METRICS ON FRACTALS AND SUB-GAUSSIAN HEAT KERNEL ESTIMATES
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ABSTRACT. It is well-known that for a Brownian motion, if we change the medium to be inhomogeneous by a measure $\mu$, then the new motion (time changed process) will diffuse according to a different metric $D(\cdot, \cdot)$. In [27], Kigami initiated a general scheme to construct such metrics through some self-similar weight functions $g$ on the symbolic space.

In this paper we give an in-depth study of this construction on the nested fractals and the generalized Sierpinski carpets; we assume further that the weight functions $g := g_a$ is generated by the “symmetric” weights $a$. Let $M$ be the set of $a$ such that $D_g$ defines a metric, and let $S$ be the boundary of $M$. We provide a recursive weight transfer construction to obtain the $M$ on the nested fractals, and illustrate the construction by the Lindstrøm snowflake; for the Sierpinski carpet we use a symmetric argument directly to obtain the $M$. Our main result is to characterize the metrics from $g_a$, $a \in S$ satisfy the metric chain condition (MCC), and are equivalent to the geodesic metrics. The MCC is used for the lower estimate of the sub-Gaussian heat kernel. This together with the upper estimate in [27] allows us to have a concrete class of metrics for time change, and for the two sided sub-Gaussian heat kernel estimate.

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1. Introduction

Metric spaces play a prominent role in various fields in mathematics. The analysis on metric spaces together with the measures (metric measure spaces) emerged as an independent research field since the 90’s. The spaces have no a priori smooth structure, but one is able to recover the infinitesimal concepts such as gradient, Laplacian, Dirichlet form, and curvature as in Euclidean function theory, geometric analysis and stochastic analysis \[11, 19, 20, 37\]. In the analysis on fractals, a wealth of exotic examples and different metrics have emerged due to self-similarity. This also provides a fertile background for the theory of metric measure spaces (see e.g., \[24, 28, 13, 14\]).

In the study of Brownian motion on the Sierpinski gasket (SG), Barlow and Perkins \[8\] first established the Li-Yau type sub-Gaussian estimate of the transition density function

\[
p_t(x, y) \approx \frac{1}{V(x, t^{1/\beta})} \exp \left( -c \left( \frac{d^\beta(x, y)}{t} \right)^{1/(\beta-1)} \right),
\]

with \( d(\cdot, \cdot) \) as the Euclidian metric, \( V(x, t^{1/\beta}) = \mu \{ z \in \text{SG} : d(x, z) \leq t^{1/\beta} \} \approx t^{\alpha/\beta} \), where \( \mu \) is the canonical \( \alpha \)-dimensional Hausdorff measure on the SG. The Sierpinski gasket has energy renormalization factor \( \rho = 3/5 \), and walk dimension \( \beta = \log 5 / \log 2 \) \[8\]. This was extended by Lindstrøm \[34\] that the Brownian motion exists on a class of self-similar sets called nested fractals, and the transition density of the Brownian motion on the nested fractal (with a technical path assumption) was shown to enjoy the two sided sub-Gaussian estimate by Kumagai \[31\]. A path breaking extension was proved on the Sierpinski carpet (SC) by Barlow and Bass in their seminal papers \[2, 4\], with \( \beta = \log 8/(\log 3 \cdot \log 2) \) (only approximate value of \( \rho \) is available).

If we change the medium to be inhomogeneous by a measure \( \mu \), then the new motion will have the same paths, but different rate of diffusion, and is associated with different metrics \( D(x, y) \); we call it a time change of the process. One of the main issues is to maintain the sub-Gaussian estimate (1.1) with the new metric \( D(x, y) \). The time change for self-similar measures \( \mu \) on p.c.f. sets that admit harmonic structures and local regular Dirichlet forms and on the SC were first studied by Barlow and Kumagai \[6\], and they showed that the time change is possible if \( \rho \mu_i < 1 \) for all \( 1 \leq i \leq N \), where \( \mu_i \)'s are the probability weights of \( \mu \).

In \[25, 27, 28\], Kigami launched a detail study of the time change problem in full generality based on the Dirichlet forms and the resistance metrics. He set up a general scheme to construct new metrics \( D(x, y) \) on fractals. In \[27\], he showed that for a class of “rationally ramified fractals” (which includes the p.c.f. fractals and the Sierpinski carpet), the upper diagonal estimate of \( p_t(x, x) \) holds if and only if the underlying self-similar measure is volume doubling. Combined with other assumptions, it was shown that the sub-Gaussian estimate (1.1) holds. From the point of view of local regular Dirichlet forms and the associated Hunt processes,
the metric $D(x, y)$ is closely connected with the resistance metric $R(x, y)$ on the Dirichlet space described by the Einstein relation

$$R(x, y)V_D(x, D(x, y)) = D(x, y)\beta.$$  

(1.2)

(see [38]). In [28] and [29], Kigami first used the quasisymmetric homeomorphism to modify the resistance metric or Euclidean metric to obtain new metrics which are more suitable to study $D(x, y)$, and heat kernel estimates. Note that quasisymmetry preserves doubling measures, it provides a convenient tool for the study.

In this paper, we adopt the same setup as [27, 28] to construct metrics by weight functions on the iterated function system (IFS) of fractals. In Kigami’s study, there were few concrete examples or discussion on the class of admissible weight functions. For this reason, we will restrict our consideration to the nested fractals and generalized Sierpinskii carpets (GSC) (see the definition in Section 2). When we do not need to distinguish the two classes, we will just call them elementary fractals for convenience. We will consider the metrics arising from the class of “symmetric self-similar weights” (Definition 1.2). The techniques used throughout the paper depend very strongly on the group of symmetries of the underlying set, which is quite different from the previous investigations. The study leads to new results on the class of admissible metrics for time change, and sharpens the sub-Gaussian heat kernel estimate.

Let $\{F_i\}_{i=1}^N$ denote the associated IFS of an elementary fractal $K$. For $n \geq 1$, let $\Sigma^n = \{1, \ldots, N\}^n$ be the collection of words with length $n$ (by convention, $\Sigma^0 = \{\emptyset\}$). For $w = w_1 \cdots w_n \in \Sigma^n$, we write $K_w = F_{w}(K) := F_{w_1} \circ \cdots \circ F_{w_n}(K)$, and call it an $n$-cell of $K$. Denote by $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$ the collection of all finite words, and by $|w|$ the length of $w$ for each $w \in \Sigma^*$. A finite sequence of words $(w(1), \ldots, w(m))$ in $\Sigma^*$ (or equivalently, cells $(K_{w(1)}, \ldots, K_{w(m)})$ in $K$) is called a chain if $K_{w(i)} \cap K_{w(i+1)} = \emptyset$ for $1 \leq i \leq m - 1$; we use $|w| = m$ to denote the length of the chain. A chain $(w(1), \ldots, w(m))$ is said to connect $x$ and $y$ if $x \in K_{w(1)}$ and $y \in K_{w(m)}$. A chain is called simple if $K_{w(i)} \cap K_{w(j)} = \emptyset$ if and only if $|i - j| \leq 1$.

**Definition 1.1** ([29, 30]). We call $g : \Sigma^* \to (0, 1]$ a weight function if it satisfies:

(i) $g(\emptyset) = 1$, $g(wj) \leq g(w)$ if $w \in \Sigma^*$ and $j \in \{1, \ldots, N\}$;

(ii) $\lim_{n \to \infty} \sup_{w \in \Sigma^n} g(w) = 0$.

We define the total weight of a chain $\gamma = (w(1), \ldots, w(m))$ by $g(\gamma) = \sum_{i=1}^m g(w(i))$, and for any $x, y \in K$,

$$D_g(x, y) = \inf \{g(\gamma) : \gamma \text{ is a chain connecting } x \text{ and } y\}.$$  

(1.3)

It is easy to see that $D_g(\cdot, \cdot)$ is finite ($D_g(x, y) = 0$), symmetric, nonnegative, $D_g(x, x) = 0$ for all $x \in K$, and satisfies the triangle inequality. However, in general, it may happen that $D_g(x, y) = 0$ for some pairs $x \neq y$ in $K$ so that $D_g$ fails to be a metric.

Let $G$ be the symmetric group associated with the elementary fractal $K$ (see Section 2). We will focus on the class of weight functions as following.
Definition 1.2. We call \( g : \Sigma^* \to (0, 1] \) a symmetric self-similar weight function if \( g \) satisfies the following two conditions:

(i). (Self-similarity) \( g(w) = \prod_{i=1}^m g(w_i) \) for \( w = w_1w_2 \cdots w_m \in \Sigma^* \).

(ii). (Symmetry) For all \( \tau \in G \), \( g \circ \tau = g \).

We remark that in the above definition (ii), for \( \tau \in G \), \( \tau \) acts on the cells \( K_w \).

Since the cells and the finite words in \( \Sigma^* \) are in 1-1 correspondence, we can define the action of \( \tau \) on \( \Sigma^* \). For any \( i, j \in \Sigma \), define \( i \sim_G j \) if there is a \( \tau \in G \) such that \( K_j = \tau(K_i) \). Let \( \Sigma \sim_G \) denote the equivalent classes and \( k = \#\Sigma \sim_G \). For example, the Sierpinski gasket and the pentagasket have \( k = 1 \); the more interesting cases are the Lindstrøm snowflake and the standard Sierpinski carpet with \( k = 2 \) (Sections 5, 6).

First by self-similarity and reflecting the cells along hyperplanes of symmetry, we prove an interesting dichotomic result.

Theorem 1.3. Let \( K \) be an elementary fractal, and let \( g \) be a symmetric self-similar weight function. Then \( D_g(\cdot, \cdot) \) is either a metric or identically equal to 0.

Let \( a := (a_1, a_2, \ldots, a_k) \in (0, 1)^k \) be the associated weights of \{\( g(i) : 1 \leq i \leq N \}\}. We write \( g = g_a \) for the weight function associate with respect to \( a \). We define

\[ M := \{a \in (0, 1)^k : D_{g_a} \text{ is a metric on } K\}, \]

and call it the set of admissible weights, and \( D_{g_a} \) an admissible metric (for time change). We have (Propositions 2.7 and 3.5).

Proposition 1.4. Let \( K \) be an elementary fractal, and let \( S = \partial M \cap (0, 1)^k \) be the boundary of \( M \). Then \( M \) is closed, and \( S \) separates \((0, 1)^k\) into two connected components \( M \) and \( M' \), with \( S \subset M \).

There is an expression for \( a \in M \), which is convenient to use in the sequel (Theorem 1.11). For \( a \in M \subset (0, 1)^k \). Consider \( a(\lambda) = (a_1^\lambda, \ldots, a_k^\lambda) \), then \( a(1) = a \), and \( \lim_{\lambda \to 0} a(\lambda) = (1, \ldots, 1) \), \( \lim_{\lambda \to \infty} a(\lambda) = (0, \ldots, 0) \). We show that \( a(\lambda) \in M \) for \( \lambda \) small, and \( a(\lambda) \in M' \) if \( \lambda \) is large (see Section 2 and Figure 1). Hence there is a unique \( \lambda_0 \) such that \( a(\lambda_0) \in S \).

Recall that the main purpose to study the admissible metrics \( D_g \) is to obtain a two-sided sub-Gaussian heat kernel estimate (1.1) with respect to \( D_g \). For the off-diagonal lower estimate in the sub-Gaussian heat kernel, one requires the metric to satisfy the metric chain condition (see e.g. [12,16]). We also remark that the two-sided sub-Gaussian heat kernel estimates does imply the metric chain condition (see [35, Corollary 1.8]).

Definition 1.5. A metric space \((M, d)\) is said to satisfy the metric chain condition \((MCC)\) if there exists a constant \( C > 0 \) such that for any two points \( x, y \in M \) and
for any positive integer \( n \), there exists a sequence \( \{x_i\}_{i=0}^n \) of points in \( M \) such that \( x_0 = x, x_n = y \) and
\[
d(x_i, x_{i+1}) \leq C \frac{d(x, y)}{n}, \quad \text{for all } i = 0, 1, \ldots, n - 1.
\] (1.4)

One of our main results is

**Theorem 1.6.** Let \( K \) be an elementary fractal, then an admissible metric \( D_{ga} \) satisfies the MCC if and only if \( a \in S \).

The proof of the theorem is in Section [4]; we employ a variant of \( D_g \) in (1.3) as follows:
\[
D_g^{(n)}(x, y) = \min \{g(\gamma) : \gamma \text{ is an } n \text{-chain connecting } x \text{ and } y\}
\]
(i.e., each word in \( \gamma \) has length \( n \)). Intuitively, \( D_g^{(n)} \) is determined by the weights of chains of \( n \)-cells of \( K \). The necessity follows from \( \lim_{n \to \infty} D_g^{(n)}(x, y) = \infty \) for \( a \in M \setminus S \) and all \( x \neq y \in K \) (Proposition [4.5]). The sufficiency is a consequence of \( \sup_{n \geq n_0} D_g^{(n)}(x, y) \leq CD_{ga}(x, y) \) for \( a \in S \) (Lemma [4.6]).

The geodesic spaces are a fundamental class of metric spaces in metric geometry. It is easy to show that a geodesic space satisfies the MCC. Conversely, if the MCC holds, then it is bi-Lipschitz equivalent to a geodesic metric (see [22, Proposition A.1]). From this, we have

**Corollary 1.7.** Let \( K \) be an elementary fractal, let \( a \in S \), then there is a geodesic metric \( D_{ga} \) on \( K \) such that \( D_{ga} \) is bi-Lipschitz equivalent to \( D_{ga} \).

It is a challenging task to identify the set of admissible weights \( M \). For nested fractals, we can use a technique of Kumagai [31] to give a constructive description. For each \( a \in M \), there is a recursive relation on the weights of the paths on each level. This allows us to formulate a finite family of “weight transfer matrices” \( \mathcal{K}(a) \). Let \( \lambda_A \) be the maximal positive eigenvalue of \( A \), we have (Theorem [5.1])

**Theorem 1.8.** For a nested fractal, the set of admissible weights is
\[
M = \{a = (a_1, \ldots, a_k) \in (0, 1)^k : \lambda_A \geq 1, \forall A \in \mathcal{K}(a)\}
\]
and its boundary \( S = \{a \in M : \exists A \in \mathcal{K}(a) : \lambda_A = 1\} \).

We use the Lindstrøm snowflake (see Section [5] and Figure [4]) as a simple example to illustrate the theorem. The Lindstrøm snowflake consists of seven maps \( \{F_i\}_{i=1}^7 \), such that \( \{K_i\}_{i=1}^6 \) compose the outer part of the snowflake and are invariant under the group of symmetry; the other one \( K_7 \) sits at the center. For \( a, b \in (0, 1) \), we define a symmetric self-similar weight function \( g_{a,b} \) by
\[
g_{a,b}(i) = \begin{cases} a, & 1 \leq i \leq 6, \\ b, & i = 7. \end{cases}
\]
Corollary 1.9. For the above $g_{a,b}$, $D_{g_{a,b}}$ is a metric if and only if $(a,b)$ satisfies $2a + b \geq 1$ and $3a \geq 1$.

Another interesting example is the standard Sierpinski carpet (SC) $K$, which requires a different technique to identify $M$. The SC is generated by 8 maps $\{F_i\}_{i=1}^8$ (see Section 6). We use $K_i, i = 1, 3, 5, 7$ to denote the four corner cells, and $K_i, i = 2, 4, 6, 8$ to denote the other four cells. These two collections are invariant under the group of symmetries on $K$. For $a, b \in (0, 1)$, we define a symmetric self-similar weight function $g_{a,b}$ by

$$g_{a,b}(i) = \begin{cases} a, & i = 1, 3, 5, 7, \\ b, & i = 2, 4, 6, 8. \end{cases}$$

Proposition 1.10. With the above $g_{a,b}$, $D_{g_{a,b}}$ is a metric if and only if $(a,b)$ satisfies $2a + b \geq 1$ and $a + 2b \geq 1$.

We remark that Proposition 1.10 was raised by Kigami in a conference. In fact he showed the necessity; the difficult part is the sufficiency. The conditions on $(a,b)$ guarantee that there exist $x, y \in K$ such that $g_{a,b}(\gamma) \geq 1$ for all paths $\gamma$ joining $x, y$ (Lemma 6.3). Then $D_{g_{a,b}}$ is a metric by Theorem 1.3.

We will apply the above results to the time change problem. It is well-known that for a nested fractal $K$, if we denote by $\alpha$ the Hausdorff dimension, and let $H^\alpha$ be the normalized $\alpha$-dimensional Hausdorff measure, then there exists a local regular Dirichlet form $(E, F)$ on $L^2(K, H^\alpha)$ satisfying the self-similar energy identity with a uniform renormalization factor $0 < \rho < 1$, that is

$$E(u) = \frac{1}{\rho} \sum_{i=1}^N E(u \circ F_i), \quad \forall u \in F,$$

and the induced process is the Brownian motion. For GSC in $\mathbb{R}^d$ with $d \leq 2$, we also have $0 < \rho < 1$. It may happen that $\rho \geq 1$ for $d \geq 3$ (see [5, Remarks 5.4]). If we let $\mu$ be a self-similar measure on these fractals with weights $\mu_i$, i.e.,

$$\mu = \sum_{i=1}^N \mu_i \circ F_i^{-1},$$

then in the case that $\mu_i \rho < 1$ for all $1 \leq i \leq N$, the measure defines a new local regular Dirichlet form $(\mathcal{E}, \mathcal{F}')$ in $L^2(K, \mu)$ with the same $\mathcal{E}$, which also induces a diffusive process [6].

We call a self-similar measure $\mu$ symmetric if $\mu_i = \mu_{\tau(i)}$ for any $\tau \in G$. It is known that symmetric self-similar measures are doubling measures under the admissible metrics [27, Theorem 3.4.5] (also see Section 7).

With this setup on elementary fractals, the time change for Brownian motion for symmetric self-similar measures, the sub-Gaussian estimate can be stated precisely.
Theorem 1.11. Let $K$ be an elementary fractal. Let $\mu$ be a symmetric self-similar measure, and let $a(\lambda)$ be the curve defined by

$$a(\lambda) = ((\mu_1(\rho))^{\lambda}, (\mu_2(\rho))^{\lambda}, \ldots, (\mu_k(\rho))^{\lambda}) \in M, \quad \lambda \in (0, \infty).$$

Let $\beta = 1/\lambda$, and $D_g$ the admissible metric defined by $a(\lambda)$. Then the time change of Brownian motion with measure $\mu$ has a transition density $p_t(x,y)$ that admits an upper sub-Gaussian estimate (UE)

$$p_t(x,y) \leq C \frac{V(x,t^{1/\beta})}{V(x,t^{1/\beta})} \exp\left(-c \left(\frac{D_g(x,y)^{\beta}}{t}\right)^{1/(\beta-1)}\right),$$

where $V(x,r) := \mu(\{z : D_g(x,z) < r\})$, and a near diagonal lower estimate (NLE): there exists small $\eta > 0$ such that

$$C \frac{V(x,t^{1/\beta})}{V(x,t^{1/\beta})} \leq p_t(x,y), \quad \forall x,y \in K, \ t > 0, \ D_g(x,y) < \eta t^{1/\beta}.$$

In particular if $\lambda = \lambda_0$ such that $a(\lambda_0) \in S$, then $p_t(x,y)$ has the two sided sub-Gaussian estimate as in (1.1).

The upper estimate and the (NLE) were proved by Kigami [27, Theorem 3.2.3] for weights under the more general situation. He also showed the off-diagonal lower estimate holds for $x,y$ if there exists a geodesic path between them. Our contribution in Theorem 1.11 is to provide concrete families of admissible metrics (Theorem 1.8, Corollary 1.9 and Proposition 1.10), and the MCC for $a \in S$ so that the off-diagonal lower estimate can be assured (Theorem 1.6 or Corollary 1.7). The proofs of the theorem in [27] (see also [15]) are quite involved and lengthy, therefore, we give an outline of their proofs incorporating with our setup. We make use of the fact that these admissible metrics are quasisymmetric to the resistance metric for nested fractal, and to the Euclidean metric for GSC [28, 29], and the classical techniques of capacity estimate and Harnack inequality in [15].

To conclude the paper, we also discuss a few examples of self-similar sets with non-symmetric weight functions, in connection to those considered in [27].

We remark that there is another setup to construct new metrics on self-similar sets which is quite different from the present one; the construction is based on certain “augmented trees”, i.e., by adding new edges to the neighboring cells of the trees of the symbolic spaces; the geodesic of these trees are defined by the graph distance. They are hyperbolic graphs (in the sense of Gromov), and there are systematic treatments for such graphs [21, 23, 32, 33]. Also in regard to the quasisymmetry of the SC, Bonk and Merenkov in [9] gave an interesting classification of quasisymmetric self-homeomorphisms for the standard $1/3$-SC and the $1/\ell$-SC.

We organize the paper as follows. In Section 2, we provide some basic definitions and preliminaries of the elementary fractals and the symmetric weight functions. We prove some basic facts of $M$, $M^c$ and the boundary $S = \partial M \cap (0,1)^\ell$. In Section 3 we verify Theorem 1.3 for $D_g$ to be identically zero in $M^c$. The main theorem (Theorem 1.6) on the MCC is proved in Section 4. In Section 5
we prove Theorem 1.8 and use it on the Lindstrøm snowflake for illustration. In Section 6, we study $\mathcal{M}$ of the Sierpinski carpet in detail. In Section 7, we combine Theorem 1.6 together with some previous known results to obtain the heat kernel estimates of the time change Brownian motion for symmetric self-similar measures (Theorem 1.11). Finally, in Section 8, we give some brief discussions on the “non-symmetric” weight functions.

2. Preliminaries and admissible metrics $D_\mathcal{R}(\cdot, \cdot)$

First we define the class of nested fractals introduced by Lindstrøm [34]. Let $K$ be the self-similar set defined by an iterated function system (IFS) \( \{F_i\}_{i=1}^{N} \) of the form $F_i(x) = g O_i x + b_i$, where $N \geq 2$, $0 < g < 1$, and for each $1 \leq i \leq N$, $O_i$ is a $d \times d$ orthogonal matrix and $b_i \in \mathbb{R}^d$. Let $P$ be the set of all fixed points of $\{F_i\}_{i=1}^{N}$. Call $p \in P$ an essential fixed point if there exist distinct $i, j \in \{1, \ldots, N\}$, and $q \in P$ such that $F_i(p) = F_j(q)$, and denote this set by $P_0$. For any distinct points $x, y \in \mathbb{R}^d$, denote the bisecting hyperplane $H_{x,y} = \{z \in \mathbb{R}^d : |x - z| = |y - z|\}$ and write $R_{x,y}$ the orthogonal reflection with respect to $H_{x,y}$; let $G$ denote the group of reflections for $x, y \in P_0$.

**Definition 2.1 (nested fractals).** Let $\{F_i\}_{i=1}^{N}$ and $K$ be as the above. We call $K$ a nested fractal if it satisfies the following conditions:

- (OSC) $\{F_i\}_{i=1}^{N}$ satisfies the open set condition;
- (Connectivity) $K$ is connected;
- (Symmetry) $K$ is invariant under $G$;
- (Nesting) For any $i, j \in \{1, \ldots, N\}$ with $i \neq j$, $F_i(K) \cap F_j(K) = F_i(P_0) \cap F_j(P_0)$.

Next we define another class of self-similar sets which are infinitely ramified, called generalized Sierpinski carpets (GSC), named and first studied by Barlow and Bass [3, 4]. Let $d \geq 2$, $\ell \geq 3$ be integers, and $H_0 = [0, 1]^d$. Set $\mathcal{Q}$ to be the mesh of closed subcubes of size $1/\ell$. For any $Q \in \mathcal{Q}$, let $F_Q : H_0 \to H_0$ be given by $F_Q(x) = x/\ell + p_Q$ where $p_Q$ is chosen so that $F_Q(H_0) = Q$. Let $\mathcal{Q}' \subseteq \mathcal{Q}$ and let $K := \text{GSC}(d, \ell, \mathcal{Q}')$ be the self-similar set associated with the iterated function system $\{F_Q\}_{Q \in \mathcal{Q}'}$. We renumber the elements in $\{F_Q\}_{Q \in \mathcal{Q}}$ by $\{F_i\}_{i=1}^{N}$ with $N = \#\mathcal{Q}'$. Set $H_1 = \bigcup_{Q \in \mathcal{Q}'} F_Q(H_0)$. Let $G$ denote the group of isometries on $H_0$.

**Definition 2.2 (generalized Sierpinski carpets).** A set $K = \text{GSC}(d, \ell, \mathcal{Q}')$ is called a generalized Sierpinski carpet (GSC) if the following conditions are satisfied:

- (Symmetry) $H_1$ is invariant under $G$;
- (Connectivity) $H_1$ is connected;
- (Non-diagonality) For any $x \in H_1$, there exists $r_0 > 0$, such that for all $0 < r < r_0$, $\text{int}(H_1 \cap B(x, r))$ is connected;
- (Borders included) The line segment $[0, 1] \times [0] \times \cdots \times [0]$ is contained in $H_1$. 
Throughout the paper we always assume that $K$ is either a nested fractal or a GSC. When we do not need to distinguish them, we will just call them elementary fractals for convenience.

A weight function $g$ will be assumed to be self-similar and symmetric as in Definition 1.2. For the weight function $a_i = g(i)$, $i \in \Sigma = \{1, 2, \ldots, N\}$, by taking quotient of symmetries, we consider $a \in (0, 1)^k$ where $k$ is the number of elements in the quotient space $\Sigma^a$. For a chain $\gamma = (w(1), \ldots, w(m))$, the weight of a chain $\gamma = (w(1), \ldots, w(m))$ is defined by

$$g(\gamma) = \sum_{i=1}^{m} g(w(i)) \quad \text{with} \quad g(w(i)) = \prod_{j=1}^{n} g(i_j), \quad w(i) = i_1 \cdots i_n.$$  

For the above chain, we also define the union of the cells in $\gamma$ by $\cup_{\gamma} = \bigcup_{i=1}^{m} K_{w(i)}$, and call $(w(i), \ldots, w(j))$ a sub-chain of $\gamma$ for any $1 \leq i \leq j \leq m$.

For any two words $w$ and $v$, if $K$ is a nested fractal, we use $K_w \sim K_v$ to denote $K_w \cap K_v \neq \emptyset$; if $K$ is a GSC in $\mathbb{R}^d$, we use $K_w \sim K_v$ to mean $\dim(F_w(H_0) \cap F_v(H_0)) \geq d - 1$, i.e., either $F_w(H_0) \cap F_v(H_0)$ is a $(d - 1)$-dimensional face or the two sets $F_w(H_0)$ and $F_v(H_0)$ are such that one is contained in the other.

**Lemma 2.3.** Let $K$ be an elementary fractal and let $g$ be defined by $a \in (0, 1)^k$. Suppose $K_{iw} \sim K_{jv}$ for some $i \neq j \in \Sigma$, and $w, v \in \Sigma^*$ with $|w| = |v| \geq 1$. Then $\tau(K_w) = K_v$ for some $\tau \in G$, and $g(\tau(w)) = \frac{a_j}{a_i} g(v)$.

**Proof.** It is known that on a nested fractal, each element in $P_0$ belongs to exactly one $n$-cell for each $n$ (Lindström [34] IV.13 Proposition). As a result, each $n$-cell contains at most one element of $P_0$ for each $n \geq 1$. By applying this property to $K_i$ (or $K_j$), we see that $K_{iw} \cap K_{jv}$ is a singleton, denote by $\{p\}$. Then there exist $p_1, p_2 \in P_0$ such that $F_i(p_1) = F_j(p_2) = p$. Let $\tau \in G$ be the orthogonal reflection with respect to $H_{p_1, p_2}$. Then $\tau(K_w) = K_v$ ([24, p.115]).

For the GSC, $K_{iw} \sim K_{jv}$ is a $(d - 1)$-dimensional face. Hence $K_w$ and $K_v$ are in the opposite face of $H_0$, and $\tau(K_w) = K_v$ for a reflection on $H_0$.

The second part follows from $g(\tau(w)) = g(i)g(w)$ and $g(v) = g(j)g(v) = g(j)g(w)$.

For $a \in (0, 1)^k$, denote $a_\ast = \min \{a_i : 1 \leq i \leq k\}$, $a^* = \max \{a_i : 1 \leq i \leq k\}$. The following simple property will be used frequently.

**Proposition 2.4.** Let $K$ be an elementary fractal, let $g := ga$, $a \in (0, 1)^k$ and let $c = a^*/a_\ast$. Suppose $K_w \sim K_v$ with $|w| = |v|$. Then we have

$$c^{-1} g(v) \leq g(w) \leq cg(v).$$

Furthermore, if we reflect a chain $\gamma$ contained in $K_v$ along the appropriate hyperplane of $K_w \sim K_v$, and denote it by $R(\gamma)$, then $R(\gamma)$ is a chain contained in $K_w$ and $c^{-1} g(\gamma) \leq g(R(\gamma)) \leq cg(\gamma)$.  


Proof. We can write \( w = u i v' \), \( v = u j v' \) for some \( i \neq j \in \Sigma, u, w', v' \in \Sigma^* \). By Lemma 2.3, the proposition follows readily. \( \square \)

Let \( \Gamma \) denote the class of chains \( \gamma = (w(1), \ldots, w(m)) \) satisfying \( K_{w(i)} \sim K_{w(i+1)} \) for all \( i \). Similar to \( D_g \) in (1.3), define

\[
D'_g(x, y) = \inf\{g(\gamma) : \gamma \in \Gamma \text{ connects } x \text{ and } y\}.
\]

Clearly for the nested fractals, \( D'_g \) is just the same as the \( D_g \).

**Corollary 2.5.** For the GSC, we have \( D'_g(\cdot, \cdot) \simeq D_g(\cdot, \cdot) \).

**Proof.** By the non-diagonality assumption in the definition of GSC, we see that if \( K_w \cap K_v \neq \emptyset \) with \( |w| = n \), then there exists a chain \( \gamma = \{w_1, \ldots, w_m\} \) of \( n \)-cells such that \( K_{w_1} \sim K_{w_{i+1}} \), \( K_{w_1} = K_w \), \( K_{w_m} \sim K_v \), and \( m \leq 2^d - 1 \). By Proposition 2.4, we have

\[
c^{-2(i-2)}g(w_i) \leq c^{-i+1}g(w_i) \leq g(w), \quad 1 \leq i \leq m.
\]

Hence we can replace the defining chains in \( D_g(x, y) \) by the chains in \( D'_g(x, y) \) and keep the above inequality. This yields \( 2^{-d}c^{-2(i-2)}D'_g(x, y) \leq D_g(x, y) \leq D'_g(x, y) \) for all \( x, y \in K \). \( \square \)

**Remark:** For the GSC, the chains in \( \Gamma \) with \( K_w \sim K_v \) are more convenient to use than \( K_w \cap K_v \neq \emptyset \). We will use it without explicitly mentioning.

Denote by \( M = \{a \in (0, 1)^k : D_{g_a} \text{ is } \text{a metric on } K\} \), and let \( M^c = (0, 1)^k \setminus M \). We call \( M \) the set of \textit{admissible weights}, and \( D_{g_a} \) the \textit{admissible metric} (determined by \( a \)). If no confusion, we also say that \( g \) is a symmetric self-similar weight function to mean \( g = g_a \) for some \( a \in (0, 1)^k \).

**Lemma 2.6.** Suppose \( a, b \in (0, 1)^k \) and \( b \geq a \) (coordinatewise). Then (i) \( a \in M \) implies \( b \in M \); (ii) \( b \in M^c \) implies \( a \in M^c \).

**Proof.** It suffices to show that (ii) holds. Suppose that \( b \in M^c \). By definition, there exist two distinct points \( x, y \in K \) and a sequence of chains \( \{\gamma_n\}_n \) between \( x \) and \( y \) such that \( g_b(\gamma_n) \to 0 \) as \( n \to \infty \). By assumption we have \( g_a(\gamma_n) \leq g_b(\gamma_n) \), hence \( g_a(\gamma_n) \to 0 \) as \( n \to \infty \), which implies that \( a \in M^c \). \( \square \)

The following is a crude estimation of \( M \) and \( M^c \).

**Proposition 2.7.** (i) For a nested fractal \( K \), there exist \( 0 < c \leq C < 1 \) such that \( [C, 1)^k \subset M \) and \( (0, c)^k \subset M^c \); (ii) For a GSC, we have \( [1/\ell, 1)^k \subset M \) and \( (0, 1/\ell)^k \subset M^c \).
Proof. (i) For two distinct $p, q \in P_0$, let $n(p, q)$ be the minimal length of the chains consisting of 1-cells between $p$ and $q$. Let $n_\ast, n^\ast$ be the minimum and maximum of $n(p, q)$ among all the pairs $p, q \in P_0$, respectively. As each $p \in P_0$ is contained in exactly one 1-cell (Lindström [34]), therefore, we have $n^\ast \geq n_\ast \geq 2$.

Let $g$ be the weight function generated by $a = (1/n_\ast, \ldots, 1/n_\ast)$. We show that $a \in \mathcal{M}$, then by Lemma 2.6, $[1/n_\ast, 1]^k \subset \mathcal{M}$. The first part of (i) follows by letting $C = 1/n_\ast$.

For this, we let $p$ and $q$ be two distinct points in $P_0$. For any given simple chain $\gamma$ between $p$ and $q$ (i.e., $K_{\gamma(i)} \cap K_{\gamma(j)} \neq \emptyset$ if and only if $|i-j| \leq 1$), choose a cell $K_w$ in $\gamma$ such that $w$ has the largest word length in $\gamma$. Let $w'$ be the parent of $w$, and $J(w')$ be the set of the $N$ children of $w'$. Then by using the definition of $n_\ast$ on $K_w$, there exist at least $n_\ast$ cells in $J(w')$ (including $w$) contained in $\gamma$. Observe that $g(u) = n^{-1}g(w')$ for any $u \in J(w')$, we have
\[ g(w') \leq \sum_{u \in J(w') \cap \gamma} g(u). \]

Hence if we replace the sub-chain of $\gamma$ in $J(w')$ by $w'$, then we get a new chain $\gamma_s$ with $g(\gamma) \geq g(\gamma_s)$. We can also assume that $\gamma_s$ is a simple chain by removing some cells in $\gamma_s$. By repeating this “merging” process to each cell with largest word length in the new chains, we finally get the trivial chain $\{0\}$. Thus $g(\gamma) \geq g(\emptyset) = 1$ and $D_{\varepsilon}(p, q) \geq 1$. For arbitrary distinct two points $p, q \in K$, we can use a similar argument to show that $D_{\varepsilon}(p, q) > 0$. Hence $(1/n_\ast, \ldots, 1/n_\ast) \in \mathcal{M}$.

Next we show that $(0, 1/n^\ast)^k \subset \mathcal{M}$. Let $b \in (0, 1/n^\ast)^k$ and fix any two distinct points $p$ and $q$ in $P_0$. For any $m \geq 0$, choose a chain $\gamma_m$ of $m$-cells between $p$ and $q$, where the length of $\gamma_m$ is not larger than $n^\ast m$. Then $D_{\varepsilon}(p, q) \leq (b^\ast)^m$ ($b^\ast = \max_i b_i$). By letting $m \to \infty$, we have $D_{\varepsilon}(p, q) = 0$ since $b^\ast < 1/n^\ast$. Hence we have $(0, 1/n^\ast)^k \subset \mathcal{M}$ and the second part of (i) follows by letting $c = 1/n^\ast$.

(ii) Let $g$ be the weight function generated by $a = (1/\ell, \ldots, 1/\ell)$. Let $p, q$ be two vertices of $H_0$ such that the line segment $\overline{pq}$ is parallel to an axis. Consider any chain $\gamma$ between $p$ and $q$. Since any cell with word length $m \geq 0$ has weight $\ell^{-m}$, the projection of $\gamma$ on the side $\overline{pq}$ covers $\overline{pq}$ so that $g(\gamma) \geq 1$.

Now let $x, y \in V_\ast = \bigcup_{n \in \mathbb{Z}} F_n(V_0)$ where $V_0$ is the set of vertices of $H_0$. Then $x, y$ can be connected by finitely many line segments parallel to the axes. By self-similarity and the above, we have $D_{\varepsilon}(x, y) \geq |x-y|$. The density of $V_\ast$ in $K$ implies that $D_{\varepsilon}(x, y) > 0$ for all distinct $x, y \in K$. By Lemma 2.6, $(1/\ell, 1)^k \subset \mathcal{M}$.

To prove the last part, let $b = (b_1, \ldots, b_k)$ such that $b^\ast < 1/\ell$. Let $p$ and $q$ be two end points on a side of the cube $H_0$. Consider a simple chain $\gamma_m$ with word length $m$ connecting $p$ and $q$ and along the edge of the GSC. Then we have $g(\gamma_m) \leq \ell^m b^\ast m \to 0$ as $m \to \infty$. Therefore $D_{\varepsilon}(p, q) = 0$ and $b \notin \mathcal{M}$. It follows that $(0, 1/\ell)^k \subset \mathcal{M}$. \qed

Let $g := g_a$ be the weight function on $K$ with $a = (a_1, a_2, \ldots, a_k) \in (0, 1)^k$. Define a curve $a(\lambda) : (0, \infty) \to (0, 1)^k$ by
\[ a(\lambda) = (a_1^\lambda, \ldots, a_k^\lambda), \quad \lambda \in (0, \infty). \]
Clearly $a(1) = a$. By Proposition 2.7, $a(\lambda) \in M$ for $\lambda$ small enough, and $a(\lambda) \in M^c$ for $\lambda$ large enough.

Since $a(\lambda_1) > a(\lambda_2)$ (coordinatewise) for $\lambda_1 < \lambda_2$, by Lemma 2.6 we have $a(\lambda_2) \in M$ implies $a(\lambda_1) \in M$. This yields a unique boundary point $\lambda_a > 0$ such that $a(\lambda) \in M$ if $\lambda > \lambda_a$, and $a(\lambda) \in M^c$ if $\lambda < \lambda_a$ (see Figure 1). We denote by $\Lambda_k = \{a \in (0,1)^k : \sum_{i=1}^k a_i = 1\}$ the set of normalized vectors of $(0,1)^k$.

Let $S = \{a(\lambda_a) : a \in \Lambda_k\}$.

It follows that $S$ separates $(0,1)^k$ into two connected components $M$ and $M^c$. We call $S$ the boundary surface of $M$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$M$, $M^c$ and $S$}
\end{figure}

Corollary 2.8. With the above notations, let $\partial M$ denote the boundary of $M$ in $(0,1)^k$. Then $S = \partial M$.

Proof. It is clear that $S \subseteq \partial M$. To prove the reverse inclusion, let $b \in \partial M$. Suppose $b \notin S$. Then there is an $a \in \Lambda_k$ such that $b = a(\lambda)$, and $a(\lambda_a) \in S$ and $\lambda \neq \lambda_a$. If $\lambda < \lambda_a$, let $\lambda_1$ be such that $\lambda < \lambda_1 < \lambda_a$, then $a(\lambda_1) \in M$. By Lemma 2.6(i), we see that $M$ contains all $b' > a(\lambda_1)$, which is a subset of $M^c$, the interior of $M$. Hence $b = a(\lambda) \in M^c$. This contradicts that $b \in \partial M$. If $\lambda_a < \lambda$, then we can use a similar method (apply Lemma 2.6(ii)) to obtain a similar contraction. \qed

3. A dichotomy for $D_g(\cdot, \cdot)$

Note that for any weight function $g$, $D_g(\cdot, \cdot)$ is nonnegative, finite, symmetric and satisfies the triangle inequality. Also $D_g(x, x) = 0$ for all $x \in K$, but in general, it may happen that $D_g(x, y) = 0$ for some pairs $x \neq y$ in $K$, so that $D_g$ fails to be a metric. In this section we prove a stronger conclusion on the elementary fractals.

Theorem 3.1. Let $K$ be an elementary fractal, and let $g$ be a symmetric self-similar weight function. Then $D_g$ is either a metric on $K$ or identically 0 on $K$. 
Equivalently, the theorem says that \( D_{g_a} \equiv 0 \) for \( a \in \mathcal{M}^c \). Since the proof involves different symmetries in the nested fractals and the GSC, we divide the proofs into two separate parts.

**Lemma 3.2.** Let \( K \) be a nested fractal, and let \( P_0 \) be the set of essential fixed points. Suppose \( D_g(q^*, s^*) = 0 \) for some distinct \( q^*, s^* \in P_0 \), then \( D_g(q, s) = 0 \) for all \( q, s \in P_0 \).

**Proof.** We define an equivalence relation in \( P_0 \) as follows: for any two points \( q \) and \( s \) in \( P_0 \), we write \( q \sim s \) if either \( q = s \) or there is a finite sequence of points \( \{q_i\}_{i=0}^l \) in \( P_0 \) with \( q_0 = q \) and \( q_l = s \), and for each \( 0 \leq i < l - 1 \), there is a \( \tau_i \in G \) satisfying \( \tau_i(q_i) = q^* \) and \( \tau_i(q_{i+1}) = s^* \). It is easy to check that \( \sim \) is indeed an equivalence relation on \( P_0 \), which is invariant under \( G \), that is, for any points \( q, s \in P_0 \), \( \tau \in G \), if \( q \sim s \), then \( \tau(q) \sim \tau(s) \). Obviously, by using the triangle inequality of \( D_g \), we have \( D_g(q, s) = 0 \) for any \( q \sim s \).

If \( q, s \in P_0 \) are two distinct points, let \( R_{q,s} \) be the orthogonal reflection along \( H_{q,s} = \{ z \in \mathbb{R}^d : |q - z| = |s - z| \} \). Let \( H_q, H_s \) be the closed half-space containing \( q, s \) respectively. Then by Sabot [36, Lemma 6.4], for any \( G \)-invariant equivalent relation on \( P_0 \), any equivalent class intersects both half-spaces. Hence there is a point \( q' \in P_0 \cap H_s \) with \( q \sim q' \). Therefore \( D_g(q, q') = 0 \), and there is a sequence of chains \( \{\eta_m\}_{m \geq 0} \) between \( q \) and \( q' \) with the total weight \( g(\eta_m) \) tending to 0 as \( m \to \infty \).

For \( m \geq 0 \), let \( \eta_m \) be the chain obtained by reflecting the part of the cells in \( \eta_m \) contained in \( H_q \) to \( H_s \) by \( R_{q,s} \). Then \( \eta_m \) connects \( q' \) and \( s \). By the symmetry of the weight function \( g \), we have \( g(\eta_m) = g(\eta_m) \to 0 \) as \( m \to \infty \), which yields \( D_g(q', s) = 0 \). Then \( D_g(q, s) \leq D_g(q, q') + D_g(q', s) = 0 \) and the lemma holds. \( \square \)

**Proof of Theorem 3.1** for nested fractals. Let \( x_0, y_0 \) be two distinct points in \( K \) with \( D_g(x_0, y_0) = 0 \). Without loss of generality, assume \( K \) is the smallest subcell which contains both \( x_0 \) and \( y_0 \). Then there exist distinct \( i, j \in \{1, 2, \ldots, N\} \) such that \( x_0 \in K_i \) and \( y_0 \in K_j \). Let \( E_{y_0} = \bigcup_{u \in \Sigma, y_0 \in K_u} F_u(P_0) \). Then \( E_{y_0} \) is a finite set and \( x_0 \not\in E_{y_0} \).

Let \( K_{v, v} \in \Sigma^* \), be a cell containing \( x_0 \) with word length large enough such that \( K_v \cap E_{y_0} = \emptyset \). Let \( n = |v| \) and \( m_0 \in \mathbb{Z}^+ \) satisfying \( m_0^{-1} < a_o^\alpha \). From \( D_g(x_0, y_0) = 0 \), for each positive integer \( m \geq m_0 \), there exists a chain \( \gamma_m \) between \( x_0, y_0 \) such that \( g(\gamma_m) < m^{-1} \). As \( m^{-1} < a_o^\alpha \), every cell in \( \gamma_m \) has length \( > n \). By the nesting property, the chain \( \gamma_m \) must pass through one of the points in \( F_v(P_0) \), and one of the points in \( E_{y_0} \). By the finiteness of both \( \#F_v(P_0) \) and \( \#E_{y_0} \), there is a subsequence of \( \{\gamma_m\} \), still denoted by \( \{\gamma_m\} \), and \( \gamma^* \in F_v(P_0) \), \( \tilde{s} \in E_{y_0} \), such that each \( \gamma_m \) passes through both \( q^* \) and \( \tilde{s} \). Let \( \{\gamma_m\} \) be sub-chains of \( \{\gamma_m\} \) connecting \( q^* \) and \( \tilde{s} \). Let

\[
E_{q^*} = \bigcup_{w \in \Sigma^n : q^* \in K_w} F_w(P_0) \setminus \{q^*\}.
\]

By taking subsequence and sub-chains again, we can find a point \( s^* \in E_{q^*} \), a word \( w \in \Sigma^n \), and a sub-chain \( \gamma'_m \) of \( \gamma_m \) between \( q^* \) and \( s^* \), contained in \( K_w \). Since our
choices of sub-chain $\overline{\gamma}_m$ and $\gamma'_m$ of $\gamma_m$ do not increase the total weight, we have $D_g(q^*, s^*) = 0$.

Using self-similarity, we can dilate $q^*$, $s^*$ to be two distinct points in $P_0$. By Lemma 3.2, we have $D_g(q, s) = 0$ for all $q, s \in P_0$. Hence by self-similarity, $D_g(q, s) = 0$ for all $q, s \in F_u(P_0), u \in \Sigma^*$. In general, for any two points $q, s$ in $K$, we can use the approximation $V_* = \bigcup_{u \in \Sigma^*} F_u(P_0)$ to show that $D_g(q, s) = 0$. This completes the proof.

For any point $p \in \mathbb{R}^d$, we denote by $x_i(p)$ the $i$-th coordinate of $p$ with $1 \leq i \leq d$. For any subset $E$ of $\mathbb{R}^d$, we denote by $\pi_i(E)$ the orthogonal projection of $E$ onto the $i$-th axis, i.e., $\pi_i(E) = \{x_i(p) : p \in E\}$. We will use $K_i(n) \sim K_{i+1}(n)$ for the connection of a chain (see Corollary 2.5). Similar to Lemma 3.2, we have the following for the GSC.

**Lemma 3.3.** Let $K$ be a GSC. Suppose $q^*$ and $s^*$ are on the two opposite faces of the cube $H_0 = [0, 1]^d$ and $D_g(q^*, s^*) = 0$, then $D_g(q, s) = 0$ for all $q, s$ in the vertices of $H_0$ with $\overline{\gamma}$ parallel to one of the coordinate axes.

**Proof.** We assume that $x_1(q^*) = 0$, $x_1(s^*) = 1$. Define $q', s'$ by changing the first coordinates of $q^*, s^*$: $x_i(q') = 1$, $x_i(s') = 0$ and $x_i(q^*) = x_i(s^*) = 1$ for $2 \leq i \leq d$. By symmetry, $D_g(q', s') = 0$.

For $0 \leq j < \ell$, let $p_j$ be the point in $K$ with coordinates $x_i(p_j) = j/\ell$, and $x_i(p_j) = 0$ for $2 \leq i \leq d$. For $1 \leq j < \ell$, let $F_j : H_0 \to H_0$ be such that $F_j(x) = x/\ell + p_{j-1}$. From the borders included condition of GSC in Definition 2.2, $\{F_j\}_{1 \leq j \leq \ell}$ is a subset of $\{F_q\}_{q \in \Sigma}$ such that each cube $F_j(H_0)$ locates along the line segment $p_0p_{\ell}$. It follows that $D_g(F_j(q'), F_j(s')) = D_g(F_j(q^*), F_j(s^*)) = 0$ for $1 \leq j \leq \ell$.

Notice that $F_j(s^*) = F_{j+1}(s^*)$ and $F_j(q^*) = F_{j+1}(q^*)$ for $1 \leq j < \ell$. If $\ell$ is an odd number, we have

$$D_g(F_j(q^*), F_j(s^*))$$

$$\leq D_g(F_1(q^*), F_1(s^*)) + D_g(F_1(q^*), F_2(q^*)) + \cdots + D_g(F_{\ell-1}(q^*), F_\ell(s^*))$$

$$= D_g(F_1(q^*), F_1(s^*)) + D_g(F_2(q^*), F_2(s^*)) + \cdots + D_g(F_{\ell-1}(q^*), F_\ell(s^*)) = 0.$$ 

By using this repeatedly, we see that $D_g(F_{\ell}(q^*), F_{\ell}(s^*)) = 0$ for all integers $n \geq 0$. By the continuity of $D_g$, we have $D_g(p_0, p_\ell) = 0$. Similarly, if $\ell$ is even, by considering $D_g(F_{\ell+1}(q^*), F_{\ell+1}(s^*))$ instead, we also have $D_g(p_0, p_\ell) = 0$. The lemma follows by symmetry.

**Proof of Theorem 3.1 for the GSC.** Let $q_0, s_0$ be two distinct points in $K$ with $D_g(q_0, s_0) = 0$. We select an $n > 0$ such that $2 \cdot \ell^{-n} < \max\{|x_1(q_0) - x_1(s_0)| : 1 \leq i \leq d\}$. Without loss of generality, we assume that $x_1(s_0) > x_1(q_0) + 2 \cdot \ell^{-n}$. Define $\alpha_n = [\ell^n(1 - q_0)] \cdot \ell^{-n}$, where $[\ell]$ is the minimum integer greater than or equal to $\ell$.

Let $m_0 \in \mathbb{Z}^+$ satisfy $m_0^{-1} < \alpha_n$. From $D_g(q_0, s_0) = 0$, then for $m \geq m_0$, there exists a chain $\gamma_m$ between $q_0$ and $s_0$ such that $g(\gamma_m) < m^{-1}$. From $m^{-1} < \alpha_n$, every
cell in $\gamma_m$ has length greater than $n$. Therefore, we can pick two points $q_m$ and $s_m$ in $K$, and a sub-chain $\tilde{\gamma}_m$ of $\gamma_m$ between $q_m$ and $s_m$, such that $x_1(q_m) = \alpha_n$, $x_1(s_m) = \alpha_n + \ell^{-n}$, and $\pi_1(\tilde{\gamma}_m) = [\alpha_n, \alpha_n + \ell^{-n}]$. By taking subsequence, we can find two $n$-cells $K_w$ and $K_v$ (independent of $m$) such that $q_m \in K_w$ and $s_m \in K_v$, and $\pi_1(K_w) = \pi_1(K_v) = [\alpha_n, \alpha_n + \ell^{-n}]$.

For each $m \geq m_0$, we replace each cell $K_w$ in $\tilde{\gamma}_m$ by $K_{w0}$ and delete the repeated ones to obtain a chain $\eta_m$ consisting of $n$-cells. Then obviously, the chain $\eta_m$ starts from $K_w$ and ends with $K_v$, and $\pi_1(\cup \eta_m) = [\alpha_n, \alpha_n + \ell^{-n}]$. Also for every two successive cells in $\eta_m$, they share a same $(d-1)$-dimensional hyperplane which is always parallel to the 1-st coordinate axis. Reflecting the chain $\tilde{\gamma}_m$ according to these $(d-1)$-dimensional hyperplanes along $\eta_m$ from $K_w$ to $K_v$ successively, we obtain a new chain $\gamma'_m$ contained in $K_v$ (see Figure 2). Note that there is a point $q'_m$ in $K_v$ with $x_1(q'_m) = \alpha_n$ such that $\gamma'_m$ is between $q'_m$ and $s_m$. By Proposition 2.4, we know that

$$g(\gamma'_m) \leq c^{\ell^{d-1}m}g(\tilde{\gamma}_m) \leq c^{\ell^{d-1}m}g(\gamma_m) \leq c^{\ell^{d-1}m}m^{-1}, \quad \text{where } c = a^* / a_s. \tag{3.1}$$

By taking subsequence if necessary, we may assume that $q'_m$ converges to $q^*$, and $s_m$ converges to $s^*$. Then $q^*, s^* \in K_v$ with $x_1(q^*) = \alpha_n$, $x_1(s^*) = \alpha_n + \ell^{-n}$, and $D_b(q^*, s^*) = 0$.

![Figure 2. Chains $\gamma_m$ and $\tilde{\gamma}_m$](image)

Using self-similarity, we can dilate $q^*$, $s^*$ to the two opposite faces of $H_0$. Then by Lemma 3.3, $D_b(q, s) = 0$ for all $q, s$ in the vertices of $H_0$ with $\overline{q}c$ parallel to one of the coordinate axes. Since any two points in $V_c$ can be connected by finitely many pairs of vertices of $F_u(H_0)$, by using the self-similarity and the triangle inequality of $D_b$, we must have that $D_b(q, s) = 0$ for any $q, s \in V_c$. By using the continuity of $D_b$ w.r.t. the Euclidean metric, we have that $D_b(q, s) = 0$ for all $q, s \in K$. This completes the proof.

We will call a process on a chain $\gamma$ splitting if it splits the cells in $\gamma$ to obtain a finer chain $\gamma'$ such that $g(\gamma') \leq g(\gamma)$.
Lemma 3.4. Let $K$ be an elementary fractal. Assume that $D_{g}$ is not a metric on $K$, then there exists $N_{0} > 0$, such that for any two points $x$ and $y$ in $K$, there is a sequence of chains $(γ_{n})_{n=0}^{∞}$ between $x$ and $y$, such that each chain is a splitting of the previous one, and

(i) $g(γ_{n}) \leq 1/4^{n+1}$;
(ii) $nN_{0} < |u| \leq (n + 1)N_{0}$ for any $u ∈ γ_{n}$.

Proof: We first claim that there exists a positive integer $N_{0}$, such that for any two points $x$ and $y$ in $K$, there is a chain $γ_{x,y}$ between $x$ and $y$ such that the following two conditions hold:

$$g(γ_{x,y}) ≤ 1/4, \quad \text{and} \quad 0 < |u| ≤ N_{0}, \; ∀ u ∈ γ_{x,y}. \quad (3.2)$$

Indeed, let $N_{1}$ be the smallest integer such that $a^{−N_{1}} ≤ 1/16$. Let $K_{w}$ and $K_{v}$ be two cells with $|w| = |v| = N_{1}$, $x ∈ K_{w}$ and $y ∈ K_{v}$. In the case that $K_{w} ∩ K_{v} ≠ ∅$, we define $γ_{w,v} = \{w, v\}$ so that

$$g(γ_{w,v}) = g(w) + g(v) ≤ 2a^{−N_{1}} ≤ 1/8.$$ 

In the case that $K_{w} ∩ K_{v} = ∅$, by Theorem 3.1 there is a chain $η_{w,v}$ connecting $K_{w}$ and $K_{v}$ such that $g(η_{w,v}) ≤ 1/8$. Let $γ_{w,v}$ be the chain constructed by adding $η_{w,v}$ in between $w$ and $v$. Then we have

$$g(γ_{w,v}) = g(w) + g(η_{w,v}) + g(v) ≤ 1/4.$$ 

Now set $N_{w,v} = \max\{|u| : u ∈ γ_{w,v}\}$, and let $N_{0}$ be the maximum of $N_{w,v}$ among all the pairs $w, v$ in $Σ^{N_{1}}$. Then for all $γ_{w,v}$, $g(γ_{w,v}) ≤ 1/4$ and $γ_{w,v}$ consists of $u$ such that $0 < |u| ≤ N_{0}$, and the claim follows. For simplicity, we write $γ_{0} ≔ γ_{x,y} ≔ \{w_{1}, \ldots, w_{m}\}$.

We now construct $γ_{1}$. For each word $w_{i}$ in $γ_{0}$, we perform a splitting as follows. Let $x' ∈ K_{w_{1}} ∩ K_{w_{i+1}}$ and $y' ∈ K_{w_{i}} ∩ K_{w_{i+1}}$ (if $i = 1$, we just take $w_{0} = w_{1}$ and $x = x'$, and similarly for $i = m$). For each $w_{i}$, consider the pull-back $F_{w_{i}}^{-1}(K_{w_{i}}) = K$, we apply the claim to $x'' = F_{w_{i}}^{-1}(x')$ and $y'' = F_{w_{i}}^{-1}(y')$ to obtain a chain $γ_{x'',y''}$ satisfying (3.2). Consider $F_{w_{i}}(γ_{x'',y''})$, which is a chain between $x'$ and $y'$ in $K_{w_{i}}$ consists of cells $F_{w_{i}}(K_{u})$ for each $u ∈ γ_{x'',y''}$. By self-similarity of $g$ and (3.2), the chain has the following property:

$$g(F_{w_{i}}(γ_{x'',y''})) = \sum_{u ∈ F_{w_{i}}(γ_{x'',y''})} g(u) = \sum_{u ∈ γ_{x'',y''}} g(u) \cdot g(w_{i}) ≤ (1/4) \cdot g(w_{i}), \quad (3.3)$$

with $|w_{i}| < |u| ≤ N_{0} + |w_{i}|$ for all $u ∈ F_{w_{i}}(γ_{x'',y''})$.

Now we replace the word $w_{i}$ in $γ_{0}$ by the chain $F_{w_{i}}(γ_{x'',y''})$ for each $i$, and obtain a new chain. We keep doing the same splitting for words with length $≤ N_{0}$ in the new chain, and after finite many times, we obtain a chain $γ_{1}$ between $x$ and $y$ such that each word in $γ_{1}$ has length $> N_{0}$. Since we are using the claim to do the splitting, we have each word in $γ_{1}$ has length $≤ 2N_{0}$ (Indeed, in the beginning of splitting,
\[|w_i| \leq N_0,\] and after the splitting, the new words has length \( \leq N_0 + N_0 = 2N_0 \). With all these, we have
\[g(\gamma_1) \leq (1/4) \cdot g(\gamma_0) \leq 1/4^2, \quad \text{with} \quad N_0 < |u| \leq 2N_0, \quad \forall \ u \in \gamma_1.\]

Inductively, we adopt the same procedure to construct \( \gamma_{n+1} \) from \( \gamma_n \): for each word \( w_i \in \gamma_n \), we use the same pull back technique to bring \( K_{w_i} \) to \( K \), and apply the claim to carry out the splitting, and the lemma follows. \( \square \)

**Proposition 3.5.** For an elementary fractal \( K \), \( \mathcal{M}^c \) is an open set in \((0, 1)^k\).

**Proof.** We adopt the same notation as in Lemma 3.4. Let \( a \in \mathcal{M}^c \) and \( \varepsilon \) be a positive real number. Consider the weight function \( g^{(\varepsilon)} \) which is defined by the vector \( a(\varepsilon) = (a_1 + \varepsilon_1, a_2 + \varepsilon_2, \ldots, a_k + \varepsilon_k) \), with \( |\varepsilon_i| \leq \varepsilon \) for \( i = 1, 2, \ldots, k \). Let \( a_\varepsilon = 1 + \frac{\varepsilon}{d} \), and let \( \gamma_n \) be as in the lemma, then \( |w| \leq (n+1)N_0 \) for all \( w \in \gamma_n \). It follows that
\[
g^{(\varepsilon)}(\gamma_n) = \sum_{w \in \gamma_n} g^{(\varepsilon)}(w) = \sum_{w \in \gamma_n} g(w) \frac{g^{(\varepsilon)}(w)}{g(w)} \leq \sum_{w \in \gamma_n} g(w) a_\varepsilon^{|w|} \leq g(\gamma_n) a_\varepsilon^{(n+1)N_0} \leq (4^{-1} a_\varepsilon^{N_0})^{n+1}, \quad \forall n \geq 0.
\]

Choose \( \varepsilon > 0 \) small enough such that \( 4^{-1} a_\varepsilon^{N_0} \leq 1/2 \), then \( g^{(\varepsilon)}(\gamma_n) \to 0 \) as \( n \to \infty \). This implies that \( a(\varepsilon) \in \mathcal{M}^c \) and thus \( \mathcal{M}^c \) is open. \( \square \)

4. Metric chain condition (MCC)

It follows from Proposition 3.5 that \( \mathcal{M} \) is closed in \((0, 1)^k\). As \( S \) is the boundary surface of \( \mathcal{M} \) (Corollary 2.3), we have \( S \subset \mathcal{M} \) and \( \mathcal{M} \setminus S = \mathcal{M}^c \), the interior of \( \mathcal{M} \). In this section, we will study \( D_{g_a} \) for \( a \in \mathcal{M} \), and in particular in \( S \) in connection with the MCC (Definition 1.5).

According to Kigami [27], we say that \( D_g \) is \( 1\text{-}adapte\)d to \( g \) if there exists a constant \( C > 0 \) such that for all \( x, y \in K \),
\[
( D_g(x, y) \leq \inf \{ g(\gamma) : \gamma \text{ connects } x \text{ and } y, |\gamma| \leq 2 \} \leq CD_g(x, y). \quad (4.1)
\]

**Remark 1.** In (4.1) for \( g = g_\alpha \), we can actually use the chains \( \gamma = \{w, v\} \) connecting \( x, y \) with \( |w| = |v| \). Indeed let \( x \in K_w, y \in K_v \), and that they do not contain each other. Suppose \( |w| > |v| \) (or \( |w| < |v| \)), we truncate the last indices of \( w \) to \( w' \) so that \( |w'| = |v| \). By Proposition 2.3 and the proof in Corollary 2.5, we have \( c^{-1} g(v) \leq g(w') \leq c g(v) \) where \( c \) depends on \( a \) and \( d \). Hence (4.1) still holds with the constant \( C' = (1 + c)C \).

As a special case of [27, Theorem 2.3.16], we have

**Proposition 4.1.** Let \( a \in \mathcal{M} \) and \( g \) be the associated weight function. Then the metric \( D_g \) is \( 1\text{-}adapte\)d to \( g \).
Lemma 4.2. Let $K$ be a nested fractal, let $\rho \in (0, 1)$ be the energy renormalization factor. For any distinct two points $x, y \in K$, let $k \geq 0$ be the maximum integer such that there are two words $w, v$ satisfying $K_w \cap K_v \neq \emptyset$ with $|w| = |v| = k$ and $x \in K_w, y \in K_v$. Then there is $C > 0$ independent of $x, y$ such that
\[
C^{-1} \rho^k \leq R(x, y) \leq C \rho^k.
\]

Thanks to Lemma 4.2, we have the following.

Lemma 4.3. Let $K$ be an elementary fractal. Let $d$ be the Euclidean metric (resistance metric) on $K$ if $K$ is a GSC (nested fractal respectively). Then $D_g$ is continuous with respect to $d$.

Proof. For the GSC, it is easy to see the following property holds. There is $C > 0$ such that for any $n \geq 0$, if two points $x, y$ satisfy $|x - y| \leq C \ell^{-n}$, then $x$ and $y$ are in the same or neighboring $n$-cells. Hence we see that $D_g(x, y) \leq 2a^n$ whenever $|x - y| \leq C \ell^{-n}$. It follows that $D_g$ is continuous with respect to the Euclidean metric.

For nested fractals, we can show that $D_g$ is continuous with respect to the resistance metric by using Lemma 4.2 and a similar argument as the GSC case.

Proposition 4.4. Let $K$ be an elementary fractal and $a \in M$. Let $D_g$ be the metric generated by $a$. Then $(K, D_g)$ is a complete metric space.

Proof. For the GSC, we let $\{x_n\}$ be a Cauchy sequence under $D_g$ in $K$. By the 1-adaptedness of $D_g$, for any fixed $m, n$, we can find a chain $\gamma_{m,n} = \{w, v\}$ such that $g(\gamma_{m,n}) \leq CD_g(x_m, x_n)$, with $|w| = |v| = k, x_m \in K_w, x_n \in K_v$. Then $|x_m - x_n| \leq Ct^{-k}$ (where $t^{-1}$ is the contractive ratio of the GSC). Since $g(\gamma_{m,n}) = g(w) + g(v) \geq a^k$, we have $k \geq \frac{\log g(\gamma_{m,n})}{\log a}$. It follows that
\[
|x_m - x_n| \leq C' \ell^k \frac{\log g(\gamma_{m,n})}{\log a} \to 0 \quad \text{as} \ m, n \to \infty,
\]

since $g(\gamma_{m,n}) \to 0$ as $m, n \to \infty$. This proves that $\{x_n\}$ is a Cauchy sequence under the Euclidean metric. Denote by $x$ the limit of $\{x_n\}$ under the Euclidean metric. Then by Lemma 4.3, we have $x$ is the limit of $\{x_n\}$ under $D_g$, and hence $(K, D_g)$ is a complete metric space.

For nest fractals, we only need to replace the above inequality for Euclidean metric by the inequality for resistance metric, then the proof for the GSC case also holds for this case.

To study the MCC on $K$, we need to use the chains $\gamma$ defined in the following. For $n \geq 0$, a chain $\gamma$ is called an $n$-chain if all the words in $\gamma$ have length equal to $n$. Define
\[
D_g^{(n)}(x, y) := \min \{g(\gamma) | \gamma \text{ is an } n\text{-chain between } x \text{ and } y\}.
\]
Remark 2. We will need the following simple fact: for \( a \in M^c \),
\[
\lim_{n \to \infty} D^{(n)}_{g_0}(x, y) = D_{g_0}(x, y) = 0, \quad \forall \, x, y \in K.
\] (4.2)

Indeed, for a given \( n \), let \( m \) be the smallest integer such that \( n \leq mN_0 \). By Lemma [3.4] there exists a chain \( \gamma_m \) connecting \( x, y \) such that \( g(\gamma_m) \leq 4^{-m-1} \), and \( mN_0 < |u| \leq (m + 1)N_0 \) for \( u \in \gamma_m \). Now we define an \( n \)-chain \( \gamma^{(n)} \) by truncating each \( u \in \gamma_m \) to \( u' \) of size \( n \). Hence the length of each word \( u' \) in \( \gamma^{(n)} \) has length differ from \( u \) by at most \( 2N_0 \). This implies \( g_a(\gamma^{(n)}) \leq C4^{-m} \) for a constant \( C \) so that (4.2) holds.

First we prove

**Proposition 4.5.** Let \( K \) be an elementary fractal, and let \( a \in M^o \) (the interior of \( M \)). Then
\[
\lim_{n \to \infty} D^{(n)}_{g_a}(x, y) = \infty, \quad \forall \, x \neq y \in K.
\]

In this case, \( D_{g_a} \) is a metric but does not satisfy the MCC.

**Proof.** For \( a \in M^c \), first we claim that for any distinct \( x, y \in K \), there exist \( C > 0 \) and \( \sigma > 1 \) such that for all \( n \geq 0 \),
\[
D^{(n)}_{g_a}(x, y) \geq C\sigma^n.
\]

Indeed, since \( a \in M^c \), there exist \( b \in \Lambda_k \) and \( 0 < \lambda < \lambda_b \), such that \( a = b(\lambda) \). Denote by \( \delta = \lambda_b - \lambda \) and let \( g_0 \) be the weight function of \( b(\lambda_b) \). Since \( b(\lambda_b) \in S \subset M \), we have \( D_{g_0}(x, y) > 0 \) and for each chain \( \gamma \) between \( x \) and \( y \),
\[
g_a(\gamma) = \sum_{w \in \gamma} (g_b(w))^\lambda = \sum_{w \in \gamma} (g_b(w))^\lambda \cdot (g_b(w))^{-\delta},
\]
and hence
\[
D^{(n)}_{g_a}(x, y) \geq b^{-\delta} \cdot D^{(n)}_{g_0}(x, y) \geq (b^{-\delta}) D_{g_0}(x, y).
\]

This proves the claim, and clearly implies that \( \lim_{n \to \infty} D^{(n)}_{g_a}(x, y) = \infty \).

To prove that \( D_{g_a} \) does not satisfy the MCC, we assume the contrary. We write \( D_{g_a} \) for \( D_{g_a} \) for simplicity. For two distinct points \( x, y \), there is \( C > 0 \) such that for any integer \( n \geq 1 \), there is a sequence \( x = x_0, x_1, \ldots, x_n = y \) such that
\[
D_{g_a}(x_i, x_{i+1}) \leq Cn^{-1}D_{g_a}(x, y), \quad \text{for } 0 \leq i \leq n - 1.
\] (4.3)

Pick \( \lambda > 1 \) close to 1 such that \( D_{g^{(\lambda)}} \) is a metric, where \( D_{g^{(\lambda)}} \) is given by the weight \( a(\lambda) = (a_1^\lambda, a_2^\lambda, \ldots, a_k^\lambda) \). By using the 1-adaptedness of \( D_{g_a} \) and \( D_{g^{(\lambda)}} \) (Proposition 4.1), we have
\[
D_{g^{(\lambda)}}(x, y) = \left( D_{g_a}(x, y) \right)^\lambda, \quad D_{g^{(\lambda)}}(x_i, x_{i+1}) = \left( D_{g_a}(x_i, x_{i+1}) \right)^\lambda, \quad \forall \, i.
\]

By using this, triangle inequality and (4.3), it follows that
\[
D_{g^{(\lambda)}}(x, y) \leq \sum_{i=0}^{n-1} D_{g^{(\lambda)}}(x_i, x_{i+1}) \leq C' \sum_{i=0}^{n-1} \left( D_{g_a}(x_i, x_{i+1}) \right)^\lambda
\]
\[ \sum_{i=0}^{n-1} (n^{-1} D_g(x,y))^i \leq C'' n^{-i-1} \left( D_g(x,y) \right)^i. \]

Letting \( n \to \infty \), we have \( D_{g^{(n)}}(x,y) = 0 \), a contradiction. Hence \( D_g \) does not satisfy the MCC.

In the following, we show that the \( D_g \) satisfies the MCC on \( S \), which is the main purpose of this section.

**Lemma 4.6.** Let \( K \) be an elementary fractal, \( x, y \in K \) and \( a \in S \). Then
\[ \sup_{n \geq 0} D_{g^{(n)}}(x,y) < \infty. \]
Moreover, there are \( n_0 \) (depends on \( x, y \)) and \( C > 0 \) (depends on \( a \)) such that
\[ \sup_{n \geq n_0} D_{g^{(n)}}(x,y) \leq CD_a(x,y). \quad (4.4) \]

The proof of this lemma is quite long, and has to be divided into two parts for the nested fractals and the GSC because of the different symmetric groups \( G \). We will prove it last. Granting this lemma, we can verify our main theorem with no difficulty.

**Theorem 4.7.** Let \( K \) be an elementary fractal, and let \( a \in M \). Then \( D_g \) is a metric satisfying the MCC if and only if \( a \in S \).

**Proof.** The necessity follows from Proposition 4.5. To prove the sufficiency, we let \( x, y \in K \), and let \( a \in S \), and write \( D_g \) for \( D_{g^{(n)}} \) for simplicity. For any \( M > 2 \), pick \( n > n_0 \) so that \( a^n \leq \frac{D_{g}(x,y)}{M} \). Consider an \( n \)-chain \( \gamma = (w(1), \ldots, w(m)) \) that attains \( D_{g^{(n)}} \). Then by (4.4),
\[ \sum_{i=1}^{m} g(w(i)) = D_{g^{(n)}}(x,y) \leq CD_a(x,y). \quad (4.5) \]

We use a “first exit time” technique to make a decomposition of \( \gamma \). Let \( s_0 = 0 \), define
\[ s_1 := \inf \left\{ j : \sum_{i=1}^{j} g(w(i)) \geq CM^{-1} D_g(x,y), \ 1 \leq j \leq m \right\}. \]
Inductively, for \( t \geq 1 \), define
\[ s_{t+1} := \inf \left\{ j : \sum_{i=s_t+1}^{j} g(w(i)) \geq CM^{-1} D_g(x,y), \ s_t + 1 \leq j \leq m \right\}. \]
Let \( \bar{t} \) be the first integer that \( s_{\bar{t}} \) can not be defined, then we assign \( s_{\bar{t}} = m \) by convention. Then we have \( \bar{t} - 1 \leq M \) (for otherwise, from the construction, \( \sum_{i=1}^{m} g(w(i)) \geq (\bar{t} - 1) CM^{-1} D_g(x,y) > CD_a(x,y) \), contradicting (4.5)). Also for each \( 0 \leq t \leq \bar{t} - 1 \),
\[ \sum_{i=s_t+1}^{s_{t+1}} g(w(i)) \leq CM^{-1} D_g(x,y) + a^n \leq (C + 1)M^{-1} D_g(x,y). \]
Now for \( 1 \leq t \leq \bar{t} - 1 \), pick each point \( z_i \in K_{w(s_t+1)} \), together with \( x = z_0 \) and \( y = z_{\bar{t}} \), we have by definition of \( D_g \),
\[ D_g(z_t, z_{t+1}) \leq \sum_{i=s_t+1}^{s_{t+1}} g(w(i)) + a^n \leq (C + 2)M^{-1} D_g(x,y), \]
where $C$ does not depend on $x, y, M$. Since $\bar{t} \leq M + 1$, this proves the MCC of $D_g$. \hfill $\Box$

**Proof of Lemma 4.6 for nested fractals.** For $p, q \in P_0$, we denote by $p \rightsquigarrow q$ if $\sup_{n \geq 0} D^{(n)}_g(p, q) < \infty$. This gives an equivalent relation on $P_0$ which is preserved under the group $G$. Observe that by using a similar argument as in Lemma 3.2, we can easily conclude that $p \rightsquigarrow q$ for some $p, q \in P_0$ if and only if $p \rightsquigarrow q$ for all $p, q \in P_0$.

We first show that $p \rightsquigarrow q$ for all $p, q \in P_0$. Suppose otherwise, $\sup_{n \geq 0} D^{(n)}_g(p, q) = \infty$ for all distinct $p, q \in P_0$, choose $N_0 \geq 1$ such that $D^{(N_0)}_g(p, q) \geq 2$ for all $p \neq q \in P_0$. It follows that for each $m \geq 1$,

$$D^{(mN_0)}_g(p, q) \geq 2^m, \quad \forall p \neq q \in P_0. \quad (4.6)$$

Indeed, let $\gamma = \{w(1), \ldots, w(\bar{t})\}$ be a chain connecting $p, q$ where $|w(i)| = (m + 1)N_0$ for all $i$. By considering $w(\bar{t})|_{N_0}$, we can use the “first exit time technique” (see the proof of Theorem 4.7) to construct a sequence of cells $\{K_{u(j)}\}_{j=1}^s$ of size $N_0$ connecting $p, q$, and decompose $\gamma$ into sub-chains $\{\gamma_i\}_{i=1}^s$. Then $F_{u(j)}^{-1}(\gamma_j)$ is an $mN_0$-chain, and induction implies $g(F_{w(i)}^{-1}(\gamma_i)) \geq 2^m$, so that $g(\gamma) \geq \sum g(u(i))2^m \geq 2^{m+1}$ and $D^{(m+1N_0)}_g(p, q) \geq 2^{m+1}$ for all $p \neq q \in P_0$.

Choose $\epsilon > 0$ small enough such that $(a_\epsilon)^{e_{N_0}} \geq 1/2$. Let $a(\epsilon) = (a_1^{1+\epsilon}, \ldots, a_k^{1+\epsilon})$, and $g^{(\epsilon)}$ be the weight function given by $a(\epsilon)$. Then for any $mN_0$-chain $\gamma_{(mN_0)}$ between $p$ and $q$, we have

$$g^{(\epsilon)}(\gamma_{mN_0}) = \sum_{w \in \gamma_{(mN_0)}} g^{(\epsilon)}(w) = \sum_{w \in \gamma_{(mN_0)}} g(w)^{1+\epsilon} \geq \sum_{w \in \gamma_{(mN_0)}} g(w) (a_\epsilon)^{e_{N_0}} \geq g(\gamma_{mN_0}) \frac{g(\gamma_{mN_0})}{2^m}.$$ 

Combining this with (4.6), we see that for all $m \geq 1$,

$$D^{(mN_0)}_g(p, q) \geq 1.$$ 

On the other hand, it follows from $a(\epsilon) \in M^e$. By Remark 2, $\lim_{m \to \infty} D^{(m)}_g(p, q) = 0$, a contraction. This proves $p \rightsquigarrow q$ for all $p, q \in P_0$.

To complete the proof of the first part of the lemma, we show further that $\sup_{n \geq 0} D^{(n)}_g(x, y) < \infty$ for any $x, y \in K$. Indeed, we can find a sequence of cells $\{K_{w(i)}\}_{i \geq 0}$ such that $w(0) = \emptyset$ and $w(i + 1)$ is a child of $w(i)$, and $\bigcap_{i \geq 1} K_{w(i)} = \{x\}$. Pick an arbitrary $x_i \in F_{w(i)}(P_0)$, we obtain a sequence of points $\{x_i\}_{i=0}^{\infty}$, such that $\lim_{i \to \infty} x_i = x$. We also see that $x_i$ and $x_{i+1}$ can be connected by a uniformly bounded pairs in $F_{w(i)}(P_0)$. Let $M = \max_{p, q \in P_0} \sup_{n \geq 0} D^{(n)}_g(p, q)$. Hence there exists a constant $C > 0$, such that for all $i$,

$$\sup_{n \geq 0} D^{(n)}_g(x_i, x_{i+1}) \leq C g(w(i)) \cdot M.$$ 

By summing up over $i$, and observing that $\sum_{i=1}^{\infty} g(w(i)) \leq \sum_{m=1}^{\infty} a^m < \infty$, we have

$$\sup_{n \geq 0} D^{(n)}_g(x_0, x) \leq C \sum_{i=1}^{\infty} g(w(i)) \cdot M \leq C' M.$$
Similarly, we pick \{y_i\}_{i=0}^\infty such that \lim_{i \to \infty} y_i = y with \ y_0 = x_0. Then we have \sup_{n \geq 0} D^{(n)}_g(x_0, y) \leq C'M, so that \sup_{n \geq 0} D^{(n)}_g(x, y) \leq 2C'M < \infty. This proves the first part.

To prove the last assertion, we observe that in the 1-adaptedness of \ D^*_g, Remark 1 allows us to assume that the two-word chain \gamma = \{w, v\} connecting \ x, y \ is such that \ |w| = |v| = n. Let \ n_0 = n_0(x, y) := \max\{n \geq 0 : \gamma = \{w, v\} connecting \ x, y \ and \ |w| = |v| = n\}. Clearly \ n_0 < \infty. We can assume that \gamma = \{w, v\} attains the maximum. Let \ z \in K_w \cap K_y. Since the two points \ x \ and \ z \ are both in \ K_w, consider \ F_w^{-1}(x), F_w^{-1}(z) \in K. By the first conclusion of the lemma, we have

\[ \sup_{n \geq 0} D^{(n)}_g\left(F_w^{-1}(x), F_w^{-1}(z)\right) \leq C_0. \]

It follows from the self-similarity that \sup_{n \geq n_0} D^{(n)}_g(x, z) \leq C_0 g(w). Similarly, \sup_{n \geq n_0} D^{(n)}_g(y, z) \leq C_0 g(v). Hence we have

\[ \sup_{n \geq n_0} D^{(n)}_g(x, y) \leq C_0 (g(w) + g(v)) \leq CD_g(x, y). \]

This completes the proof of the lemma. \qed

To prove Lemma 4.6 for the GSC, we need yet another fact: Let \ p_1 = (0, 0, \ldots, 0), \ p_2 = (1, 0, \ldots, 0). Let \ L = \{0\} \times [0, 1]^{d-1} \ and \ R = \{1\} \times [0, 1]^{d-1} \ be the left and right face of \ H_0. Define

\[ D^{(n)}_g(L, R) = \min\{|g(\gamma)| : \gamma \ is \ an \ n-chain \ between \ L \ and \ R\}, \]

where an \ n-chain \ \gamma = (w(1), \ldots, w(m)) \ is called an \ n-chain between \ L \ and \ R \ if \ K_{w(1)} \cap L \neq \emptyset \ and \ K_{w(m)} \cap R \neq \emptyset. The sub-lemma we need is

\[ (**) \quad \text{For} \ a \in (0, 1)^k, \ \sup_{n \geq 0} D^{(n)}_g(L, R) < \infty \quad \Rightarrow \quad \sup_{n \geq 0} D^{(n)}_g(p_1, p_2) < \infty. \]

We will prove it at the end as it requires another technique that is quite complicated. Assuming this, the proof for the GSC is similar to the previous case of nested fractals.

**Proof of Lemma 4.6 for GSC.** The same argument in the previous proof is also valid for the second assertion.

For the first assertion, using the same argument again as in the previous proof, we can also see that \sup_{n \geq 0} D^{(n)}_g(p_1, p_2) < \infty implies \sup_{n \geq 0} D^{(n)}_g(x, y) < \infty for all \ x, y \in K \ (the \ x_i \ we \ choose \ is \ an \ arbitrary \ vertex \ of \ the \ cell \ K_{w(i)}, \ and \ note \ that \ x_i, x_{i+1} \ can \ be \ connected \ by \ a \ finite \ number \ of \ pairs \ that \ can \ be \ expressed \ as \ affine \ combination \ of \ some \ F_w \ and \ \tau \in G). Thus, it suffices to show that \ \ a \in S \ implies \sup_{n \geq 0} D^{(n)}_g(p_1, p_2) < \infty. Suppose otherwise, then by (**),

\[ \sup_{n \geq 0} D^{(n)}_g(L, R) = \infty. \]
Let $m \geq 2$ and $\gamma^{(m)}$ be an $m\mathbb{N}_0$-chain between $p_1$ and $p_2$. We will decompose $\gamma^{(m)} = \{w(1), w(2), \ldots, w(j)\}$ into a sequence of sub-chains as follows. Let $w(1)^{-}$ be the truncation of $w(1)$ to word length $(m-1)\mathbb{N}_0$, and let

$$N_1 = \cup\{K_v : |v| = (m-1)\mathbb{N}_0, K_{w(1)^{-}} \cap K_v \neq \emptyset\}$$

be the neighborhood of $K_{w(1)^{-}}$ which is contained in a hypercube having three times the size as $K_{w(1)^{-}}$. Set $t_0 = 1$ and define

$$t_1 = \inf\{1 \leq t \leq j : K_{w(t)} \not\subseteq N_1\},$$

and let $\gamma^{(m)}_1 = \{w(1), w(2), \ldots, w(t_1-1)\}$. We also adjust $\{w(1)^{-}, w(2)^{-}, \ldots, w(t_1-1)^{-}\}$ to a simple $(m-1)\mathbb{N}_0$-chain $\eta_1$ starting at $w(1)^{-}$ and ending at $w(t_1-1)^{-}$, then $|\eta_1| \leq 3^d$. From Lemma 2.3, $g(w(1)^{-}) \geq c_1 g(\eta_1)$, where $c_1 = 3^{-d}(a_*/a*)^3d$.

By using the “first exit time” technique, we can define $\gamma^{(m)}_i := \{w(t_{i-1}), \ldots, w(t_i-1)\}$ and the associate $\eta_i$ with $g(w(t_i-1)^{-}) \geq c_1 g(\eta_i)$, $1 \leq i \leq s$.

First we show that there is $c > 0$ (only depending on $a$ and $K$) such that for $1 \leq i \leq s$,

$$g\left(\gamma^{(m)}_i\right) \geq cC_0 g\left(w(t_i-1)^{-}\right) \geq cc_1 C_0g(\eta_i), \quad (4.7)$$

where $C_0 = D^{(\mathbb{N}_0)}(L, R)$.

Indeed, we let $H_i := F_{w(t_{i-1})}(H_0)$ be a hypercube and $\partial H_i$ be its boundary. For $i \geq 2$, let

$$N_i = \cup\{K_v : |v| = (m-1)\mathbb{N}_0, K_{w(t_{i-1})^{-}} \cap K_v \neq \emptyset\}.$$

Let $3H_i$ be the hypercube having the same center as $H_i$ but with side length 3 times larger than $H_i$, then the definition of $N_i$ implies $N_i$ is the union of cells in $3H_i$, with $K_{w(t_{i-1})^{-}}$ in the center. Note that $w(t_i-1)$ is a cell in $\gamma^{(m)}_i$ hitting a hyperplane $A_1$ of $3H_i$. As $\gamma^{(m)}_i$ hits the center cell $K_{w(t_i-1)^{-}}$, $\gamma^{(m)}_i$ has a sub-chain $\gamma^{(m)}_{i'}$ through another parallel hyperplane $A_2$ determined by a face of $K_{w(t_i-1)^{-}}$. Then using reflection finitely times (only depends on the dimension $d$), we can obtain a new chain $\gamma^{(m)}_{i'}$ crossing through $A_1, A_2$ in an $(m-1)\mathbb{N}_0$-cell. Hence

$$g(\gamma^{(m)}_i) \geq g(\gamma^{(m)}_{i'}) \geq c' g(\gamma^{(m)}_{i'}) \geq cg(w(t_i-1)^{-}) \cdot D^{(\mathbb{N}_0)}(A_1, A_2).$$

This proves (4.7) since $D^{(\mathbb{N}_0)}(A_1, A_2) = D^{(\mathbb{N}_0)}(L, R)$.

Next we observe that $\eta_1 \cup \cdots \cup \eta_s$ is an $(m-1)\mathbb{N}_0$-chain connecting $p_1$ and $p_2$, hence

$$\sum_{i=1}^{s} g(\eta_i) \geq D^{((m-1)\mathbb{N}_0)}(p_1, p_2). \quad (4.8)$$

Therefore from (4.7) and (4.8) that

$$g(\gamma^{(m)}) = \sum_{i=1}^{s} g(\gamma^{(m)}_i) \geq cc_1 C_0 \sum_{i=1}^{s} g(\eta_i) \geq cc_1 C_0 D^{((m-1)\mathbb{N}_0)}(p_1, p_2).$$
Pick $C_0$ large enough such that $cc_1C_0 \geq 2$. Since $\gamma^{(m)}$ is an $mN_0$-chain, we have

$$D^{(mN_0)}_g(p_1, p_2) \geq 2D^{(m-1)N_0}_g(p_1, p_2) \geq 2^{m-1}cc_1C_0 \geq 2^m.$$  

Now we can use $a(e) = (a_1^{+e}, \ldots, a_k^{+e})$ as in the proof in the nested fractal case to obtain a contradiction, hence $\sup_{n \geq 0} D^{(n)}_g(p_1, p_2) < \infty$. □

**Proof of (**)**. By assumption, for each $n \geq 0$, there is an $n$-chain $\gamma^{(n)}$ starting from $L$ and ending at $R$ such that $g(\gamma^{(n)}) \leq M = \sup_{n \geq 0} D^{(n)}_g(L, R) < \infty$. We then show that there is $C > 0$ independent of $n$ such that $\gamma^{(n)}$ can be transformed to another $n$-chain $\gamma^{(n)}_{\text{new}}$ joining $p_1$ and $p_2$ such that $g(\gamma^{(n)}_{\text{new}}) \leq CM$. Then $\sup_{n \geq 0} D^{(n)}_g(p_1, p_2) < \infty$. We will divide the construction in two steps.

First we specify the type of transformations we will use. For an $n$-chain $\vartheta$ from $L$ to $R$, call $\vartheta'$ a **G-image** of $\vartheta$ in $K$ if it is one of the following:

(i) **(Symmetric image)** $\vartheta' = \sigma(\vartheta)$ for $\sigma \in G$;

(ii) **(Deflected image)** if $K_w$ is a cell in $\vartheta$, with its center lying on a diagonal hyperplane separating $L$ and $R$, then we can keep one portion of the chain before $K_w$ fixed, and reflect the other portion along the diagonal hyperplane (see Figure 3(a));

(iii) **(Half-reflected image)** keep half of $\vartheta$ up to the bisecting (or diagonal) hyperplane separating $L$ and $R$, and replace the other half with the reflected image as in Figure 3(b).

![Figure 3. Transformations](image)

Likewise, we can define the $G$-image of $\vartheta$ in $F_u(K)$ to $F_v(K)$ where $|v| = |u|$ by translating $\vartheta'$ to $F_v(K)$. We still call the $\vartheta'$ in $F_v(K)$ a **G-image** of $\vartheta$; we also refer the above process as a $G$-transform of $\vartheta$. Note that the $G$-transform of (i), (ii) of $\gamma$ in $F_u(K)$ to $\gamma'$ in $F_v(K)$ satisfies

$$g(\gamma') = \frac{g(v)}{g(u)}g(\gamma). \quad (4.9)$$

**Step 1.** Let $\gamma^{(n)}$ be the fixed $n$-chain as the above. For convenience, we omit all the superscript $n$. Denote by $a_{\text{min}} = \min(a_i : F_i(K) \cap L \neq \emptyset, i \in S)$. We may assume that $i_0$ is such that $a_{i_0} = a_{\text{min}}$ and $F_{i_0}(K) \cap L \neq \emptyset$. We denote by $p_{i_0}$ the fixed point of $F_{i_0}$. Let $\tau \in G$ be the reflection of $L$ onto $R$. 

We will construct an n-chain \( \gamma' \) between \( p_{i_0} \) and \( \tau(p_{i_0}) \) such that \( g(\gamma') \leq C' g(\gamma) \) for some \( C' > 0 \) independent of \( n \). This gives \( \sup_{n \geq 0} D_{g}^{(i)}(p_{i_0}, \tau(p_{i_0})) < \infty \).

Write \( \gamma = (w(1), \ldots, w(m)) \) with \( |w(j)| = n \) for all \( j \). Let \( \gamma_L \) and \( \gamma_R \) be the two sub-chains determined by first hitting the bisecting hyperplane of \( L \) and \( R \). Without loss of generality, we assume that \( g(\gamma_L) \leq \frac{1}{2}g(\gamma) \) and consider the left side of \( \gamma \) (otherwise, we consider the \( G \)-image \( \tau(\gamma) \) and start from the left side of \( \tau(\gamma) \)). Let \( K_{w(1)} \subset K_{i_1} \) for some \( i_1 \in \Sigma \). Let

\[
L_1 = \bigcup \{ K_v : |v| = 1, K_v \cap L \neq \emptyset \},
\]

and let \( j_0 = \inf \{ 1 \leq j \leq m : K_{w(j)} \not\subset L_1 \} - 1 \). Let \( \gamma_1 = (w(1), w(2), \ldots, w(j_0)) \). Then \( g(\gamma_1) \leq \frac{1}{2}g(\gamma) \). Note that \( \gamma_1 \) reaches the left side and the right side of \( L_1 \). Using reflection, we can move \( \gamma_1 \) to the 1-cell \( K_{i_1} \), and denote it by \( \tilde{\gamma}_1 \); it is an \( n \)-chain in \( K_{i_1} \) from \( F_{u_1}(L) \) to \( F_{u_1}(R) \). Note also that if we denote the two end \( n \)-cells of \( \gamma \) and \( \tilde{\gamma}_1 \) by \( \alpha_0, \beta_0 \), and \( \alpha_1, \beta_1 \) respectively, then \( \alpha_1 = \alpha_0 \) or \( \beta_0 \) depends on the choice of using \( \gamma_L \) or \( \gamma_R \).

By similarity, we can perform the same process on \( \tilde{\gamma}_1 \) in \( K_{i_1} \) to obtain \( \tilde{\gamma}_2 \) in a 2-cell \( K_{u_1u_2} \) for some \( u_2 \in \Sigma \), and cross the left and right face of \( K_{u_1u_2} \); inductively, we obtain \( \tilde{\gamma}_\ell \) in \( K_{u_{\ell-1}u_{\ell}} \), \( 1 \leq \ell \leq n \) with the same property. Note that if we denote the end \( n \)-cells in \( \tilde{\gamma}_\ell \) by \( \alpha_\ell, \beta_\ell \), then we have \( \alpha_{\ell+1} = \alpha_\ell \) or \( \beta_\ell \); also we have \( a_{u_\ell} \geq a_{\min} \) for each \( \ell \).

Consider a sequence of distinct 1-cells connecting \( K_{i_0} \) and \( K_{\tau(i_0)} \),

\[
K_{i_0} \sim K_{i_1} \sim \cdots \sim K_{i_\ell} \sim K_{\tau(i_0)}, \quad \ell \in \Sigma.
\]

We use the \( G \)-images of \( \tilde{\gamma}_1 \) to obtain an \( n \)-chain \( \xi_1 \) to connect these 1-cells \( \{ K_{i_j} \}_{j=1}^\ell \) inductively (leave out the first one and the last one). On \( j = 1 \), we will translate \( \tilde{\gamma}_1 \) or the reverse of \( \tilde{\gamma}_1 \) to \( K_{i_1} \) according to \( \alpha_2 = \alpha_1 \) or \( \beta_1 \), so that the starting \( n \)-cell is in the \( K_{i_1\sigma(u_2)} \) for some \( \sigma \in G \), the same relative position as \( \alpha_2 \) in \( K_{u_1u_2} \). The construction of \( \xi_1 \) from \( K_{i_1} \) to \( K_{i_\ell 1} \) can be handled by the combination of the \( G \)-images (i) and (ii) and is straightforward.

Denote the reflected image of \( K_{i_1\sigma(u_2)} \) in \( K_{i_0} \) by \( K_{i_0u_2}' \). Consider a sequence of distinct 2-cells

\[
K_{i_0u_2} \sim K_{i_0u_1} \sim \cdots \sim K_{i_0s_{\ell-1}} \sim K_{i_0s_\ell} = K_{i_0u_2}', \quad s_\ell \in \Sigma.
\]

We use the same procedure as the above to construct an \( n \)-chain \( \xi_2 \) in \( \bigcup_{j=1}^\ell K_{i_0s_{\ell-j}} \), of \( \xi_2 \) to start the first cell associate with \( \alpha_3 \); in addition we have to connect the last cell of \( \xi_2 \) to the first cell of \( \xi_1 \): if the last cell is \( \alpha_2 \), then we are done; otherwise, we can use the \( G \)-transform (iii) to convert it to \( \alpha_2 \).

We apply this construction through \( 1 \leq \ell \leq n \), then \( \xi' = \xi_n \cup \cdots \cup \xi_1 \) is an \( n \)-chain in \( K_{i_0} \cup \{ K_{i_j} \}_{j=1}^\ell \) starting at \( p_{i_0} \). To extend the chain to reach \( \tau(p_{i_0}) \), we use the \( G \)-transform (iii) to reflect half of it to the \( R \) side, and denote this new \( n \)-chain by \( \gamma' \).

It is direct to check that \( g(\tilde{\gamma}_1) \leq \frac{a_{u_1}}{a_{\min}} g(\gamma_1) \leq \frac{a_{u_1}}{2a_{\min}} g(\gamma) \). As the same construction is used in each step for \( \ell = 1, \ldots, n \), we have \( g(\tilde{\gamma}_\ell) \leq \frac{a_{u_\ell}}{2a_{\min}} g(\tilde{\gamma}_{\ell-1}) \) so that
Combining with (4.10), we obtain

\[ g(\overline{\gamma}_\ell) \leq \frac{1}{2^\ell} \cdot \frac{a_{u_1 \cdots u_\ell}}{a_{\min}} g(\gamma). \]  

(4.10)

Since \( \zeta_\ell \) consists of finitely many \( G \)-images of \( \overline{\gamma}_\ell \) and together using (4.9), we see that there exists \( C > 0 \) such that for all \( \ell = 1, \ldots, n \),

\[ g(\zeta_\ell) \leq C \cdot \frac{g(\ell)}{g(u_1 \cdots u_\ell)} \cdot g(\overline{\gamma}_\ell) = C \cdot \frac{a_{\min}}{a_{u_1 \cdots u_\ell}} \cdot g(\overline{\gamma}_\ell). \]

Combining with (4.10), we obtain \( g(\zeta_\ell) \leq \frac{C}{2^\ell} \cdot g(\gamma) \) for \( \ell = 1, \ldots, n \). It follows that

\[ g(\gamma) \leq C' \sum_{\ell=1}^n g(\zeta_\ell) \leq C' \sum_{\ell=1}^n \frac{C}{2^\ell} g(\gamma) \leq C'' g(\gamma), \]

where \( C'' \) is independent of \( n \). So we obtain \( \sup_{n \geq 0} D^{(n)}_g(p_{i_0}, \tau(p_{i_0})) < \infty \).

**Step 2.** For each \( n \geq 0 \), let \( \gamma^{(n)} \) be the \( n \)-chain constructed in Step 1 (the superscript \( n \) was suppressed there for simplicity, but we will keep it here). Now for each \( n \geq 2 \), and \( 1 \leq i \leq n \), define \( q_i = F_1(\tau(p_{i_0})) \) where \( F_1 \) is the map with fixed point \( p_1 \). We will use the \( \{\gamma^{(n)}\}_{n \geq 0} \) (between \( p_{i_0} \) and \( \tau(p_{i_0}) \)) in Step 1 to construct an \( (n+2) \)-chain \( \xi^{(n+2)} \) connecting \( q_i \)'s consecutively for \( i = 1, 2, \ldots, n-1 \) (and then \( p_1 \) and \( p_2 \)) such that

\[ g(\xi^{(n+2)}) \leq CM \]

for some \( C > 0 \) independent of \( n \). This yields

\[ \sup_{n \geq 0} g(\xi^{(n)}) \leq CM. \]

First, let \( q_1 \) and \( q_2 \) be connected by some 2-cells in \( K_1 \) between \( K_{12} \) and \( K_{1\tau(i_0)} \) (as in Step 1). Considering \( F_{11}(\gamma^{(n)}) \) as an \( (n+2) \)-chain in \( K_{12} \), we can use \( G \)-images of it to construct an \( (n+2) \)-chain \( \xi^{(n+2)}_1 \) in those 2-cells between \( q_1 \) and \( q_2 \) (the same construction of \( \xi^{(n)}_1 \) as in Step 1). Then as before, there is \( C' > 0 \) (only dependent on \( K \) and \( a \) ) such that

\[ g(\xi^{(n+2)}_1) \leq C' \cdot g(1) \cdot g(\gamma^{(n)}) \leq C_1 a_1 M. \]  

(4.11)

Next consider \( q_2, q_3 \) connected by some 3-cells in \( K_{11} \) between \( K_{111} \) and \( K_{11\tau(i_0)} \). By using the \( (n+2) \)-chain \( F_{11}(\gamma^{(n-1)}) \) and the previous construction, we can find \( \xi^{(n+2)}_2 \) between \( q_2 \) and \( q_3 \) satisfying

\[ g(\xi^{(n+2)}_2) \leq C_1 \cdot g(11) \cdot g(\gamma^{(n-1)}) \leq C_1 a_1^2 M. \]

Inductively, for \( 1 \leq i \leq n-1 \) we get an \( (n+2) \)-chain \( \xi^{(n+2)}_i \) between \( q_i \) and \( q_{i+1} \) inside \( K_{1i} \), and

\[ g(\xi^{(n+2)}_i) \leq C_1 \cdot g(1^i) \cdot g(\gamma^{(n+1-i)}) \leq C_1 a_1^i M. \]

Finally, \( p_1 \) and \( q_n \) can be connected by some \( (n+2) \)-cells with bounded total weight. Hence \( p_1 \) can be connected to \( q_1 \) through these \( \xi^{(n+2)}_i \) and

\[ g(\xi^{(n+2)}) \leq C_1 M \sum_{i=1}^{n-1} a_1^i + C'. \]
We can do the same for \( p_2 \) to the symmetric \( q'_1 = \tau(q_1) \). Also by using \( F_{12}(y^{n_0}) \), following the same argument as the above, we can connect \( q_1 \) and \( q'_1 \) by an \((n+2)\)-chain with the same bound (with a different constant). Combining this three parts, we obtain an \((n+2)\)-chain \( \xi_{a}^{(n+2)} \) joining \( p_1 \) and \( p_2 \) with bound \( \leq CM \), so that

\[
\sup_{n \geq 0} D_{g}^{(n)}(p_1, p_2) < \infty,
\]

which proves our assertion. □

5. Nested fractals

In this section, we give a constructive expression of \( M \) for the class of nested fractals. The main idea of representing the recursive weight transfer into matrix is from \([31]\).

Let \( \ell_1, \ell_2, \ldots, \ell_r \) be such that \( 0 < \ell_1 < \ell_2 < \cdots < \ell_r \) and \( \{\ell_1, \ldots, \ell_r\} := \{|x - y| : x, y \in P_0, x \neq y\} \). For \( n \geq 1 \), denote by \( P_n = \bigcup_{w \in \Sigma} F_w(P_0) \). For each \( x \in P_n \) with \( n \geq 0 \) and for \( i = 1, \ldots, r \), let \( N_i^n(x) \) be the set of all \( y \in P_n \) belonging to the same \( n \)-cell of \( x \) and \(|x - y| = g^{-n}\ell_i\); for \( y \in N_i^n(x) \), we call the one step move from \( x \) to \( y \) an \( n \)-move of type \( (i) \), \( 1 \leq i \leq r \). A sequence \( x_0, \ldots, x_m \in P_n \) is called an \( n \)-walk if \( x_j \) and \( x_{j+1} \) are joined in the same \( n \)-cell for all \( 0 \leq j \leq m - 1 \).

Next we fix a symmetric self-similar weight function \( g \) generated by some \( a = (a_1, \ldots, a_k) \in (0, 1)^k \). Pick any \( x \in P_0 \) and \( y \in N_i^n(x) \) for some \( 1 \leq i \leq r \), consider all the \( 1 \)-walk \( x_0, x_1, \ldots, x_m \) such that \( x_0 = x \) and \( x_m = y \) with \( x_1, \ldots, x_{m-1} \in P_1 \setminus P_0 \) which do not pass through the same point twice. Fix such \( 1 \)-walk, we count all the \( \langle j \rangle \)-type \( 1 \)-moves, \( 1 \leq j \leq r \), in this walk. For each \( \langle j \rangle \)-type \( 1 \)-move, it can be assigned to a unique \( 1 \)-cell, and we say that this \( \langle j \rangle \)-type move has weight \( a_i \) if the weight of this \( 1 \)-cell is \( a_i \). Then we sum up all the weights of these \( \langle j \rangle \)-type moves and denote it by \( c_j^i \). Let \( c^i = (c_1^i, \ldots, c_r^i) \) be the weight of the \( 1 \)-walk. By the symmetric assumption of \( g = g_a \), it is clear that \( c^i \) does not depend on the choice of \( x \in P_0 \) and \( y \in N_i^n(x) \). Let \( S_{a}^i \) be the set of \( c^i \) for all these finite number of \( 1 \)-walks. Then \( S_{a}^i, 1 \leq i \leq r \) is a collection of finite number of \( r \)-dimensional vectors, and each one is a nonnegative linear combination of weights in \( a \in (0, 1)^k \) with integer coefficients.

Let

\[
\mathcal{K}(a) := \{A : A \text{ is a } r \times r \text{-matrix} \exists \text{ for } 1 \leq i \leq r, (i \text{-th row of } A) \in S_{a}^i\}.
\]

We call \( A \) a weight transfer matrix. For \( A \in \mathcal{K}(a) \), \( A \) has nonnegative entries and each row is nonzero. Let \( \lambda_A \) be the spectral radius of \( A \), i.e., the largest positive eigenvalue of \( A \). Then it is clear that \( \lambda_A \) is a solution of some polynomial with coefficients generated by \( \{a_1, a_2, \ldots, a_k\} \).

**Theorem 5.1.** For a nested fractal, we have

\[
\mathcal{M} = \{a = (a_1, \ldots, a_k) \in (0, 1)^k : \lambda_A \geq 1 \text{ for all } A \in \mathcal{K}(a)\},
\]

and the boundary \( S = \{a \in \mathcal{M} : \text{ there exists } A \in \mathcal{K}(a) \text{ such that } \lambda_A = 1\} \).
Proof: We first show that if \(a = (a_1, \ldots, a_k) \in (0, 1)^k\) is such that \(\lambda_A < 1\) for some \(A \in \mathcal{K}(\alpha)\), then \(D_{\lambda_A}\) is not a metric. The idea is that we use this \(A\) to recursively construct \(n\)-chains \(\{G_n\}_{n \geq 0}\) between two distinct points \(x, y \in P_0\) such that \(\lim_{n \to \infty} g(\gamma_n) \to 0\). By assumption, for \(1 \leq i \leq r\), the \(i\)-th row of \(A\) is determined by a \(1\)-walk \(\xi_i\) between some pair \(x, y \in P_0\) with \(|x - y| = \ell_i\). We pick \(x, y \in P_0\) such that \(|x - y| = \ell_1\) and let \(\eta_1 = \xi_1\) be the \(1\)-walk between \(x\) and \(y\). Let \(\gamma_1\) be the associated \(1\)-chain of \(\eta_1\). For \(n \geq 1\), we define recursively an \(n\)-walk \(\eta_n\) between \(x, y\) and denote by \(\gamma_n\) the associated \(n\)-chain. Define \(\eta_2\) by replacing each \(\langle i \rangle\)-type \(1\)-move in \(\eta_1\) by the \(2\)-walk \(F_w(\tau(\xi_i))\) for some \(\tau \in G\), where \(w\) is the assigned \(1\)-cell of the \(1\)-move. Recursively, we define the \(n\)-walk \(\eta_n\) from \(\eta_{n-1}\) in a similar manner. For the weight \(c^1\) of \(\eta_1\) and \(1 = (1, \ldots, 1)\), we have

\[
g(\gamma_n) = c^1 A^{n-1} 1' \to 0 \quad \text{as } n \to \infty,
\]

by \(\lambda_A < 1\). This implies that \(D_{\lambda_A}(x, y) = 0\) and consequently \(D_{\lambda_A}\) is not a metric.

Conversely, We use contradiction to show that if \(\lambda_A \geq 1\) for all \(A \in \mathcal{K}(\alpha)\), then \(D_{\lambda_A}\) is a metric on \(K\). We fix \(a\) and define the operator \(G_a : [0, \infty)^r \to [0, \infty)^r\) by

\[
(G_a(x))_i = \min_{e^i \in S_a} \{ \sum_{j=1}^r c^j x_j \} = \min_{e^i \in S_a} \{ (e^i, x') \}, \quad 1 \leq i \leq r,
\]

where \(x = (x_1, \ldots, x_r)\). Note that for each \(1 \leq i \leq r\), there exists \(e^i \in S_a\) such that \((G_a(x))_i = \langle e^i, x' \rangle\). This defines a matrix \(A_{\min} \in \mathcal{K}(\alpha)\) (depends on \(x\)) such that \(G_a(x) = A_{\min} x\).

By using the same technique as in \([31]\) Lemma 3.3, we have \(G_a(B) \subseteq B\) for \(B = \{x \in \mathbb{R}^r : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_r\}\). For completeness, we provide a proof for this with a slightly modification.

Fix \(p \in P_0\), \(q_i \in N(p)\) and \(q'_i \in N^{i-1}(p)\) for \(2 \leq i \leq r\). Let \(R_{q_iq'_i}\) be the reflection between \(q_i\) and \(q'_i\). Let \(V_i = \{z \in \mathbb{R}^n : |z - q'_i| \leq |z - q_i|\}\) and \(\tilde{z} = R_{q_iq'_i}z\). Let \(T_i : \mathbb{R}^n \to \mathbb{R}^n\) be given by

\[
T_i(z) = \begin{cases} 
  z, & z \in V_i, \\
  \tilde{z}, & \text{otherwise}.
\end{cases}
\]

Given \(x \in B\) and \(i \geq 2\), consider \(e^i \in A_{\min}\), then there is a \(1\)-walk \(\xi_i\) between \(p \) and \(q_i\) with weight \(e^i\). Express \(\xi_i\) by \(p = x_0, x_1, \ldots, x_m = q_i\) into \(1\)-moves \(\{(x_j, x_{j+1})\}_{j=0}^{m-1}\). Then we see that \(T_i(\xi_i)\) is a \(1\)-walk between \(p \) and \(q'_i\), and \((T_i(x_j), T_i(x_{j+1}))\) is a \(1\)-move in the same cell as \((x_j, x_{j+1})\), which has type smaller than or equal to \((x_j, x_{j+1})\) because \(|T_i(x_j) - T_i(x_{j+1})| \leq |x_j - x_{j+1}|\). Denote by \(t = (t_1, \ldots, t_s)\) the weight of the \(1\)-walk \(T_i(\xi_i)\). Then we have

\[
(G_a(x))_{i-1} \leq \langle t, x' \rangle \leq \langle e^i, x' \rangle = (G_a(x))_i,
\]

since \(x \in B\). This proves \(G_a(B) \subseteq B\).

Consider the normalization \(\tilde{G}(x) = G(x)/\sum_i G(x)_i\) on \(B_\varepsilon = \{x \in B : \sum_i x_i, \ x_i \geq \varepsilon\}\). Then \(\tilde{G}(x)(B_\varepsilon) \subset B_\varepsilon\). By using the Brouwer fixed point theorem, there is a fixed point \(\tilde{G}(x) = x\). It follows that \(G(x) = \lambda x = A_{\min}x\) where \(\lambda = \sum_i (G(x))_i\), and is the maximum eigenvalue of \(A_{\min}\) (for detail, see \([31]\) Proposition 3.4)).
Finally, for $n \geq 0$, let $z_{n} = (z_{n,1},z_{n,2},\ldots,z_{n,r})$ be a vector of positive real numbers such that $z_{n,i} = D_{n}(p,q)$, where $p,q \in P_{0}$ and $|p - q| = \ell_{i}$, $1 \leq i \leq r$. Then it is easy to see that

$$z_{n} = G_{a}(z_{n-1}), \quad n \geq 1.$$  

Denote $C' = \min \{ \chi^{-1} : 1 \leq i \leq r \}$. Then $z_{0} = (1, 1) \geq C'x$ so that $z_{1} \geq G_{a}(C'x) = C'.\lambda x$. In general, we have $z_{n} \geq G_{a}(C'.\lambda^{n-1}x) = C'.\lambda^{n}x$ for all $n \geq 1$. On the other hand, if $D_{g_{a}}$ is not a metric on $K$, then by \cite{[2]}, $\lim_{n \to \infty} D_{g_{a}}^{(n)}(x,y) = 0$ for any $x,y \in P_{0}$. This contradicts the fact that $\lambda \geq 1$ and $x$ is positive. Thus $a \in M$ and the proof of the first assertion is complete.

To prove the second assertion, we denote by $S':= \{ a \in M : \exists A \in \mathcal{K}(a) \ni \lambda_{A} = 1 \}$. For any $a = (a_{1},\ldots,a_{k}) \in S'$, we may assume $A(a) \in \mathcal{K}(a)$ such that $\lambda_{A(a)} = 1$. For any $\delta > 0$, denote by $a^{\delta} = (a_{1}^{\delta},\ldots,a_{k}^{\delta})$, each nonzero entry of $A(a^{\delta})$ is strictly smaller than that of $A(a)$. Denote by

$$c_{0} = \max \{ c > 1 : \text{each entry of } cA(a^{\delta}) \text{ is smaller than or equal to that of } A(a) \}.$$ 

Thus $c_{0} > 1$ and by using the Perron-Frobenius theorem, we have

$$c_{0}\lambda_{A(a^{\delta})} = \lambda_{c_{0}A(a^{\delta})} \leq \lambda_{A(a)} = 1,$$

and hence $\lambda_{A(a^{\delta})} \leq c_{0}^{-1} < 1$, which implies that $a^{\delta} \in M$ for any $\delta > 1$. Hence $a \in S'$, which implies $S' \subseteq S$. On the other hand, for any $a = (a_{1},\ldots,a_{k}) \in S$, we show that $a \in S'$. If $a \notin S'$, then $\lambda_{A(a)} > 1$ for all $A(a) \in \mathcal{K}(a)$. It is clear that $\lambda_{A(a)}$ is continuous with respect to $a$ and the cardinality of $\mathcal{K}(a)$ is finite. Thus for all $s$ close to 1, for all $A(a^{s}) \in \mathcal{K}(a^{s})$, we have $\lambda_{A(a^{s})} > 1$. Hence $a \in M^{o}$, a contradiction. This shows that $a \in S'$ and $S \subseteq S'$. Thus we have $S = S'$.

**Remark.** The size of the families $S_{a}^{i},1 \leq i \leq r$, can be cut down substantially for calculation. We define the essential class $\tilde{S}_{a}^{i},1 \leq i \leq r$ to be the set of $e := (c_{1},\ldots,c_{r}) \in S_{a}$ that is smallest (in the sense of coordinatewise ordering) for $S_{a}^{i}$. Then for $e' \in \tilde{S}_{a}^{i}$ with $e \leq e'$, by positive matrix theory if we replace the row in the weight transfer matrix $A$ containing $e$ by $e'$, then the maximal eigenvalue increases.

Let $\mathcal{K}(a)$ be the $r \times r$ matrices $A$ such that the $i$-th row is in $\tilde{S}_{a}^{i}$, then clearly

$$M = \{ a = (a_{1},\ldots,a_{k}) \in (0,1)^{k} : \lambda_{A} \geq 1 \text{ for all } A \in \mathcal{K}(a) \}.$$ 

In the following, we use the *Lindström snowflake* to give an illustration of the theorem and the remark.

Let $p_{i} = (\cos(\pi/3),\sin(\pi/3)), i = 1, 2, \ldots, 6$ and $p_{7} = (0,0)$. Define $F_{i} : \mathbb{R}^{2} \to \mathbb{R}^{2}$ by $F_{i}(x) = (x - p_{i})/3 + p_{i}$ for $i = 1, 2, \ldots, 7$. The Lindström snowflake $K$ is the self-similar set generated by the IFS $\{F_{i}\}_{i=1}^{7}$ (see Figure 4). It has boundary $V_{0} = \{p_{1},\ldots,p_{6}\}$.
In this case, $\Sigma = \{1, 2, \ldots, 7\}$. We consider a symmetric self-similar weight function $g = g_{a,b}$ on $\Sigma^*$ defined by

$$g(i) = \begin{cases} 
  a, & 1 \leq i \leq 6, \\
  b, & i = 7, 
\end{cases} \quad (5.1)$$

where $a, b \in (0, 1)$. (See Figure 4.)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lindstrom_snowflake.png}
\caption{Lindstrøm snowflake, and the weight function $g_{a,b}(w)$ for $|w| = 1, 3$.}
\end{figure}

**Corollary 5.2.** Let $g = g_{a,b}$ be defined as the above. Then $D_{g_{a,b}}$ is a metric if and only if $3a \geq 1$ and $2a + b \geq 1$. Moreover, $D_g$ satisfies the MCC if and only if $(a, b) \in \{3a = 1, b \geq \frac{1}{3}\} \cup \{2a + b = 1, b \leq \frac{1}{3}\}$.

**Proof:** The last conclusion is a consequence of the boundary of $M$ in the first part and Theorem 4.7.

To prove the first part, we adopt the notations and setup preceding Theorem 5.1. It is easy to see that the Lindstrøm snowflake has three types, that is $\ell_1 = \overline{p_1p_2}$, $\ell_2 = \overline{p_1p_3}$ and $\ell_3 = \overline{p_1p_4}$. Let $a = (a, b)$ as in (5.1), and consider $p_2 \in N_0^1(p_1)$.

For a 1-walk of $p_1$ to $p_2$ in $P_1 \setminus P_0$, by using the above Remark and elementary checking case by case, we obtain two vectors in the essential classes $\tilde{S}_1^a$. Similarly, we can calculate $\tilde{S}_2^a$ and $\tilde{S}_3^a$ (see Figure 5).

\begin{align*}
\tilde{S}_1^a &= \{(0, 2a, 0), (b, 0, 2a)\}; \\
\tilde{S}_2^a &= \{(a + b, a, a), (0, b, 2a), (a, a, a + b), (0, 3a, 0)\}; \\
\tilde{S}_3^a &= \{(0, 4a, 0), (2a + b, 2a, 0), (a + b, 2a, a), (a + b, a + b, a), (0, 0, 2a + b)\}.
\end{align*}

The $A \in \mathcal{K}(a)$ are formed by picking each vector in each of the $\tilde{S}_i^a$. It can be checked directly that $\lambda_A \geq 1$ for all $A \in \mathcal{K}(a)$ is equivalent to

$$\begin{cases} 
  3a \geq 1, \\
  2a + b \geq 1. 
\end{cases} \quad (5.2)$$
In particular, the two determining matrices for $M$ and $S$ are

$$A_1 = \begin{pmatrix} 0 & 2a & 0 \\ 0 & 3a & 0 \\ 0 & 4a & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2a & 0 \\ 0 & b & 2a \\ 0 & 0 & 2a + b \end{pmatrix},$$

with the two spectral radii $\lambda_{A_1} = 3a$ and $\lambda_{A_2} = 2a + b$ respectively.

**Remark.** In fact, it is easy to see that the two conditions in (5.2) are necessary for $D_g$ to be a metric by looking at the straight lines contained in the fractal. For example, the line between $p_1$ and $p_3$ gives $3a \geq 1$ and the line between $p_1$ and $p_4$ gives $2a + b \geq 1$. The sufficiency can also be proved by showing that for $a, b$ in (5.2), every chain $\gamma$ connecting $p_1$ and $p_4$ satisfies $g_a(\gamma) \geq 1$. We omit the detail.

### 6. Sierpinski carpet

In this section, we consider the standard Sierpinski carpet $GSC(2, 3, Q')$, where $Q' = Q \setminus ([1/3, 2/3]^2)$. Let $F_i(x) = (x + 2p_i)/3, i = 1, \ldots, 8$, be contractive maps on $\mathbb{R}^2$, with the $p_i$ specified as in Figure 6.

For any $(a, b) \in (0, 1)^2$, define $g := g_{a,b} : \Sigma^* \to (0, 1]$ by

$$g(i) = \begin{cases} a, & \text{if } i \in \{1, 3, 5, 7\}, \\ b, & \text{if } i \in \{2, 4, 6, 8\}, \end{cases}$$

(6.1)
see Figure 6 for the case $|w| = 3$.

Let $\sigma = \{(a, b) \in (0, 1)^2 : 2a + b \geq 1 \text{ and } a + 2b \geq 1\}$, and let $\sigma_I := \{(a, b) \in \sigma : a + 2b = 1 \text{ or } 2b + a = 1\}$ be the boundary of $\sigma$ in $(0, 1)^2$.

**Theorem 6.1.** $D_{g_{a,b}}$ is a metric on $K$ if and only if $(a, b) \in \sigma$. Consequently, $D_{g_{a,b}}$ satisfies the MCC if and only if $(a, b) \in \sigma_I$.

The MCC follows directly from the first part and Theorem 4.7. The necessity that $D_{g_{a,b}}$ is a metric is due to Kigami. It is rather straightforward. We provide a proof in the following for completeness.

**Proof of the necessity.** For $n \geq 1$, let $\gamma_n$ be a chain of $n$-cells connecting $p_1$ and $p_3$ and each cell intersects the line $p_1p_3$ (i.e., $\gamma_1 = \{1, 2, 3\}$, $\gamma_2 = \{11, 12, 13; 21, 22, 23; 31, 32, 33\}$, and so on). By elementary calculations, we obtain

$$g(\gamma_n) = (2a + b)^n.$$  

Since $D_{g}$ is a metric, $\inf_{n \in \mathbb{Z}^+} (2a + b)^n \geq D_{g}(p_1, p_3) > 0$ so that $2a + b \geq 1$.

Next, we consider the chains $\gamma'_n$ of $n$-cells connecting $p_2$ and $p_4$, and each cell intersects the line segment $p_2p_4$ (i.e., $\gamma'_1 = \{2, 3, 4\}$, $\gamma'_2 = \{22, 23, 24; 38, 37, 36; 42, 43, 44\}$, and so on). Then

$$g(\gamma'_n) = (a + 2b)^n.$$  

Same as the above, we have $a + 2b \geq 1$.

The proof of the sufficiency of the theorem is more complicate. The following corollary is a simple consequence of Lemma 2.3 adapted to the situation we need.
Corollary 6.2. Suppose \( K_u \sim K_v \) and \( |u| = |v| \). Then
\[
\frac{g(u)}{g(v)} = \frac{a}{b} \text{ or } \frac{b}{a}.
\]
Furthermore, if \( \gamma \) is a chain contained in \( K_u \), if we reflect it along the intersection line \( L \), and denote the reflected chain by \( \gamma' \), then \( \gamma' \) is a chain in \( K_v \), and \( \frac{g(\gamma')}{g(\gamma)} = \frac{a}{b} \text{ or } \frac{b}{a} \).

In view of Theorem 3.1 and (4.2), we only need to consider the \( n \)-chains between arbitrary two fixed distinct points in \( K \). As in the proof, we will use Corollary 6.2 frequently for reflection. We need a few more lemmas.

Lemma 6.3. Let \( (a, b) \in \sigma \) and let \( n \geq 0 \). Suppose \( \gamma \) is an \( n \)-chain satisfying either
(i) \( \gamma \) connects \( p_1 \) and \( p_3 \) and all its cells intersect the line segment \( p_1p_3 \); or
(ii) \( \gamma \) connects \( p_2 \) and \( p_8 \) and all its cells intersect the line segment \( p_2p_8 \).

Then
\[
g(\gamma) \geq 1.
\]

Proof. See the proof of necessity. \( \square \)

Let
\[
\sigma_1 = \{ (a, b) \in \sigma : a \leq b \} \quad \text{and} \quad \sigma_2 = \{ (a, b) \in \sigma : a \geq b \}.
\]
Since the reflections we use are different on the two regions, we will separate the proofs into two parts for clarity.

The case for \( \sigma_1 \). For \( n \geq 1 \), we let \( P_n = F_{1^{n-1}}(0, 1/2) \) and \( P'_n = F_{1^{n-1}}(0, 1/3) \), and use \( L_n \) and \( L'_n \) to denote the lines \( y = \frac{1}{2^{3^{n-1}}} \) and \( y = \frac{1}{3^n} \), respectively. Then \( L_n \) passes through \( P_n \), and \( L'_n \) passes through \( P'_n \). We set \( D'_0 = [0, 1]^2 \), and denote by \( D_n \) (or \( D'_n \)) the rectangle enclosed by the lines \( x = 0, 1, y = 0, \) and \( y = L_n \) (or \( y = L'_n \) respectively) (see Figure 7).

![Figure 7](image-url)
For a cell $K_w$, we define its center to be $F_w(1/2, 1/2)$. We also define an $(L_n)$-reflection to be reflecting a cell along the line $L_n$, and similarly an $(L_n')$-reflection.

**Lemma 6.4.** Let $K_w$ be a cell with $|w| \geq n$.

(i) If $K_w$ has center in $D'_{n-1} \setminus D_n$, and let $K_u$ be the reflected cell of $K_w$ along $L_n$, then $K_u$ is centered in $D_n$ and $g(u) \leq g(w)$.

(ii) If $K_w$ has center in $D_n \setminus D'_n$, and let $K_u$ be the reflected cell of $K_w$ along $L'_n$, then $K_u$ is centered in $D'_n$ and $g(u) \leq g(w)$.

**Proof.** In the first case, the center of $K_{w_{n-1}}$ is on $L_n$, and the $(L_n)$-reflection sends $K_{w_{n-1}}$ to itself. It follows that $K_u$ is centered in $D_n$, and $g(u) = g(w)$ by symmetry. In the second case, one of the line segment of $\partial K_{w_n}$ is on $L'_n$, hence the $(L'_n)$-reflected cell is in $D'_n$, and $g(u) = (a/b) \cdot g(w) \leq g(w)$ by Corollary 6.2.

**Lemma 6.5.** Suppose $g = g_{a,b}$ is not a metric. For $n \geq 0$, let $\gamma$ be an $n$-chain contained in $K$ connecting $p_1$ and $p_3$. Then there exists another $n$-chain $\tilde{\gamma}$ connecting $p_1$ and $p_3$, with $\tilde{p}_1\tilde{p}_3 \subset \bigcup \tilde{\gamma}$, and

$$g(\tilde{\gamma}) \leq g(\gamma).$$

**Proof.** Denote $\gamma_1 := \gamma$, the given chain contained in $K$ connecting $p_1$ and $p_3$. For any cell $K_w$ in $\gamma_1$, $K_w$ has center in $D'_0$. We then apply the $(L_1)$-reflection to those cells centered in $D'_0 \setminus D_1$ to obtain another chain that the cells are centered in $D_1$ (by Lemma 6.4). We do the $(L'_1)$-reflection for those cells in the new chain centered in $D_1 \setminus D'_1$ to obtain another chain, denoted by $\gamma_2$. Then the cells $K_w$ of $\gamma_2$ are centered in $D'_1$.

Inductively we apply this process for $n$ times with $(L_n)$-reflection and $(L'_n)$-reflection for $i = 1, \ldots, n$. Then we obtain a chain $\gamma_{n+1}$ such that each $K_w$ in $\gamma_{n+1}$ has center in $D'_{n}$. By Lemma 6.4,

$$g(\gamma_{n+1}) \leq \cdots \leq g(\gamma_2) \leq g(\gamma_1).$$

Denote $\gamma_{n+1}$ by $\tilde{\gamma}$. Now all cells $K_u$ in $\tilde{\gamma}$ have word length $|u| = n$ and center in $D'_n$. Then one of the line segment of $\partial K_u$ is on $\tilde{p}_1\tilde{p}_3$. Note that all the reflections keep $p_1, p_3$ in the two end cells. This yields $\tilde{p}_1\tilde{p}_3 \subset \bigcup \tilde{\gamma}$. Since

$$g(\tilde{\gamma}) \leq g(\gamma_1),$$

the proof is completed. □

**Proof of “sufficiency” for $\sigma_1$.** Let $(a, b) \in \sigma_1$, then $2a + b \geq 1$ and $a \leq b$. Assume that $D_{b}$ is not a metric. Then by Theorem 5.1 and 5.2, for each $n \geq 1$, there exists an $n$-chain $\gamma_n$ connecting $p_1$ and $p_3$ such that $\lim_{n \to \infty} g(\gamma_n) = 0$. By Lemma 6.5, we obtain a new chain $\tilde{\gamma}_n$ such that $\tilde{p}_1\tilde{p}_3 \subset \bigcup \tilde{\gamma}_n$, and

$$g(\tilde{\gamma}_n) \to 0 \quad \text{as } n \to \infty.$$  

On the other hand, by Lemma 6.3 (i), we see that $g(\gamma_n) \geq 1$ for all $n \geq 1$, a contradiction. Hence $D_{b}$ is a metric. □
The case for $\sigma_2$. The idea of proof is the same as the case $\sigma_1$, but the geometry of the reflection is slightly more complicated. We will concentrate on the set

$$\Omega = K_{37} \cup K_{36} \cup K_{35} \cup K_{41} \cup K_{42} \cup K_{43}.$$ 

For $n \geq 1$, we denote $q_n = F_{31^n}(1/2, 1/2)$ and $q_n' = F_{42^n}(1/2, 1/2)$. We define $\ell_n$ and $\ell_n'$ to be the two lines passing through $q_n$ and $q_n'$ with slope $-1$. Clearly, whenever $\ell_n (\ell_n')$ intersects the “interior” of a cell $K_w, |w| \geq n$, the center of $K_w$ lies in $\ell_n (\ell_n'$ respectively).

Let $\ell_*, \ell_n$ be the lines passing through the points $q_1$ and $q_1'$ with the same slope $1$. Let $M_1$ denote the hexagon enclosed by lines $y = 2/9$, $y = 4/9$, $\ell_1$, $\ell_1'$, $\ell_*$, $\ell_n$ and $\ell_n'$; for $n \geq 2$, let $M_n$ denote the rectangle enclosed by the lines $\ell_*, \ell_n$, $\ell_n'$ and $\ell_n''$, see Figure 8. Also we let $\ell$ be the line passing through the point $q = F_{3}(1/2, 1) = F_{4}(1/2, 0)$ (in the center of $\Omega$) with slope $-1$.

![Figure 8. $\Omega$ and lines for reflection.](image)

The following are some simple geometric properties of the notions we defined.

(i) Let $d_n = \frac{\sqrt{3}}{2} 3^{-n}$. Then $d_n = d(\ell_n, \ell_n') = d(\ell_n, \ell_{n-1}) = d(\ell_n', \ell_{n-1}')$. Therefore, $\ell_n'$ is the reflection of $\ell_{n-1}$ through $\ell_n$, and $\ell_n$ is the reflection of $\ell_{n-1}'$ through $\ell_n'$.

(ii) $\ell \cap (F_{3}(K) \cup F_{4}(K))$ is a line segment contained in $K$, and lies in between $\ell_n$ and $\ell_n'$ for all $n \geq 1$. If $K_w$ is an $n$-cell centered in $\ell_n$ or $\ell_n'$, then by $\diam(K_w) = 2d_n$, $K_w$ intersects $\ell$.

(iii) Let $K_w$ be an $n$-cell. If its center is in $M_{n-1}$, then the center must be in one of the $\ell_{n-1}$, $\ell_n$, $\ell_{n-1}'$ and $\ell_n'$. If its center is in $M_n$, then the center must be in $\ell_n$ or $\ell_n'$. Hence if the center of $K_w$ is in $M_{n-1} \setminus M_n$, then it must lie in $\ell_{n-1}$ or $\ell_{n-1}'$.

(iv) For $K_w$ with $|w| \geq n$, if $K_w$ has center in $M_n$, then $K_w$ also has center in $M_{n-1} \setminus M_n$. If $K_w$ has center in $M_{n-1} \setminus M_n$, then $K_w$ has center in $M_{n-1} \setminus M_n$.

Suppose $n \geq 2$, $K_w$ is a cell with $|w| \geq n$ and has center in $M_{n-1}$. From (iii)-(iv), $K_w$ has center in $\ell_{n-1} \cup \ell_n$ (or $\ell_{n-1}' \cup \ell_n'$). We define the $(n)$-reflection to be the reflection of $K_w$ with respect to $\ell_n$ (or $\ell_n'$ respectively).
Lemma 6.6. Suppose $K_w$ with $|w| \geq n \geq 2$ has center in $M_{n-1}$. Let $K_u$ be the $(n)$-reflected cell of $K_w$, then $K_u$ is centered in $M_n$, and

$$g(u) \leq g(w).$$

(6.2)

Proof. We assume that $w = w_1 \cdots w_m$ and $u = u_1 \cdots u_m$, where $m \geq n$. The assumption on $K_w$ implies the center of $K_{w|n}$ also lies in $M_{n-1}$ (by (iv)), and hence lies in $\ell_n$ or $\ell_{n-1}$. For the first case, the $(n)$-reflection of $K_{w|n}$ is itself, then $K_u$ is also a subcell of $K_{w|n}$. By the symmetry, we have $r_{u_i} = r_{w_i}$ for all $i \geq n + 1$ (the $r_j$ is defined in (6.1)), so that $g(u) = g(w)$.

For the second case, let us assume that the center of $K_{w|n}$ lies on $\ell_{n-1}$, then the center of $K_{w|n-1}$ also lies in $\ell_{n-1}$, and $w_n = 3$ or 7, so that $r_{w_n} = a$. It is easy to check that $K_{u|n-1}$ and $K_{w|n-1}$ share the same line segment so that $g(u_{n-1}) = g(w_{n-1})$ is either $a$ or $b$.

Furthermore $u_n = 2$ if $w_n = 7$, and $u_n = 8$ if $w_n = 3$ so that $r_{u_n} = b$. Thus $g(u_{n-1})$ is either $b$ or $\frac{a}{2}$ (Corollary 6.2). By using that $a \geq b$, we have $g(u_{n-1}) \leq g(w_{n-1})$. By symmetry, we have $r_{u_i} = r_{w_i}$ for all $i \geq n + 1$, so that $g(u) \leq g(w)$. □

Lemma 6.7. Let $n \geq 2$ and let $\gamma$ be an $n$-chain contained in $K_{36} \cup K_{42}$, connecting $p_0 = F_{36}(1,0)$ and $p'_0 = F_{42}(0,1)$. Then there exists another $n$-chain $\tilde{\gamma}$ with the two end cells touching $\ell_2$ and $\ell'_2$, all its cells intersecting the line segment $\ell \cap \Omega$, and

$$g(\tilde{\gamma}) \leq g(\gamma).$$

Proof. Let $\gamma_1 := \gamma$ be a given $n$-chain contained in $K_{36} \cup K_{42}$, connecting $p_0$ and $p'_0$. Then each cell has word length $n$ and has center in $M_1$.

By applying the (2)-reflection, we obtain another $n$-chain whose cells have centers in $M_2$ (by Lemma 6.6). We denote this chain by $\gamma_2$. We carry out the operations $m$-reflection for $m$ from 2 to $n$ inductively, and obtain a chain $\gamma_{n+1}$, such that each $K_w$ in $\gamma_{n+1}$ has center in $M_{n+1}$.

We denote by $\tilde{\gamma} = \gamma_{n+1}$ the same as Lemma 6.5 for $\sigma_1$ with some obvious adjustments (we use (ii) to guarantee the cells in $\tilde{\gamma}$ intersects $\ell$), and arrive the conclusion. □

Proof of the “sufficiency” for $\sigma_2$. The proof is the same as for the case $\sigma_1$, using Lemma 6.7 and Lemma 6.3(ii) instead. □

7. Application to time change

In this section we consider the time change and the sub-Gaussian heat kernel estimates by summarizing the techniques in [27, 28, 15]. We show that the admissible metrics $D_g$ defined by weights on $S$ allow us to give a concrete class of time-changed metrics that admit a two-sided sub-Gaussian estimates.

Recall that for two metrics $d_1$ and $d_2$ on $M$, $d_1$ is said to be quasi-symmetric to $d_2$ if there exists a homeomorphism $h$ from $[0, \infty)$ to itself with $h(0) = 0$ such that
for any \( t > 0 \) and \( x, y, z \in M \), \( d_2(x, z) < h(t)d_2(x, y) \) whenever \( d_1(x, z) < td_1(x, y) \) [19]. In [29] Theorem 15.7], Kigami proved the following proposition.

**Proposition 7.1.** Let \( d \) be the resistance metric on \( K \) if \( K \) is nested fractal, or the Euclidean metric if \( K \) is a GSC. Let \( a \in M \). Then \( D_g \) is quasi-symmetric to \( d \).

By the 1-adaptedness of \( D_g \) in Proposition 4.1, it follows that the diameter of each cell \( K_w \) under \( D_g \) is comparable to \( g(w) \). If we write \( a = (a_1, a_2, \ldots, a_N) \) (where the \( a_i \) in the same equivalent class takes the same value), it follows from a similar proof as [10] Theorem 9.3] that the Hausdorff dimension of \( K \) under the metric \( D_g := D_{ga} \) is the unique real number \( \alpha \) satisfying

\[
\sum_{i=1}^{N} a_i^\alpha = 1.
\]

Moreover, \( 0 < \mathcal{H}^\alpha(K) < \infty \).

We call a measure \( \mu \) satisfies the (volume) doubling condition \((VD)\) if there exists \( C > 0 \) such that \( \mu(2B) \leq C\mu(B) \) for any ball \( B \). For a symmetric self-similar measure (i.e., \( \mu_\tau = \mu_\tau(w) \) for \( \tau \in G \)), from [27] Theorem 3.4.5], we know that \( \mu \) is volume doubling under \( d \) in Proposition 7.1 [27] Theorems 1.6.6 and 3.4.5], hence \( \mu \) is also volume doubling under \( D_g \) since they are quasi-symmetric equivalent.

In the following, we consider the time change of the standard Brownian motion on \( K \) with respect to the symmetric self-similar measures. Let \( \rho \) be the renormalization factor of the associated Dirichlet form in \( L^2(K, \mathcal{H}^\alpha) \) (see the paragraph of (1.5) in the Introduction). We define the capacity \( \text{cap}(A, \Omega)(= R(A, \Omega)^{-1}) \) between two open sets \( A, \Omega \) with \( A \Subset \Omega \) by

\[
\text{cap}(A, \Omega) := \inf \{ \mathcal{E}(u) : u \in \mathcal{F}, u|_A = 1, u = 0 \text{ on } \Omega^c \}.
\]

Let \( B = B(x, r) := \{ y \in K : |x - y| < r \} \) be a metric ball in \( K \) under the metric \( d \) in Proposition 7.1. We use \( \text{cap}(B, 2B) \) to denote the capacity of two concentric balls \( B \) and \( 2B \). Then, we have [1] [4]

\[
\text{cap}(B, 2B) \sim r^{\xi}.
\]

(7.1)

where \( \xi = -\log \rho / \log \ell \) for GSC, and \( \xi = -1 \) for nested fractals. We denote this property by \((\text{cap})\_d\).

Next let us consider a symmetric self-similar measure \( \mu \) with weights \( \{\mu_i\}_{i=1}^N \) and \( \mu_i\rho < 1 \). Let \( g := g_a \) be the symmetric weight function with

\[
a := a(\lambda) = \left( (\mu_1\rho)^l, (\mu_2\rho)^l, \ldots, (\mu_N\rho)^l \right) / \sim_G,
\]

where \( \lambda > 0 \) is such that \( a \in \mathcal{M} \), and let \( D_g \) be the associated admissible metric. Then \( D_g \) is quasi-symmetric to \( d \) as in Proposition 7.1, hence \( \mu \) is also a doubling measure with respect to \( D_g \). We use \( B_D = B_D(x, r) \) to denote the balls for \( D_g \), and we express (7.1) in terms of \( \mu \).
Lemma 7.2. With the above $D_g$, there exists $C > 0$ such that
\[ C^{-1} \frac{\mu(B_D)}{r^\beta} \leq \text{cap}(B_D, 2B_D) \leq C \frac{\mu(B_D)}{r^\beta}, \]
where $r$ is the radius of $B_D$, and $\beta = \lambda^{-1}$.

We denote this property by $(\text{cap})_D$. It is used in proving the sub-Gaussian estimation [15, Theorem 1.1].

Proof. We note that by a standard covering argument, the capacity conditions are equivalent if we change $(B_D, 2B_D)$ into concentric balls $(B_D, \eta B_D)$ with some $\eta > 1$.

We first show the upper estimate in $(\text{cap})_D$. Assume $B_D = B_D(x, r)$ for some $x \in K$ and $r \in (0, 1)$. Let $Q := \{ w \in \Sigma^* : g(w) < r \leq g(w^-) \}$. Pick $w \in Q$ such that $K_w \cap B_D \neq \emptyset$. Assume that $|w| = n$. By the 1-adaptedness and doubling property of $\mu$, it is clear that $\mu(B_D) \asymp \mu_w$. By using the quasi-symmetric of $D_g$ and $d$, there is $\delta > 1$ independent of $x$, $r$ and a metric ball $B_d = B_d(x, r_1)$ such that $B_D \subseteq B_d$ and $\delta B_d \supseteq 2B_d$. Clearly, $r_1 \asymp \ell^{-n}$ for GSC, and $r_1 \asymp \rho^n$ for nested fractals. It follows from the definition of capacity and $(\text{cap})_d$ that
\[ \text{cap}(B_D, \eta B_D) \leq C \mu(B_D) \frac{r^\beta}{r^\beta}. \]

On the other hand, we have
\[ \frac{\mu(B_D)}{r^\beta} \asymp \frac{\mu_w}{g(w)^\beta} = \frac{\mu_w}{(\mu_w\rho^n)^\beta} = \rho^{-n}. \]

Hence we obtain
\[ \text{cap}(B_D, \eta B_D) \leq C \frac{\mu(B_D)}{r^\beta}. \]

This completes the proof of the upper estimate in $(\text{cap})_D$. The lower estimate can be proved in a similar manner. Hence we have that condition $(\text{cap})_d$ holds. \qed

We say that the elliptic Harnack inequality holds if there is $C > 0$ such that for any nonnegative harmonic function $u$ on $2B$,
\[ \sup_B u \leq C \inf_B u, \]
where the ball $B$ is with respect to some reference metric; we denote by $(H)_d$ if the metric balls are under metric $d$, and $(H)_D$ if the metric balls are under $D_g$. It is known that for the standard Dirichlet forms constructed on the elementary fractals, condition $(H)_d$ holds. Since $D_g$ is quasi-symmetric to $d$, we have that $(H)_D$ also holds, see [7, Lemma 5.3].

Outline of proof of Theorem 1.11. Notice that $\mu$ is a volume doubling measure. By [15, Theorem 1.1], we know that under conditions $(VD)$ and $(RVD)$ (reversed $VD$), we have $(H)_D + (\text{cap})_D \Leftrightarrow (UE) + (\text{NLE})$, (Note that the $(RVD)$ follows from $(VD)$ if $K$ is unbounded, and here we can extend $K$ to infinity by self-similarity.) As the two conditions on the left side are satisfied (Lemma 7.2), the conditions on the right side are also satisfied. This implies the first part of the theorem.
Since $D_g$ satisfies the MCC if and only if $\lambda = \lambda_0$ such that $a(\lambda_0) \in S \subset M$ (Theorem 4.7), by a standard chain argument (see [13, p.39-41]), we see that $(NLE)$ implies the off-diagonal lower estimate for such $a$ and $D_g$ with $\lambda_0$.

**Remark 1.** Note that the renormalization factors for the elementary fractals are $0 < \rho < N$ (see $(\ast)$ below), we can conclude that each $a \in M$ can be express as in (1.6),

$$a = a(\lambda) := ((\rho \mu_1)^{1/\lambda}, \ldots, (\rho \mu_k)^{1/\lambda})$$

for some $\lambda > 0$, and hence Theorem 1.11 applies to all $a$ in $M$ and in $S$. Indeed, with a slight abuse of notation, we write $a = (a_1, \ldots, a_N)$ where $a_i = a_j$ for $i \sim_G j$. Then there exists $\lambda > 0$ such that $\sum_{i=1}^N a_i^{1/\lambda} = \rho$ (since the sum goes to 0 as $\lambda \to 0$, and goes to $N$ as $\lambda \to \infty$). Let $\mu_i = a_i^{1/\lambda} / \rho$, it defines a symmetric self-similar measure, and $a$ has the expression as asserted.

$(\ast)$ For nested fractal, $\rho < 1$; for GSC, we can even have the bound $\rho \leq N / \ell^2$.

The proof in [3, Proposition 5.2] derived by a crude shorting argument on the SC can be extended to the GSC; notice their $\rho$, $k$ and $(k^2 - R)$ (in $\mathbb{R}^2$) corresponding to our $\rho^{-1}$, $\ell$ and $(\ell^d - R) = N$ respectively.

**Remark 2.** The theorem can be viewed as an extension of the works [8] for Sierpinski gasket, [31] for nested fractals and [4] for GSCs with Hausdorff measures, which are special cases of the symmetric self-similar measure when we assign the same weight on each cell. In [31], a technical path condition is assumed to ensure that resistance metric is Hölder equivalent to the Euclidean metric, and then the heat kernel estimate can be expressed using a new metric constructed via the Euclidean metrics. Here the quasisymmetry of the admissible metrics and the resistance metrics avoids the technical path assumption.

Finally, we use the standard Sierpinski carpet $K$ in Section 6 to give an illustration of Theorem 1.11. Recall that the boundary $S$ of $M$ is

$$\{(a,b) : 2a + b = 1, b \geq a\} \cup \{(a,b) : a + 2b = 1, a \geq b\}.$$

Let $\rho \in (0,1)$ be the renormalization factor of the associated Dirichlet form. Let $\mu = (\mu_1, \ldots, \mu_8)$ be a self-similar measure on $K$ with $\mu_1 = \mu_3 = \mu_5 = \mu_7$, $\mu_2 = \mu_4 = \mu_6 = \mu_8$. Let $\beta$ be the unique positive number satisfying

$$(\max\{\rho \mu_1, \rho \mu_2\})^{1/\beta} + 2 (\min\{\rho \mu_1, \rho \mu_2\})^{1/\beta} = 1.$$

Let $D_{g_{a,b}}$ be the metric associated with weights $a = (\rho \mu_1)^{1/\beta}$ and $b = (\rho \mu_2)^{1/\beta}$. Then $(a,b) \in S$, and the time change Brownian motion on $K$ via $\mu$ has a continuous heat kernel $p_t(x,y)$ satisfying the estimate as in (1.1), where $d(\cdot, \cdot)$ is replaced by $D_{g_{a,b}}(\cdot, \cdot)$. 

8. Remarks on non-symmetric weights

Not much is known about the set of admissible weights $\mathcal{M}$ and the admissible metrics if we do not assume the symmetric condition for $a \in \mathcal{M}$. In the following, we examine some simple examples of non-symmetric case that different situations can occur in comparison with the symmetric cases.

The unit interval. Let $K = [0, 1]$ be the unit interval. For $N \geq 2$, let the IFS $\{F_i\}_{i=1}^N$ be given by $F_i(x) = \frac{x}{2} + \frac{1}{N}$. Let $\Sigma = \{1, \ldots, N\}$, we consider a self-similar weight function $g$ on $\Sigma^*$ given by $g(i) = a_i \in (0, 1)$ for all $i = 1, \ldots, N$. It is straightforward to see that $D_g$ is a metric if and only if $\sum_{i=1}^N g(i) \geq 1$, and $D_g$ satisfies MCC if and only if $\sum_{i=1}^N g(i) = 1$. Note that in this example, the result in Theorem 1.11 holds even without the symmetric assumption of $g$.

We note that $D_g$ gives a metric quasi-symmetric to the Euclidean metric on $K$ if and only if $g(1) = g(N)$. Let $\mu$ be a self-similar measure on $K$ with probability weights $\{\mu_i\}_{i=1}^N$ satisfying $\mu_1 = \mu_N$. Let $\beta > 0$ be defined by $\sum_{i=1}^N (\mu_i/N)^{1/\beta} = 1$, and let $D_g$ be the metric defined by $a_i = g(i) = (\mu_i/N)^{1/\beta}$. Then the time change of the standard Brownian motion on $K$ via the measure $\mu$ has the same heat kernel estimate as in Theorem 1.11.

For the self-similar measure with an IFS which does not satisfy the open set condition (i.e., with overlaps), we refer to [18], in which the infinite Bernoulli convolution associated with the golden ratio, and a family of convolutions of Cantor-type measures are studied. They make use of Strichartz’s second-order identities defined by auxiliary IFS’s to construct a new metric quasisymmetric to the Euclidean one, and under the new metric, the time change Brownian motion enjoys two-sided sub-Gaussian estimates.

Sierpinski gasket. Let $p_1 = 0, p_2 = 1, p_3 = \exp(i\pi/3)$ be the three vertices of the Sierpinski gasket in $\mathbb{R}^2$ with IFS $\{F_i\}_{i=1}^3$, where $F_i(x) = \frac{1}{2}(x - p_i) + p_i, i = 1, 2, 3$.

For $(a, b, c) \in (0, 1)^3$, a self-similar weight function $g$ on $\Sigma^*$ can be defined as:

$$g(w) = r_{w_1}r_{w_2} \cdots r_{w_n}, \quad w = w_1w_2 \cdots w_n,$$

where

$$r_\theta = \begin{cases} a, & \theta = 1; \\ b, & \theta = 2; \\ c, & \theta = 3. \end{cases}$$

Clearly, if $g$ is a symmetric self-similar weight function as defined in Definition 1.2, we must have $a = b = c$. This trivially gives the critical point with $a(b = c) = 1/2$, and the metric $D_g$ is equivalent to the Euclidean metric on $K$.

If $g$ is not symmetric, by using a similar argument as in the proof of Corollary 5.2 for the Lindstrøm snowflake, we can show that $D_g$ is a metric if and only if

$$\begin{cases} a + b \geq 1, \\ b + c \geq 1, \\ c + a \geq 1. \end{cases}$$
In particular, when \( a, b, c \) are not all equal, we claim that \( D_g \) does not satisfy the MCC. Indeed, in this case, one of \( \{a + b, b + c, c + a\} \) must be strictly larger than 1. We assume for instance \( a + b > 1 \), then \( \lim_{n \to \infty} D_g^{(n)}(p_1, p_2) = \infty \). Using the argument in Proposition 4.5 we know that the MCC fails. Even worse, if \( a, b, c \) are not all equal, the metric \( D_g \) is not quasi-symmetric to the Euclidean metric. For example, if \( a \neq b \), then one can see this from that the metric \( D_g \) at the point \( F_1(K) \cap F_2(K) \) having uncontrollable ratio in two different areas \( F_1(K) \) and \( F_2(K) \).

In [26], Kigami presented an example of time change diffusion on the Sierpinski gasket via the Kusuoka measure (a non-self-similar singular measure defined through the energies on cells). It also has a heat kernel that enjoys the two-sided Li-Yau estimates, with the walk dimension \( \beta = 2 \). The metric used there is the so-called harmonic geodesic metric, which is defined by the shortest path under harmonic coordinates. This structure of the volume, metric and energy are shown to be very similar to that in Riemannian geometry, and it is called the “measurable Riemannian geometry” on the Sierpinski gasket.

In another direction on the Sierpinski gasket, we refer to [17] for the time change of the standard Brownian motion on the unbounded Sierpinski gasket via a class of measures with polynomial growth, where the heat kernel has a two-sided sub-Gaussian estimates with the standard walk dimension \( \beta = \log 5/\log 2 \) under a new metric constructed according to the measure.

**Sierpinski carpet.** Let \( K \) be the standard Sierpinski carpet as in Section 6. We will show that if the self-similar weight is not symmetric, then we cannot require \( D_g \) to satisfy the MCC, even in the case that \( g \) has enough symmetry to make \( D_g \) quasi-symmetric to the Euclidean metric. This example also shows that in general, Theorem 3.1 does not hold if \( g \) is not symmetric.

Let \( g \) be the weight function given by three weights \( (a, b, c) \in (0, 1)^3 \), where \( a_1 = a_3 = a_5 = a_7 = a; a_2 = a_6 = b; \) and \( a_4 = a_8 = c \). We note that if \( D_g \) is a metric on \( K \), then it is quasi-symmetric to the Euclidean metric by a result of Kigami [29, Theorem 15.7].

We consider a special situation of the above weight that \( a < b < c \) and \( 2a + b = 1, 2a + c > 1 \), (e.g., take \( a = \frac{1}{3}, b = \frac{1}{2} \) and \( c = \frac{3}{4} \)). By Lemma 2.6 we also see that \( (a, b, c) \) with conditions \( a < b < c, 2a + b = 1 \) and \( 2a + c > 1 \) are in \( \mathcal{M} \). We claim that this \( (a, b, c) \) is a critical point on the curve \( a(\lambda) = (\lambda, b^\lambda, c^\lambda) \) for \( \lambda \in (0, \infty) \).

Indeed, for any \( \lambda > 1 \), we must have \( 2\lambda + b^{1} < 2a + b = 1 \). Then we can apply the proof of the necessity part of Theorem 6.1 to see that \( D_g(p_1, p_3) = 0 \). This shows that \( (\lambda, b^\lambda, c^\lambda) \notin \mathcal{M} \) for any \( \lambda > 1 \) and hence \( (a, b, c) \) is a critical point. On the other hand, by the assumption \( 2a + c > 1 \), and noting that the chains between \( p_1 \) and \( p_7 \) having smallest weight must intersect the line \( p_1p_7 \), we have \( D_g^{(n)}(p_1, p_7) \geq (2a + c)^n \to \infty \) as \( n \to \infty \). This implies that \( D_g \) with the weight \( (a, b, c) \) does not satisfy MCC for the two points \( p_1 \) and \( p_7 \) in view of Proposition 4.5.
Finally, we remark that except the standard Sierpinski carpet, we do not know the explicit expression for $S$ associated with symmetric self-similar weight functions on the other GSC. It seems to be a difficult problem.

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References


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