} \kappa \mathcal{H}_w^s(K \cap \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U))$, it follows from (6.2) that

$$\eta \geq \frac{1}{2} \kappa \mathcal{H}_w^s(K \cap U) > \frac{1}{2} \kappa \frac{(\text{diam}_w U)^s}{1 - \kappa} > 0.$$

For $\eta > 0$, we can choose a sequence of sets $\{U_i\}_i$ with $\bigcup_i U_i \supseteq K \setminus \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)$ and $\text{diam}_w(U_i) < \delta$ such that

$$\begin{aligned} \sum_i (\text{diam}_w U_i)^s &\leq \mathcal{H}_{w, \delta}^s(K \setminus \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \eta \\ &\leq \mathcal{H}_w^s(K \setminus \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \eta. \end{aligned} \quad (6.3)$$

The family $\{f_{\mathbf{i}}(U)\}_{\mathbf{i} \in \Sigma^m} \cup \{U_i\}_i$ is clearly a δ -cover of K w.r.t. $w(x)$. Using the fact that $\sum_{\mathbf{i} \in \Sigma^m} r_{\mathbf{i}}^s = 1$ and (6.3), we obtain that

$$\begin{aligned} \mathcal{H}_{w, \delta}^s(K) &\leq \sum_{\mathbf{i} \in \Sigma^m} (\text{diam}_w f_{\mathbf{i}}(U))^s + \sum_i (\text{diam}_w U_i)^s \\ &\leq (\text{diam}_w U)^s + \mathcal{H}_w^s(K \setminus \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \eta \\ &\leq (1 - \kappa) \mathcal{H}_w^s(K \cap U) + \mathcal{H}_w^s(K \setminus \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \eta \end{aligned}$$

Taking the inequality (6.2) into account, this yields

$$\begin{aligned} \mathcal{H}_{w, \delta}^s(K) &\leq (1 - \kappa) \mathcal{H}_w^s(K \cap \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \mathcal{H}_w^s(K \setminus \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \eta \\ &\leq \mathcal{H}_w^s(K) - \kappa \mathcal{H}_w^s(K \cap \bigcup_{\mathbf{i} \in \Sigma^m} f_{\mathbf{i}}(U)) + \eta \\ &= \mathcal{H}_w^s(K) - \eta \\ &\leq \mathcal{H}_w^s(K) - \frac{1}{2} \kappa \mathcal{H}_w^s(K \cap U). \end{aligned}$$

Letting $\delta \rightarrow 0$, we get

$$\mathcal{H}_w^s(K) \leq \mathcal{H}_w^s(K) - \frac{1}{2}\kappa\mathcal{H}_w^s(K \cap U),$$

which is a contradiction since $0 < \mathcal{H}_w^s(K) < \infty$ and $\frac{1}{2}\kappa\mathcal{H}_w^s(K \cap U) > 0$. \square

Lemma 3.2 shows that if the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfies the OSC, then the probability measure σ in (3.1) is a multiple of the restriction of the s -dimensional pseudo Hausdorff measure \mathcal{H}_w^s to the set K , with $s = \dim_w^v K = n \ln(\#\mathcal{D})/\ln(q)$, i.e.

$$\sigma = (\mathcal{H}_w^s(K))^{-1}\mathcal{H}_w^s \upharpoonright K. \quad (6.4)$$

Combining the formula (6.4), Lemma 6.1, Theorem 3.1 and Theorem 4.1, we obtain the following lemma.

Lemma 6.2. *Let $K := K(A, \mathcal{D})$ be a self-affine set and let the IFS $\{f_d\}_{d \in \mathcal{D}}$ satisfy the OSC. Then for any $r_0 > 0$,*

$$(\mathcal{H}_w^s(K))^{-1} = \sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s},$$

where s is the pseudo similarity dimension of K , σ is defined by (3.1) and the supremum is over all convex sets U with $U \cap K \neq \emptyset$ and $0 < \text{diam}_w U \leq r_0$.

Proof. By applying Theorem 3.1, K is an s -set w.r.t. $w(x)$. From Theorem 4.1, we can pick a point $x \in K$ such that $D_{w,c}^s(K, x) = 1$. Then there exists a positive sequence $\{r_n\}_n$ with $r_n \leq r_0$, $r_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\sup_{0 < \text{diam}_w U \leq r_n, x \in U} \frac{\mathcal{H}_w^s(K \cap U)}{(\text{diam}_w U)^s} - \frac{1}{n} \leq 1 \leq \sup_{0 < \text{diam}_w U \leq r_n, x \in U} \frac{\mathcal{H}_w^s(K \cap U)}{(\text{diam}_w U)^s} + \frac{1}{n}.$$

For each n , there exists a convex set U_n containing x with $0 < \text{diam}_w U_n \leq r_n$ such that

$$\sup_{0 < \text{diam}_w U \leq r_n, x \in U} \frac{\mathcal{H}_w^s(K \cap U)}{(\text{diam}_w U)^s} \leq \frac{\mathcal{H}_w^s(K \cap U_n)}{(\text{diam}_w U_n)^s} + \frac{1}{n}.$$

Thus

$$\frac{\mathcal{H}_w^s(K \cap U_n)}{(\text{diam}_w U_n)^s} - \frac{1}{n} \leq 1 \leq \frac{\mathcal{H}_w^s(K \cap U_n)}{(\text{diam}_w U_n)^s} + \frac{2}{n},$$

which yields that $\frac{\mathcal{H}_w^s(K \cap U_n)}{(\text{diam}_w U_n)^s} \rightarrow 1$ as $n \rightarrow \infty$. Moreover, by Lemma 6.1, for each convex set U with $K \cap U \neq \emptyset$, we have $\frac{\mathcal{H}_w^s(K \cap U)}{(\text{diam}_w U)^s} \leq 1$. Hence $\sup_{0 < \text{diam}_w U \leq r_0} \frac{\mathcal{H}_w^s(K \cap U)}{(\text{diam}_w U)^s} = 1$. By applying the formula (6.4) to the above equality, the lemma follows. \square

We have the following representation for the pseudo Hausdorff measure of self-affine sets.

Theorem 6.3. *Let $K := (A, \mathcal{D})$ be a self-affine set and let $s := n \ln(\#\mathcal{D})/\ln(q)$ be the pseudo similarity dimension of K . Then $\mathcal{H}_w^s(K) = (\mathcal{E}_{w,s}^+(\mu))^{-1}$, where μ is defined by (1.1).*

Proof. Let us assume first that $\mathcal{H}_w^s(K) > 0$ and thus that the OSC holds by Theorem 1.1. By Lemma 6.2, it is sufficient to prove that

$$\mathcal{E}_{w,s}^+(\mu) = \sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s}$$

for some $r_0 > 0$, where the supremum is over all convex sets U with $U \cap K \neq \emptyset$ and $0 < \text{diam}_w U \leq r_0$.

Fix $r_0 > 0$. It follows from Lemma 6.2 that $\sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s}$ is finite. Then, for any given $\varepsilon > 0$, there exists a convex set U_0 with $\text{diam}_w U_0 \leq r_0$ and $U_0 \cap K \neq \emptyset$ such that

$$\frac{\sigma(U_0)}{(\text{diam}_w U_0)^s} \geq \sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s} - \varepsilon. \quad (6.5)$$

For any $N \geq 1$, define $\mu_N = \sum_{d_0, \dots, d_{N-1} \in \mathcal{D}} \delta_{d_0 + A d_1 + \dots + A^{N-1} d_{N-1}}$. Using Lemma 5.1 and Lemma 5.2, we have

$$\begin{aligned} \frac{\sigma(U_0)}{(\text{diam}_w U_0)^s} &= \frac{\sigma * \mu_N(A^N U_0)}{(\text{diam}_w(A^N U_0))^s} = \lim_{N \rightarrow \infty} \frac{\sigma * \mu_N(A^N U_0)}{(\text{diam}_w(A^N U_0))^s} \\ &\leq \lim_{r \rightarrow \infty} \sup_{\text{diam}_w U \geq r > 0} \frac{\sigma * \mu(U)}{(\text{diam}_w U)^s} \\ &= \mathcal{E}_{w,s}^+(\sigma * \mu) = \mathcal{E}_{w,s}^+(\mu). \end{aligned} \quad (6.6)$$

It follows from (6.5) and (6.6) that

$$\sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s} \leq \mathcal{E}_{w,s}^+(\mu) + \varepsilon.$$

By letting $\varepsilon \rightarrow 0$, we get

$$\sup_{0 < \text{diam}_w U \leq r_0} \frac{\sigma(U)}{(\text{diam}_w U)^s} \leq \mathcal{E}_{w,s}^+(\mu).$$

Conversely, for any given convex set U , using Lemma 5.2, we have,

$$\begin{aligned} \frac{\sigma * \mu(U)}{(\text{diam}_w U)^s} &= \lim_{N \rightarrow \infty} \frac{\sigma * \mu_N(U)}{(\text{diam}_w U)^s} = \lim_{N \rightarrow \infty} \frac{\sigma(A^{-N} U)}{(\text{diam}_w(A^{-N} U))^s} \\ &\leq \sup_{0 < \text{diam}_w V \leq r_0} \frac{\sigma(V)}{(\text{diam}_w V)^s}. \end{aligned}$$

Using Lemma 5.1 again, we have thus that

$$\mathcal{E}_{w,s}^+(\mu) = \mathcal{E}_{w,s}^+(\mu * \sigma) \leq \sup_{0 < \text{diam}_w V \leq r_0} \frac{\sigma(V)}{(\text{diam}_w V)^s}.$$

Thus we have proved the desired result in the case that $\mathcal{H}_w^s(K) > 0$.

On the other hand, if $\mathcal{H}_w^s(K) = 0$, then the IFS $\{f_d\}_{d \in \mathcal{D}}$ does not satisfy the OSC by Theorem 1.1. Thus, by Theorem 1.1, either the $(\#\mathcal{D})^M$ expansions in \mathcal{D}_M are not distinct for some M or \mathcal{D}_∞ is not uniformly discrete. For $z \in \mathbb{R}^n$, we will use

$$I_k(z) = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |y_i - z_i| \leq \frac{k}{2}, i = 1, 2, \dots, n\}$$

to denote the cube centered at $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ with side length k .

Assume first that there exists some M such that the $(\#\mathcal{D})^M$ expansions in \mathcal{D}_M are not distinct. Then there exists $a \in \mathcal{D}_M$ which can be represented in two different ways in terms of the digits in \mathcal{D} , i.e.

$$a = \sum_{j=0}^{M-1} A^j d_j = \sum_{j=0}^{M-1} A^j d'_j, \quad d_j, d'_j \in \mathcal{D},$$

with $d_j \neq d'_j$ for at least one $0 \leq j \leq M-1$. Then $a + A^M a$ has at least four distinct expansions in \mathcal{D}_{2M} . More generally, for $k \geq 1$, $\sum_{j=0}^{k-1} A^{Mj} a$ has at least 2^k distinct expansions

in \mathcal{D}_{kM} . Then, if $a_k = \sum_{j=0}^{k-1} A^{Mj} a$, then $\mu(\{a_k\}) \geq 2^k$. Then, for any $r > 0$, we have

$$\frac{\mu(I_r(a_k))}{(\text{diam}_w I_r(a_k))^s} \geq \frac{2^k}{(\text{diam}_w I_r(0))^s} \rightarrow \infty, \quad k \rightarrow \infty,$$

This implies that $\sup_{\text{diam}_w U \geq r > 0} \frac{\mu(U)}{(\text{diam}_w U)^s} = \infty$ for any $r > 0$, and in particular, $\mathcal{E}_{w,s}^+(\mu) = \infty$.

Next, assume that $\#\mathcal{D}_M = (\#\mathcal{D})^M$ holds for each $M \geq 1$, but \mathcal{D}_∞ is not a uniformly discrete set. Then there exists $M_1 \geq 1$ and $x_1, y_1 \in \mathcal{D}_{M_1} \subseteq \mathcal{D}_\infty$ with $x_1 \neq y_1$ such that $\|x_1 - y_1\| < \frac{1}{2}$. Write $F_1 = \{x_1, y_1\}$ and $w_1 = x_1$. Then $F_1 \subset \mathcal{D}_{k_1} \subset \mathcal{D}_\infty$ and $\|z_1 - w_1\| < \frac{1}{2}$ for any $z_1 \in F_1$. Let $S_1 = 0$. Inductively, for $k \geq 2$, assume that M_j, S_j and $x_j, y_j \in \mathcal{D}_{M_j}$, $F_j \subset \mathcal{D}_{S_j + M_j}$ have been defined for $1 \leq j \leq k-1$. Let $S_k = \sum_{j=1}^{k-1} M_j$. Choose M_k and $x_k, y_k \in \mathcal{D}_{M_k} \subset \mathcal{D}_\infty$ with $x_k \neq y_k$ and $\|x_k - y_k\| < \frac{1}{2^k \|A\|^{S_k}}$. Write

$$\begin{aligned} F_k &= \{z_1 + A^{S_2} z_2 + \dots + A^{S_k} z_k : z_i \in \{x_i, y_i\}, 1 \leq i \leq k\}, \\ w_k &= x_1 + A^{S_2} x_2 + \dots + A^{S_k} x_k. \end{aligned}$$

Then $F_k \subset \mathcal{D}_{S_k + M_k} \subset \mathcal{D}_\infty$, $w_k \in \mathcal{D}_{S_k + M_k}$. Thus for any $k \geq 1$, $z \in F_k$, we have

$$\begin{aligned} \|z - w_k\| &= \|(z_1 - x_1) + A^{S_2}(z_2 - x_2) + \dots + A^{S_k}(z_k - x_k)\| \\ &\leq \frac{1}{2} + \|A\|^{S_2} \frac{1}{2^2 \|A\|^{S_2}} + \dots + \|A\|^{S_k} \frac{1}{2^k \|A\|^{S_k}} \\ &< 1. \end{aligned}$$

This shows that $\mu(I_2(w_k)) \geq 2^k$. Hence, for any $r \geq 2$, we have $I_2(w_k) \subset I_r(w_k)$ and

$$\frac{\mu(I_r(w_k))}{(\text{diam}_w I_r(w_k))^s} \geq \frac{2^k}{(\text{diam}_w I_r(0))^s} \rightarrow \infty, \quad k \rightarrow \infty.$$

So $\mathcal{E}_{w,s}^+(\mu) = \infty$ as before.

Therefore, we always have $\mathcal{H}_w^s(K) = (\mathcal{E}_{w,s}^+(\mu))^{-1}$. \square

Acknowledgements The authors would like to thank the anonymous referees for many valuable comments and suggestions which are helpful to improve the presentation of this manuscript.

References

- [1] E. Ayer, R. S. Strichartz, Robert S. Exact Hausdorff measure and intervals of maximum density for Cantor sets. *Trans. Amer. Math. Soc.* 351 (1999), no. 9, 3725–3741.
- [2] C. Bandt, N. V. Hung, H. Rao, On the open set condition for self-similar fractals. *Proc. Amer. Math. Soc.* 134 (2006), no. 5, 1369–1374.
- [3] C. J. Bishop, Y. Peres, *Fractals in probability and analysis*. Cambridge Studies in Advanced Mathematics, 162. Cambridge University Press, Cambridge, 2017.
- [4] A. Cohen, K. Gröchenig, and L. F. Villemoes, Regularity of multivariate refinable functions. *Constr. Approx.* 15 (1999), no. 2, 241–255.
- [5] K. J. Falconer, *Fractal geometry. Mathematical foundations and applications*. John Wiley & Sons, Ltd., Chichester, 1990.
- [6] X. Y. Fu, J.-P. Gabardo, Measure of self-affine sets and associated densities. *Constr. Approx.* 40 (2014), no. 3, 425–446.
- [7] J.-P. Gabardo, X. J. Yu, Natural tiling, lattice tiling and Lebesgue measure of integral self-affine tiles. *J. London Math. Soc.* (2) 74 (2006), no. 1, 184–204.
- [8] K. Gröchenig, A. Haas, Self-similar lattice tilings. *J. Fourier Anal. Appl.* 1 (1994), no. 2, 131–170.
- [9] K. Gröchenig, W. R. Madych, Multiresolution analysis, Haar bases, and self-similar tilings of \mathbf{R}^n . *IEEE Trans. Inform. Theory* 38 (1992), no. 2, part 2, 556–568.
- [10] X.-G. He, K.-S. Lau, On a generalized dimension of self-affine fractals. *Math. Nachr.* 281 (2008), no. 8, 1142–1158.
- [11] X. G. He, K. S. Lau, H. Rao, Self-affine sets and graph-directed systems. *Constr. Approx.* 19 (2003), no. 3, 373–397.
- [12] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* 30 (1981), no. 5, 713–747.
- [13] B. G. Jia, Bounds of Hausdorff measure of the Sierpinski gasket. *J. Math. Anal. Appl.* 330 (2007), no. 2, 1016–1024.
- [14] B. G. Jia, Maximum density for the Sierpinski carpet. *Comput. Math. Appl.* 57 (2009), no. 10, 1615–1621.
- [15] B. G. Jia, Z. L., Zhou, Z. W. Zhu, A lower bound for the Hausdorff measure of the Sierpinski gasket. *Nonlinearity* 15 (2002), no. 2, 393–404.
- [16] C. K. Lai, K. S. Lau, H. Rao, Spectral structure of digit sets of self-similar tiles on \mathbb{R}^1 . *Trans. Amer. Math. Soc.* 365 (2013), no. 7, 3831–3850.
- [17] C. K. Lai, K. S. Lau, H. Rao, Classification of tile digit sets as product-forms. *Trans. Amer. Math. Soc.* 369 (2017), no. 1, 623–644.
- [18] J. C. Lagarias, Y. Wang, Self-affine tiles in \mathbb{R}^n . *Adv. Math.* 121 (1996), no. 1, 21–49.

- [19] J. C. Lagarias, Y. Wang, Integral self-affine tiles in \mathbf{R}^n . I. Standard and nonstandard digit sets. *J. London Math. Soc.* (2) 54 (1996), no. 1, 161–179.
- [20] J. C. Lagarias, Y. Wang, Integral self-affine tiles in \mathbf{R}^n . II. Lattice tilings. *J. Fourier Anal. Appl.* 3 (1997), no. 1, 83–102.
- [21] J. C. Lagarias, Y. Wang, Haar type orthonormal wavelet bases in \mathbf{R}^2 . *J. Fourier Anal. Appl.* 2 (1995), no. 1, 1–14.
- [22] H. Landau, Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Math.* 117 1967 37–52.
- [23] K. S. Lau, S. M. Ngai, Multifractal measures and a weak separation condition. *Adv. Math.* 141 (1999), no. 1, 45–96.
- [24] K. S. Lau, S. M. Ngai, H. Rao, Iterated function systems with overlaps and self-similar measures. *J. London Math. Soc.* (2) 63 (2001), no. 1, 99–116.
- [25] P. G. Lemarié-Rieusset, Projecteurs invariants, matrices de dilatation, ondelettes et analyses multi-résolutions. *Rev. Mat. Iberoamericana* 10 (1994), no. 2, 283–347.
- [26] S. M. Ngai, Y. Wang, Hausdorff dimension of self-similar sets with overlaps. *J. London Math. Soc.* (2) 63 (2001), no. 3, 655–672.
- [27] L. Olsen, Density theorems for Hausdorff and packing measures of self-similar sets. *Aequationes Math.* 75 (2008), no. 3, 208–225.
- [28] H. Qiu, Exact Hausdorff and packing measures of Cantor sets with overlaps. *Ergodic Theory Dynam. Systems* 35 (2015), no. 8, 2632–2668.
- [29] A. Schief, Separation properties for self-similar sets. *Proc. Amer. Math. Soc.* 122 (1994), no. 1, 111–115.
- [30] K. Simon, B. Solomyak, On the dimension of self-similar sets. *Fractals* 10 (2002), no. 1, 59–65.
- [31] Y. H. Xiong, J. Zhou, The Hausdorff measure of a class of Sierpinski carpets. *J. Math. Anal. Appl.* 305 (2005), no. 1, 121–129.
- [32] Y. C. Zhu, J. Lou, The Hausdorff measure of generalized Sierpinski carpets. *Approx. Theory Appl.* (N.S.) 16 (2000), no. 2, 13–18.
- [33] Z. L. Zhou, Li Feng, A new estimate of the Hausdorff measure of the Sierpinski gasket. *Nonlinearity* 13 (2000), no. 3, 479–491.