ON A RECURSIVE CONSTRUCTION OF DIRICHLET FORM ON THE SIERPIŃSKI GASKET

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Abstract. Let Γ_n denote the n-th level Sierpiński graph of the Sierpiński gasket K. We consider, for any given conductance (a_0, b_0, c_0) on Γ_0, the Dirchlet form $\mathcal{E}$ on K obtained from a recursive construction of compatible sequence of conductances (a_n, b_n, c_n) on $\Gamma_n, n \geq 0$. We prove that there is a dichotomy situation: either $a_0 = b_0 = c_0$ and $\mathcal{E}$ is the standard Dirichlet form, or $a_0 > b_0 = c_0$ (or the two symmetric alternatives), and $\mathcal{E}$ is a non-self-similar Dirichlet form independent of a_0, b_0. The second situation has been studied in [9, 10] as a one-dimensional asymptotic diffusion. The analytic approach here is more direct and yields sharper results; in particular, for the spectral property, we give a precise estimate of the eigenvalue distribution of the associated Laplacian, which improves a similar result in [10].

1. Introduction

Dirichlet forms play a central role in the analysis on fractals. There is a large literature on the topic based on Kigami's analytic approach on the *post critically finite (p.c.f.) self-similar sets*, and the probabilistic approach of Lindstrøm on the nested fractals as well as Barlow and Bass on the Sierpiński carpet (see [1, 2, 3, 6, 12, 15, 16, 17, 24, 25, 27, 29] and the references therein). In those studies, the Sierpiński gaskets and carpets are always served as fundamental examples, and are a source of inspiration.

Recall that a *Sierpiński gasket* (SG) is the unique nonempty compact set K in $\mathbb{R}^2$ satisfying $K = \bigcup_{i=1}^3 F_i(K)$ for an iterated function system (IFS) $\{F_i\}_{i=1}^3$ on $\mathbb{R}^2$ such that $F_i(x) = \frac{1}{2}(x - p_i) + p_i$ with non-collinear p_i's. For convenience, we fix $p_1 = 0$, $p_2 = 1$, $p_3 = \exp\left(\frac{\pi\sqrt{-1}}{3}\right)$. Denote by $V_0 = \{p_1, p_2, p_3\}$ the *boundary* of K, and let $F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}$ for a word $\omega \in W_n = \{1, 2, 3\}^n$. The *standard Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on the SG is well-known [18, 29]: the energy $\mathcal{E}$ and the domain $\mathcal{F}$ are given by

$$\mathcal{E}(u) = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \sum_{p\sim_n q} (u(p) - u(q))^2, \quad \mathcal{F} = \{u \in C(K) : \mathcal{E}(u) < \infty\} \tag{1.1}$$

where $p \sim_n q$ means $p \neq q$ and $p, q \in F_\omega(V_0)$ for some $\omega \in W_n$. The domain $\mathcal{F}$ is known to be some Besov type space [15]. In [28], Sabot classified all the Dirichlet forms on the

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SG which satisfy the energy self-similar identity

$$\mathcal{E}(u) = \sum_{i=1}^{3} \frac{1}{r_i} \mathcal{E}(u \circ F_i), \tag{1.2}$$

where r_i, $i = 1, 2, 3$ are some positive numbers called the *renormalization factors* of the energy form. The energy self-similar identity for the p.c.f fractals and nested fractals is also studied in detail in [18].

More generally, one can also consider Dirichlet form without satisfying the energy self-similar identity. Let Γ_0 be the complete graph on V_0 and for $n \geq 1$, Γ_n the graph on V_n which is defined inductively by $V_n = \bigcup_{i=1}^{3} F_i(V_{n-1})$ with the edge relation $\sim_n$ defined as in (1.1). Let $l(V_n)$ be the collection of functions defined on V_n, and let $(\mathcal{E}_n, l(V_n))$ be defined by

$$\mathcal{E}_n(u) = \sum\nolimits_{p \sim_n q} c_{pq}^{(n)} (u(p) - u(q))^2, \quad u \in \ell(V_n), \tag{1.3}$$

where $c_{pq}^{(n)} \geq 0$, call it the *conductance* of p and q in Γ_n. In the case that $\mathcal{E}(u) := \lim_{n\to\infty} \mathcal{E}_n(u) < \infty$ exists for u on $V_* = \bigcup_{n=0}^{\infty} V_n$, it will allow us to define a Dirichlet form on the SG. For the limit to exist, the key issue is that the sequence of $\mathcal{E}_n$'s are compatible (see[18] for details): *the restriction (or trace) of $\mathcal{E}_n$ on $\ell(V_{n-1})$ must be equal to $\mathcal{E}_{n-1}$, $n \geq 1$.*

In an attempt to produce all the Dirichlet forms (include the non-self-similar ones), Meyers, Strichartz and Teplyaev [22] used the compatibility condition to solve a system of linear equations of conductances on V_1 (9 of them) in terms of those on V_0 as well as the given values of the harmonic functions on $V_1 \setminus V_0$, then extend this inductively. However the setup is too general and the expressions are rather complicated, thus it does not give much information on the structure of the limiting Dirichlet form. Recently two of the authors studied some anomalous p.c.f. fractals in regard to the domains of the Dirichlet forms and the associated Besov spaces [8]. In their investigation, a construction of the non-self-similar energy form was considered, and some interesting properties were found (see Section 4). In this note we intend to use the SG to study this construction in greater detail so as to give more insight to the general cases.

For this class of Dirichlet form on the SG, we require the conductances of the cells $F_\omega(V_0)$ on the same level $|\omega| = n$ have the same expression (note that the conductances on the edges of $F_\omega(V_0)$ may be different), and we will give a necessary and sufficient condition for the existence of a compatibility sequence $\{\mathcal{E}_n\}_n$. The tool we use is the well-known electrical network theory. The energy $\mathcal{E}_n(u)$ in (1.3) corresponds to an electrical network $R(\Gamma_n)$ with *resistance* $r_{pq}^{(n)} = (c_{pq}^{(n)})^{-1}$, and u is the *potential* on V_n. The sequence of networks $\{R(\Gamma_n)\}_{n=0}^{\infty}$ are said to be *compatible* if the trace of $R(\Gamma_n)$ on V_{n-1} equals $R(\Gamma_{n-1})$, $n \geq 1$. Note that this is equivalent to the compatibility of the sequence of energy forms $\mathcal{E}_n, n \geq 0$.

Let (a_0, b_0, c_0) be the conductance on V_0, and let (a_n, b_n, c_n) be the conductances of $F_\omega(V_0), |\omega| = n, n \geq 1$ to be determined. By the well-known $\Delta - Y$ transform [18, 29], the resistances $(a_n^{-1}, b_n^{-1}, c_n^{-1})$ on the Δ-side is equivalent to a set of resistances (x_n, y_n, z_n)

on the Y-side. It is direct to show (use (2.1) and refer to Figure 2) that $\{R(\Gamma_n)\}_{n=0}^{\infty}$ are compatible can be reduced to $\{(x_n, y_n, z_n)\}_{n\geq 0}$ satisfy

$$\begin{cases} x_{n-1} = x_n + \phi(x_n; y_n, z_n), \\ y_{n-1} = y_n + \phi(y_n; z_n, x_n), \\ z_{n-1} = z_n + \phi(z_n; x_n, y_n), \end{cases} \qquad n \geq 1, \tag{1.4}$$

where $\phi(x_n; y_n, z_n) := \frac{(x_n+y_n)(x_n+z_n)}{2(x_n+y_n+z_n)}$, and symmetrically for the other two. We will refer to finding the solution of (x_n, y_n, z_n) from $(x_{n-1}, y_{n-1}, z_{n-1})$ as a *recursive construction* of the energy form $\mathcal{E}_n$. Necessarily, (x_n, y_n, z_n) has to be positive, and the following is a necessary and sufficient condition for this to hold.

Proposition 1.1. *For $a_0, b_0, c_0 > 0$, in order for* (1.4) *to have positive solutions $(x_n, y_n, z_n), n \geq 1$, it is necessary and sufficient that $x_0 \geq y_0 = z_0 > 0$ (or the symmetric alternates).*

In this case, $x_n \geq y_n = z_n > 0,\ n \geq 0$ and $\{(x_n, y_n, z_n)\}_{n\geq 0}$ is uniquely determined by the initial data (x_0, y_0, z_0).

The proposition will be proved in Lemmas 2.1, 2.2. We let μ be the normalized α-Hausdorff measure on K with $\alpha = \frac{\log 3}{\log 2}$. For two functions $f, g \geq 0$, we use $f \asymp g$ to mean that they dominate each other by a positive constant. As a consequence of Proposition 1.1, we have the following theorem.

Theorem 1.2. *For the case $x_0 > y_0 = z_0 > 0$ in the above proposition, we have $a_0 > b_0 = c_0$ and*

$$a_n = \frac{x_n}{y_n(2x_n + y_n)} \asymp 2^n, \qquad b_n = c_n = \frac{1}{2x_n + y_n} \asymp \left(\frac{3}{2}\right)^n.$$

Moreover $\mathcal{E}^{(a_0,b_0)}(u) = \lim\limits_{n\to\infty} \mathcal{E}_n^{(a_0,b_0)}(u)$ defines a strongly local regular Dirichlet form on $L^2(K,\mu)$ with domain $\mathcal{F}$ independent of (a_0, b_0); it satisfies

$$\mathcal{E}^{(a_0,b_0)}(u) = \sum_{i=1}^{3} \mathcal{E}^{(a_1,b_1)}(u \circ F_i), \tag{1.5}$$

but does not satisfy the energy self-similar identity. (Here $\mathcal{E}_n^{(a_0,b_0)} := \mathcal{E}_n$ is defined as in (1.3) *with conductances (a_n, b_n, c_n) on each n-level subcells).*

It follows that for initial data $x_0 \geq y_0 = z_0 > 0$ on Γ_0, the recursive construction gives a dichotomy result on the Dirchlet forms: when $a_0 = b_0 = c_0 > 0$, then $\mathcal{E}^{(a_0,b_0)}$ is the standard Dirichlet form in (1.1); when $a_0 > b_0 (= c_0) > 0$, then by the above estimation of a_n and $b_n (= c_n)$, we have

$$\mathcal{E}^{(a_0,b_0)}(u) \asymp$$
$$\sup_{n\geq 0} \left\{ 2^n \sum_{\omega\in W_n} \left(\left(u_\omega(p_2) - u_\omega(p_3)\right)^2 + \left(\frac{3}{4}\right)^n \left(u_\omega(p_1) - u_\omega(p_2)\right)^2 + \left(\frac{3}{4}\right)^n \left(u_\omega(p_1) - u_\omega(p_3)\right)^2 \right) \right\},$$

where $u_\omega(x) = u \circ F_\omega(x)$. It is seen that there are two scaling factors in $\mathcal{E}^{(a_0,b_0)}$. The renormalizing factor is 2^n, and the energy is basically concentrated on the $\overline{p_2 p_3}$ direction.

For this Dirichlet form $\mathcal{E}^{(a_0,b_0)}$ with $a_0 > b_0$, we can give a sharp estimate of the distribution of the eigenvalues (Section 3). Let $\Delta^{(a_0,b_0)}$ be the *Laplacian*, the infinitesimal generator of $\left(\mathcal{E}^{(a_0,b_0)}, \mathcal{F}\right)$ on $L^2(K,\mu)$. Denote by $\rho^{(a_0,b_0)}(t)$ the eigenvalue count with the *Dirichlet boundary condition* (D.B.C), that is

$$\rho^{(a_0,b_0)}(t) = \#\Big\{\lambda \le t : \lambda \text{ is an eigenvalue of } -\Delta^{(a_0,b_0)} \text{ with D.B.C.}\Big\}, \tag{1.6}$$

Theorem 1.3. *Assume that $a_0 > b_0 = c_0$, and let $t_0 = \inf\{t : \rho^{(a_0,b_0)}(t) > 0\}$, then*

$$\rho^{(a_0,b_0)}(t) \asymp t^{\frac{\log 3}{\log(9/2)}}, \qquad t > t_0.$$

We remark that in another investigation, K. Hattori, T. Hattori and Watanabe [9] studied the asymptotically one-dimensional diffusion processes on the SG (see also Hambly and Jones [10], Hambly and Yang [13]). The random walk they considered is in fact the normalized probability of (a_n, b_n, b_n) as transition probability on the three sides of the n-level cells of the SG. (They used this as an assumption, and in fact it is one of the dichotomy cases from Theorem 1.2 (or Proposition 1.1).) We will give a brief comparison of these two approaches in Section 2. For the estimate of the eigenvalue distribution in Theorem 1.3, it improves the lower bound of $\rho^{(a_0,b_0)}(t)$ in [10, Theorem 13] where it was shown to be $C^{-1}t^{\log 3/\log(9/2)}(\log t)^{-\beta}$ with $\beta > \log 3/\log 2$, using a heat kernel technique in the estimation.

The recursive construction can be extended to more general p.c.f. sets (see [8] for some examples), but it also have limitation. In Section 4, we give two other examples that this construction have abnormality. The first one is the *twisted SG* introduced by Mihai and Strichartz [23], it is a modification of the IFS of the SG that reflecting the three subcells of the SG along the angle bisectors at the three vertices. We show that for $a_0 > b_0 = c_0$, the closure of V_* under the (effective) resistance metric has interesting topology different from the SG; the second one is from [8], it is called a *Sierpinski sickle*, which is the attractor of an IFS of 17 similitudes and three boundary points, of which the recursive construction does not yield a compatible sequence for a Dirichlet form.

2. **Proof of Theorem 1.2**

Let (a,b,c) denote the conductance of a Δ-shape network. Recall the Δ-*Y transform* (see e.g., [18], [29]) states that the Δ-shaped network with resistance (a^{-1}, b^{-1}, c^{-1}) and the Y-shaped network with resistance (x,y,z) (see Figure 1) are equivalent by the following relation

$$x = \frac{a}{\eta}, \quad y = \frac{b}{\eta}, \quad z = \frac{c}{\eta}\,, \tag{2.1}$$

with $\eta = ab + bc + ca$, and conversely,

$$a = \frac{x}{r}, \quad b = \frac{y}{r}, \quad c = \frac{z}{r}\,, \tag{2.2}$$

where $r = xy + yz + zx = \eta^{-1}$.

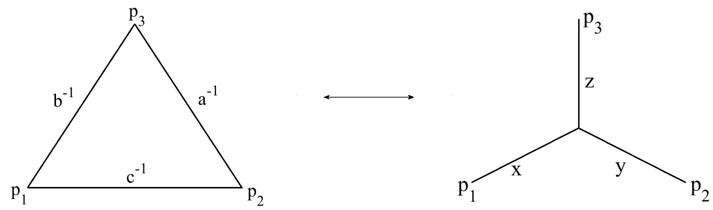

FIGURE 1. $\Delta - Y$-transform

Assume the conductances on the edges of the n-th level cells are given by (a_n, b_n, c_n) for $n \geq 0$. The compatibility of the n-th and $(n-1)$-th resistance networks on the Y-side reduces to trace the left side graph in Figure 2 on V_0, which yields (1.4).

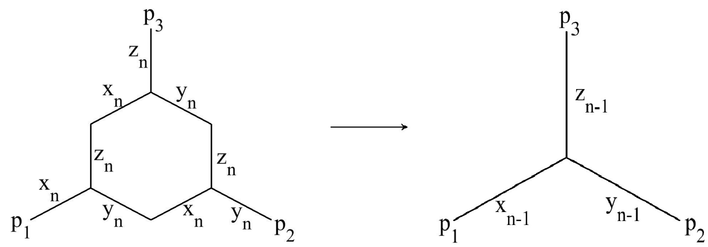

FIGURE 2. Consistence of the n-th and $(n-1)$-th resistance networks

Our first lemma is to characterize all compatible resistance sequences $\{(x_n, y_n, z_n)\}_{n\geq 0}$.

Lemma 2.1. *In order for (1.4) to have positive solutions* (x_n, y_n, z_n) *for all* $n \geq 1$, *it is necessary and sufficient that* $x_0 \geq y_0 = z_0 > 0$ *(or its symmetric alternatives).*

Proof. Sufficiency. Without loss of generality, assume that $x_0 \geq y_0 = z_0 > 0$. Then the equations (1.4) becomes

$$\begin{cases} x_0 = x_1 + \frac{(x_1+y_1)^2}{2(x_1+2y_1)}, \\ y_0 (= z_0) = y_1 + \frac{y_1(x_1+y_1)}{x_1+2y_1}. \end{cases} \tag{2.3}$$

Using the second equation in (2.3), we obtain

$$x_1 = \frac{y_1^2}{2y_1 - y_0} - 2y_1, \tag{2.4}$$

(we can exclude the case that $2y_1 = y_0$). Substituting (2.4) back to the first equation in (2.3), we obtain

$$5y_1^2 + (4x_0 - 2y_0)y_1 - 2x_0y_0 - y_0^2 = 0,$$

which gives y_1, then x_1 (by substituting into (2.4) again) as the following:

$$\begin{cases} x_1 = \frac{1}{15}\left(14x_0 + 3y_0 - 2\sqrt{4x_0^2 + 6x_0y_0 + 6y_0^2}\right), \\ y_1 (= z_1) = \frac{1}{5}\left(-2x_0 + y_0 + \sqrt{4x_0^2 + 6x_0y_0 + 6y_0^2}\right), \end{cases} \tag{2.5}$$

is a pair of positive solution of (1.4). Also by $x_0 \geq y_0$, we have

$$x_1 - y_1 = \frac{1}{15}\left(20x_0 - 5\sqrt{4x_0^2 + 6x_0y_0 + 6y_0^2}\right) \geq 0.$$

Hence, $x_1 \geq y_1 = z_1$. We can repeat this process inductively, and obtain the sequence $\{(x_n, y_n, z_n)\}_{n\geq 0}$ as positive solution of (1.4).

Necessity. Without loss of generality, let $x_0 \geq y_0 \geq z_0 > 0$, we will show that $y_0 = z_0$. Assume otherwise, $y_0 > z_0$. Let (x_1, y_1, z_1) be positive solution of (1.4) for $n = 1$, we first prove the following claims in regard to (x_1, y_1, z_1):

(i) $x_1 \geq y_1 > z_1$: For if $x_1 < y_1$, then clearly, $x_1 + \phi(x_1; y_1, z_1) < y_1 + \phi(y_1; z_1, x_1)$, which is $x_0 < y_0$, a contradiction. Hence $x_1 \geq y_1$; by the same argument, we have $y_1 > z_1$ from $y_0 > z_0$.

(ii) $\dfrac{y_1}{z_1} > \dfrac{y_0}{z_0}$: Indeed, if this were not true, letting $\frac{y_0}{z_0} = \mu_0 > 1$, we have

$$\frac{y_1}{z_1} \leq \mu_0 = \frac{y_0}{z_0} = \frac{y_1 + \phi(y_1; z_1, x_1)}{z_1 + \phi(z_1; x_1, y_1)}.$$

Therefore $\frac{y_1}{z_1} \leq \frac{y_1+x_1}{z_1+x_1}$, that is $y_1 \leq z_1$, which contradicts the fact that $y_1 > z_1$ in (i).

(iii) Let $\lambda_0 = \frac{2x_0}{y_0+z_0} > 1$, and let $\rho = \frac{1}{5}\left(6 - \lambda_0^{-1}\right) (> 1)$, we claim that

$$\frac{2x_1}{y_1 + z_1} \geq \lambda_0\rho. \tag{2.6}$$

If otherwise, then

$$\frac{2x_1}{y_1 + z_1} < \lambda_0\rho. \tag{2.7}$$

By (1.4), we have

$$\lambda_0 = \frac{2x_0}{y_0 + z_0} = \frac{2x_1 + 2\phi(x_1; y_1, z_1)}{(y_1 + z_1) + \frac{(y_1+z_1)(2x_1+y_1+z_1)}{2(x_1+y_1+z_1)}}. \tag{2.8}$$

Observe that

$$\frac{2(x_1 + y_1 + z_1)}{(2x_1 + y_1 + z_1)} < 2.$$

This, together with (2.7), (2.8) and a simple calculation, yields

$$\frac{2(x_1 + y_1)(x_1 + z_1)}{(y_1 + z_1)(2x_1 + y_1 + z_1)} > \lambda_0 \cdot (3 - 2\rho). \tag{2.9}$$

On the other hand, by using $\rho = \frac{1}{5}\left(6 - \lambda_0^{-1}\right)$, we have

$$\begin{aligned}\frac{2(x_1 + y_1)(x_1 + z_1)}{(y_1 + z_1)(2x_1 + y_1 + z_1)} &\leq \frac{1}{2} \cdot \frac{(2x_1 + y_1 + z_1)^2}{(y_1 + z_1)(2x_1 + y_1 + z_1)} = \frac{1}{2}\left(1 + \frac{2x_1}{y_1 + z_1}\right) \\ &< \frac{1}{2}(1 + \lambda_0 \cdot \rho) = \lambda_0 \cdot (3 - 2\rho).\end{aligned}$$

This contradicts (2.9), and (iii) follows.

By (i), we can carry out the estimate in (iii) inductively and obtain

$$\frac{2x_n}{y_n + z_n} \geq \lambda_0 \cdot \rho^n \to \infty, \quad \text{as } n \to \infty. \tag{2.10}$$

Also using (ii), we have $\frac{y_n}{z_n} \geq \frac{y_0}{z_0} = \mu_0 > 1$ for any $n \geq 1$, and a similar argument as in (iii) yields

$$\frac{y_n}{z_n} \to \infty, \quad \text{as } n \to \infty \tag{2.11}$$

(for example, one can take $\rho = (5 - \mu_0^{-1})/4 > 1$, and show that there is n_0 such that for all $n \geq n_0$, $\frac{y_n}{z_n} > \rho \frac{y_{n-1}}{z_{n-1}}$ holds).

Now consider

$$\begin{cases} y_{n-1} = y_n + \phi(y_n; z_n, x_n), \\ z_{n-1} = z_n + \phi(z_n; x_n, y_n), \end{cases}$$

for n and $\frac{x_n}{y_n+z_n}$ sufficiently large. By $\frac{x_n}{y_n+z_n} \to \infty$, it reduces to

$$\begin{cases} y_{n-1} = (\frac{3}{2} y_n + \frac{1}{2} z_n)(1 + o(1)), \\ z_{n-1} = (\frac{3}{2} z_n + \frac{1}{2} y_n)(1 + o(1)), \end{cases}$$

where $o(1)$ is an error term that tends to 0 as $n \to \infty$. Therefore we obtain

$$\begin{cases} y_n \asymp \frac{3}{4} y_{n-1} - \frac{1}{4} z_{n-1}, \\ z_n \asymp \frac{3}{4} z_{n-1} - \frac{1}{4} y_{n-1}. \end{cases}$$

This together with (2.11) contradicts the assumption that $\{z_n\}_{n \geq 0}$ are positive. Therefore we must have $y_0 = z_0$, and this completes the proof. □

Lemma 2.2. *Let $x_0 \geq y_0 = z_0 > 0$ be fixed, then for $n \geq 1$,*

$$\begin{cases} x_n = \frac{1}{15}\left(14x_{n-1} + 3y_{n-1} - 2\sqrt{4x_{n-1}^2 + 6x_{n-1}y_{n-1} + 6y_{n-1}^2}\right), \\ y_n = \frac{1}{5}\left(-2x_{n-1} + y_{n-1} + \sqrt{4x_{n-1}^2 + 6x_{n-1}y_{n-1} + 6y_{n-1}^2}\right). \end{cases}$$

Also for $x_0 = y_0 = z_0$, then $x_n = y_n = z_n = \left(\frac{3}{5}\right)^n x_0$, and for $x_0 > y_0 = z_0$,

$$x_n \asymp \left(\frac{2}{3}\right)^n, \quad y_n = z_n \asymp \left(\frac{1}{2}\right)^n.$$

Proof. Similar to (2.5), we can solve equations (1.4) for x_n and y_n as the above. It follows that if $x_0 = y_0 = z_0$, then $x_n = y_n = z_n = \left(\frac{3}{5}\right)^n x_0$. By Lemma 2.1, we see that $x_0 > y_0 = z_0 > 0$ implies $x_n > y_n = z_n$ inductively. Also from (2.10), we see that for all $n \geq 0$, $\frac{y_n}{x_n} \leq C\delta^n$ for some constant $C > 0$ and $0 < \delta < 1$ (depending only on $\frac{y_0}{x_0}$). Combining this with

$$\frac{x_n}{x_{n-1}} = \frac{1}{15}\left(14 + 3\frac{y_{n-1}}{x_{n-1}} - 2 \cdot \sqrt{4 + 6\frac{y_{n-1}}{x_{n-1}} + 6\left(\frac{y_{n-1}}{x_{n-1}}\right)^2}\right),$$

we can find $C_1 > 0$ such that for large n,

$$\frac{2}{3} - C_1\delta^n \leq \frac{x_n}{x_{n-1}} \leq \frac{2}{3} + C_1\delta^n.$$

Therefore we have$x_n \asymp \left(\frac{2}{3}\right)^n$. Similarly, we have

$$\begin{aligned}\frac{y_n}{y_{n-1}} &= \frac{1}{5}\left(-2\frac{x_{n-1}}{y_{n-1}} + 1 + \sqrt{4\left(\frac{x_{n-1}}{y_{n-1}}\right)^2 + 6\frac{x_{n-1}}{y_{n-1}} + 6}\right),\\ &= \frac{2\frac{x_{n-1}}{y_{n-1}} + 1}{2\frac{x_{n-1}}{y_{n-1}} - 1 + \sqrt{4\left(\frac{x_{n-1}}{y_{n-1}}\right)^2 + 6\frac{x_{n-1}}{y_{n-1}} + 6}},\\ &= \frac{2 + \frac{y_{n-1}}{x_{n-1}}}{2 - \frac{y_{n-1}}{x_{n-1}} + \sqrt{4 + 6\frac{y_{n-1}}{x_{n-1}} + 6\left(\frac{y_{n-1}}{x_{n-1}}\right)^2}},\end{aligned}$$

and we can find $C_2 > 0$ such that for large n,

$$\frac{1}{2} - C_2\delta^n \le \frac{y_n}{y_{n-1}} \le \frac{1}{2} + C_2\delta^n.$$

This implies that $y_n \asymp \left(\frac{1}{2}\right)^n$.

□

Proof of Proposition 1.1. It follows readily from Lemmas 2.1 and 2.2. □

It follows from the compatibility of $\{(x_n, y_n, z_n)\}_{n\ge0}$ and the Δ-Y transform that $\{\mathcal{E}_n^{(a_0,b_0)}\}_{n\ge0}$ are compatible. Hence for a function $u \in \ell(V_n)$, we can construct inductively harmonic extensions u_m on V_m, $m > n$ and $\mathcal{E}_m^{(a_0,b_0)}(u_m) = \mathcal{E}_n^{(a_0,b_0)}(u)$; also for $u \in \ell(V_*)$, $\mathcal{E}_n(u|_{V_n})$ is an increasing sequence. We define $\mathcal{E}(u) := \mathcal{E}^{(a_0,b_0)}(u) = \lim_{n\to\infty} \mathcal{E}_n^{(a_0,b_0)}(u|_{V_n})$ for $u \in \ell(V_*)$. Recall that the *(effective) resistance metric* $R := R^{(a_0,b_0)}$ on $V_* \times V_*$ is defined by $R^{(a_0,b_0)}(x, x) = 0$ for any $x \in V_*$, and for any two distinct points $x, y \in V_*$,

$$R(x, y)^{-1} := \inf\{\mathcal{E}(u) : \ u \in \ell(V_*),\ u(x) = 1, u(y) = 0\}.$$

Note that for $a_0 = b_0 = c_0$, then $R(x, y) \asymp |x - y|^\gamma$ where $\gamma = \frac{\log(5/3)}{\log 2}$ [18, 29].

Proposition 2.3. *For $a_0 > b_0 = c_0$, the completion of the $(V_*, R^{(a_0,b_0)})$ is K, and*

$$C^{-1}|x - y| \le R^{(a_0,b_0)}(x, y) \le C|x - y|^{\gamma'}, \qquad x, y \in K \tag{2.12}$$

where $\gamma' = \frac{\log 3}{\log 2} - 1$ and $C > 0$ is a constant depending on a_0 and b_0.

Furthermore $R^{(a_0,b_0)}$ is a bounded metric with

$$\sup\left\{R^{(a_0,b_0)}(x, y) : x, y \in K\right\} \le C' b_0^{-1}. \tag{2.13}$$

where $C' > 0$ is independent of a_0 and b_0.

Proof. Fix $x_0 > y_0 = z_0 > 0$, then $x_n > y_n = z_n$. As in (2.2), $r_n = x_ny_n + y_nz_n + z_nx_n = 2x_ny_n + y_n^2$. By (2.2) and Lemma 2.2,

$$a_n = \frac{x_n}{r_n} = \frac{x_n}{2x_ny_n + y_n^2} \asymp 2^n, \qquad b_n = c_n = \frac{y_n}{r_n} = \frac{y_n}{2x_ny_n + y_n^2} \asymp \left(\frac{3}{2}\right)^n.$$

Let us write $R(x,y) = R^{(a_0,b_0)}(x,y)$. To estimate $R(x,y)$ on V_*, we first consider $x \sim_n y$, and let $\psi_x^{(n)}(z) = \delta_x(z),\ x,z \in V_n$ where δ_x is the Dirac measure on V_n. It follows that

$$R^{-1}(x,y) \le \mathcal{E}(\psi_x^{(n)}) \le C_1 2^n = C_1|x-y|^{-1}.$$

On the the other hand, we have

$$R^{-1}(x,y) \ge \min\{a_n, b_n\} \ge C_2\Big(\frac{3}{2}\Big)^n = C_2|x-y|^{-\log(3/2)/\log 2}.$$

For the estimate of $R(x,y)$ with any distinct $x,y \in V_*$. Let n be the maximal integer such that both x,y belong to either an n-level cell or a union of two adjacent n-level cells. Then using a similar argument as above, we have $R(x,y) \le C\Big(\frac{2}{3}\Big)^n$ and $R(x,y) \ge C^{-1}2^{-n}$. This gives that $R(x,y)$ satisfies the required estimate since $|x-y| \asymp 2^{-n}$. This completes the proof of (2.12) for $x,y \in V_*$, thus it follows that the completion of (V_*, R) is K, and the same estimate holds for $x,y \in K$.

To prove (2.13), we only need to estimate $R(x,p_1)$ from above with $x \in K$ since for any two points x,y in K, $R(x,y) \le R(x,p_1) + R(y,p_1)$. We can find a chain of points $\{x_n\}_{n=0}^{\infty}$ in V_* with $x_0 = p_1$ and $x_n \to x$ as $n \to \infty$ such that x_n, x_{n+1} are two of the boundary points of some $(n+1)$-cell. Thus by triangle inequality, we have

$$R(x,p_1) \le \sum_{n=0}^{\infty} R(x_n, x_{n+1}) \le \sum_{n=0}^{\infty} b_n^{-1}. \tag{2.14}$$

On the other hand, we see that

$$\begin{aligned}
&\frac{b_{n-1}}{b_n} = \frac{2x_n + y_n}{2x_{n-1} + y_{n-1}} \le \max\left\{\frac{x_n}{x_{n-1}}, \frac{y_n}{y_{n-1}}\right\} \\
&= \max\left\{\frac{1}{15}\left(14 + 3\frac{y_{n-1}}{x_{n-1}} - 2\sqrt{4 + 6\frac{y_{n-1}}{x_{n-1}} + 6\Big(\frac{y_{n-1}}{x_{n-1}}\Big)^2}\right),\ \frac{1}{5}\left(-2\frac{x_{n-1}}{y_{n-1}} + 1 + \sqrt{4\Big(\frac{x_{n-1}}{y_{n-1}}\Big)^2 + 6\frac{x_{n-1}}{y_{n-1}} + 6}\right)\right\} \\
&\le \max\left\{\frac{1}{15}(14 + 3 - 2\sqrt{4}),\ \frac{1}{5}\left(-2\frac{x_{n-1}}{y_{n-1}} + 1 + 2\frac{x_{n-1}}{y_{n-1}} + 3\right)\right\} = \frac{13}{15} < 1.
\end{aligned}$$

Therefore the series in (2.14) converges and is bounded above by Cb_0^{-1}.

□

It follows that under the resistance metric, $u \in \ell(V_n)$ can be extended harmonically on V_*, then continuously on K, and we call this an *n-piecewise harmonic function* on K. As a special case, consider the harmonic function that takes value $1,0,0$ on p_1,p_2,p_3. It is direct to check, using the harmonicity of u at $V_1 \setminus V_0$, that

$$u(p_{12}) = u(p_{13}) = \frac{a_1 + b_1}{3a_1 + 2b_1}, \qquad u(p_{23}) = \frac{b_1}{3a_1 + 2b_1}. \tag{2.15}$$

For the special case that $a_0 = b_0 = c_0$, it is the $\frac{1}{5}$-$\frac{2}{5}$-law in the standard Dirichlet form on SG [18, 29].

Proof of Theorem 1.2. Fix $x_0 > y_0 = z_0 > 0$, it follows from the proof in Proposition 2.3 that $a_n \asymp 2^n,\ b_n = c_n \asymp \left(\frac{3}{2}\right)^n$. For $u \in C(K)$ and $n \geq 0$, let

$$\mathcal{E}_n^{(a_0,b_0)}(u) = \sum_{\omega \in W_n} b_n\Big(u_\omega(p_1) - u_\omega(p_2)\Big)^2 + b_n\Big(u_\omega(p_1) - u_\omega(p_3)\Big)^2 + a_n\Big(u_\omega(p_2) - u_\omega(p_3)\Big)^2,$$

where $u_\omega(x) = u \circ F_\omega(x)$. Define

$$\mathcal{E}^{(a_0,b_0)}(u) = \lim_{n\to\infty} \mathcal{E}_n^{(a_0,b_0)}(u|_{V_n}), \quad \mathcal{F}(= \mathcal{F}^{(a_0,b_0)}) = \{u \in C(K) : \mathcal{E}^{(a_0,b_0)}(u) < \infty\}.$$

In view of the compatibility of the sequence $\{(a_n, b_n, c_n)\}_n$, we have $\mathcal{E}_n^{(a_0,b_0)}(u) = \sum_{i=1}^3 \mathcal{E}_{n-1}^{(a_1,b_1)}(u\circ F_i)$. By taking limit, we obtain

$$\mathcal{E}^{(a_0,b_0)}(u) = \sum_{i=1}^{3} \mathcal{E}^{(a_1,b_1)}(u \circ F_i).$$

It is standard to check that $(\mathcal{E}^{(a_0,b_0)}, \mathcal{F})$ is a Dirichlet form on $L^2(K,\mu)$. It is regular by observing that the *piecewise harmonic functions* are continuous functions in $\mathcal{F}$ and are dense in $C(K)$, and $C(K) \cap \mathcal{F}(= \mathcal{F})$ is trivially $(\mathcal{E}^{(a_0,b_0)})^{1/2} + \|\cdot\|_{L^2(K,\mu)}$-dense in $\mathcal{F}$. By using the above identity repeatedly, we obtain that for any $n \geq 1$,

$$\mathcal{E}^{(a_0,b_0)}(u) = \sum_{\omega\in W_n} \mathcal{E}^{(a_n,b_n)}(u \circ F_\omega), \tag{2.16}$$

which leads to the strong locality of $(\mathcal{E}^{(a_0,b_0)}, \mathcal{F})$.

Finally, we see that $(\mathcal{E}^{(a_0,b_0)}, \mathcal{F})$ does not satisfy the energy self-similar identity (1.2). It is because if it satisfies the identity for some r_i, then by our construction, all the r_i in (1.1) should be equal. However, by the uniqueness result of Sabot [28], $(\mathcal{E}^{(a_0,b_0)}, \mathcal{F})$ should be the standard one defined by (1.1), a contradiction. □

The following dichotomic result follows directly from Theorem 1.2.

Corollary 2.4. *For the recursive construction of the Dirichlet form with initial data* (a_0, b_0, c_0)*, there are only two cases, either*

(i) $a_0 = b_0 = c_0$*, and in this case* $\mathcal{E}$ *is the standard Dirichlet form as in* (1.1)*,*

or

(ii) $a_0 > b_0 = c_0$ *(or the symmetric alternates), and the Dirichlet form satisfies*

$$\mathcal{E}(u) \asymp \sup_{n\geq 0} \left\{2^n \sum_{\omega\in W_n} \left(\Big(u_\omega(p_2) - u_\omega(p_3)\Big)^2 + \left(\frac{3}{4}\right)^n \Big(u_\omega(p_1) - u_\omega(p_2)\Big)^2 + \left(\frac{3}{4}\right)^n \Big(u_\omega(p_1) - u_\omega(p_3)\Big)^2\right)\right\}.$$

It is well-known that a regular strongly local Dirichlet form associates with a continuous diffusion process [5]. In fact, this probability counter part of $\mathcal{E}^{(a_0,b_0)}$ had been studied by Hattori *et al* [9] as an asymptotically one-dimensional diffusion processes on the SG. To conclude this section, we give a brief discussion of their study in comparison with our consideration.

For a random walk $\{Z_k^{(n,\alpha)}\}_k$ on V_n with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, the probability that the walk goes to the four neighbors (except at V_0) in the three directions (counting the opposite direction as one), define the $(n-1)$-*decimated walk* $\{Z'_\ell\}_\ell$ on V_{n-1} that records the visit of $Z_k^{(n,\alpha)}$ in V_{n-1} in the ℓ-th time (with a state distinct from $Z'_{\ell-1}$). Then it is direct to show that for $\{Z_k^{(n,\alpha)}\}_k$ with starting point on V_{n-1}, $\{Z'_\ell\}_\ell$ obeys the same law as $\{Z_k^{(n-1,T\alpha)}\}_k$ where

$$T\alpha = C\Big(\alpha_1 + \frac{\alpha_2\alpha_3}{3},\ \alpha_2 + \frac{\alpha_3\alpha_1}{3},\ \alpha_3 + \frac{\alpha_1\alpha_2}{3}\Big),$$

and C is a normalized constant [9]. This sets up the compatible condition by letting $\alpha_{n-1} = T\alpha_n$ (renormalization group), the exact analog of (1.4). Then they define the random walk using

$$\alpha_n = (\alpha_{n,1},\ \alpha_{n,2},\ \alpha_{n,3}) := C'(1, w_n, w_n),$$

where $0 < w_0 < 1$, C' is a normalized constant, and $w_n,\ n \ge 1$, are defined inductively by

$$w_n = \Big(-2 + 3w_{n-1} + \sqrt{4 + 6w_{n-1} + 6w_{n-1}^2}\Big) / (6 - w_{n-1}).$$

For x_n, y_n in Lemma 2.2, it can be shown that y_n/x_n has the same expression as the above w_n.

Note that in this case $\lim_{n\to\infty} \alpha_n = (1,0,0)$. Let $X_t(n) = Z_{[6^n t]}(n, \alpha_n)$, then with some more work, they proved that $\{X_t(n)\}_{n=0}^\infty$ converges weakly to a continuous, strongly Markov processes X_t on K, and the moves are asymptotically one-dimensional, dominated in the direction parallel to $\overline{p_2p_3}$, and of order $O(3/4)^n$ in the other two directions. This is in line with the expression of $\mathcal{E}^{(a_0,b_0)}$ in Corollary 2.4(ii), as the energy has two scaling exponents and is concentrated in the $\overline{p_2p_3}$ direction.

3. **Spectral asymptotics**

Let $\Delta^{(a_0,b_0)}$ be the *Laplacian*, the infinitesimal generator of the Dirichlet form $\big(\mathcal{E}^{(a_0,b_0)}, \mathcal{F}\big)$ on $L^2(K,\mu)$. In both cases $a_0 = b_0$ and $a_0 > b_0$, by using the resistance estimates in Proposition 2.3 and a standard argument, we see that $\mathcal{F}$ is compactly imbedded in $C(K)$ and hence in $L^2(K,\mu)$. Therefore the eigenvalues of $-\Delta^{(a_0,b_0)}$ with the Dirichlet or Neumann boundary condition are nonnegative, countable and have no limit point. Denote by $\rho^{(a_0,b_0)}(t)$ the *eigenvalue counting function* of $-\Delta^{(a_0,b_0)}$ with the *Dirichlet boundary condition* as in (1.6), and by $\rho_N^{(a_0,b_0)}(t)$ the eigenvalue counting function of $-\Delta^{(a_0,b_0)}$ with the *Neumann boundary condition*, where in both cases, each eigenvalue is counted according to its multiplicity. We are interested in the asymptotic growth rate of $\rho^{(a_0,b_0)}(t)$ and $\rho_N^{(a_0,b_0)}(t)$ as $t \to \infty$. It is known that (see [19, Lemma 2.3(2)])

$$\rho^{(a_0,b_0)}(t) \le \rho_N^{(a_0,b_0)}(t) \le \rho^{(a_0,b_0)}(t) + 3, \tag{3.1}$$

where 3 is the dimension of the space of all the harmonic functions on K. Hence $\rho^{(a_0,b_0)}(t)$ and $\rho_N^{(a_0,b_0)}(t)$ have the same asymptotic behavior.

In the case $a_0 = b_0 = c_0$ for the standard Dirichlet form, it is known that (e.g. [6], [19])

$$\rho^{(a_0,b_0)}(t) \asymp t^{\log 3/\log 5}, \qquad t \to \infty.$$

In the following, our concentration is on the case $a_0 > b_0 = c_0$. First we provide a general result on the dimension of some linear subspaces. Recall that a linear subspace $\mathcal{L}$ of $L^2(K,\mu)$ is called a *sublattice* if $u \in \mathcal{L}$ implies $|u| \in \mathcal{L}$.

Proposition 3.1. *Let K be a compact connected set and μ be a Borel measure on K with full support, and let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(K,\mu)$ with $\mathcal{F} \subset C(K)$. Denote by $\{P_t\}_{t\geq 0}$ the associated semigroup of operators of $(\mathcal{E},\mathcal{F})$. Suppose $\mathcal{L} \subset \mathcal{F}$ is a closed linear sublattice of $L^2(K,\mu)$, and there exists $C > 0$ such that*

$$P_t u \leq Cu, \qquad \forall\, t > 0,\ \ u \geq 0,\ \ u \in \mathcal{L}. \tag{3.2}$$

Then $\mathcal{L}$ has dimension at most one.

Proof. The essentially idea of the proof comes from [4, Theorems 7.2, 7.3]. Suppose $\mathcal{L}$ is nontrivial, let $u \geq 0$ be any non-zero element in $\mathcal{L}$, then $u \in C(K)$. Let $U = \{x \in K : u(x) \neq 0\}$. We claim that $U = K$, modulo a μ-null set. If $v \in C(K)$ and $|v| \leq \alpha u$ for some $\alpha \geq 0$, then by the Markovian property of $\{P_t\}_{t>0}$ and (3.2), we have

$$|P_t v| \leq P_t|v| \leq \alpha P_t u \leq \alpha C u.$$

Hence for

$$\mathcal{G} = \{v \in C(K) : |v| \leq \alpha u \ \text{ for some } \alpha \geq 0\},$$

then $P_t(\mathcal{G}) \subseteq \mathcal{G}$ for all $t \geq 0$. As U is an open set by definition, $\mathcal{G}$ contains all the continuous functions that are compactly supported in U. The L^2-closure of $\mathcal{G}$ is the set of all $v \in L^2(K,\mu)$ with $v = 0$ on $K \setminus U$. So U is an *invariant* set of the semigroup $\{P_t\}_{t>0}$. (A μ-measurable set $B \subset K$ is said to be P_t–invariant if $P_t(1_B f) = 1_B P_t f$ μ-a.e. for any $f \in L^2$ and $t > 0$.) Hence by [5, Theorem 1.6.1], $1_U \in \mathcal{F}$. However, as K is connected, this holds if and only if $U = K$ or $U = \emptyset$. Since u is nonzero, we conclude that $U = K$, and the claim follows.

Now, if $u \in \mathcal{L}$, then u^+ and u^- are in $\mathcal{L}$ and have disjoint supports. It follows from the claim that one of them must vanish. Hence $u \in \mathcal{L}$ implies $u \geq 0$ or $(-u) \geq 0$. If u, v are two distinct positive elements of $\mathcal{L}$, then $u + \eta v$ is either positive or negative for all $\eta \in \mathbb{R}$. But the sum must change sign as η increases through $\mathbb{R}$. Hence there is η such that $u + \eta v = 0$. This is a contradiction, and hence $\mathcal{L}$ is one dimensional. □

Lemma 3.2. *Let K be the Sierpiński gasket and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E}^{(a,b)},\mathcal{F})$ be the Dirichlet form defined in Theorem 1.2. Let Λ_1 be the eigenfunction space of λ_1, the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. Then Λ_1 is of dimension one.*

Proof. We make use of the Rayleigh quotient for the first eigenvalue:

$$\lambda_1 = \inf_{u\in\mathcal{F}_0, u\neq 0} \frac{\mathcal{E}(u)}{\|u\|_2^2}, \tag{3.3}$$

where $\mathcal{F}_0 := \{u \in \mathcal{F} : u|_{V_0} = 0\}$. There exists a function $u \in \mathcal{F}$ attains the infimum, and all such functions must be eigenfunctions with eigenvalue λ_1. Therefore by the Markovian property of the Dirichlet form, we see that Λ_1 is a closed *sublattice*, hence also u^+, u^- are contained in Λ_1. For any $u \in \Lambda_1$, we have

$$P_t u = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Delta^n u = \sum_{n=0}^{\infty} \frac{t^n}{n!}(-\lambda_1)^n\, u = e^{-t\lambda_1} u \leq u.$$

By using Proposition 3.1 with $\mathcal{L} = \Lambda_1$, we see that Λ_1 is of dimension at most one, and thus Λ_1 is one dimensional since Λ_1 is nontrivial. □

Lemma 3.3. *There exists $C > 0$ such that for any initial data $a > b = c > 0$ on Γ_0, we have*

$$C^{-1}b \leq \lambda_1^{(a,b)} \leq Cb, \tag{3.4}$$

where $\lambda_1^{(a,b)}$ is the first eigenvalues of $-\Delta^{(a,b)}$ with the Dirichlet boundary condition.

Proof. We will make use of the Rayleigh quotient in (3.3) again. Let u_1 be the *1-piecewise harmonic function on K* with prescribed values $u_1(p_1) = u_1(p_2) = u_1(p_3) = u_1(p_{23}) = 0$, $u_1(p_{12}) = u_1(p_{13}) = 1$, where p_{ij} is the vertex in V_1 opposite to p_k for distinct $i, j, k \in \{1, 2, 3\}$ (see Figure 3 for the values of u_1). Then by (2.14)

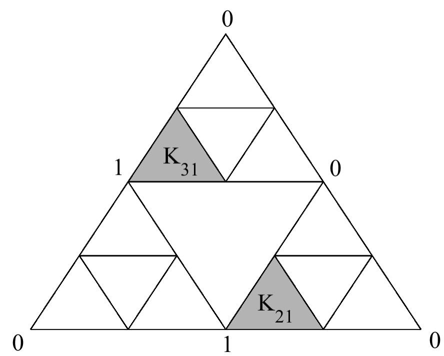

Figure 3. The value of u_1

$$\|u_1\|_2^2 \geq \int_{F_{21}(K)\cup F_{31}(K)} u_1^2 d\mu \geq \frac{2}{9} \cdot \left(\frac{a_2 + b_2}{3a_2 + 2b_2}\right)^2 \geq \frac{2}{81},$$

where a_2, b_2 are the second iterations of a $(= a_0)$, b $(= b_0)$ respectively. Also observe that $\mathcal{E}^{(a,b)}(u_1) = 6b_1$. Therefore

$$\lambda_1^{(a,b)} \leq \frac{\mathcal{E}^{(a,b)}(u_1)}{\|u_1\|_2^2} \leq C'b_1 \leq Cb$$

for some $C', C > 0$.

To estimate the lower bound, we let $u \in \mathcal{F}$, then

$$|u(x) - u(y)|^2 \leq R^{(a,b)}(x, y)\mathcal{E}^{(a,b)}(u), \quad x, y \in K.$$

It follows that for $u \in \mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$, $u \neq 0$, by choosing $y = p_3$, we have

$$|u(x)|^2 \leq R^{(a,b)}(x, p_3)\mathcal{E}^{(a,b)}(u), \quad \forall x \in K.$$

Integrating both sides with respect to μ, we obtain

$$\|u\|_2^2 \leq \int_K R^{(a,b)}(x, p_3)d\mu(x) \cdot \mathcal{E}^{(a,b)}(u).$$

Recall that the resistance $R(x, y)$, $x, y \in K$ has the expression $R(x, y) = \sup\{\frac{|u(x)-u(y)|^2}{\mathcal{E}(u)} : u \in \mathcal{F}, \mathcal{E}(u) \neq 0\}$. Using (2.13), we have $C_1 > 0$ such that

$$C_1 b \leq \frac{\mathcal{E}^{(a,b)}(u)}{\|u\|_2^2}.$$

Since u is arbitrary, this implies that $C_1 b \le \lambda_1^{(a,b)}$. This completes the proof of the lemma. $\square$

Lemma 3.4. *Let $a_0 > b_0 = c_0$, then for all $t \ge 0$ and $n \ge 0$,*

$$3^n \rho^{(a_n,b_n)}\left(\frac{t}{3^n}\right) \le \rho^{(a_0,b_0)}(t), \quad \textit{and} \quad \rho_N^{(a_0,b_0)}(t) \le 3^n \rho_N^{(a_n,b_n)}\left(\frac{t}{3^n}\right). \tag{3.5}$$

Here $\rho^{(a_n b_n)}(t)$ is the eigenvalue counting function using $a_n > b_n = c_n$ as initial data on V_0. We refer to the similar proof in [19, Propsitions 6.2, 6.3]. The technique is that first we restrict $\mathcal{E}^{(a_0,b_0)}$ on the sub-domain $\mathcal{F}_1 := \{u \in \mathcal{F} : u|_{V_1} = 0\}$. Denote by $\rho\left(t; \mathcal{E}^{(a_0,b_0)}, \mathcal{F}_1\right)$ the corresponding eigenvalue counting function, then by making use of the identity (1.5) we have the following relation

$$\rho\left(t; \mathcal{E}^{(a_0,b_0)}, \mathcal{F}_1\right) = 3\rho^{(a_1,b_1)}\left(\frac{t}{3}\right).$$

where $\frac{1}{3}$ in the bracket is the scaling factor of μ. Using this repeatedly and that $\rho\left(t; \mathcal{E}_0^{(a,b)}, \mathcal{F}_1\right) \le \rho^{(a,b)}(t)$, we obtain the first inequality in (3.5). The second inequality can be shown by constructing another Dirichlet form which has domain $\mathcal{F}_2 := \{u : K \setminus V_1 \to \mathbb{R} : u \circ F_i = f_i \text{ on } K \setminus V_0 \text{ for some } f_i \in \mathcal{F}, i = 1,2,3\}$ and using a similar argument.

Theorem 3.5. *Assume that $a_0 > b_0 = c_0$ on Γ_0, then for $t_0 = \inf\{t : \rho^{(a_0,b_0)}(t) > 0\}$,*

$$\rho^{(a_0,b_0)}(t) \asymp t^{\frac{\log 3}{\log(9/2)}}, \qquad t > t_0.$$

Similarly, the same inequality holds when $\rho^{(a_0,b_0)}(t)$ is replaced by $\rho_N^{(a_0,b_0)}(t)$ and for any $t_0 > 0$.

Proof. By Lemma 3.2, we see that if we use $a_n > b_n = c_n$ as initial data on Γ_0, then we have $\rho^{(a_n,b_n)}\left(\lambda_1^{(a_n,b_n)}\right) = 1$, and $\rho_N^{(a_n,b_n)}\left(\lambda_1^{(a_n,b_n)}\right) \le \rho^{(a_n,b_n)}\left(\lambda_1^{(a_n,b_n)}\right) + 3 = 4$. Then by Lemma 3.4, we have

$$\rho_N^{(a_0,b_0)}\left(3^n \lambda_1^{(a_n,b_n)}\right) \le 4 \cdot 3^n.$$

Letting $t = 3^n \lambda_1^{(a_n,b_n)}$, by Lemma 3.3, we have $t \asymp 3^n b_n \asymp 3^n (3/2)^n = (9/2)^n$ and $3^n \asymp t^{\log 3/\log(9/2)}$. It follows that

$$\rho^{(a_0,b_0)}(t) \le \rho_N^{(a_0,b_0)}(t) \le C t^{\frac{\log 3}{\log(9/2)}}$$

for some $C > 0$. The same argument yields the other inequality. $\square$

Recall that the *spectral dimension* d_s of a Dirichlet form is defined to be $\lim\limits_{t\to\infty} \frac{2\log\rho(t)}{\log t}$ if the limit exists. Heuristically, $d_s/2 = d_f/d_w$ where d_f is the Hausdorff dimension of K, and d_w is the *walk dimension* of K. The walk dimension is the space-time relation $E_x(|X_t - x|^2) \approx t^{2/d_w}$ of the associate diffusion process [29], which is also the critical exponent of the Besov space corresponding to the domain of the Dirichlet form [7, 8, 15] (see Section 4). From Theorem 3.5, we see that for $a_0 > b_0$,

$$d_s = \frac{\log 9}{\log(9/2)}, \qquad d_w = \frac{\log 9}{\log 2} - 1.$$

4. **Other examples and remarks**

In this section, we consider two more examples. The first one is a modification of the SG such that for $a_0 > b_0 = c_0$, the closure of V_* under the resistance metric is different from the SG; the second one is detailed in [8], it is a p.c.f. set constructed by an IFS of 17 maps with three boundary points, of which the recursive construction does not yield a compatible sequence of Dirichlet forms $\mathcal{E}_n,\ n \geq 0$.

Let $p_0 = 0, p_1 = 1, p_3 = \exp\left(\frac{\pi\sqrt{-1}}{3}\right)$. We define the *twisted Sierpiński gasket* [23] to be the unique nonempty compact set K on $\mathbb{R}^2$ with the contractions $\{T_i\}_{i=1}^3$ such that $F_1(x) = \frac{\overline{x-p_1}}{2} \cdot \exp\left(\frac{\pi\sqrt{-1}}{3}\right) + p_1,\ F_2(x) = \frac{\overline{x-p_2}}{2} \cdot \exp\left(-\frac{\pi\sqrt{-1}}{3}\right) + p_2$, and $F_3(x) = -\frac{x-p_3}{2} + p_3$, (i.e., F_i reflects the sub-triangle K_i along the angle bisection at p_i). Then the attractor K is still the Sierpiński gasket. In [23], Mihai and Strichartz investigated the self-similar energy forms on this twisted SG.

Similar to the standard SG, by using the Δ-Y transform (see Figure 4), the compatibility of $\{(x_n, y_n, z_n)\}_{n\geq 0}$ must satisfy the following equations:

$$\begin{cases} x_{n-1} = x_n + \psi(x_n; y_n, z_n), \\ y_{n-1} = y_n + \psi(y_n; z_n, x_n), \\ z_{n-1} = z_n + \psi(z_n; x_n, y_n), \end{cases} \qquad n \geq 1, \tag{4.1}$$

where $\psi(x_n; y_n, z_n) = \frac{2y_nz_n}{x_n+y_n+z_n}$, and symmetrically for $\psi(y_n; z_n, x_n)$ and $\psi(z_n; x_n, y_n)$.

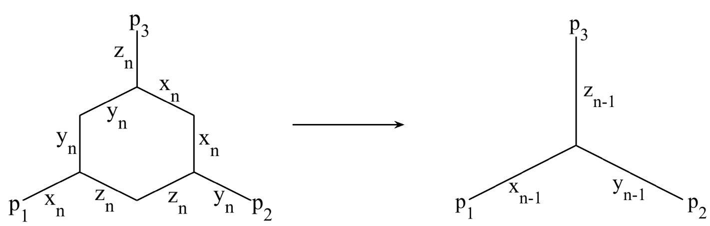

FIGURE 4. Δ-Y transform for the twisted maps

Lemma 4.1. *For $x_0, y_0, z_0 > 0$, in order for* (4.1) *to have positive solution $(x_n, y_n, z_n), n \geq 1$, it is necessary and sufficient that $x_0 \geq y_0 = z_0 > 0$ (or the symmetric alternates). In this case, $\{(x_n, y_n, z_n)\}_{n=0}^\infty$ is uniquely determined by (x_0, y_0, z_0).*

Furthermore, for $x_0 > y_0 = z_0$, we have the estimate $x_n \asymp 1, y_n = z_n \asymp \left(\frac{1}{3}\right)^n$, and hence $a_n \asymp 3^n,\ b_n = c_n \asymp 1$.

Proof. The proof for the first part is the same as Lemma 2.1. For the sufficiency, assuming that $x_0 \geq y_0 = z_0$, we can solve

$$\begin{cases} x_1 = \frac{1}{10}\left(x_0 + 2y_0 + 3\sqrt{9x_0^2 - 4x_0y_0 - 4y_0^2}\right), \\ y_1 (= z_1) = \frac{1}{5}\left(3x_0 + y_0 - \sqrt{9x_0^2 - 4x_0y_0 - 4y_0^2}\right), \end{cases} \tag{4.2}$$

and $x_1 \geq y_1 = z_1$, then proceed inductively. For the necessity, we need to change $\rho = 2 - \frac{1}{\lambda_0}$ for the estimation in (2.6), and make some obvious readjustments on the calculations.

For the second part, we note that x_n, y_n can be expressed in terms of x_{n-1}, y_{n-1} as in (4.2). By the same estimation as in Lemma 2.2, we have $x_n \asymp 1, y_n \asymp (\frac{1}{3})^n$, and the estimate of a_n, b_n follows. □

It follows from the estimation of the $a_n \asymp 3^n,\ b_n = c_n \asymp 1$ that

$$\mathcal{E}_n(u) \asymp \sum_{\omega \in W_n} \Big\{ 3^n \Big(u_\omega(p_2) - u_\omega(p_3)\Big)^2 + \Big(u_\omega(p_1) - u_\omega(p_2)\Big)^2 + \Big(u_\omega(p_1) - u_\omega(p_3)\Big)^2 \Big\} \qquad (4.3)$$

and the compatibility of $\{\mathcal{E}_n\}_{n\geq 0}$ implies that for $u \in \ell(V_*)$, $\mathcal{E}(u) = \lim_{n\to\infty} \mathcal{E}_n(u|_{V_n})$ exists.

Let $R := R^{(a_0,b_0)}$ denote the resistance metric on $V_* \times V_*$, and let Ω be the completion of V_* with respect to R. We will give a description of the topology and the completion of (Ω, R). Let $U_0 = \{p_2, p_3\}$, $U_n = \bigcup_{i=1}^3 F_i(U_{n-1})$ for $n \geq 1$ and $U_* = \bigcup_{n\geq 0} U_n$; also let $W_0 = \{p_1\}$, $W_n = \bigcup_{i=1}^3 F_i(W_{n-1})$ for $n \geq 1$ and $W_* = \bigcup_{n\geq 0} W_n$ (see Figure 5).

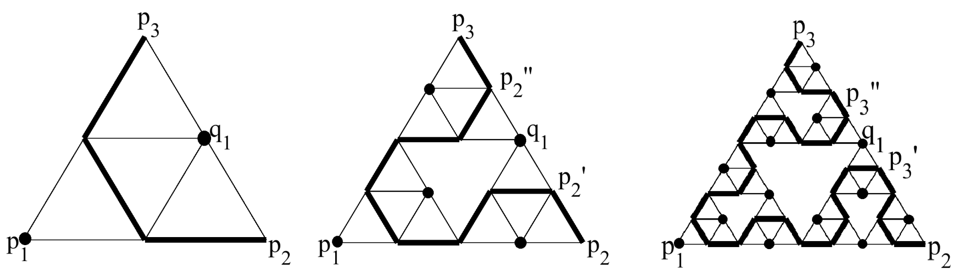

Figure 5. $U_n, W_n, n = 1, 2, 3$; bold lines are the edges joining the neighboring points in U_n, and the bold dots forms W_n.

Proposition 4.2. *On the twisted SG,*

(i) the resistance metric R is a uniform discrete metric on W_*, and $\mathrm{dist}_R(U_*, W_*) > 0$*;*

(ii) On U_* , for $x \in U_*$, let K_x be the largest subcell of K that has x as a vertex, then

$$R(x,y) \asymp |x-y|^{\log 3/\log 2} \quad \textit{for} \quad y \in U_* \cap K_x; \qquad (4.4)$$

on the other hand, let $q \in W_k \setminus W_{k-1}$ with two adjacent cells K'_q, K''_q (as defined above), then

$$R(x,y) \asymp \Big(\frac{1}{3}\Big)^k \quad \textit{for} \quad x \in U_n \cap K'_q,\ y \in U_n \cap K''_q,\ n \geq k. \qquad (4.5)$$

Consequently, the completion $\Omega = \overline{U_*} \cup W_*$, where $\overline{U_*}$ is pathwise connected and locally connected, and is such that for each $q \in W_k \setminus W_{k-1}$, $\overline{U_*}$ has two limit points p'_q, p''_q with $R(p'_q, p''_q) \asymp \big(\frac{1}{3}\big)^k$.

Remark 4.1: The completion $\overline{U_*}$ can be realized as cutting up the SG at each $q \in W_*$, and bend the two subcells K'_q, K''_q apart at the cut points with the appropriate distance without breaking the SG (see Figure 6 at q_1).

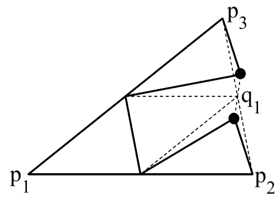

FIGURE 6. Completion of U_* at q_1; similarly at other $q \in W_n$

Proof. Recall that $R(x,y)^{-1} := \inf\{\mathcal{E}(u) : \ u \in \ell(V_*), \ u(x) = 1, u(y) = 0\}$. For $x, y \in W_n$, by using the tend functions, and observe that the effective resistance of the two nodes is $\geq b_n^{-1}/4 \asymp 1$, we conclude the resistance metric R on W_* has a uniform lower bound > 0. This implies W_* is a uniform discrete metric space. Also by the same reason, we see that $\mathrm{dist}_R(W_*, U_*) > 0$.

Next we consider the resistance metric on U_n. Let $K_n \subset K_x$ be the smallest subcell of K that contains both x, y. By using $a_n^{-1} \asymp \left(\frac{1}{3}\right)^n$, and the path property of U_n, it is direct to prove the estimation of (4.4).

To prove (4.5), it suffices to consider the case $q = q_1 \in W_1 \setminus W_0$, the midpoint of the line segment $\overline{p_2p_3}$, then use the IFS to move the argument to other $q \in W_*$. Let $p_n' \in U_n \cap F_2(K)$ that is a neighbor (in V_n) of q_1. Similarly, let $p_n'' \in U_n \cap F_3(K)$ that is a neighbor (in V_n) of q_1. Then we have

$$R(p_n', p_n'') \asymp R(p_2, p_3) \asymp 1.$$

(For the above estimation, note that the geodesic in V_n joining $p_n'(\in F_2(K))$ and $p_n''(\in F_3(K))$ must pass through $F_1(K)$ (see Figure 6), so that $R(p_2, p_3) \geq R(p_n', p_n'') \geq \frac{1}{3}R(p_2, p_3)$.)

Finally the statement of the completion $\overline{U_*}$ follows from (4.4) and (4.5). □

From the probabilistic point of view, it will be interesting to understand the corresponding diffusion process on $\overline{U_*}$ and Ω. We also note that in [11], Hambly and Kumagai studied this type of diffusions on several types of nested fractals(e.g. the higher dimensional and higher level Sierpinski gaskets and also the Vicsek sets). On the variations of gaskets they showed that there exist such Dirichlet forms, while on the Vicsek set, they observed that if one assigns resistance a and b on the side and diagonal edges with $a > b$, then the resistance between two diagonal lines on any n-cell has a uniform lower bound, and thus a similar situation as the twisted SG occurs.

Next we list another example in [8] on which the recursive construction does not work. Let K be a p.c.f. set as in Figure 7, which has three boundary points, and is generated by an IFS of 17 similitudes with contraction ratio $1/7$. We call it a *Sierpiński Sickle*.

Proposition 4.3. [8] *For the Sierpiński sickle, the recursive construction for any* (a_0, b_0, c_0) *does not give a compatible sequence of* $\{(a_n, b_n, c_n)\}_{n=0}^{\infty}$. *However, one can construct a Dirichlet form satisfying the energy self-similar identity.*

For the recursive construction, the basic reason for no compatible sequence is that the solution similar to the system of equations (1.4) fails to be positive. For the self-similar

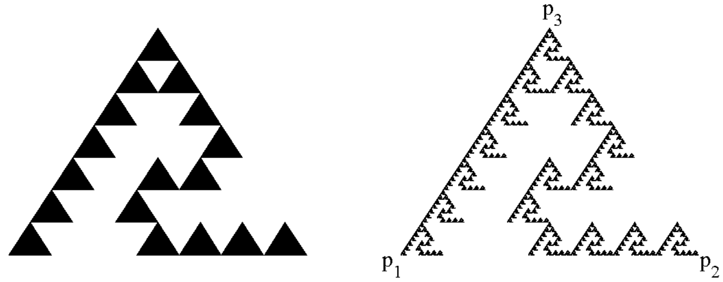

FIGURE 7. The Sierpiński sickle K

case, we find the explicit renormalizing factors for $\mathcal{E}(u) = \sum_{i=1}^{17} r_i^{-1}\mathcal{E}(u \circ F_i)$. Let r_L, r_R, r_T on the cells of K be defined as follows:

$r_1 = r_2 = \cdots = r_5 = r_L$ on the left 5 sub-triangles $F_1(K), F_2(K), \cdots, F_5(K)$;

$r_6 = r_7 = \tau_8 = r_T$ on the 3 top sub-triangles $F_6(K), F_7(K), F_8(K)$;

$r_9 = \tau_{10} = \cdots =_{17}= r_R$ on the right 9 sub-triangles $F_9(K), F_{10}(K), \cdots, F_{17}(K)$.

Then we can solve a system of equations and obtain, for $k \geq 2$,

$$r_L = \frac{k(k-1)}{5(k^2+6k+3)}, \quad r_R = \frac{k^2-1}{(13k+5)(k^2+6k+3)}, \quad r_T = \frac{2(2k+1)}{k^2+6k+3}.$$

Remark 4.2. Note that if the Dirichlet form from the recursive construction is self-similar, then all the renormalizing factors r_i's are equal. In [8, Section 4.2], we have a p.c.f. set (the Vicsek eyebolted cross) that the Dirichlet forms from the recursive construction cannot satisfy the condition, hence they are all non-self-similar. Nevertheless, the self-similar Dirichlet forms can still be obtained similar to the above example.

Remark 4.3. We do not know to what extend the recursive construction and the dichotomy result can be extended to the other p.c.f. sets; also in view of the different situations on the SG and the previous examples, it will be nice to have some specific criteria on the more general fractals.

The original usage of the above mentioned p.c.f sets was to study the critical exponents σ^* of the Besov spaces $B^\sigma_{2,\infty} \subset L^2(K,\mu)$, $\sigma > 0$ (where $K \subset \mathbb{R}^d$ is closed, and μ is an α-Ahlfors regular measure on K) in connection with the domain of the Dirichlet forms and the walk dimension ([8, 20, 21]). Recall that $B^\sigma_{2,\infty}$ has norm $\|u\|_{B^\sigma_{2,\infty}} = \|u\|_2 + [u]_{B^\sigma_{2,\infty}}$ where

$$[u]^2_{B^\sigma_{2,\infty}} = \sup_{0<r<1} r^{-2\sigma} \int_K \Big(\frac{1}{r^\alpha}\int_{B(x,r)} |u(x)-u(y)|^2 d\mu(y)\Big)\, d\mu(x).$$

The critical exponent of the Besov spaces $\{B^\sigma_{2,\infty}\}_{\sigma>0}$ is defined to be

$$\sigma^* = \sup\{\sigma : \ B^\sigma_{2,\infty} \cap C(K) \text{ is dense in } C(K)\}.$$

For example, for the SG, $\sigma^* = \log 5/\log 2$ and $B^{\sigma^*}_{2,\infty} = \mathcal{F}$ [15] (see also [27] for some nested fractals and [14] for p.c.f. fractals). In those cases, when $\sigma > \sigma^*$, then $B^\sigma_{2,\infty}$ contents only constant functions. In [8], we asked whether this is necessary, and investigate the relevance with the Dirichlet forms. For this we introduce another critical exponent

$$\sigma^\# = \sup\{\sigma : \ B^\sigma_{2,\infty} \text{ contains non-constant functions}\}$$

and constructed the above example. It was shown that $\sigma^* < \sigma^\#$ with the explicit expressions of σ^* and $\sigma^\#$. Also $B^{\sigma^*}_{2,\infty}$ ($\subset C(K)$) is dense in $C(K)$, and $B^{\sigma^\#}_{2,\infty}$ is dense in $L^2(K,\mu)$ (but not dense in $C(K)$). This Besov space $B^{\sigma^*}_{2,\infty}$ does not support a local regular Dirichlet for $(\mathcal{E},\mathcal{F})$ with $\mathcal{E}(u) \asymp \|u\|^2_{B^{\sigma^*}_{2,\infty}}$ for all $u \in \mathcal{F}$.

In all the known examples, the critical exponent σ^* of the Besov spaces on K equals to the walk dimension d_w of K. This heuristic relation is not very intuitive as σ^* is defined through the geometry of K, and the walk dimension is certain space-time exponent of the walk. Some of these aspects had been studied in [7] in terms of the heat kernel. It will be interesting to find out the more natural and direct connection of these exponents.

The existence of a Dirichlet form on a fractal set still posts a fundamental and challenging question. In [25], Peirone proved there is a large class of p.c.f. self-similar sets (not necessary symmetric) possess self-similar energy forms, and more recently, he claimed an example of a p.c.f. set that does not admit such energy forms [26]. It might be worthwhile to see if the recursive construction will produce a non-self-similar energy form in his example. For the non-p.c.f sets, it remains largely unknown for the existence of the Dirichlet forms. Even for the Sierpiński carpet, the construction is to use a probabilistic approach [2], which is technically quite complicated, and surprisingly, there is no clear analytic approach on the discrete approximations yet.

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