METRICS ON FRACTALS AND SUB-GAUSSIAN HEAT KERNEL ESTIMATES

QINGSONG GU, KA-SING LAU, HUA QIU, AND HUO-JUN RUAN

ABSTRACT. It is well-known that for a Brownian motion, if we change the medium to be inhomogeneous by a measure \( \mu \), then the new motion (time changed process) will diffuse according to a different metric \( D(\cdot, \cdot) \). In [22], Kigami initiated a general scheme to construct such metrics through some self-similar weight functions \( g \) on the symbolic space.

In order to provide concrete models to Kigami’s theoretical construction, in this paper, we give a thorough study of his metric on two classes of fractals of primary importance: the nested fractals and the generalized Sierpinski carpets; we assume further that the weight functions \( g := g_a \) are generated by “symmetric” weights \( a \). Let \( M \) be the domain of \( a \) such that \( D_a \) defines a metric, and let \( S \) be the boundary of \( M \). One of our main results is that the metrics from \( g_a \) satisfy the metric chain condition (MCC) if and only if \( a \in S \). To determine \( M \) and \( S \), we provide a recursive weight transfer construction on the nested fractals, and a basic symmetric argument on the Sierpinski carpet. As an application, we use the MCC to obtain the lower estimate of the sub-Gaussian heat kernel. This together with the upper estimate in [22] allows us to have a concrete class of metrics for time change, and the two sided sub-Gaussian heat kernel estimate on the fundamental fractals.

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1. Introduction

Metric spaces play a prominent role in various fields in mathematics. The analysis on metric spaces together with measures (metric measure spaces) emerged as an independent research field since the 90’s. The spaces have no a priori smooth structure, but one is able to recover the infinitesimal concepts such as gradient, Laplacian, Dirichlet form, and curvature as in Euclidean function theory, geometric analysis and stochastic analysis [10, 16, 17, 32]. In the analysis on fractals, a wealth of exotic examples and different metrics have emerged due to self-similarity. This also provides a fertile background for the theory of metric measure spaces (see e.g. [20, 23, 13]).

In the study of Brownian motion on the Sierpinski gasket (SG), Barlow and Perkins [8] first established the Li-Yau type sub-Gaussian estimate of the transition density function

\[ p_t(x, y) \approx \frac{1}{V(x, t^{1/\beta})} \exp \left( -c \frac{d^{\beta}(x, y)}{t} \right)^{1/(\beta - 1)} \]  

(1.1)

with \( d(\cdot, \cdot) \) as the Euclidian metric, \( V(x, t^{1/\beta}) = \mu \{ z \in \text{SG} : d(x, z) \leq t^{1/\beta} \} \approx t^{\alpha/\beta} \), where \( \mu \) is the canonical \( \alpha \)-dimensional Hausdorff measure on the SG. The Sierpinski gasket has energy renormalization factor \( \rho = 3/5 \), and walk dimension \( \beta = \log 5/\log 2 \) [8]. This was extended by Lindstrøm [29] by showing that the Brownian motion exists on a class of self-similar sets called nested fractals, and the transition density of the Brownian motion on the nested fractal (with a technical path assumption) was shown to enjoy the two sided sub-Gaussian estimate by Kumagai [26]. A path breaking extension was proved on the Sierpinski carpet (SC) by Barlow and Bass in their seminal papers [2, 4], with \( \beta = \log 8 / \log 3 \) (only approximate value of \( \rho \) is available).

If we change the medium to be inhomogeneous by a measure \( \mu \), then the new motion will have the same paths, but different rate of diffusion, and is associated with a different metric \( D(x, y) \); we call it a time change of the process. One of the main issues is to maintain the sub-Gaussian estimate (1.1) with the new metric \( D(x, y) \). The time change for self-similar measures \( \mu \) on p.c.f. sets that admit harmonic structures and local regular Dirichlet forms and on the SC were first studied by Barlow and Kumagai [6], and they showed that the time change is possible if \( \rho_i \mu_i < 1 \) for all \( 1 \leq i \leq N \), where \( \mu_i \)'s are the probability weights of \( \mu \).

In [21, 22, 23], Kigami launched a detail study of the time change problem in full generality based on the Dirichlet forms and the resistance metrics. He set up a general scheme to construct new metrics \( D(x, y) \) on fractals. From the point of view of local regular Dirichlet forms and the associated Hunt processes, the metric \( D(x, y) \) is closely connected with the resistance metric \( R(x, y) \) on the Dirichlet space described by the Einstein relation (see [33]).

\[ R(x, y) V_D(x, D(x, y)) \approx D(x, y)^\beta. \]
In this paper, we adopt the same setup as [22, 23] to construct metrics by weight functions on the iterated function system (IFS) of fractals. In Kigami’s study, there were few concrete examples or discussion on the class of admissible weight functions. For this reason, we will restrict our consideration to the two most basic classes of fractals: the nested fractals and generalized Sierpinski carpets (GSC) (see the definition in Section 2). When we do not need to distinguish the two classes, we will just call them elementary fractals for convenience. We will consider the metrics arising from the class of “symmetric self-similar weights” (Definition 1.2). The techniques used throughout the paper depend very strongly on the group of symmetries of the underlying set, which is quite different from the previous investigations. The study leads to new results on the class of admissible metrics for time change, and sharpens the sub-Gaussian heat kernel estimate.

Let \( \{F_i\}_{i=1}^N \) denote the associated IFS of an elementary fractal \( K \). For \( n \geq 1 \), let \( \Sigma^n = \{1, \ldots, N\}^n \) be the collection of words with length \( n \) (by convention, \( \Sigma^0 = \{\emptyset\} \)). For \( w = w_1 \cdots w_n \in \Sigma^n \), we write \( K_w = F_{w_1}(K) \circ \cdots \circ F_{w_n}(K) \), and call it an \( n \)-cell of \( K \). Denote by \( \Sigma^* = \bigcup_{n \geq 0} \Sigma^n \) the collection of all finite words, and by \( |w| \) the length of \( w \) for each \( w \in \Sigma^* \). A finite sequence of words \( (w(1), \ldots, w(m)) \) in \( \Sigma^* \) (or equivalently, cells \( (K_{w(1)}, \ldots, K_{w(m)}) \) in \( K \)) is called a chain if \( K_{w(i)} \cap K_{w(i+1)} \neq \emptyset \) for \( 1 \leq i \leq m-1 \); we use \( |\gamma| = m \) to denote the length of the chain. A chain \( (w(1), \ldots, w(m)) \) is said to connect \( x \) and \( y \) if \( x \in K_{w(1)} \) and \( y \in K_{w(m)} \). A chain is called simple if \( K_{w(i)} \cap K_{w(j)} \neq \emptyset \) if and only if \( |i - j| \leq 1 \).

**Definition 1.1** ([23, 24]). *We call \( g : \Sigma^* \to (0, 1) \) a weight function if it satisfies:

(i) \( g(\emptyset) = 1 \), \( g(wj) \leq g(w) \) if \( w \in \Sigma^* \) and \( j \in \{1, \ldots, N\} \);

(ii) \( \lim_{n \to \infty} \sup_{w \in \Sigma^*} g(w) = 0 \).

We define the total weight of a chain \( \gamma = (w(1), \ldots, w(m)) \) by \( g(\gamma) = \sum_{i=1}^m g(w(i)) \), and for any \( x, y \in K \),

\[
D_g(x, y) = \inf \{ g(\gamma) : \gamma \text{ is a chain connecting } x \text{ and } y \}. \tag{1.2}
\]

It is easy to see that \( D_g(\cdot, \cdot) \) is finite (\( D_g \leq g(\emptyset) = 1 \)), symmetric, nonnegative, \( D_g(x, x) = 0 \) for all \( x \in K \), and satisfies the triangle inequality. However, in general, it may happen that \( D_g(x, y) = 0 \) for some pairs \( x \neq y \) in \( K \) so that \( D_g \) fails to be a metric.

Let \( G \) be the group of symmetries associated with the elementary fractal \( K \) (see Section 2). We will focus on the class of weight functions as following.

**Definition 1.2.** *We call \( g : \Sigma^* \to (0, 1) \) a symmetric self-similar weight function if \( g \) satisfies the following two conditions:

(i). (Self-similarity) \( g(w) = \prod_{i=1}^m g(w_i) \) for \( w = w_1w_2 \cdots w_m \in \Sigma^* \).

(ii). (Symmetry) For all \( \sigma \in G \), \( g \circ \sigma = g \).

We remark that in the above definition (ii), for \( \sigma \in G \), \( \sigma \) acts on the cells \( K_w \). Since the cells and the finite words in \( \Sigma^* \) are in 1–1 corresponding, we can define the procedure of \( \sigma \) on \( \Sigma^* \). For any \( i, j \in \Sigma \), define \( i \sim_G j \) if there is a \( \sigma \in G \) such
that \( K_j = \sigma(K_i) \). Let \( \Sigma \sim G \) denote the equivalent classes and \( k = \#\Sigma \sim G \). For example, the Sierpinski gasket and the pentagasket have \( k = 1 \); the more interesting cases are the Lindstrøm snowflake and the standard Sierpinski carpet with \( k = 2 \) (Sections 5).

First by self-similarity and reflecting the cells along hyperplanes of symmetry, we prove an interesting dichotomic result.

**Theorem 1.3.** Let \( K \) be an elementary fractal, and let \( g \) be a symmetric self-similar weight function. Then \( D_g(\cdot, \cdot) \) is either a metric or identically equal to 0.

Let \( a := (a_1, a_2, \ldots, a_k) \in (0, 1)^k \) be the associated weights of \( \{g(i) : 1 \leq i \leq N\} \). We write \( g = g_a \) for the weight function associated with respect to \( a \). We define

\[ M := \{a \in (0, 1)^k : D_{g_a} \text{ is a metric on } K\}, \]

and call it the set of admissible weights, and \( D_{g_a} \) an admissible metric (for time change). We have (Propositions 2.7 and 3.5).

**Proposition 1.4.** Let \( K \) be an elementary fractal, and let \( S = \partial M \cap (0, 1)^k \) be the boundary of \( M \). Then \( M \) is closed, and \( S \) separates \((0, 1)^k\) into two connected components \( M \) and \( M' \), with \( S \subset M \).

There is an expression for \( a \in M \), which is convenient to use in the sequel (Theorem 1.8). For \( a \in M \subset (0, 1)^k \) and \( \lambda \in (0, \infty) \), consider \( a(\lambda) = (a_1^\lambda, \ldots, a_k^\lambda) \), then \( a(1) = a \), and \( \lim_{\lambda \to 0} a(\lambda) = (1, \ldots, 1) \), \( \lim_{\lambda \to \infty} a(\lambda) = (0, \ldots, 0) \). We show that \( a(\lambda) \in M \) for \( \lambda \) small, and \( a(\lambda) \in M' \) if \( \lambda \) is large (see Section 2 and Figure 1). Hence there is a unique \( \lambda_0 \) such that \( a(\lambda_0) \in S \).

Recall that the main purpose to study the admissible metrics \( D_g \) is to obtain a two-sided sub-Gaussian heat kernel estimate (1.1) with respect to \( D_g \). For the off-diagonal lower estimate in the sub-Gaussian heat kernel, one requires the metric to satisfy the **metric chain condition** (see e.g. [11, 15]). We also remark that the two-sided sub-Gaussian heat kernel estimate does imply the metric chain condition (see [30, Corollary 1.8]).

**Definition 1.5.** A metric space \((M, d)\) is said to satisfy the **metric chain condition** (MCC) if there exists a constant \( C > 0 \) such that for any two points \( x, y \in M \) and for any positive integer \( n \), there exists a sequence \( \{x_i\}_{i=0}^n \) of points in \( M \) such that \( x_0 = x, x_n = y \) and

\[ d(x_i, x_{i+1}) \leq C \frac{d(x, y)}{n}, \text{ for all } i = 0, 1, \ldots, n - 1. \]

The MCC plays an important role in the lower estimate of the sub-Gaussian heat kernel (see e.g. [12]). The following theorem is one of our main results (Lemma 4.2, Theorem 4.3).

**Theorem 1.6.** Let \( K \) be an elementary fractal, then an admissible metric \( D_{g_a} \) satisfies the MCC if and only if \( a \in S \).
It is a challenging task to identify the set of admissible weights $M$, and there is no results or non-trivial examples in literature. Our next goal is to give a detail study of this for the elementary fractals. For nested fractals, we use a technique of Kumagai [26] to give a constructive algorithm: for each $a \in M$, there is a recursive relation on the weights of the paths on each level. This allows us to formulate a finite family of “weight transfer matrices” $K(a)$. Let $\lambda_A$ be the maximal positive eigenvalue of a matrix $A$, we have (Theorem 5.1)

**Theorem 1.7.** For a nested fractal, the set of admissible weights is

$$M = \{a = (a_1, \ldots, a_k) \in (0, 1)^k : \lambda_A \geq 1, \forall A \in K(a)\}$$

and its boundary $S = \{a \in M : \exists A \in K(a) : \lambda_A = 1\}$.

We use the Lindstrøm snowflake (see Section 5 and Figure 3) as an example to illustrate the theorem. For the Sierpinski carpet, it requires a different technique to identify $M$. We will give a detail consideration of this in Section 6.

We then apply the above results to the time change problem. It is well-known that for a nested fractal $K$, if we denote by $\alpha$ the Hausdorff dimension, and let $H^{\alpha}$ be the normalized $\alpha$-dimensional Hausdorff measure, then there exists a local regular Dirichlet form $(E, F)$ on $L^2(K, H^{\alpha})$ satisfying the self-similar energy identity with a uniform renormalization factor $0 < \rho < 1$, that is

$$E(u) = \frac{1}{\rho} \sum_{i=1}^N E(u \circ F_i), \quad \forall u \in F,$$ (1.3)

and the induced process is the Brownian motion. For a GSC in $\mathbb{R}^d$ with $d \leq 2$, we also have $0 < \rho < 1$. It may happen that $\rho \geq 1$ for $d \geq 3$ (see Remarks 5.4). If we let $\mu$ be a self-similar measure on these fractals with weights $\mu_i$, i.e., $\mu = \sum_{i=1}^N \mu_i \circ F_i^{-1}$, then in the case that $\mu_i \rho < 1$ for all $1 \leq i \leq N$, the measure defines a new local regular Dirichlet form $(E, F')$ in $L^2(K, \mu)$ with the same $E$, which also induces a diffusive process [6].

We call a self-similar measure $\mu$ symmetric if $\mu_i = \mu_{\sigma(i)}$ for any $\sigma \in G$ and $i \in \Sigma$. It is known that symmetric self-similar measures are doubling measures under the admissible metrics [22, Theorem 3.4.5] (also see Section 7).

With this setup on elementary fractals, the sub-Gaussian heat kernel estimate of the time change of Brownian motion for symmetric self-similar measures can be stated precisely.

**Theorem 1.8.** Let $K$ be an elementary fractal. Let $\mu$ be a symmetric self-similar measure, and let $a(\lambda)$ be the curve defined by

$$a(\lambda) = (\mu_1 \rho)^{\lambda}, (\mu_2 \rho)^{\lambda}, \ldots, (\mu_k \rho)^{\lambda}) \in M, \quad \lambda \in (0, \infty).$$ (1.4)

Let $\beta = 1/\lambda$, and $D_\beta$ be the admissible metric defined by $a(\lambda)$. Then the time change of Brownian motion with measure $\mu$ has a transition density $p_t(x, y)$ that admits an
upper sub-Gaussian estimate (UE)

\[ p_t(x, y) \leq \frac{C}{V(x, t^{1/\beta})} \exp \left( -c \left( \frac{D_g(x, y)^\beta}{t} \right)^{1/(\beta - 1)} \right), \]

where \( V(x, r) := \mu \{ z : D_g(x, z) < r \} \), and a near diagonal lower estimate (NLE): there exists small \( \eta > 0 \) such that

\[ \frac{C}{V(x, t^{1/\beta})} \leq p_t(x, y), \quad \forall \ x, y \in K, \ t > 0, \ D_g(x, y) < \eta t^{1/\beta}. \]

In particular if \( \lambda = \lambda_0 \) such that \( a(\lambda_0) \in S \), then \( p_t(x, y) \) has the two sided sub-Gaussian estimate as in (1.1).

The upper estimate and the near diagonal lower estimate (NLE) were proved by Kigami [22, Theorem 3.2.3] for weights under the more general situation. He also showed the off-diagonal lower estimate holds for \( x, y \) if there exists a geodesic path between them. Our contribution in Theorem 1.8 is to provide concrete families of admissible metrics with MCC for \( a \in S \) so that the off-diagonal lower estimate can be assured. The proof in [22] (see also [14]) are quite involved and lengthy, therefore, we give an outline of their proofs incorporating with our setup. We make use of the fact that these admissible metrics are quasisymmetric to the resistance metric for nested fractal, and to the Euclidean metric for GSC [23, 24], and the classical techniques of capacity estimate and Harnack inequality in [14].

We remark that there are few examples on non-symmetric weight functions in connection to those considered in [22]. We also remark that there is another setup to construct new metrics on self-similar sets which is quite different from the present one; the construction is based on certain “augmented trees”, i.e., by adding new edges to the neighboring cells of the trees of the symbolic spaces; the geodesic of these trees are defined by the graph distance. They are hyperbolic graphs (in the sense of Gromov), and there are systematic treatments for such graphs [18, 19, 27, 28]. Also in regard to the quasisymmetry of the SC, Bonk and Merenkov in [9] gave an interesting classification of quasisymmetric self-homeomorphisms for the standard \( 1/3 \)-SC and the \( 1/\ell \)-SC.

We organize the paper as follows. In Section 2 we provide some basic definitions and preliminaries of the elementary fractals and the symmetric weight functions. We prove some basic facts of \( M, M^c \) and the boundary \( S = \partial M \cap (0, 1)^k \). In Section 3 we verify Theorem 1.3 for \( D_g \) to be identically zero in \( M^c \). The main theorem (Theorem 1.6) on the MCC is proved in Section 4. In Section 5 we prove Theorem 1.7 and use it on the Lindström snowflake for illustration. In Section 6 we study \( M \) of the Sierpinski carpet in detail. Finally in Section 7 we combine Theorem 1.6 together with some previous known results to obtain the heat kernel estimates of the time change Brownian motion for symmetric self-similar measures (Theorem 1.8).
2. Preliminaries and admissible metrics $D_g(\cdot, \cdot)$

First we define the class of nested fractals introduced by Lindstrøm [29]. Let $K$ be the self-similar set defined by an iterated function system (IFS) $\{F_i\}_{i=1}^N$ of the form $F_i(x) = g O_i x + b_i$, where $N \geq 2$, $0 < q < 1$, and for each $1 \leq i \leq N$, $O_i$ is a $d \times d$ orthogonal matrix and $b_i \in \mathbb{R}^d$. Let $P$ be the set of all fixed points of $\{F_i\}_{i=1}^N$. Call $p \in P$ an essential fixed point if there exist distinct $i, j \in \{1, \ldots, N\}$, and $q \in P$ such that $F_i(p) = F_j(q)$, and denote this set by $P_0$. For any distinct points $x, y \in \mathbb{R}^d$, denote the bisecting hyperplane $H_{x,y} = \{z \in \mathbb{R}^d : |x - z| = |y - z|\}$ and write $R_{x,y}$ the orthogonal reflection with respect to $H_{x,y}$; let $G$ denote the group of reflections for $x, y \in P_0$.

**Definition 2.1 (nested fractals).** Let $\{F_i\}_{i=1}^N$ and $K$ be as the above. We call $K$ a nested fractal if it satisfies the following conditions:

- (OSC) $\{F_i\}_{i=1}^N$ satisfies the open set condition;
- (Connectivity) $K$ is connected;
- (Symmetry) $K$ is invariant under $G$;
- (Nesting) For any $i, j \in \{1, \ldots, N\}$ with $i \neq j$, $F_i(K) \cap F_j(K) = F_i(P_0) \cap F_j(P_0)$.

Next we define another class of self-similar sets which are infinitely ramified, called generalized Sierpinski carpets (GSC), named and first studied by Barlow and Bass [2, 4]. Let $d \geq 2$, $\ell \geq 3$ be integers, and $H_0 = [0,1]^d$. Set $Q$ to be the mesh of closed subcubes of size $1/\ell$. For any $Q \in Q$, let $F_Q : H_0 \to H_0$ be given by $F_Q(x) = x/\ell + p_Q$ where $p_Q$ is chosen so that $F_Q(H_0) = Q$. Let $Q' \subseteq Q$ and let $K := \text{GSC}(d, \ell, Q')$ be the self-similar set associated with the iterated function system $\{F_Q\}_{Q \in Q'}$. We renumber the elements in $\{F_Q\}_{Q \in Q'}$ by $\{F_i\}_{i=1}^N$ with $N = \#Q'$. Set $H_1 = \bigcup_{Q \in Q'} F_Q(H_0)$. Let $G$ denote the group of isometries on $H_0$.

**Definition 2.2 (generalized Sierpinski carpets).** A set $K = \text{GSC}(d, \ell, Q')$ is called a generalized Sierpinski carpet (GSC) if the following conditions are satisfied:

- (Symmetry) $H_1$ is invariant under $G$;
- (Connectivity) $H_1$ is connected;
- (Non-diagonality) For any $x \in H_1$, there exists $r_0 > 0$, such that for all $0 < r < r_0$, $\text{int}(H_1 \cap B(x, r))$ is connected;
- (Borders included) The line segment $[0,1] \times \{0\} \times \cdots \times \{0\}$ is contained in $H_1$.

Throughout the paper we always assume that $K$ is either a nested fractal or a GSC. When we do not need to distinguish them, we will just call them elementary fractals for convenience.

A weight function $g$ will be assumed to be self-similar and symmetric as in Definition 1.2. For the weight function $a_i = g(i)$, $i \in \Sigma = \{1, 2, \ldots, N\}$, by taking quotient of symmetries, we consider $a \in (0,1)^k$ where $k$ is the number of elements in the quotient space $\Sigma^{\sim \sigma}$. For a chain $\gamma = (w(1), \ldots, w(m))$, the weight of a chain $\gamma = (w(1), \ldots, w(m))$ is defined by

$$g(\gamma) = \sum_{i=1}^m g(w(i)) \quad \text{with} \quad g(w(i)) = \prod_{j=1}^\sigma g(i_j), \quad w(i) = i_1 \cdots i_\sigma.$$
For the above chain, we also define the union of the cells in $\gamma$ by $\cup \gamma = \bigcup_{i=1}^{m} K_{w(i)}$, and call $(w(i), \ldots, w(j))$ a sub-chain of $\gamma$ for any $1 \leq i \leq j \leq m$.

For any two words $w$ and $v$, if $K$ is a nested fractal, we use $K_w \sim K_v$ to denote $K_w \cap K_v \neq \emptyset$; if $K$ is a GSC in $\mathbb{R}^d$, we use $K_w \sim K_v$ to mean $\dim(F_w(H_0) \cap F_v(H_0)) \geq d - 1$, i.e., either $F_w(H_0) \cap F_v(H_0)$ is a $(d - 1)$-dimensional face or the two sets $F_w(H_0)$ and $F_v(H_0)$ are such that one is contained in the other.

**Lemma 2.3.** Let $K$ be an elementary fractal and let $g$ be defined by $a \in (0, 1)^k$. Suppose $K_{iw} \sim K_{jv}$ for some $i \neq j \in \Sigma$, and $w, v \in \Sigma^*$ with $|w| = |v| \geq 1$. Then $\sigma(K_w) = K_v$ for some $\sigma \in G$, and $g(iw) = \frac{a_i}{a_j} g(jv)$.

**Proof.** It is known that on a nested fractal, each element in $P_0$ belongs to exactly one $n$-cell for each $n \geq 1$. By applying this property to $K_i$ (or $K_j$), we see that $K_{iw} \cap K_{jv}$ is a singleton, denoted by $\{p\}$. Then there exist $p_1, p_2 \in P_0$ such that $F_i(p_1) = F_j(p_2) = p$. Let $\sigma \in G$ be the orthogonal reflection with respect to $H_{p_1, p_2}$. Then $\sigma(K_w) = K_v$ (Lindstrøm [29, IV.13 Proposition]). As a result, each $n$-cell contains at most one element of $P_0$ for each $n$. By applying this property to $K_i$ and $K_j$, we see that $K_{iw} \cap K_{jv}$ is a singleton, denoted by $\{p\}$. Then there exist $p_1, p_2 \in P_0$ such that $F_i(p_1) = F_j(p_2) = p$. Let $\sigma \in G$ be the orthogonal reflection with respect to $H_{p_1, p_2}$. Then $\sigma(K_w) = K_v$ (Lindstrøm [29, IV.13 Proposition]).

For the GSC, $K_{iw} \cap K_{jv}$ is a $(d - 1)$-dimensional face. Hence $K_w$ and $K_v$ are in the opposite face of $H_0$, and $\sigma(K_w) = K_v$ for a reflection on $H_0$.

The second part follows from $g(iw) = g(i)g(w)$ and $g(jv) = g(j)g(v) = g(j)g(w)$. □

For $a \in (0, 1)^k$, denote $a_* = \min\{a_i : 1 \leq i \leq k\}$, $a^* = \max\{a_i : 1 \leq i \leq k\}$. By Lemma 2.3, we obtain the following simple property which will be used frequently.

**Proposition 2.4.** Let $K$ be an elementary fractal, let $g := g_a$, $a \in (0, 1)^k$ and let $c = a^*/a_*$. Suppose $K_w \sim K_v$ with $|w| = |v|$. Then we have

$$c^{-1} g(v) \leq g(w) \leq cg(v).$$

Furthermore, if we reflect a chain $\gamma$ contained in $K_v$ along the appropriate hyperplane of $K_w \sim K_v$, and denote it by $R(\gamma)$, then $R(\gamma)$ is a chain contained in $K_w$ and $c^{-1} g(\gamma) \leq g(R(\gamma)) \leq cg(\gamma)$.

Let $\Gamma$ denote the class of chains $\gamma = (w(1), \ldots, w(m))$ satisfying $K_{w(i)} \sim K_{w(i+1)}$ for all $i$. Similar to $D_g$ in (1.2), define

$$D'_g(x, y) = \inf\{g(\gamma) : \gamma \in \Gamma \text{ connects } x \text{ and } y\}.$$ 

Clearly for the nested fractals, $D'_g$ is just the same as the $D_g$.

**Corollary 2.5.** For the GSC, we have $D'_g(\gamma, \cdot) \equiv D_g(\gamma, \cdot)$.

**Proof.** By the non-diagonality assumption in the definition of GSC, we see that if $K_w \cap K_v \neq \emptyset$ with $|w| = n$, then there exists a chain $\gamma = \{w_1, \ldots, w_m\}$ of $n$-cells such that $K_{w_1} \sim K_{w_{i+1}}$, $K_{w_1} = K_w$, $K_{w_m} \sim K_v$, and $m \leq 2^d - 1$. By Proposition 2.4

$$c^{-(2^d-2)} g(w_i) \leq c^{-1} g(w_i) \leq g(w), \quad 1 \leq i \leq m.$$
Hence we can replace the defining chains in $D_g(x, y)$ by the chains in $D'_g(x, y)$ and keep the above inequality. This yields $2^{-d}c^{-(2d-2)}D'_g(x, y) \leq D_g(x, y) \leq D'_g(x, y)$ for all $x, y \in K$. 

\[ \square \]

**Remark.** For the GSC, the chains in $\Gamma$ with $K_w \sim K_v$ are more convenient to use than $K_w \cap K_v \neq \emptyset$. We will use it without explicitly mentioning.

Denote by $\mathcal{M} = \{a \in (0, 1)^k : D_{g_a} \text{ is a metric on } K\}$, and let $\mathcal{M}^c = (0, 1)^k \setminus \mathcal{M}$. We call $\mathcal{M}$ the set of admissible weights, and $D_{g_a}$ the admissible metric (determined by $a$). If no confusion, we also say that $g$ is a symmetric self-similar weight function to mean $g = g_a$ for some $a \in (0, 1)^k$.

**Lemma 2.6.** Suppose $a, b \in (0, 1)^k$ and $b \geq a$ (coordinatewise). Then (i) $a \in \mathcal{M}$ implies $b \in \mathcal{M}$; (ii) $b \in \mathcal{M}^c$ implies $a \in \mathcal{M}^c$.

**Proof.** It suffices to show that (ii) holds. Suppose that $b \in \mathcal{M}^c$. By definition, there exist two distinct points $x, y \in K$ and a sequence of chains $\{\gamma_n\}_n$ between $x$ and $y$ such that $g_b(\gamma_n) \to 0$ as $n \to \infty$. By assumption we have $g_a(\gamma_n) \leq g_b(\gamma_n)$, hence $g_a(\gamma_n) \to 0$ as $n \to \infty$, which implies that $a \in \mathcal{M}^c$. 

The following is a crude estimation of $\mathcal{M}$ and $\mathcal{M}^c$.

**Proposition 2.7.** (i) For a nested fractal $K$, there exist $0 < c \leq C < 1$ such that $[C, 1)^k \subset \mathcal{M}$ and $(0, c)^k \subset \mathcal{M}^c$; (ii) For a GSC, we have $[1/\ell, 1)^k \subset \mathcal{M}$ and $(0, 1/\ell)^k \subset \mathcal{M}^c$.

**Proof.** (i) For two distinct $p, q \in P_0$, let $n(p, q)$ be the minimal length of the chains consisting of 1-cells between $p$ and $q$. Let $n_s, n^*$ be the minimum and maximum of $n(p, q)$ among all the pairs $p, q \in P_0$, respectively. As each $p \in P_0$ is contained in exactly one 1-cell (Lindstrøm [29]), therefore, we have $n^* \geq n_s \geq 2$.

Let $g$ be the weight function generated by $a = (1/n_s, \ldots, 1/n_s)$. We show that $a \in \mathcal{M}$, then by Lemma 2.6 $[1/n_s, 1)^k \subset \mathcal{M}$. The first part of (i) follows by letting $C = 1/n_s$.

For this, we let $p$ and $q$ be two distinct points in $P_0$. For any given simple chain $\gamma$ between $p$ and $q$ (i.e., $K_{w(i)} \cap K_{w(j)} \neq \emptyset$ if and only if $|i - j| \leq 1$), choose a cell $K_w$ in $\gamma$ such that $w$ has the largest word length in $\gamma$. Let $w'$ be the parent of $w$, and $J(w')$ be the set of the $N$ children of $w'$. Then by using the definition of $n_s$ on $K_w$, there exist at least $n_s$ cells in $J(w')$ (including $w$) contained in $\gamma$. Observe that $g(u) = n_s^{-1}g(w')$ for any $u \in J(w')$, we have
\[
g(w') \leq \sum_{u \in J(w') \cap \gamma} g(u).
\]

Hence if we replace the sub-chain of $\gamma$ in $J(w')$ by $w'$, then we get a new chain $\gamma_s$ with $g(\gamma) \geq g(\gamma_s)$. We can also assume that $\gamma_s$ is a simple chain by removing some cells in $\gamma_s$. By repeating this “merging” procedure to each cell with largest word length in the new chains, we finally get the trivial chain $\{0\}$. Thus $g(\gamma) \geq g(\{0\}) = 1$ and $D_g(p, q) \geq 1$. For arbitrary distinct two points $p, q \in K$, we can use a similar argument to show that $D_g(p, q) > 0$. Hence $(1/n_s, \ldots, 1/n_s) \in \mathcal{M}$. 


Next we show that \((0, 1/n^*)^k \subset M^c\). Let \(b \in (0, 1/n^*)^k\) and fix any two distinct points \(p\) and \(q\) in \(P_0\). For any \(m \geq 0\), choose a chain \(\gamma_m\) of \(m\)-cells between \(p\) and \(q\), where the length of \(\gamma_m\) is not larger than \(n^m\). Then \(D_g(p, q) \leq (b^*n^*)^m\) \((b^* = \max_i |b_i|)\). By letting \(m \to \infty\), we have \(D_g(p, q) = 0\) since \(b^* < 1/n^*\). Hence we have \((0, 1/n^*)^k \subset M^c\) and the second part of (i) follows by letting \(c = 1/n^*\).

(ii) Let \(g\) be the weight function generated by \(a = (1/\ell, \ldots, 1/\ell)\). Let \(p, q\) be two vertices of \(H_0\) such that the line segment \(\overline{pq}\) is parallel to an axis. Consider any chain \(\gamma\) between \(p\) and \(q\). Since any cell with word length \(m \geq 0\) has weight \(\ell^{-m}\), the projection of \(\gamma\) on the side \(\overline{pq}\) covers \(\overline{pq}\) so that \(g(\gamma) \geq 1\).

Now let \(x, y \in V_e = \bigcup_{w \in \Sigma^*} F_w(V_0)\) where \(V_0\) is the set of vertices of \(H_0\). Then \(x, y\) can be connected by finitely many line segments parallel to the axes. By self-similarity and the above, we have \(D_{g_a}(x, y) \geq |x-y|\). The density of \(V_e\) in \(K\) implies that \(D_{g_a}(x, y) > 0\) for all distinct \(x, y \in K\). By Lemma 2.6, \([1/\ell, 1/\ell, 1]^k \subset M\).

To prove the last part, let \(b = (b_1, \ldots, b_k)\) such that \(b^* < 1/\ell\). Let \(p\) and \(q\) be two end points on a side of the cube \(H_0\). Consider a simple chain \(\gamma_m\) with word length \(m\) connecting \(p\) and \(q\) and along the edge of the GSC. Then we have \(g(\gamma_m) \leq \ell^m(b^*)^m \to 0\) as \(m \to \infty\). Therefore \(D_g(p, q) = 0\) and \(b \notin M\). It follows that \((0, 1/\ell)^k \subset M^c\).

Let \(g := g_a\) be the weight function on \(K\) with \(a = (a_1, \ldots, a_k) \in (0, 1)^k\). Define a curve \(a(\lambda) : (0, \infty) \to (0, 1)^k\) by
\[
a(\lambda) = (a_1^\lambda, \ldots, a_k^\lambda), \quad \lambda \in (0, \infty).
\]
Clearly \(a(1) = a\). By Proposition 2.7, \(a(\lambda) \in M\) for \(\lambda\) small enough, and \(a(\lambda) \in M^c\) for \(\lambda\) large enough. Since \(a(\lambda_1) > a(\lambda_2)\) (coordinatewise) for \(\lambda_1 < \lambda_2\), by Lemma 2.6, \(a(\lambda_2) \in M\) implies \(a(\lambda_1) \in M\). This yields a unique boundary point \(\lambda_a > 0\) such that \(a(\lambda) \in M\) if \(\lambda > \lambda_a\), and \(a(\lambda) \in M^c\) if \(\lambda < \lambda_a\) (see Figure 1). Denote by \(\Lambda_k = \{a \in (0, 1)^k : \sum_{i=1}^k a_i = 1\}\) the set of normalized vectors of \((0, 1)^k\). Let
\[
S = \{a(\lambda_a) : a \in \Lambda_k\}.
\]
It follows that \(S\) separates \((0, 1)^k\) into two connected components \(M\) and \(M^c\). We call \(S\) the boundary surface of \(M\).

**Figure 1.** \(M, M^c\) and \(S\).
Corollary 2.8. With the above notations, let $\partial M$ denote the boundary of $M$ in $(0, 1)^k$. Then $S = \partial M$.

Proof: It is clear that $S \subseteq \partial M$. To prove the reverse inclusion, let $b \in \partial M$. Suppose $b \notin S$. Then there is an $a \in \Lambda_k$ such that $b = a(\lambda)$, and $a(\lambda_1) \in S$ and $\lambda \neq \lambda_1$. If $\lambda < \lambda_1$, let $\lambda_1$ be such that $\lambda < \lambda_1 < \lambda_\alpha$, then $a(\lambda_1) \in M$. By Lemma 2.6(i), we see that $M$ contains all $b' > a(\lambda_1)$, which is a subset of $M^\circ$, the interior of $M$. Hence $b = a(\lambda) \in M^\circ$. This contradicts that $b \in \partial M$. If $\lambda_\alpha < \lambda$, then we can use a similar method (apply Lemma 2.6(ii)) to obtain a similar contradiction. □

3. A dichotomy for $D_g(\cdot, \cdot)$

Note that in general, for any weight function $g$, either $D_g$ is a metric or there exists a pair $x \neq y$ in $K$ such that $D_g(x, y) = 0$. In this section we prove a stronger conclusion for symmetric self-similar weights on the elementary fractals.

Theorem 3.1. Let $K$ be an elementary fractal, and let $g$ be a symmetric self-similar weight function. Then $D_g$ is either a metric or identically 0 on $K$.

Equivalently, the theorem says that $D_g \equiv 0$ for $a \in M^\circ$. Since the proof involves different symmetries in the nested fractals and the GSC, we divide the proofs into two separate parts.

Lemma 3.2. Let $K$ be a nested fractal, and let $P_0$ be the set of essential fixed points. Suppose $D_g(q^*, s^*) = 0$ for some distinct $q^*, s^* \in P_0$, then $D_g(q, s) = 0$ for all $q, s \in P_0$.

Proof: We define an equivalence relation in $P_0$ as follows: for any two points $q$ and $s$ in $P_0$, we write $q \sim s$ if either $q = s$ or there is a finite sequence of points $\{q_i\}_{i=0}^l$ in $P_0$ with $q_0 = q$ and $q_l = s$, and for each $0 \leq i \leq l - 1$, there is a $\sigma_i \in G$ satisfying $\sigma_i(q_i) = q_{i+1}$ and $\sigma_i(q_{i+1}) = s$. It is easy to check that “$\sim$” is indeed an equivalence relation on $P_0$, which is invariant under $G$, that is, for any points $q, s \in P_0, \sigma \in G$, if $q \sim s$, then $\sigma(q) \sim \sigma(s)$. Obviously, by using the triangle inequality of $D_g$, we have $D_g(q, s) = 0$ for any $q \sim s$.

If $q, s \in P_0$ are two distinct points, let $R_{q,s}$ be the orthogonal reflection along $H_{q,s} = \{z \in \mathbb{R}^d : |q - z| = |s - z|\}$. Let $H_q, H_s$ be the closed half-space containing $q, s$ respectively. Then by Sabot [31, Lemma 6.4], for any $G$-invariant equivalent relation on $P_0$, any equivalent class intersects both half-spaces. Hence there is a point $q'$ in $P_0 \cap H_s$ with $q \sim q'$, where “$\sim$” is the relation defined as above. Therefore we have $D_g(q, q') = 0$, and there is a sequence of chains $\{\eta_m\}_{m \geq 0}$ between $q$ and $q'$ with the total weight $g(\eta_m)$ tending to 0 as $m \to \infty$.

For $m \geq 0$, let $\eta_m$ be the chain obtained by reflecting the part of the cells in $\eta_m$ contained in $H_q$ to $H_s$ by $R_{q,s}$. Then $\eta_m$ connects $q'$ and $s$. By the symmetry of the weight function $g$, we have $g(\eta_m) = g(\eta_m) \to 0$ as $m \to \infty$, which yields $D_g(q', s) = 0$. Then $D_g(q, s) \leq D_g(q, q') + D_g(q', s) = 0$ and the lemma holds. □
Proof of Theorem 3.1 for nested fractals. Let \( x_0, y_0 \) be two distinct points in \( K \) with \( D_g(x_0, y_0) = 0 \). Without loss of generality, assume \( K \) is the smallest subcell which contains both \( x_0 \) and \( y_0 \). Then there exist distinct \( i, j \in \{1, 2, \ldots, N\} \) such that \( x_0 \in K_i \) and \( y_0 \in K_j \). Let \( E_{y_0} = \bigcup_{u \in \Sigma \cap y_0 \in K_u} F_u(P_0) \). Then \( E_{y_0} \) is a finite set and \( x_0 \not\in E_{y_0} \).

Let \( K_v, v \in \Sigma^* \), be a cell containing \( x_0 \) with word length large enough such that \( K_v \cap E_{y_0} = \emptyset \). Let \( n = |v| \) and \( m_0 \in \mathbb{Z}^+ \) satisfying \( m_0^{-1} < a_0^n \). From \( D_g(x_0, y_0) = 0 \), for each positive integer \( m \geq m_0 \), there exists a chain \( \gamma_m \) between \( x_0, y_0 \) such that \( g(\gamma_m) < m^{-1} \). As \( m^{-1} < a_0^n \), every cell in \( \gamma_m \) has length \( > n \). By the nesting property, the cell \( \gamma_m \) must pass through one of the points in \( F_v(P_0) \), and one of the points in \( E_{y_0} \). By the finiteness of both \( \#F_v(P_0) \) and \( \#E_{y_0} \), there is a subsequence of \( \{\gamma_m\} \), still denoted by \( \{\gamma_m\} \), and \( q^* \in F_v(P_0), \tilde{s} \in E_{y_0} \), such that each \( \gamma_m \) passes through both \( q^* \) and \( \tilde{s} \). Let \( \{\gamma_m\} \) be sub-chains of \( \{\gamma_m\} \) connecting \( q^* \) and \( \tilde{s} \).

\[
E_{q^*} = \bigcup_{w \in \Sigma^*: q^* \in K_w} F_w(P_0) \setminus \{q^*\}.
\]

By taking subsequence and sub-chains again, we can find a point \( s^* \in E_{q^*} \), a word \( w \in \Sigma^* \), and a sub-chain \( \gamma'_m \) of \( \gamma_m \) between \( q^* \) and \( s^* \), contained in \( K_w \). Since our choices of sub-chains \( \gamma'_m \) and \( \gamma'_m \) of \( \gamma_m \) do not increase the total weight, we have \( D_g(q^*, s^*) = 0 \).

Using self-similarity, we can dilate \( q^* \), \( s^* \) to be two distinct points in \( P_0 \). By Lemma 3.2, we have \( D_g(q, s) = 0 \) for all \( q, s \in P_0 \). Hence by self-similarity, \( D_g(q, s) = 0 \) for all \( q, s \in F_u(P_0), u \in \Sigma^* \). In general, for any two points \( q, s \) in \( K \), we can use the approximation by points in \( V_s = \bigcup_{u \in \Sigma} F_u(P_0) \) to show that \( D_g(q, s) = 0 \). This completes the proof.

We then turn to the GSC. For any point \( p \in \mathbb{R}^d \), we denote by \( x_i(p) \) the \( i \)-th coordinate of \( p \) with \( 1 \leq i \leq d \). For any subset \( E \) of \( \mathbb{R}^d \), we denote by \( \pi_i(E) \) the orthogonal projection of \( E \) onto the \( i \)-th axis, i.e. \( \pi_i(E) = \{x_i(p) : p \in E\} \). We will use \( K_{w(i)} \sim K_{w(i+1)} \) for the connection of a chain (see Corollary 2.5). Similar to Lemma 3.2, we have the following for the GSC.

Lemma 3.3. Let \( K \) be a GSC. Suppose \( q^* \) and \( s^* \) are on the two opposite faces of the cube \( H_0 = [0, 1]^d \) and \( D_g(q^*, s^*) = 0 \), then \( D_g(q, s) = 0 \) for all \( q, s \) in the vertices of \( H_0 \) with \( \vec{qs} \) parallel to one of the coordinate axes.

Proof: We assume that \( x_1(q^*) = 0, x_1(s^*) = 1 \). Define \( q', s' \) by changing the first coordinates of \( q^* \), \( s^* \): \( x_i(q') = 1, x_i(s') = 0 \) and \( x_i(q^*) = x_i(q'), x_i(s') = x_i(s^*) \) for \( 2 \leq i \leq d \). By symmetry, \( D_g(q', s') = 0 \).

For \( 0 \leq j \leq \ell \), let \( p_j \) be the point in \( K \) with coordinates \( x_1(p_j) = j/\ell \), and \( x_i(p_j) = 0 \) for \( 2 \leq i \leq d \). For \( 1 \leq j \leq \ell \), let \( F_j : H_0 \rightarrow H_0 \) be such that \( F_j(x) = x/\ell + p_{j-1} \). From the borders included condition of GSC in Definition 2.2, \( \{F_j\}_{1 \leq j \leq \ell} \) is a subset of \( \{F_0\}_{0 \in \Sigma} \) such that each cube \( F_j(H_0) \) locates along the line segment \( \overrightarrow{p_{j-1}p_j} \). It follows that \( D_g(F_j(q^*), F_j(s^*)) = D_g(F_j(q'), F_j(s')) = 0 \) for \( 1 \leq j \leq \ell \).
Notice that \( F_j(s^*) = F_{j+1}(s') \) and \( F_j(q') = F_{j+1}(q') \) for \( 1 \leq j < \ell \). If \( \ell \) is an odd number, we have
\[
D_g(F_1(q^*), F_\ell(s^*)) \\
\leq D_g(F_1(q^*), F_1(s^*)) + D_g(F_1(s^*), F_2(q')) + \cdots + D_g(F_{\ell-1}(q'), F_\ell(s^*)) \\
= D_g(F_1(q^*), F_1(s^*)) + D_g(F_2(s'), F_2(q')) + \cdots + D_g(F_\ell(q^*), F_\ell(s^*)) = 0.
\]
By using this repeatedly, we see that \( D_g(F_1(q^*), F_n(s^*)) = 0 \) for all integers \( n \geq 0 \). Without loss of generality, we assume that \( x_1(q_0) > x_1(q_0) + 2 \cdot \ell^{-n} \). Define \( \alpha_n = [\ell^n x_1(q_0)] \cdot \ell^{-n} \), where \( \lceil \cdot \rceil \) is the minimal integer no smaller than \( t \).

Let \( m_0 \in \mathbb{Z}^+ \) satisfy \( m_0^{-1} < a^n \). From \( D_g(q_0, s_0) = 0 \), then for \( m \geq m_0 \), there exists a chain \( \gamma_m \) between \( q_0 \) and \( s_0 \) such that \( g(\gamma_m) < m^{-1} \). From \( m^{-1} < a^n \), every cell in \( \gamma_m \) has length greater than \( n \). Therefore, we can pick two points \( q_m \) and \( s_m \) in \( K \), and a sub-chain \( \gamma_m \) of \( \gamma_m \) between \( q_m \) and \( s_m \), such that \( x_1(q_m) = \alpha_m \), \( x_1(s_m) = \alpha_n + \ell^{-n} \), and \( \pi_1(\gamma_m) \cup \gamma_m = [\alpha_n, \alpha_n + \ell^{-n}] \). By taking subsequence, we can find two \( \alpha \)-cells \( K_m \) and \( K_n \) (independent of \( m \)) such that \( q_m \in K_m \) and \( s_m \in K_n \), and \( \pi_1(K_m) = \pi_1(K_n) = [\alpha_n, \alpha_n + \ell^{-n}] \).

For each \( m \geq m_0 \), we replace each cell \( K_m \) in \( \gamma_m \) by \( K_m_u \) and delete the repeated ones to obtain a chain \( \eta_m \) consisting of \( \alpha \)-cells. Then obviously, the chain \( \eta_m \) starts from \( K_m \) and ends with \( K_n \), and \( \pi_1(\eta_m) = [\alpha_n, \alpha_n + \ell^{-n}] \). Also for every two successive cells in \( \eta_m \), they share a same \( (d - 1) \)-dimensional hyperplane which is always parallel to the 1-st coordinate axis. Reflecting the chain \( \gamma_m \) according to these \( (d - 1) \)-dimensional hyperplanes along \( \eta_m \) from \( K_m \) to \( K_n \) successively, we obtain a new chain \( \gamma_m' \) contained in \( K_n \) (see Figure 2). Note that there is a point \( q_m' \) in \( K_n \) with \( x_1(q_m') = \alpha_n \) such that \( \gamma_m' \) is between \( q_m' \) and \( s_m \). By Proposition 2.4,
\[
g(\gamma_m') \leq c^{(d-1)m} g(\gamma_m) \leq c^{(d-1)m} g(\gamma_m) < c^{(d-1)m} m^{-1}, \quad \text{where} \ c = a^n/\alpha_n.
\]
By taking subsequence if necessary, we may assume that \( q_m' \) converges to \( q^* \), and \( s_m \) converges to \( s^* \). Then \( q^*, s^* \in K \) with \( x_1(q^*) = \alpha_n, x_1(s^*) = \alpha_n + \ell^{-n} \), and \( D_g(q^*, s^*) = 0 \).

Using self-similarity, we can dilate \( q^*, s^* \) to the two opposite faces of \( H_0 \). Then by Lemma 3.3, \( D_g(q, s) = 0 \) for all \( q, s \) in the vertices of \( H_0 \) with \( \overline{q s} \) parallel to one of the coordinate axes. Since any two points in \( V_+ \) can be connected by finitely many pairs of vertices of \( F_\infty(H_0) \), by using the self-similarity and the triangle inequality of \( D_g \), we must have that \( D_g(q, s) = 0 \) for any \( q, s \in V_+ \). By using the continuity of \( D_g \) w.r.t. the Euclidean metric, we have that \( D_g(q, s) = 0 \) for all \( q, s \in K \). This completes the proof.

We will call a procedure on a chain \( \gamma \) splitting if it splits the cells in \( \gamma \) to obtain a finer chain \( \gamma' \) such that \( g(\gamma') \leq g(\gamma) \).
Lemma 3.4. Let $K$ be an elementary fractal. Assume that $D_g$ is not a metric on $K$, then there exists $N_0 > 0$, such that for any two points $x$ and $y$ in $K$, there is a sequence of chains $\{\gamma_n\}_{n=0}^\infty$ between $x$ and $y$, such that each chain is a splitting of the previous one, and

(i) $g(\gamma_n) \leq 1/4^{n+1}$;
(ii) $nN_0 < |u| \leq (n + 1)N_0$ for any $u \in \gamma_n$.

Proof. We first claim that there exists a positive integer $N_0$, such that for any two points $x$ and $y$ in $K$, there is a chain $\gamma_{x,y}$ between $x$ and $y$ such that the following two conditions hold:

$$g(\gamma_{x,y}) \leq 1/4, \quad \text{and} \quad 0 < |u| \leq N_0, \quad \forall \ u \in \gamma_{x,y}. \quad (3.1)$$

Indeed, let $N_1$ be the smallest integer such that $a^{*N_1} \leq 1/16$. Let $K_w$ and $K_v$ be two cells with $|w| = |v| = N_1$, $x \in K_w$ and $y \in K_v$. In the case that $K_w \cap K_v \neq \emptyset$, we define $\gamma_{w,v} = \{w, v\}$ so that

$$g(\gamma_{w,v}) = g(w) + g(v) \leq 2a^{*N_1} \leq 1/8.$$ 

In the case that $K_w \cap K_v = \emptyset$, by Theorem 3.1, there is a chain $\eta_{w,v}$ connecting $K_v$ and $K_v$ such that $g(\eta_{w,v}) \leq 1/8$. Let $\gamma_{w,v}$ be the chain constructed by adding $\eta_{w,v}$ in between $w$ and $v$. Then we have

$$g(\gamma_{w,v}) = g(w) + g(\eta_{w,v}) + g(v) \leq 1/4.$$ 

Now set $N_{w,v} = \max\{|u| : u \in \gamma_{w,v}\}$, and let $N_0$ be the maximum of $N_{w,v}$ among all the pairs $w, v$ in $\Sigma N_1$. Then for all $\gamma_{w,v}$, $g(\gamma_{w,v}) \leq 1/4$ and $\gamma_{w,v}$ consists of $u$ such that $0 < |u| \leq N_0$, and the claim follows. For simplicity, we write $\gamma_0 := \gamma_{x,y} := (w(1), \ldots, w(m))$.

We now construct $\gamma_1$. For each word $w(i)$ in $\gamma_0$, we perform a splitting as follows. Let $x' \in K_{w(i)} \cap K_{w(i+1)}$ and $y' \in K_{w(i)} \cap K_{w(i+1)}$ (if $i = 1$, we just take $w(0) = w(1)$ and $x = x'$, and similar for $i = m$). For each $w(i)$, consider the pullback $F_{w(i)}^{-1}(K_{w(i)}) (= K)$, we apply the claim to $x'' = F_{w(i)}^{-1}(x')$ and $y'' = F_{w(i)}^{-1}(y')$ to obtain a chain $\gamma_{x'', y''}$ satisfying (3.1). Consider $F_{w(i)}(\gamma_{x'', y''})$, which is a chain

![Figure 2. Chains $\gamma_m$ and $\tilde{\gamma}_m$](image)
with all these, obtain a new chain. We keep doing the same splitting for words with length \( \leq M \). For an elementary fractal \( K \), Proposition 3.5.

We adopt the same notation as in Lemma 3.4. Let \( g \) be as in the lemma, then \( g(\gamma_1) \leq (1/4) \cdot g(\gamma_0) \leq 1/4^2 \), with \( N_0 < |u| \leq 2N_0 \), \( \forall u \in \gamma_1 \).

Inductively, we adopt the same procedure to construct \( \gamma_{n+1} \) from \( \gamma_n \): for each word \( w_i \in \gamma_n \), we use the same pull-back technique to bring \( K_{w(i)} \) to \( K \), and apply the claim to carry out the splitting, and the lemma follows.

**Proposition 3.5.** For an elementary fractal \( K \), \( \mathcal{M}^c \) is an open set in \((0, 1)^k\).

**Proof.** We adopt the same notation as in Lemma 3.4. Let \( a \in \mathcal{M}^c \) and \( \varepsilon > 0 \). Consider the weight function \( g^{(\varepsilon)} \) which is defined by the vector \( a(\varepsilon) = (a_1 + \varepsilon_1, a_2 + \varepsilon_2, \ldots, a_k + \varepsilon_k) \), with \( |\varepsilon_i| \leq \varepsilon \) for \( i = 1, 2, \ldots, k \). Let \( a_\varepsilon = 1 + \frac{\varepsilon}{a_\varepsilon} \), and let \( \gamma_n \) be as in the lemma, then \( |w| \leq (n + 1)N_0 \) for all \( w \in \gamma_n \). It follows that

\[
g^{(\varepsilon)}(\gamma_n) = \sum_{w \in \gamma_n} g^{(\varepsilon)}(w) = \sum_{w \in \gamma_n} g(w) \frac{g^{(\varepsilon)}(w)}{g(w)} \leq \sum_{w \in \gamma_n} g(w) a_{\varepsilon}^{|w|}
\]

\[
\leq g(\gamma_n) a_\varepsilon^{(n+1)N_0} \leq \left(4^{-1} a_\varepsilon^{N_0}\right)^{n+1}, \quad \forall n \geq 0.
\]

Choose \( \varepsilon > 0 \) small enough such that \( 4^{-1} a_\varepsilon^{N_0} \leq \frac{1}{2} \), then \( g^{(\varepsilon)}(\gamma_n) \to 0 \) as \( n \to \infty \). This implies that \( a(\varepsilon) \in \mathcal{M}^c \) and thus \( \mathcal{M}^c \) is open. \( \square \)

### 4. Metric chain condition (MCC)

It follows from Proposition 3.5 that \( \mathcal{M} \) is closed in \((0, 1)^k\). As \( S \) is the boundary surface of \( \mathcal{M} \) (Corollary 2.8), we have \( S \subset \mathcal{M} \) and \( \mathcal{M} \setminus S = \mathcal{M}^o \), the interior of \( \mathcal{M} \). In this section, we will study \( D_{g_a} \) for \( a \in \mathcal{M} \), and in particular in \( S \) in connection with the MCC (Definition 1.5).

According to Kigami [22], we say that \( D_g \) is 1-adapted to \( g \) if there exists a constant \( C > 0 \) such that for all \( x, y \in K \),

\[
(D_g(x, y) \leq \inf \{g(\gamma) : \gamma \text{ connects } x \text{ and } y, |\gamma| \leq 2\} \leq CD_g(x, y).
\]

**Remark 1.** In [22], the terminology of “\( m \)-adapted” is defined for any integer \( m \geq 1 \) by replacing the “\( |\gamma| \leq 2 \)” in (4.1) by “\( |\gamma| \leq 1 + m \)”. For \( g = g_a \), we can
actually use chains of the form $\gamma = \{w, v\}$ with $|w| = |v|$ to connect $x, y$. Indeed let $x \in K_\kappa, y \in K_\gamma$, and assume that they do not contain each other. Suppose $|w| > |v|$ (or $|w| < |v|$), we truncate the last indices of $w$ to $w'$ so that $|w'| = |v|$. By Proposition 2.4 and the proof in Corollary 2.5, we have $c^{-1}g(v) \leq g(w') \leq cg(v)$ where $c$ depends on $a$ and $d$. Hence (4.1) still holds with the constant $C' = (1+c)C$.

As a special case of [22, Theorem 2.3.16], we have

**Proposition 4.1.** Let $a \in \mathcal{M}$ and $g$ be the associated weight function. Then the metric $D_g$ is 1-adapted to $g$.

To study the MCC on $K$, we use the $n$-chains defined in the following. For $n \geq 0$, a chain $\gamma$ is called an $n$-chain if all the words in $\gamma$ have length equal to $n$. Define

$$D^n_g(x, y) := \min \{g(\gamma) | \gamma \text{ is an } n\text{-chain between } x \text{ and } y\}. \tag{4.2}$$

Clearly, $D_g(x, y) \leq D^n_g(x, y)$ for all $x, y \in K$.

**Remark 2.** We will need the following simple fact: for $a \in \mathcal{M}$,

$$\lim_{n \to \infty} D^n_{ga}(x, y) = D_{ga}(x, y) = 0, \quad \forall x, y \in K. \tag{4.2}$$

Indeed, for a given $n$, let $m$ be the smallest integer such that $n \leq mN_0$. By Lemma 3.4 there exists a chain $\gamma_m$ connecting $x, y$ such that $g(\gamma_m) \leq 4^{-m-1}$, and $mN_0 < |u| \leq (m + 1)N_0$ for $u \in \gamma_m$. Now we define an $n$-chain $\gamma^{(n)}$ by truncating each $u \in \gamma_m$ to $u'$ of size $n$. Hence the length of each word $u'$ in $\gamma^{(n)}$ has length differ from $u$ by at most $2N_0$. This implies $g_a(\gamma^{(n)}) \leq C4^{-m}$ for a constant $C$ so that (4.2) holds.

First we prove

**Proposition 4.2.** Let $K$ be an elementary fractal, and let $a \in \mathcal{M}^o$ (the interior of $\mathcal{M}$). Then

$$\lim_{n \to \infty} D^n_{ga}(x, y) = \infty, \quad \forall x \neq y \in K.$$ 

In this case, $D_{ga}$ is a metric but does not satisfy the MCC.

**Proof.** For $a \in \mathcal{M}^o$, first we claim that for any distinct $x, y \in K$, there exist $C > 0$ and $\sigma > 1$ such that for all $n \geq 0$,

$$D^n_{ga}(x, y) \geq C\sigma^n.$$ 

Indeed, since $a \in \mathcal{M}^o$, there exist $b \in \Lambda_k$ and $0 < \lambda < \lambda_b$, such that $a = b(\lambda)$. Denote by $\delta = \lambda_b - \lambda$ and let $g_0$ be the weight function of $b(\lambda_b)$. Since $b(\lambda_b) \in \mathcal{S} \subset \mathcal{M}$, we have $D_{g_0}(x, y) > 0$ and for each chain $\gamma$ between $x$ and $y$,

$$g_a(\gamma) = \sum_{w \in \gamma} (g_{b}(w))^\lambda = \sum_{w \in \gamma} (g_b(w))^{\lambda_b} \cdot (g_b(w))^{-\delta},$$

and hence

$$D^n_{ga}(x, y) \geq b^{\sigma-\delta n} \cdot D^n_{g_0}(x, y) \geq (b^{\sigma-\delta})^n \cdot D_{g_0}(x, y).$$

This proves the claim, and clearly implies that $\lim_{n \to \infty} D^n_{ga}(x, y) = \infty.$
To prove that \( D_{Sg} \) does not satisfy the MCC, we assume the contrary. We write \( D_g \) for \( D_{Sg} \) for simplicity. For two distinct points \( x \) and \( y \), there is \( C > 0 \) such that for any integer \( n \geq 1 \), there is a sequence \( x = x_0, x_1, \ldots, x_n = y \) such that
\[
D_g(x_i, x_{i+1}) \leq Cn^{-1}D_g(x, y), \quad \text{for } 0 \leq i \leq n - 1. \tag{4.3}
\]
Pick \( \lambda > 1 \) close to 1 such that \( D_{g,\lambda} \) is a metric, where \( D_{g,\lambda} \) is given by the weight \( a(\lambda) = (a_1^\lambda, a_2^\lambda, \ldots, a_k^\lambda) \). By using the 1-adaptedness of \( D_g \) and \( D_{g,\lambda} \) (Proposition 4.1), we have
\[
D_{g,\lambda}(x, y) = \left(D_g(x, y)^{\lambda}\right)^{\lambda}, \quad D_{g,\lambda}(x_i, x_{i+1}) = \left(D_g(x_i, x_{i+1})^{\lambda}\right)^{\lambda}, \quad \forall i.
\]
By using this, triangle inequality and (4.3), it follows that
\[
D_{g,\lambda}(x, y) \leq \sum_{i=0}^{n-1} D_{g,\lambda}(x_i, x_{i+1}) \leq C'\sum_{i=0}^{n-1} \left(D_g(x_i, x_{i+1})\right)^{\lambda} \leq C''\sum_{i=0}^{n-1} \left(n^{-1}D_g(x, y)\right)^{\lambda} \leq C''n^{\lambda-1}\left(D_g(x, y)\right)^{\lambda}.
\]
Letting \( n \to \infty \), we have \( D_{g,\lambda}(x, y) = 0 \), a contradiction. Hence \( D_g \) does not satisfy the MCC.

The main purpose of the section is to prove Theorem 4.4. We need a lemma.

**Lemma 4.3.** Let \( K \) be an elementary fractal, \( x, y \in K \) and \( a \in S \). Then
\[
\sup_{n \geq 0} D_{Sg}^{(n)}(x, y) < \infty.
\]
Moreover, there are \( n_0 \) (depends on \( x, y \)) and \( C > 0 \) (depends on \( a \)) such that
\[
\sup_{n \geq n_0} D_{Sg}^{(n)}(x, y) \leq CD_{Sg}(x, y). \tag{4.4}
\]

**Proof.** We first proof the case for the nested fractals. For \( p, q \in P_0 \), we denote by \( p \bowtie q \) if \( \sup_{n \geq 0} D_{Sg}^{(n)}(p, q) < \infty \). This gives an equivalent relation on \( P_0 \) which is preserved under the group \( G \). Observe that by using a similar argument as in Lemma 3.2, we conclude that \( p \bowtie q \) for some distinct \( p, q \in P_0 \) if and only if \( p \bowtie q \) for all \( p, q \in P_0 \).

We first show that \( p \bowtie q \) for all \( p, q \in P_0 \). Suppose otherwise, \( \sup_{n \geq 0} D_{Sg}^{(n)}(p, q) = \infty \) for all distinct \( p, q \in P_0 \), choose \( N_0 \geq 1 \) such that \( D_{Sg}^{(N_0)}(p, q) \geq 2 \) for all \( p \neq q \in P_0 \). It follows that for each \( m \geq 1 \),
\[
D_{Sg}^{(mN_0)}(p, q) \geq 2^m, \quad \forall p \neq q \in P_0. \tag{4.5}
\]
Indeed, let \( \gamma = \{w(1), \ldots, w(t)\} \) be a chain connecting \( p, q \) where \( |w(i)| = (m + 1)N_0 \) for all \( i \). By considering \( w(i)|_{N_0} \), we obtain a sequence of \( N_0 \)-cells \( \{K_u(i)\}_{j=1}^t \) connecting \( p, q \), and decompose \( \gamma \) into sub-chains \( \{\gamma_j\} \) with \( \gamma_j \) contained in \( K_u(i) \). Then \( F_{t-1}^{-1}(\gamma_j) \) is an \( mN_0 \)-chain, and induction implies \( g(F_{t-1}^{-1}(\gamma_i)) \geq 2^m \), so that \( g(\gamma) \geq \sum g(u(i))2^m \geq 2^{m+1} \) and \( D_{Sg}^{(m+1)N_0}(p, q) \geq 2^{m+1} \) for all \( p \neq q \in P_0 \).
Choose \( \epsilon > 0 \) small enough such that \( (a_\epsilon)^{2N_0} \geq 1/2 \). Let \( a(\epsilon) = (a_1^{1+\epsilon}, \ldots, a_k^{1+\epsilon}) \), and \( g(\epsilon) \) be the weight function given by \( a(\epsilon) \). Then for any \( mN_0 \)-chain \( \gamma \) between \( p \) and \( q \), we have

\[
g(\epsilon)(\gamma(mN_0)) = \sum_{w \in \gamma(mN_0)} g(\epsilon)(w) = \sum_{w \in \gamma(mN_0)} g(w)^{1+\epsilon} \geq \sum_{w \in \gamma(mN_0)} g(w)(a_\epsilon)^{mN_0} \geq \frac{g(\gamma(mN_0))}{2^m}.
\]

Combining this with (4.5), we see that for all \( m \geq 1 \), \( D^{(mN_0)}(p, q) \geq 1 \). On the other hand, it follows from \( a \in S \) that \( a(\epsilon) \in M^c \), which gives \( \lim_{n \to \infty} D^{(n)}(p, q) = 0 \) by Remark 2, a contradiction. This proves \( p \not\sim q \) for all \( p, q \in P_0 \).

To complete the proof of the first part of the lemma, we show further that \( \sup_{n \geq 0} D^{(n)}(x, y) < \infty \) for any \( x, y \in K \). Indeed, we can find a sequence of cells \( \{K_{w(i)}\}_{i=0}^{\infty} \) such that \( w(0) = \emptyset \) and \( w(i + 1) \) is a child of \( w(i) \), and \( \bigcap_{i=1}^{\infty} K_{w(i)} = \{x\} \). Pick an arbitrary \( x_i \in F_{w(i)}(P_0) \), we obtain a sequence of points \( \{x_i\}_{i=0}^{\infty} \) such that \( \lim_{i \to \infty} x_i = x \). We also see that \( x_i \) and \( x_{i+1} \) can be connected by a uniformly bounded number of pairs in \( F_{w(i+1)}(P_0) \). Let \( M = \max_{p \in P_0} \sup_{n \geq 0} D^{(n)}(p, q) \). Hence by self-similarity, there exists a constant \( C > 0 \), such that for all \( i \),

\[
\sup_{n \geq 0} D^{(n)}(x_i, x_{i+1}) \leq Cg(w(i + 1)) \cdot M.
\]

By summing up over \( i \), and observe that \( \sum_{i=1}^{\infty} g(w(i)) \leq \sum_{m=1}^{\infty} a^m < \infty \), we have

\[
\sup_{n \geq 0} D^{(n)}(x_0, x) \leq C \sum_{i=1}^{\infty} g(w(i)) \cdot M \leq C'M.
\]

Similarly, we pick \( \{y_i\}_{i=0}^{\infty} \) such that \( \lim_{i \to \infty} y_i = y \) with \( y_0 = x_0 \). Then we have

\[
\sup_{n \geq 0} D^{(n)}(x_0, y) \leq C'M, \text{ so that } \sup_{n \geq 0} D^{(n)}(x, y) \leq 2C'M < \infty. \text{ This proves the first part.}
\]

To prove the last assertion, we observe that in the 1-adaptedness of \( D_g \), Remark 1 allows us to assume that the two-word chain \( \gamma = \{w, v\} \) connecting \( x, y \) is such that \( |w| = |v| \). Let

\[
n_0 = n_0(x, y) := \max\{n \geq 0 : \gamma = \{w, v\} \text{ connecting } x, y \text{ and } |w| = |v| = n\}.
\]

Clearly \( n_0 < \infty \). We can assume that \( \gamma = \{w, v\} \) attains the maximum. Let \( z \in K_w \cap K_v \). Since the two points \( x \) and \( z \) are both in \( K_w \), consider \( F_w^{-1}(x), F_w^{-1}(z) \in K \). By the first conclusion of the lemma, we have

\[
\sup_{n \geq 0} D^{(n)}(F_w^{-1}(x), F_w^{-1}(z)) \leq C_0.
\]

It follows from the self-similarity that \( \sup_{n \geq n_0} D^{(n)}(x, z) \leq C_0g(w) \). Similarly, \( \sup_{n \geq n_0} D^{(n)}(y, z) \leq C_0g(v) \). Hence we have

\[
\sup_{n \geq n_0} D^{(n)}(x, y) \leq C_0(g(w) + g(v)) \leq CD_g(x, y).
\]

This completes the proof of the lemma for the nested fractals.

To prove the case for the GSC, we need some new notations. Let \( p_1 = (0, 0, \ldots, 0) \), \( p_2 = (1, 0, \ldots, 0) \). Let \( L = \{0\} \times [0, 1]^{d-1} \) and \( R = \{1\} \times [0, 1]^{d-1} \) be the left and
right face of \( H_0 \). Define
\[
D_g^{(n)}(L, R) = \min\{g(\gamma) : \gamma \text{ is an } n\text{-chain between } L \text{ and } R\},
\]
where an \( n\)-chain \( \gamma = (w(1), \ldots, w(m)) \) is called an \( n\)-chain between \( L \) and \( R \) if \( K_{w(1)} \cap L \neq \emptyset \) and \( K_{w(m)} \cap R \neq \emptyset \). We need another lemma.

**Sublemma.** For \( a \in (0, 1)^k \), \( \sup_{n \geq 0} D_g^{(n)}(L, R) < \infty \iff \sup_{n \geq 0} D_g^{(n)}(p_1, p_2) < \infty \).

Since the proof of this sublemma uses more notions and another technique, and is quite long, we will prove it in the Appendix in order not to distract the main proof.

For the first assertion, using the same argument again as in the previous proof of the nested fractals, we can also see that \( \sup_{n \geq 0} D_g^{(n)}(p_1, p_2) < \infty \) implies that \( \sup_{n \geq 0} D_g^{(n)}(x, y) < \infty \) for all \( x, y \in K \) (the \( x_i \) we choose is an arbitrary vertex of the cube \( F_{w(j)}(H_0) \), and note that \( x_i, x_{i+1} \) can be connected by a finite number of pairs that can be expressed as affine combination of \( p_1, p_2 \) under some \( F_w \) and \( \sigma \in G \). Thus, it suffices to show that \( a \in S \) implies \( \sup_{n \geq 0} D_g^{(n)}(p_1, p_2) < \infty \). Suppose otherwise, then by sublemma,
\[
\sup_{n \geq 0} D_g^{(n)}(L, R) = \infty.
\]
From this, we see that for any given \( C_0 > 1 \), there exists \( N_0 \) large enough such that \( D_g^{(N_0)}(L, R) \geq C_0 \), i.e., for any two points \( x \in L \) and \( y \in R \), \( D_g^{(N_0)}(x, y) \geq C_0 \).

Note that by symmetry, \( D_g^{(N_0)}(L, R) = D_g^{(N_0)}(L', R') \) for any two opposite faces \( L', R' \) of the unit cube. We will pick some large \( C_0 \) (specified later) and show that
\[
D_g^{(N_0)}(p_1, p_2) \geq 2^m.
\]
(4.6)
Let \( m \geq 2 \) and \( \gamma^{(m)} \) be an \( mN_0 \)-chain between \( p_1 \) and \( p_2 \). We decompose \( \gamma^{(m)} = \{w(1), w(2), \ldots, w(j)\} \) into a sequence of sub-chains as follows. Let \( w(1)^- \) be the truncation of \( w(1) \) to word length \( (m - 1)N_0 \), and let
\[
N_1 = \cup\{K_v : |v| = (m - 1)N_0, K_{w(1)^-} \cap K_v \neq \emptyset\}
\]
be the neighborhood of \( K_{w(1)^-} \) which is contained in a cube having the size three times as \( K_{w(1)^-} \). Set \( t_0 = 1 \) and define
\[
t_1 = \inf\{t : 1 \leq t \leq j, K_{w(t)} \subseteq N_1\},
\]
and let \( \gamma^{(m)}_1 = \{w(1), w(2), \ldots, w(t_1 - 1)\} \). Follow this “first exit time” technique, we can define \( N_i \) and \( \gamma^{(m)}_i = \{w(t_{i-1}), \ldots, w(t_i - 1)\} \), \( 1 \leq i \leq s \) as the above. Then \( \gamma^{(m)} \) is decomposed into subchains \( \{\gamma^{(m)}_i\}_{i=1}^s \).

We first consider \( \gamma^{(m)}_1 \). Note that \( H_{w(1)^-} := F_{w(1)^-}(H_0) \) is the subcube containing \( K_{w(1)^-} \). Consider a subchain \( \gamma^{(m)}_1 \) of \( \gamma^{(m)}_1 \) such that for \( w(i) \in \gamma^{(m)}_1 \), \( K_{w(i)^-} \subset \tilde{N}_1 := N_1 \setminus H_{w(1)^-} \); it corresponds to a chain of \( mN_0 \)-cells starts from an \( (m - 1)N_0 \)-cell \( T_0 \), which touches the boundary \( \partial H_{w(1)^-} \), and reaches the outer border of \( \tilde{N}_1 \) with an exit face contained in a \( (d - 1) \)-dimensional hyperplane \( H \). We can take the union of translates of \( T_0 \) towards \( H \) and obtain a straight tube-like set \( T \) composed of
We now adjust \( \{w(1)^{−}, w(2)^{−}, \ldots, w(t_1−1)^{−}\} \) to a simple \((m−1)N_0\)-chain \( \eta_1 \) starting at \( w(1)^{−} \) and ending at \( w(t_1−1)^{−} \), then \( |\eta_1| \leq 3^d \). From Lemma 2.3, \( g(w(1)^{−}) \geq c_2 g(\eta_1) \), where \( c_2 = 3^{−d}(a_s/a_s^∗)^{3^d} \). Thus using (4.7), we have

\[
 g(\gamma_i^{(m)}) \geq c_1 c_2 C_0 g(\eta_1).
\]

The same estimate holds for \( \gamma_i^{(m)}, 2 \leq i \leq s \), we obtain \( g(\gamma_i^{(m)}) \geq c_1 c_2 C_0 g(\eta_i) \).

Next we observe that the concatenation of \( \{\eta_1, \ldots, \eta_s\} \) is an \((m−1)N_0\)-chain connecting \( p_1 \) and \( p_2 \), hence \( \sum_{i=1}^{s} g(\eta_i) \geq D_g^{(m−1)N_0}(p_1, p_2) \). Therefore, we have

\[
 g(\gamma^{(m)}) = \sum_{i=1}^{s} g(\gamma_i^{(m)}) \geq c_1 c_2 C_0 \sum_{i=1}^{s} g(\eta_i) \geq c_1 c_2 C_0 D_g^{(m−1)N_0}(p_1, p_2).
\]

Pick \( C_0 \) such that \( C_0 \geq 2 + 2c_1^{−1}c_2^{−1} \). Since \( \gamma^{(m)} \) is an arbitrary \( mN_0 \)-chain between \( p_1 \) and \( p_2 \), we have

\[
 D_g^{(mN_0)}(p_1, p_2) \geq 2D_g^{(m−1)N_0}(p_1, p_2) \geq 2^{m−1} C_0 \geq 2^m.
\]

This proves (4.6).

Finally we use \( a(\epsilon) = (a_1^{1+\epsilon}, \ldots, a_k^{1+\epsilon}) \) as in the proof of the nested fractal case to obtain a contradiction, hence \( \sup_{\nu \geq 0} D_g^{(n)}(p_1, p_2) < \infty \).

The proof of the second part of the lemma is similar to the case of nested fractals.

\[\square\]

**Theorem 4.4.** Let \( K \) be an elementary fractal, and let \( a \in \mathcal{M} \). Then \( D_{S_a} \) is a metric satisfying the MCC if and only if \( a \in S \).

**Proof.** The necessity follows from Proposition 4.2. To prove the sufficiency, we let \( x, y \in K \), and let \( a \in S \), and write \( D_g \) for \( D_{S_a} \) for simplicity. Fix any integer \( M > 2 \), pick \( n > n_0 \) and so that \( a^{−n} \leq \frac{D_g(x,y)}{M} \). Consider an \( n \)-chain \( \gamma = (w(1), \ldots, w(m)) \) that attains \( D_g^{(n)} \). Then by (4.4),

\[
 \sum_{i=1}^{m} g(w(i)) = D_g^{(n)}(x,y) \leq CD_g(x,y).
\]

We make a decomposition of \( \gamma \) by a “first exit time” technique. Let \( s_0 = 0 \), define

\[
 s_1 := \inf \left\{ j : \sum_{i=1}^{j} g(w(i)) \geq CM^{-1}D_g(x,y), 1 \leq j \leq m \right\},
\]

\[\square\]
where $C$ is the same as in (4.8). Inductively, for $t \geq 1$, define
\[
s_{t+1} := \inf \left\{ j : \sum_{i=s_t+1}^{j} g(w(i)) \geq CM^{-1}D_g(x,y), \quad s_t + 1 \leq j \leq m \right\}.
\]
Let $\bar{t}$ be the first integer that $s_{\bar{t}}$ can not be defined, and we assign $s_{\bar{t}} = m$ by convention. Then we have $\bar{t} - 1 \leq M$ (for otherwise, from the construction, $\sum_{i=1}^{m} g(w(i)) \geq (\bar{t} - 1)CM^{-1}D_g(x,y) > CD_g(x,y)$, contradicting (4.8)). Also for each $0 \leq t \leq \bar{t} - 1$,
\[
\sum_{i=s_t+1}^{s_{t+1}} g(w(i)) \leq CM^{-1}D_g(x,y) + a^n \leq (C + 1)M^{-1}D_g(x,y).
\]
Now for $1 \leq t \leq \bar{t} - 1$, pick each point $z_t \in K_{w(s_t+1)}$, together with $x = z_0$ and $y = z_\bar{t}$, we have by definition of $D_g$,
\[
D_g(z_t, z_{t+1}) \leq \sum_{i=s_t+1}^{s_{t+1}} g(w(i)) + a^n \leq (C + 2)M^{-1}D_g(x,y),
\]
where $C$ is independent of $x, y, M$. Since $\bar{t} \leq M + 1$, we obtain the MCC of $D_g$. \[ \square \]

5. Nested fractals

In this section, we give a description of $\mathcal{M}$ for the class of nested fractals. The main idea of representing the recursive weight transfer into matrix is from [26].

Let $\ell_1, \ell_2, \ldots, \ell_r$ be such that $0 < \ell_1 < \ell_2 < \cdots < \ell_r$ and \(\{\ell_1, \ldots, \ell_r\} := \{x - y : x, y \in P_0, x \neq y\}\). For $n \geq 1$, denote by $P_n = \bigcup_{w \in \Sigma^n} F_w(P_0)$. For each $x \in P_n$ with $n \geq 0$ and for $i = 1, \ldots, r$, let $N_n^i(x)$ be the set of all $y \in P_n$ belonging to the same $n$-cell of $x$ and $|x - y| = g^{-n}\ell_i$; for $y \in N_n^i(x)$, we call the one step move from $x$ to $y$ an $n$-move of type $(i)$, $1 \leq i \leq r$. A sequence $x_0, \ldots, x_m \in P_n$ is called an $n$-walk if $x_j$ and $x_{j+1}$ are joined in the same $n$-cell for all $0 \leq j \leq m - 1$.

Next we fix a symmetric self-similar weight function $g$ generated by some $a = (a_1, \ldots, a_k) \in (0, 1)^k$. Pick any $x \in P_0$ and $y \in N_{0}^{i}(x)$ for some $1 \leq i \leq r$, consider all the 1-walk $x_0, x_1, \ldots, x_m$ such that $x_0 = x$ and $x_m = y$ with $x_1, \ldots, x_{m-1} \in P_1 \setminus P_0$ which do not pass through the same point twice. Fix such a 1-walk, for $1 \leq j \leq r$, we count all the $\langle j \rangle$-type 1-moves in this walk. For each $\langle j \rangle$-type 1-move, it can be assigned to a unique 1-cell, and we say that this $\langle j \rangle$-type move has weight $a_i$ if the weight of this 1-cell is $a_i$. Then we sum up all the weights of these $\langle j \rangle$-type moves and denote it by $c_j$. Let $c^i = (c_{i1}, \ldots, c_{ik})$ be the weight of the 1-walk. By the symmetric assumption of $g = g_a$, it is clear that $c^i$ does not depend on the choice of $x \in P_0$ and $y \in N_{0}^{i}(x)$. Let $S_a^i$ be the set of $c^i$ for all these finite number of 1-walks. Then $S_a^i, 1 \leq i \leq r$ is a finite collection of $r$-dimensional vectors, and each one is a nonnegative linear combination of weights in $a \in (0, 1)^k$ with integer coefficients.

Let
\[
\mathcal{K}(a) := \{ A : A \text{ is a } r \times r \text{-matrix } \exists \text{ for } 1 \leq i \leq r, (i \text{-th row of } A) \in S_a^i \}.
\]

We call $A$ a weight transfer matrix. For $A \in \mathcal{K}(a)$, $A$ has nonnegative entries and each row is nonzero. Let $\lambda_A$ be the largest positive eigenvalue of $A$. Then it is clear that $\lambda_A$ is a solution of some polynomial with coefficients generated by $\{a_1, \ldots, a_k\}$.
Theorem 5.1. For a nested fractal, we have
\[ M = \{ a = (a_1, \ldots, a_k) \in (0, 1)^k : \lambda_A \geq 1 \text{ for all } A \in \mathcal{K}(a) \}, \]
and the boundary \( S = \{ a \in M : \text{there exists } A \in \mathcal{K}(a) \text{ such that } \lambda_A = 1 \}. \)

Proof. We first show that if \( a = (a_1, \ldots, a_k) \in (0, 1)^k \) is such that \( \lambda_A < 1 \) for some \( A \in \mathcal{K}(a) \), then \( D_{sA} \) is not a metric. The idea is that we use this \( A \) to recursively construct \( n \)-chains \( \{y_n\}_{n \geq 0} \) between two distinct points \( x, y \in P_0 \) such that \( \lim_{n \to \infty} g(y_n) \to 0 \). By assumption, for \( 1 \leq i \leq r \), the \( i \)-th row of \( A \) is determined by a 1-walk \( \xi_i \) between some pair \( x, y \in P_0 \) with \( |x - y| = \ell_i \). We pick \( x, y \in P_0 \) such that \( |x - y| = \ell_1 \) and let \( \eta_1 = \xi_1 \) be the 1-walk between \( x \) and \( y \). Let \( \gamma_1 \) be the associated 1-chain of \( \eta_1 \). For \( n \geq 1 \), we define recursively an \( n \)-walk \( \eta_n \) between \( x, y \) and denote by \( \gamma_n \) the associated \( n \)-chain. Define \( \eta_2 \) by replacing each \( \langle i \rangle \)-type 1-walk in \( \eta_1 \) by the 2-walk \( F_w(\sigma(\xi_j)) \) for some \( \sigma \in G \), where \( w \) is the assigned 1-cell of the 1-move. Recursively, we define the \( n \)-walk \( \eta_n \) from \( \eta_{n-1} \) in a similar manner. For the weight \( c^1 \) of \( \eta_1 \) and \( 1 = (1, \ldots, 1) \), we have
\[ g(\gamma_n) = c^1 A^{n-1} 1' \to 0 \quad \text{as } n \to \infty, \]
by \( \lambda_A < 1 \). This implies that \( D_s(x, y) = 0 \) and consequently \( D_s \) is not a metric.

Conversely, We use contradiction to show that if \( \lambda_A \geq 1 \) for all \( A \in \mathcal{K}(a) \), then \( D_{sA} \) is a metric on \( K \). We fix \( a \) and define the operator \( G_a : [0, \infty)^r \to [0, \infty)^r \) by
\[ (G_a(x))_i = \min \left\{ \sum_{j=1}^r c^j_j x_j \right\} = \min\{\langle c^i, x' \rangle\}, \quad 1 \leq i \leq r, \]
where \( x = (x_1, \ldots, x_r) \). Note that for each \( 1 \leq i \leq r \), there exists \( c^i \in S^i_g \), such that \( (G_a(x))_i = \langle c^i, x' \rangle \). This defines a matrix \( A_{\min} \in \mathcal{K}(a) \) (depends on \( x \)) such that \( G_a(x) = A_{\min} x' \).

By using the same technique as in [26] Lemma 3.3, we have \( G_a(B) \subseteq B \) for \( B = \{ x \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_r \} \). For completeness, we provide a proof for this with a slight modification.

Fix \( p \in P_0 \), \( q_i \in N_0^i(p) \) and \( q'_i \in N_0^{i-1}(p) \) for \( 2 \leq i \leq r \). Let \( R_{q_i q'_i} \) be the reflection between \( q_i \) and \( q'_i \). Let \( V_i = \{ z \in \mathbb{R}^n : |z - q_i| \leq |z - q'_i| \} \) and \( z = R_{q_i q'_i} z \). Let \( T_i : \mathbb{R}^n \to \mathbb{R}^n \) be given by
\[ T_i(z) = \begin{cases} z, & z \in V_i, \\ \bar{z}, & \text{otherwise}. \end{cases} \]

Given \( x \in B \) and \( i \geq 2 \), consider \( c^i \in A_{\min} \), then there is a 1-walk \( \xi_i \) between \( p \) and \( q_i \) with weight \( c^i \). Express \( \xi_i \) by \( p = x_0, x_1, \ldots, x_m = q_i \). Then we see that \( T_i(\xi_i) \) is a 1-walk between \( p \) and \( q'_i \), and \( (T_i(x_j), T_i(x_{j+1})) \) is a 1-move in the same cell as \( (x_j, x_{j+1}) \), which has type smaller than or equal to \( (x_j, x_{j+1}) \) because \( |T_i(x_j) - T_i(x_{j+1})| \leq |x_j - x_{j+1}| \). Denote by \( t = (t_1, \ldots, t_r) \) the weight of the 1-walk \( T_i(\xi_i) \). Then we have
\[ (G_a(x))_{i-1} \leq \langle t, x' \rangle \leq \langle c^i, x' \rangle = (G_a(x))_i, \]
since \( x \in B \). This proves \( G_a(B) \subseteq B \).
Consider the normalization \( \widetilde{G}(x) = G(x)/\sum_i G(x)_i \) on \( B_\varepsilon = \{ x \in B : \sum_i x_i, x_1 \geq \varepsilon \} \). Then \( \widetilde{G}(x)(B_\varepsilon) \subset B_\varepsilon \). By using the Brouwer fixed point theorem, there is a fixed point \( \widetilde{G}(x) = x \). It follows that \( G(x) = \lambda x = A_{\min} x \) where \( \lambda = \sum_i G(x)_i \), and is the maximum eigenvalue of \( A_{\min} \) (see detail, see [26, Proposition 3.4]).

Finally, for \( n \geq 0 \), let \( z_n = (z_{n,1}, \ldots, z_{n,r}) \) be a vector of positive real numbers such that \( z_{n,i} = D^{(n)}_a(p_i, q), \) where \( p, q \in P_0 \) and \( |p - q| = \ell_i, \ 1 \leq i \leq r \). Then

\[
z_{n+1} = G_a(z_{n-1}), \quad n \geq 1.
\]

Denote \( C' = \min \{ x_i^{-1} : 1 \leq i \leq r \} \). Then \( z_0 = (1, \ldots, 1) \geq C'x \) so that \( z_1 \geq G_a(C'x) = C'\lambda x \). In general, we have \( z_n \geq G_a(C'\lambda^n x) = C'\lambda^nx \) for all \( n \geq 1 \). On the other hand, if \( D_{ga} \) is not a metric on \( K \), then by (4.2), \( \lim_{n \to \infty} D^{(n)}_{sa}(x, y) = 0 \) for any \( x, y \in P_0 \). This contradicts the fact that \( \lambda \geq 1 \) and \( x \) is positive. Thus \( a \in M \) and the proof of the first assertion is complete.

To prove the second assertion, we denote by \( S' := \{ a \in M : \exists A \in K(a) \ni \lambda_A = 1 \} \). For any \( a = (a_1, \ldots, a_k) \in S' \), we may assume \( A(a) \in K(a) \) such that \( \lambda_{A(a)} = 1 \). For any \( \delta > 0 \), denote by \( a^\delta = (a_1^\delta, \ldots, a_k^\delta) \), each nonzero entry of \( A(a^\delta) \) is strictly smaller than that of \( A(a) \). Denote by

\[
c_0 = \max \{ c > 1 : \text{each entry of } cA(a^\delta) \text{ is smaller than or equal to that of } A(a) \}.
\]

Thus \( c_0 > 1 \) and by using the Perron-Frobenius theorem, we have

\[
c_0 \lambda_{A(a^\delta)} = \lambda_{c_0 A(a^\delta)} \leq \lambda_{A(a)} = 1,
\]

and hence \( \lambda_{A(a^\delta)} \leq c_0^{-1} < 1 \), which implies that \( a^\delta \in M' \) for any \( \delta > 1 \). Hence \( a \in S \), which implies \( S' \subseteq S \). On the other hand, for any \( a = (a_1, \ldots, a_k) \in S \), we show that \( a \in S' \). If \( a \notin S' \), then \( \lambda_{A(a)} > 1 \) for all \( A(a) \in K(a) \). It is clear that \( \lambda_{A(a)} \) is continuous with respect to \( a \) and the cardinality of \( K(a) \) is finite. Thus for all \( s \) close to 1, for all \( A(a^s) \in K(a^s) \), we have \( \lambda_{A(a^s)} > 1 \). Hence \( a \in M' \), a contradiction. This shows that \( a \in S' \) and \( S' \subseteq S \). Thus we have \( S = S' \).

**Remark.** The size of the families \( S_{a_i}' \), \( 1 \leq i \leq r \), can be cut down substantially for calculation. We define the essential class \( \tilde{S}_{a_i}' \), \( 1 \leq i \leq r \) to be the set of \( \mathbf{e} := (c_1, \ldots, c_r) \in S_{a_i}' \) that is smallest (in the sense of coordinatewise ordering) for \( S_{a_i}' \). This is because for \( \mathbf{e}' \in S_{a_i}' \) with \( \mathbf{e} \leq \mathbf{e}' \), if we replace the row in the weight transfer matrix \( A \) containing \( e \) by \( e' \), then the maximal eigenvalue increases.

Let \( \tilde{K}(a) \) be the \( r \times r \) matrices \( A \) such that the \( i \)-th row is in \( \tilde{S}_{a_i}' \), then clearly

\[
M = \{ a = (a_1, \ldots, a_k) \in (0, 1)^k : \lambda_A \geq 1 \} \text{ for all } A \in \tilde{K}(a).
\]

In the following, we use the Lindström snowflake to give an illustration of the theorem and the remark.

Let \( p_i = (\cos(i\pi/3), \sin(i\pi/3)), \ i = 1, \ldots, 6 \) and \( p_7 = (0, 0) \). Define \( F_i : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( F_i(x) = (x - p_i)/3 + p_i \) for \( i = 1, \ldots, 7 \). The Lindström snowflake \( K \) is the self-similar set generated by the IFS \( \{ F_i \}_{i=1}^7 \) (see Figure 3). It has boundary \( V_0 = \{ p_1, \ldots, p_6 \} \).
In this case, \( \Sigma = \{1, \ldots, 7\} \). We consider a symmetric self-similar weight function \( g = g_{a,b} \) on \( \Sigma^* \) defined by

\[
g(i) = \begin{cases} 
a, & 1 \leq i \leq 6, 
\ b, & i = 7,
\end{cases}
\]

where \( a, b \in (0, 1) \). (See Figure 3.)

**Corollary 5.2.** Let \( g = g_{a,b} \) be defined as the above. Then \( D_{g,a,b} \) is a metric if and only if \( 3a \geq 1 \) and \( 2a + b \geq 1 \). Moreover, \( D_g \) satisfies the MCC if and only if \( (a, b) \in (3a = 1, b \geq \frac{1}{3}) \cup \{2a + b = 1, b \leq \frac{1}{3}\} \).

**Proof.** The second conclusion is a consequence of the first part and Theorem 4.4.

To prove the first part, we adopt the notations and setup preceding Theorem 5.1. It is easy to see that the Lindstrøm snowflake has three types of 1-move, that is \( \ell_1 = |p_1p_2|, \ell_2 = |p_1p_3| \) and \( \ell_3 = |p_1p_4| \). Let \( a = (a,b) \) be as in (5.1), and consider \( p_2 \in N_0^1(p_1) \). For a 1-walk of \( p_1 \) to \( p_2 \) in \( P_1 \setminus P_0 \), by using the above remark and elementarily checking case by case, we obtain two vectors in the essential classes \( \tilde{S}_a^1 \). Similarly, we can calculate \( \tilde{S}_a^2 \) and \( \tilde{S}_a^3 \) (see Figure 4).

\[
\tilde{S}_a^1 = \{(0, 2a, 0), (b, 0, 2a)\};
\tilde{S}_a^2 = \{(a + b, a, a), (0, b, 2a), (a, a, a + b), (0, 3a, 0)\};
\tilde{S}_a^3 = \{(0, 4a, 0), (2a + b, 2a, 0), (a + b, 2a, a), (a, a + b, a), (0, 0, 2a + b)\}.
\]

The \( A \in \mathcal{K}(a) \) are formed by picking one vector in each of the \( \tilde{S}_a^i \). It can be checked directly that \( \lambda_A \geq 1 \) for all \( A \in \mathcal{K}(a) \) is equivalent to

\[
\begin{cases} 
3a \geq 1, \\
2a + b \geq 1.
\end{cases}
\]

In particular, the two determining matrices for \( M \) and \( S \) are

\[
A_1 = \begin{pmatrix} 0 & 2a & 0 \\ 0 & 3a & 0 \\ 0 & 4a & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2a & 0 \\ 0 & b & 2a \\ 0 & 0 & 2a + b \end{pmatrix},
\]
Figure 4. The essential classes $\tilde{S}^1_a$ (two elements), $\tilde{S}^2_a$ (four elements), $\tilde{S}^3_a$ (five elements)

with the two spectral radii $\lambda_{A_1} = 3a$ and $\lambda_{A_2} = 2a + b$ respectively.

Remark. In fact, it is easy to see that the two conditions in (5.2) are necessary for $D_g$ to be a metric by looking at the chains along the straight lines contained in the fractal. For example, the line $p_1p_3$ gives $3a \geq 1$ and the line $p_1p_4$ gives $2a + b \geq 1$.

6. Sierpinski carpet

In this section, we consider the standard Sierpinski carpet $GSC(2, 3, Q')$, where $Q' = Q \setminus \{(1/3, 2/3)\}$. Let $F_i(x) = (x + 2p_i)/3, i = 1, \ldots, 8$, be contractive maps on $\mathbb{R}^2$, with the $p_i$ specified as in Figure 5.

For any $(a, b) \in (0, 1)^2$, define $g := g_{a,b} : \Sigma^* \to (0, 1]$ by

$$
g(i) = \begin{cases} 
a, & \text{if } i \in \{1, 3, 5, 7\}, 
\ b, & \text{if } i \in \{2, 4, 6, 8\}, 
\end{cases}
$$

(6.1)

see Figure 5 for the case $|w| = 3$.

Let $\Pi = \{(a, b) \in (0, 1)^2 : 2a + b \geq 1 \text{ and } a + 2b \geq 1\}$, and let $\partial \Pi := \{(a, b) \in \Pi : a + 2b = 1 \text{ or } 2b + a = 1\}$ be the boundary of $\Pi$ in $(0, 1)^2$.

Theorem 6.1. $D_{g_{a,b}}$ is a metric on $K$ if and only if $(a, b) \in \Pi$. Consequently, $D_{g_{a,b}}$ satisfies the MCC if and only if $(a, b) \in \partial \Pi$. 

Figure 5. Sierpinski carpet, and the weight function $g_{a,b}(w)$ for $|w| = 3$.

The MCC follows directly from the first part and Theorem 4.4. The necessity that $D_{g,a,b}$ is a metric is due to Kigami. It is rather straightforward. We provide a proof in the following for completeness.

**Proof of the necessity.** For $n \geq 1$, let $\gamma_n$ be a chain of $n$-cells connecting $p_1$ and $p_3$ and each cell intersects the line $p_1p_3$ (i.e., $\gamma_1 = \{1, 2, 3\}$, $\gamma_2 = \{11, 12, 13; 21, 22, 23;\ 31, 32, 33\}$, and so on). By elementary calculations, we obtain

$$g(\gamma_n) = (2a + b)^n.$$  

Since $D_g$ is a metric, $\inf_{n \in \mathbb{Z}^+} (2a + b)^n \geq D_g(p_1, p_3) > 0$ so that $2a + b \geq 1$.

Next, we consider the chains $\{\gamma'_n\}_{n \geq 1}$ of $n$-cells connecting $p_2$ and $p_4$, and each cell intersects the line segment $\overline{p_2p_4}$ (i.e., $\gamma'_1 = \{2, 3, 4\}$, $\gamma'_2 = \{22, 23, 24; 38, 37, 36; 42, 43, 44\}$, and so on). Then

$$g(\gamma'_n) = (a + 2b)^n.$$  

Same as the above, we have $a + 2b \geq 1$. \hfill \Box

The proof of the sufficiency of the theorem is more complicate. The following corollary is a simple consequence of Lemma 2.3 adapted to the situation we need.

**Corollary 6.2.** Suppose $K_u \sim K_v$ and $|u| = |v|$. Then

$$\frac{g(u)}{g(v)} = \frac{a}{b} \text{ or } \frac{b}{a}.$$  

Furthermore, if $\gamma$ is a chain contained in $K_u$, if we reflect it along the intersection line $L$, and denote the reflected chain by $\gamma'$, then $\gamma'$ is in $K_v$, and $\frac{g(\gamma')}{g(\gamma)} = \frac{a}{b} \text{ or } \frac{b}{a}.$

In view of Theorem 3.1 and (4.2), we only need to consider the $n$-chains between arbitrary two fixed distinct points in $K$. As in the proof, we will use Corollary 6.2 frequently for reflection. We need a few more lemmas.

The following lemma is easy in view of the proof of necessity.
Lemma 6.3. Let \((a, b) \in \Pi\) and let \(n \geq 0\). Suppose \(\gamma\) is an \(n\)-chain satisfying either
(i) \(\gamma\) connects \(p_1\) and \(p_3\) and all its cells intersect the line segment \(\overline{p_1 p_3}\); or
(ii) \(\gamma\) connects \(p_2\) and \(p_8\) and all its cells intersect the line segment \(\overline{p_2 p_8}\).
Then
\[ g(\gamma) \geq 1. \]

We separate the proof into two cases. Define
\[ \Pi_1 = \{(a, b) \in \Pi : a \leq b\} \quad \text{and} \quad \Pi_2 = \{(a, b) \in \Pi : a \geq b\}. \]

The case for \(\Pi_1\). For \(n \geq 1\), we let \(P_n = F_{1^{n-1}}(0, 1/2)\) and \(P'_n = F_{1^{n-1}}(0, 1/3)\), and use \(L_n\) and \(L'_n\) to denote the lines \(y = \frac{1}{2} \cdot 3^{n-1}\) and \(y = \frac{1}{3^n}\), respectively. Then \(L_n\) passes through \(P_n\), and \(L'_n\) passes through \(P'_n\). We set \(D'_0 = [0, 1]^2\), and denote by \(D_n\) (or \(D'_n\)) the rectangle enclosed by the lines \(x = 0, 1, y = 0,\) and \(y = L_n\) (or \(y = L'_n\) respectively) (see Figure 6).

![Figure 6. Lines for reflection in the case \(\Pi_1\).](image)

For a cell \(K_w\), we define its center to be \(F_w(1/2, 1/2)\). We also define an \((L_n)\)-reflection to be reflecting a cell along the line \(L_n\), and similarly an \((L'_n)\)-reflection.

Lemma 6.4. Suppose \((a, b) \in \Pi_1\). Let \(K_w\) be a cell with \(|w| \geq n\).
(i) If \(K_w\) has center in \(D'_{n-1} \setminus D_n\), and let \(K_u\) be the reflected cell of \(K_w\) along \(L_n\), then \(K_u\) is centered in \(D_n\) and \(g(u) \leq g(w)\).
(ii) If \(K_w\) has center in \(D_n \setminus D'_{n-1}\), and let \(K_u\) be the reflected cell of \(K_w\) along \(L'_n\), then \(K_u\) is centered in \(D'_n\) and \(g(u) \leq g(w)\).

Proof. In the first case, the center of \(K_{w_{n-1}}\) is on \(L_n\), and the \((L_n)\)-reflection sends \(K_{w_{n-1}}\) to itself. It follows that \(K_u\) is centered in \(D_n\), and \(g(u) = g(w)\) by symmetry. In the second case, one of the line segment of \(\partial K_{w_{n-1}}\) is on \(L'_n\), hence the \((L'_n)\)-reflected cell is in \(D'_n\), and \(g(u) = (a/b) \cdot g(w) \leq g(w)\) by Corollary 6.2. \(\square\)
Lemma 6.5. Suppose \((a, b) \in \Pi_1\). For \(n \geq 0\), let \(\gamma\) be an \(n\)-chain contained in \(K\) connecting \(p_1\) and \(p_3\). Then there exists another \(n\)-chain \(\tilde{\gamma}\) connecting \(p_1\) and \(p_3\), with \(\overline{p_1p_3} \subset \bigcup \tilde{\gamma}\), and
\[
g(\tilde{\gamma}) \leq g(\gamma).
\]

Proof. Denote \(\gamma_1 := \gamma\), the given chain contained in \(K\) connecting \(p_1\) and \(p_3\).

For any cell \(K_w\) in \(\gamma_1\), \(K_w\) has center in \(D'_0\). We then apply the \((L_1)\)-reflection to those cells centered in \(D'_0\) to obtain another chain that the cells are centered in \(D_1\) (by Lemma 6.4). We do the \((L'_i)\)-reflection for those cells in the new chain centered in \(D_1 \setminus D'_1\) to obtain another chain, denoted by \(\gamma_2\). Then the cells \(K_w\) of \(\gamma_2\) are centered in \(D'_1\).

Inductively we apply this procedure for \(n\) times with \((L_i)\)-reflection and \((L'_i)\)-reflection for \(i = 1, \ldots, n\). Then we obtain a chain \(\gamma_{n+1}\) such that each \(K_w\) in \(\gamma_{n+1}\) has center in \(D'_n\). By Lemma 6.4

\[
g(\gamma_{n+1}) \leq \cdots \leq g(\gamma_2) \leq g(\gamma_1).
\]

Denote \(\gamma_{n+1}\) by \(\tilde{\gamma}\). Now all cells \(K_w\) in \(\tilde{\gamma}\) have word length \(|u| = n\) and center in \(D'_n\). Then one of the line segments of \(\partial K_w\) is on \(\overline{p_1p_3}\). Note that all the reflections keep \(p_1, p_3\) in the two end cells. This yields \(\overline{p_1p_3} \subset \bigcup \tilde{\gamma}\). Since
\[
g(\tilde{\gamma}) \leq g(\gamma_1),
\]
the proof is completed. \(\square\)

Proof of “sufficiency” for \(\Pi_1\). Let \((a, b) \in \Pi_1\), then \(2a + b \geq 1\) and \(a \leq b\). Assume that \(D_k\) is not a metric. Then by Theorem 3.1 and (4.2), for each \(n \geq 1\), there exists an \(n\)-chain \(\gamma_n\) connecting \(p_1\) and \(p_3\) such that \(\lim_{n \to \infty} g(\gamma_n) = 0\). By Lemma 6.5 we obtain a new chain \(\tilde{\gamma}_n\) such that \(\overline{p_1p_3} \subset \bigcup \tilde{\gamma}_n\), and
\[
g(\tilde{\gamma}_n) \to 0 \quad \text{as } n \to \infty.
\]
On the other hand, by Lemma 6.3 (i), we see that \(g(\tilde{\gamma}_n) \geq 1\) for all \(n \geq 1\), a contradiction. Hence \(D_k\) is a metric. \(\square\)

The case for \(\Pi_2\). The idea of proof is the same as the case \(\Pi_1\), but the geometry of the reflection is slightly more complicated. We will concentrate on the set

\[
\Omega = K_37 \cup K_36 \cup K_{35} \cup K_{41} \cup K_{42} \cup K_{43}.
\]

For \(n \geq 1\), we denote \(q_n = F_{36n-1}(1/2, 1/2)\) and \(q'_n = F_{42n-1}(1/2, 1/2)\). We define \(\ell_n\) and \(\ell'_n\) to be the two parallel lines passing through \(q_n\) and \(q'_n\) with slope \(-1\). Clearly, whenever \(\ell_n\) \((\ell'_n)\) intersects the “interior” of a cell \(K_w\), \(|w| \geq n\), the center of \(K_w\) lies in \(\ell_n\) \((\ell'_n)\) respectively.

Let \(\ell_s\) and \(\ell'_s\) be the lines passing through the points \(q_1\) and \(q'_1\) with the same slope 1. Let \(M_1\) denote the hexagon enclosed by lines \(y = 2/9\), \(y = 4/9\), \(\ell_1\), \(\ell'_1\), \(\ell_1\), and \(\ell'_1\); for \(n \geq 2\), let \(M_n\) denote the rectangle enclosed by the lines \(\ell_n\), \(\ell'_n\), \(\ell_n\), and \(\ell'_n\), see Figure 7. Also we let \(\ell\) be the line passing through the point \(q = F_3(1/2, 1) = F_4(1/2, 0)\) (in the center of \(\Omega\)) with slope \(-1\).

The following are some simple geometric properties of the notions we defined.
By symmetry, we have $r_u b$ either reflection of $K$ center of Lemma 6.6. Proof. Suppose $w \geq n$, then $K_w$ has center in $M_n$. Let $K_w$ be the (n)-reflected cell of $K$, we have $g(u) \leq g(w)$. 

**Lemma 6.6.** Let $(a, b) \in \Pi_2$. Suppose $K_w$ with $\pi \geq n \geq 2$ has center in $M_{n-1}$. Let $K_a$ be the (n)-reflected cell of $K_w$, then $K_a$ is centered in $M_n$, and $g(u) \leq g(w)$. 

**Proof.** We assume that $w = w_1 \cdots w_m$ and $u = u_1 \cdots u_m$, where $m \geq n$. The assumption on $K_w$ implies the center of $K_{w_1}$ also lies in $M_{n-1}$ (by (iv)), and hence lies in $\ell_n \cup \ell_n'$ or $\ell_n \cup \ell_n'$ (by (iii)). For the first case, the (n)-reflection of $K_{w_1}$ itself, then $K_{w_1}$ is also a subcell of $K_{w_1}$. By the symmetry, we have $r_{u_i} = r_{w_i}$ for all $i \geq n + 1$ (the $r_j$ is defined in (6.1)), so that $g(u) = g(w)$.

For the second case, let us assume that the center of $K_{w_1}$ lies in $\ell_n \cup \ell_n'$, then the center of $K_{w_1}$ also lies in $\ell_n \cup \ell_n'$, and $w_n = 3$ or $7$, so that $r_{w_n} = a$. It is easy to check that $K_{u_1}$ and $K_{w_1}$ share the same line segment so that $g(u_{i-1}) = g(w_{i-1})$ is either $\frac{a}{b}$ or $\frac{b}{a}$. Furthermore $u_n = 2$ if $w_n = 3$ and $a_n = 8$ if $w_n = 3$ so that $a_n = b$. Thus $g(u_{i-1}) = g(w_{i-1})$ is either $\frac{b}{a}$ or $\frac{b}{a}$ (Corollary [6.2]). By using that $a \geq b$, we have $g(u_{i-1}) \leq g(w_{i-1})$. By symmetry, we have $r_{u_i} = r_{w_i}$ for all $i \geq n + 1$, so that $g(u) \leq g(w)$. \[\square\]
Lemma 6.7. Suppose \((a, b) \in \Pi_2\). Let \(n \geq 2\) and let \(\gamma\) be an \(n\)-chain contained in \(K_{36} \cup K_{42}\), connecting \(p_\ast = F_{36}(1, 0)\) and \(p'_\ast = F_{42}(0, 1)\). Then there exists another \(n\)-chain \(\bar{\gamma}\) with the two end cells touching \(\ell_\ast\) and \(\ell'_\ast\), all its cells intersecting the line segment \(\ell \cap \Omega\), and

\[ g(\bar{\gamma}) \leq g(\gamma). \]

Proof. Let \(\gamma_1 := \gamma\) be a given \(n\)-chain contained in \(K_{36} \cup K_{42}\), connecting \(p_\ast\) and \(p'_\ast\). Then each cell has word length \(n\) and has center in \(M_1\).

By applying the \((2)\)-reflection, we obtain another \(n\)-chain whose cells have centers in \(M_2\) (by Lemma 6.6). We denote this chain by \(\gamma_2\). We carry out the operations \(m\)-reflection for \(m\) from 2 to \(n\) inductively, and obtain a chain \(\gamma_{n+1}\), such that each \(K_w\) in \(\gamma_{n+1}\) has center in \(M_{n+1}\).

We denote by \(\bar{\gamma} = \gamma_{n+1}\) the same as Lemma 6.5 for \(\Pi_1\) with some obvious adjustments (use (ii) to guarantee the cells in \(\bar{\gamma}\) intersect \(\ell\)), and arrive the conclusion. \(\square\)

Proof of the “sufficiency” for \(\Pi_2\). The proof is the same as for the case \(\Pi_1\), using Lemma 6.7 and Lemma 6.3(ii) instead. \(\square\)

7. Application to time change

In this section we consider the time change and the sub-Gaussian heat kernel estimates by summarizing the techniques in \([22, 23, 14]\). We show that the admissible metrics \(D_g\) defined by weights on \(S\) allow us to give a concrete class of geodesic metrics that admit a two-sided sub-Gaussian estimates.

Recall that for two metrics \(d_1\) and \(d_2\) on \(M\), \(d_1\) is said to be quasisymmetric to \(d_2\) if there exists a homeomorphism \(h\) from \([0, \infty)\) to itself with \(h(0) = 0\) such that for any \(t > 0\) and \(x, y, z \in M\), \(d_2(x, z) < h(t)d_2(x, y)\) whenever \(d_1(x, z) < td_1(x, y)\) \([16]\). In \([24\text{ Theorem 15.7}]\), Kigami proved the following proposition.

Proposition 7.1. Let \(d\) be the resistance metric on \(K\) if \(K\) is a nested fractal, or the Euclidean metric if \(K\) is a GSC. Let \(a \in M\). Then \(D_g := D_{ga}\) is quasisymmetric to \(d\).

We call a measure \(\mu\) satisfies the (volume) doubling condition (\(VD\)) if there exists \(C > 0\) such that \(\mu(2B) \leq C\mu(B)\) for any ball \(B\). For a symmetric self-similar measure (i.e., \(\mu_i = \mu_{r(i)}\) for \(\sigma \in G\) and \(i \in \Sigma\), from \([22\text{ Theorems 1.6.6 and 3.4.5}]\), we know that \(\mu\) is volume doubling under \(d\) in Proposition 7.1, hence by quasisymmetry, \(\mu\) is also volume doubling under \(D_g\).

In the following, we consider the time change of the standard Brownian motion on \(K\) with respect to the symmetric self-similar measures. Let \(\rho\) be the renormalization factor of the associated Dirichlet form in \(L^2(K, \mathcal{H})\) (see the paragraph of \([1.3]\) in the Introduction). We define the capacity \(\text{cap}(A, \Omega)(:= R(A, \Omega)^{-1})\) between two open sets \(A, \Omega\) with \(A \subset \Omega\) by

\[ \text{cap}(A, \Omega) := \inf \{ E(u) : u \in \mathcal{F}, u|_A = 1, u = 0 \text{ on } \Omega^c \}. \]
Let $B = B(x, r) := \{ y \in K : |x - y| < r \}$ be a metric ball in $K$ under the metric $d$ in Proposition 7.1. We use $\text{cap}(B, 2B)$ to denote the capacity of two concentric balls $B$ and $2B$. Then, we have \[ \text{cap}(B, 2B) \asymp r^\xi, \tag{7.1} \]
where $\xi = -\log \rho / \log \ell$ for GSC, and $\xi = -1$ for nested fractals. We denote this property by $(\text{cap})_d$.

Next let us consider a symmetric self-similar measure $\mu$ with weights $\{\mu_i\}_{i=1}^N$ and $\mu_i \rho < 1$. Let $g := g_a$ be the symmetric weight function with $a := a(\lambda) = \left( (\mu_1 \rho)^{l^1}, (\mu_2 \rho)^{l^2}, \ldots, (\mu_N \rho)^{l^N} \right) / \sim_G$, where $\lambda > 0$ is such that $a \in M$, and let $D_g$ be the associated admissible metric. Then $D_g$ is quasisymmetric to $d$ as in Proposition 7.1, hence $\mu$ is also a doubling measure with respect to $D_g$. We use $B_D = B_D(x, r)$ to denote the balls for $D_g$, and express (7.1) in terms of $\mu$.

By (7.1) and quasisymmetry, we can easily obtain the following capacity estimate. Let $B_D$ be a ball with radius $r$ under $D_g$, then \[ \text{cap}(B_D, 2B_D) \asymp \mu(B_D) r^{1/\beta}. \]
We denote this property by $(\text{cap})_D$.

We say that the *elliptic Harnack inequality* holds if there is $C > 0$ such that for any nonnegative harmonic function $u$ on $2B$, \[ \sup_B u \leq C \inf_B u, \]
where the ball $B$ is with respect to some reference metric; we denote by $(H)_d$ if the metric balls are under metric $d$, and $(H)_D$ if the metric balls are under $D_g$. It is known that for the standard Dirichlet forms constructed on the elementary fractals, condition $(H)_d$ holds. By quasisymmetry and [7, Lemma 5.3], we obtain $(H)_D$.

**Outline of the proof of Theorem 1.8.** Notice that $\mu$ is a volume doubling measure. By [14, Theorem 1.1], we know that under conditions $(VD)$ and $(RVD)$ (reversed $VD$), we have $(H)_D + (\text{cap})_D \Leftrightarrow (UE) + (NLE)$. (Note that the $(RVD)$ follows from $(VD)$ if $K$ is unbounded, and here we can extend $K$ to infinity by self-similarity.) As the two conditions on the left side are satisfied, the right side also hold. This implies the first part of the theorem.

Since $D_g$ satisfies the MCC if and only if $\lambda = \lambda_0$ such that $a(\lambda_0) \in S \subset M$ (Theorem 4.4), by a standard chain argument (see [12, p.39-41]), we see that $(NLE)$ implies the off-diagonal lower estimate for such $a$ and $D_g$ with $\lambda_0$.

**Remark.** Note that the renormalization factors for the elementary fractals are $0 < \rho < N$ (see ($*$) below), we can conclude that each $a \in M$ can be expressed as in [1.4], \[ a = a(\lambda) := ((\rho \mu_1)^l, \ldots, (\rho \mu_k)^l) \quad \text{for some} \quad \lambda > 0, \]
and hence Theorem 1.8 applies to all $a$ in $M$ and in $S$. Indeed, with a slight abuse of notation, we write $a = (a_1, \ldots, a_N)$ where $a_i = a_j$ for $i \sim_G j$. Then there exists $\lambda > 0$ such that $\sum_{i=1}^N a_i^{1/\lambda} = \rho$ (since the sum goes to 0 as $\lambda \to 0$, and goes to $N$ as $\lambda \to \infty$). Let $\mu_i = a_i^{1/\lambda}/\rho$, it defines a symmetric self-similar measure, and $a$ has the expression as asserted.

(*) For nested fractal, $\rho < 1$; for GSC, we even have $\rho \leq N/\ell^2$, see [2] Proposition 5.2] for SC and it can be generalized to the GSC; notice their $\rho, k$ and $k^2 - R$ (in $\mathbb{R}^2$) correspond to our $\rho^{-1}, \ell$ and $\ell^d - R = N$ respectively.

Finally, we use the standard Sierpinski carpet $K$ in Section 5.2 to give an illustration of Theorem 1.8. Recall that the boundary $S$ of $M$ is

$$\{(a, b) : 2a + b = 1, b \geq a\} \cup \{(a, b) : a + 2b = 1, a \geq b\}.$$  

Let $\rho \in (0, 1)$ be the renormalization factor of the associated Dirichlet form. Let $\mu = (\mu_1, \ldots, \mu_8)$ be a self-similar measure on $K$ with $\mu_{2i+1} = \mu_1$, $\mu_{2i+2} = \mu_2$ for $i = 1, 2, 3$. Let $\beta$ be the unique positive number satisfying

$$\left(\max(\rho \mu_1, \rho \mu_2)\right)^2 + 2(\min(\rho \mu_1, \rho \mu_2))^2 = 1.$$  

Let $D_{\rho \mu \beta}$ be the metric associated with weights $a = (\rho \mu_1)^{1/\beta}$ and $b = (\rho \mu_2)^{1/\beta}$. Then $(a, b) \in S$, and the time change Brownian motion on $K$ via $\mu$ has a continuous heat kernel $p_t(x, y)$ satisfying the estimate as in (1.1) with the metric $d(\cdot, \cdot)$ given by $D_{\rho \mu \beta}(\cdot, \cdot)$.

8. Completing the Proof of Lemma 4.3

Sublemma. For $a \in (0, 1)^k$, $\sup_{n \geq 0} D_8^{(n)}(L, R) < \infty \iff \sup_{n \geq 0} D_8^{(n)}(p_1, p_2) < \infty$.

Without loss of generality, we will consider the GSC is in $\mathbb{R}^2$; since we are mainly considering the $n$-chains here, we will omit the superscript $n$ on $\gamma^{(n)}$ when there is no confusion. The direction “$\Rightarrow$” is trivial, let us prove the direction “$\Leftarrow$”.

Assume $\sup_{n \geq 0} D_8^{(n)}(L, R) < \infty$, and denote the value by $M$. Hence for each $n \geq 0$, there is an $n$-chain $\gamma$ starting from $L$ and ending at $R$ such that $g(\gamma) \leq M$. We will show that there is another $n$-chain $\gamma'$ joining $p_1$ and $p_2$ such that $g(\gamma') \leq CM$ and $C$ is independent of $n$. Then $\sup_{n \geq 0} D_8^{(n)}(p_1, p_2) < \infty$. We divide our proof into two steps in the sequel.

First we specify three types of transformations we will need in the construction. Let $S$ denote the unit square, and consider $F_w(S)$; we use the same $L$ and $R$ to denote the left and right side of $F_w(S)$. Let $\vartheta$ be an $n$-chain in $F_w(S)$ cross over $L$ to $R$ (or the other two sides), we call $\vartheta'$ a $G$-image of $\vartheta$ if it is one of the following:

(i) (Symmetric image) $\vartheta' = \sigma(\vartheta)$ for $\sigma \in G$. In particular, we use $\tau$ to denote reflecting the chain through to the vertical bisector of $F_w(S)$.

(ii) (Deflected image) Let $\vartheta$ get across a diagonal separating $L$ and $R$ of $F_w(S)$, then we can keep one portion of the chain before the cross, and deflect the other
portion along the diagonal to get a new chain \( \theta' \) that reaches the upper or lower sides of \( F_w(S) \).

(iii) (reflected image to neighbor) Let \( F_w(S) \) be the neighbor of \( K_w \) that touches \( \theta \), we can reflect the chain in \( F_w(S) \) to \( F_w(S) \) along the intersecting edge.

Note that in (i), (ii), \( \theta' \) and \( \theta \) in \( K_w \) satisfies
\[
g(\theta') = g(\theta);
\] (8.1)
and in (iii), we have (by Lemma 2.3),
\[
g(\theta) = \frac{a}{d}g(\theta').
\] (8.2)

**Remark 1.** In the proof, we also allow the \( G \)-transform to act on \( F_w(S) \) to some other \( F_w'(S) \) by repeatedly using (i)-(iii) and a scaling. In this case, the \( n \)-chain is transformed to be an \( n + (|w'| - |w|) \)-chain.

**Remark 2.** We need one more technique in the construction: Let \( \theta \) be an \( n \)-chain in \( F_w(K) \) with left and right end cells \( x \) and \( y \) respectively; let \( \theta' \) be another \( n \)-chain with right end cell \( y' \). Consider the intersection of the two chains (use (ii) starting form \( y' \) if necessary), we can produce a new \( n \)-chain \( \theta'' \) by taking \( \theta \) before the intersection and \( \theta' \) after the intersection. Then \( \theta'' \) starts from \( x \) and ends at \( y' \), and

\[
g(\theta'') \leq g(\theta) + g(\theta').
\]

Let \( \gamma \) be an \( n \)-chain in \( K \). Let \( \gamma_L \) and \( \gamma_R \) denote the left half and right half of the chain, divided by the vertical bisector of \( S \). Then
\[
g(\gamma_L) \leq \frac{1}{2}g(\gamma) \quad \text{or} \quad g(\gamma_R) \leq \frac{1}{2}g(\gamma).
\]
We denote the two cases by Case (A) and Case (B) respectively. By similarity, this also applied to \( F_w(S) \).

Let \( \gamma = (w(1), w(2), \ldots) \) be the fixed \( n \)-chain as above. Let \( \Sigma_1 := \{ i \in \Sigma : F_i(K) \cap L \neq \emptyset \} \) be the indices of 1-cells at the outer border of \( K \). Denote by \( a_{\min} = \min[a_i : i \in \Sigma_1] \), and let \( i_0 \) be the index that attains the minimum. We denote by \( p_{i_0} \) the fixed point of \( F_{i_0}, \) i.e., \( F_{i_0}(x) = \rho(x - p_{i_0}) + p_{i_0} \).

**Step 1.**

We will construct an \( n \)-chain \( \gamma' \) between \( p_{i_0} \) and \( \tau(p_{i_0}) \) such that \( g(\gamma') \leq C'g(\gamma) \) for some \( C' > 0 \) independent of \( n \). This gives \( \sup_{n \geq 0} D_k(\tau(p_{i_0}), \tau(p_{i_0})) < \infty \).

By applying \( \tau \) on \( S \), we can assume that \( \gamma \) satisfies Case (A). Let \( K_{u_1} \) be the 1-cell contain \( K_{w(1)} \); let \( S_1 \) be the rectangle that is the union of the 1-cells that intersects \( L \). Let \( j_0 \) be the first \( w(j) \) that exits \( S_1 \), and let \( \gamma_1 \) be the segment before \( j_0 \) that is in \( S_1 \). Using the \( G \)-transform (iii), we can “fold up” the \( \gamma_1 \) into to \( K_{u_1} \), and denote it by \( \overline{\gamma}_1 \); obviously \( \overline{\gamma}_1 \) is a \( n \)-chain in \( K_{u_1} \) from \( F_{u_1}(L) \) to \( F_{u_1}(R) \) and satisfies
\[
g(\overline{\gamma}_1) \leq \frac{a_{u_1}}{a_{\min}}g(\gamma_1) \leq \frac{a_{u_1}}{2a_{\min}}g(\gamma).
\] (8.3)
On \( K_{u_1} \), similar to the situation of \( \gamma \) in \( K \), there are also Case(A) and Case (B) for \( \overline{\gamma}_1 \).
Now let
\[ K_{i_0} \sim K_{i_1} \sim \cdots \sim K_{i_{\kappa_1}} \sim K_{\sigma(i_0)}, \quad t_j \in \Sigma \] (8.4)
be a finite sequence of distinct 1-cells connecting \( K_{i_0} \) and \( K_{\sigma(i_0)} \). Consider the \( G \)-images of \( \gamma_1 \) to \( K_{i_0} \). In Case (A), we use this \( G \)-image in each of the 1-cell of \( \{K_{i_j}\}_{j=1}^{\kappa_1} \) (not include the first and the last cell), and apply the suitable \( G \)-transform (i) or (ii) to paste up these segments to form a connected \( n \)-chain \( \zeta_1 \) in \( \{K_{i_j}\}_{j=1}^{\kappa_1} \); for Case (B), we use \( \tau \) on \( K_{i_1} \) (i.e., reflecting the \( G \)-image on the vertical bisector of \( K_{i_1} \)), and use the same construction to get \( \zeta_1 \). In either cases we have the estimate (using (iii))
\[ g(\zeta_1) \leq \kappa_1 \frac{g(i_0)}{g(u_1)} g(\gamma_1) = \kappa_1 \frac{a_{\min}}{a_{u_1}} g(\gamma_1). \]

To fix our mind, let us assume our \( \zeta_1 \) comes from Case (A). We reflect the first 2-cell of \( \zeta_1 \) to the left and call it \( K_{i_0u_2} \), and the \( n \)-chain there as \( \gamma_2 \). Note that by our choice of the two cases, we have
\[ g(\gamma_2) \leq \frac{1}{2\sigma} \frac{a_{12}}{a_{\min}} g(\gamma). \]

We use it to construct a 2-chain \( \zeta_2 \) to reach \( K_{i_0} \) (running in the opposite direction). Consider a chain of distinct 2-cells
\[ K_{i_0u_2} = K_{i_0s_1} \sim K_{i_0s_2} \sim \cdots \sim K_{i_0s_{s_2}} \sim K_{i_0s_0}, \quad s_j \in \Sigma. \]

We apply the same construction to obtain an \( n \)-chain in \( \{K_{i_0s_j}\}_{j=1}^{s_2} \), and an estimate
\[ g(\zeta_2) \leq \kappa_2 \frac{g(i_0)}{g(u_1u_2)} g(\gamma_1) = \kappa_2 \frac{a_{\min}}{a_{u_1u_2}} g(\gamma_1). \]

Next we continue extending \( \zeta_2 \) to \( \zeta_3 \) in \( K_{i_0} \), we will face the same Case (A), Case (B) situation in \( K_{i_0} \). We proceed as the above to choose \( \gamma_3 \) and construct \( \zeta_3 \). But we need to be caution in Case (B), the reflecting case. Nevertheless, an application of Remark 2 on \( K_{i_0} \) will allow us to connect \( \zeta_2 \) and \( \zeta_3 \).

We apply the same construction through \( 3 \leq \ell \leq n \), we have
\[ g(\gamma_\ell) \leq \frac{1}{2\sigma} \frac{a_{u_1\cdots u_\ell}}{a_{\min}} g(\gamma). \]

Note that the \( \kappa_\ell \), \( 1 \leq \ell \leq n \) is uniformly bounded, denote the bound by \( C \). Then
\[ g(\zeta_\ell) \leq C \frac{g(i_0)}{g(u_1 \cdots u_\ell)} \cdot g(\gamma_\ell) = C \frac{a_{\min}}{a_{u_1 \cdots u_\ell}} \cdot g(\gamma_\ell). \]

It follows that
\[ g(\zeta_{\ell_n}, \cdots, \zeta_1) \] to form an \( n \)-chain in \( K_{i_0} \cup \{K_{i_j}\}_{j=1}^{\kappa_1} \). By reflection the part before the vertical bisector of \( S \) to the right side, and put these
two parts together, we get a new $n$-chain starting at $p_{i_0}$ and ending at $p_{i_0}$. We denote it by $\gamma'$, it satisfies
\[
 g(\gamma') \leq C' \sum_{\ell=1}^{n} g(\xi_\ell) \leq C' \sum_{\ell=1}^{n} \frac{1}{2^\ell} g(\gamma) \leq C'' g(\gamma)
\]
for some $C'$ independent of $n$. Hence $\sup_{n \geq 0} D_g^n(p_{i_0}, \sigma(p_{i_0})) < \infty$.

**Step 2.** Let $M = \sup_{n \geq 0} g(\gamma^{(n)})$, where $\gamma^{(n)}$ is the $n$-chain constructed in Step 1 (the superscript $n$ was suppressed there for simplicity, but we will keep it here). For each $n$, we will use $\{\gamma^{(n-i)}\}_{i=2}^{n-1}$ in the following construction; note that they are all start from $p_0$ and ends at $\tau(p_0)$.

Define $q_i = F_1(\tau(p_{i_0}))$ where $F_1$ is the similitude with fixed point $p_1$. We will use the $\{\gamma^{(n-i)}\}_{i=2}^{n-1}$ to construct an $n$-chain $\xi^{(n)}$ connecting $q_i$’s consecutively for $i = 1, 2, \ldots, n-2$ (and then $p_1$) such that $g(\xi^{(n)}) \leq CM$ for some $C > 0$ independent of $n$. This yields
\[
 \sup_{n \geq 0} g(\xi^{(n)}) \leq CM.
\]

First, let $q_1$ and $q_2$ be connected by some 2-cells in $K_1$ between $K_{12}$ and $K_{1\tau(i_0)}$ (as in [8,4]). Considering $F_{12}(\gamma^{(n-2)})$ as an $n$-chain in $K_{12}$, we can use $G$-images of it to construct an $n$-chain $\xi_1^{(n)}$ in those the 2-cells between $q_1$ and $q_2$ (the same construction of $\xi_1^{(n)}$ as in Step 1), and use Remark 2 to ensure $\xi_1$ reaches $q_1$. Then as before, there is $C' > 0$ (only dependent on $K$ and $a$) such that
\[
 g(\xi_1^{(n)}) \leq C' \cdot g(1) \cdot g(\gamma^{(n-2)}) \leq C_1 a_1 M.
\]

Inductively, for $1 \leq i \leq n-2$, by using $\gamma^{(n-2-i)}$, we get an $n$-chain $\xi_i^{(n)}$ between $q_i$ and $q_{i+1}$ inside $K_{i1}$, and
\[
 g(\xi_i^{(n)}) \leq C' \cdot g(1) \cdot g(\gamma^{(n-i)}) \leq C_1 a_1^i M.
\]

Finally, $q_1$ and $q_{n-2}$ can be connected by some $n$-cells with bounded total weight; we can trivially add several $n$-cells to connect $q_{n-2}$ to $p_1$. Hence $p_1$ can be connected to $q_1$ and satisfies
\[
 g(\xi^n) \leq C_1 M \sum_{i=1}^{n-2} a_1^i + C'.
\]

Denote this chain by $\xi$. Let $q'_1 = \tau(q_1)$, we can reflect $\xi$ to obtain an $n$-chain to connect $q'_1$ and $p_2$. Also we can show that $q_1$ connects $q'_1$ by using the $G$-transform of $\gamma^{(n-1)}$ (with a different constant). Combining the three parts, we obtain an $n$-chain $\xi^{(n)}$ joining $p_1$ and $p_2$ with bound $\leq CM$. This implies
\[
 \sup_{n \geq 0} D_g^n(p_1, p_2) < \infty,
\]
which proves the sublemma. \hfill \Box

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Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China

Email address: 001gqs@163.com

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China, & Department of Mathematics, University of Pittsburgh, Pittsburgh, PA. 15217, U.S.A.

Email address: kslau@math.cuhk.edu.hk

Department of Mathematics, Nanjing University, Nanjing 210093, China

Email address: huaqiu@nju.edu.cn

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

Email address: ruanhj@zju.edu.cn