TRANSLATION INVARIANT DIRICHLET FORMS AND THEIR SPECTRAL ASYMPTOTICS ON P.C.F. FRACTALS

QINGSONG GU, KA-SING LAU, AND HUA QIU

Abstract. We consider translation invariant Dirichlet forms (not necessarily self-similar) on a class of p.c.f. self-similar sets possessing certain homogeneity. This class of forms may have distinct conductance growth ratio on different directions. Such examples include the asymptotically one-dimensional diffusions on the Sierpinski gasket [18] and its natural generalizations on certain nested fractals [14]. We obtain sharp eigenvalue asymptotic law of the associated Laplacians. For the examples considered in [14], we show that the construction succeeds on all the generalized Sierpinski gaskets but fails on the Vicsek checkerboards. Another example is the translation invariant Dirichlet form on the eyebolted Vicsek cross constructed in [11]. Further, we construct a new class of Dirichlet forms with 3 different types of conductances on the 3-dimensional Sierpinski gasket. As application, we obtain the explicit spectral asymptotics for these examples.

1. Introduction

Dirichlet forms play a central role in the analysis on fractals. There is a large literature on the topic based on Kigami's analytic approach on the *post-critically finite (p.c.f.) self-similar sets*, and the probabilistic approach of Lindstrøm on the nested fractals as well as Barlow and Bass on the Sierpinski carpet (see [1, 2, 16, 24, 29, 30, 31, 34] and the references therein).

For the p.c.f. fractals, usually we obtain a local Dirichlet form through finding a harmonic structure, which is some non-degenerate fixed point of certain nonlinear map between consecutive levels of resistance networks approximating the fractal. Thus the form has a self-similar property[30, 31]. The detailed spectral distribution for regular self-similar Dirichlet forms on p.c.f. fractals with a self-similar measure was obtained by Kigami and Lapidus [26]. In particular, the spectral dimension d_S is given as the unique solution of

$$\sum_{i=1}^{N}(r_i\mu_i)^{\frac{d_S}{2}} = 1, \tag{1.1}$$

where r_i is the resistance renormalization factor and μ_i is the weight of the self-similar measure. The explicit heat kernel estimates for self-similar Dirichlet forms on p.c.f. fractals with the canonical self-similar measure (i.e., the measure with weights $\mu_i = r_i^{\delta}$) was obtained by Hambly and Kumagai [15].

The situation is quite different if one considers degenerate fixed point of the resistance map. The first consideration is by Hattori, Hattori and Watanabe [18]. The basic idea is to use the resistance renormalization map to iterate backwards to construct a sequence

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of compatible networks on the approximating graphs converging to some diffusion on the Sierpinski gasket (SG) (and more general *abc*-gaskets). The Dirichlet forms associated with these diffusions have the translation invariant property but do not have the self-similar property. Recently, the authors showed that there are only two kinds of translation invariant energy forms on the SG [12], one is the standard fully symmetric self-similar Dirichlet form and the other is the forms given in [18]. Hambly and Kumagai [14] explored the construction on a class of nested fractals including the class of generalized SGs and Vicsek checkerboards, they studied the corresponding heat kernel estimates. Later, Hambly and Jones [13] gave a more detailed heat kernel estimate of the asymptotically one-dimensional diffusions on the SG. As a consequence, they obtained the spectral dimension of the associated Laplacian, which was recently improved by the authors [12] by showing that the lower bound of the spectral has the same asymptotic as upper bound (removing a logarithmic term in the lower bound obtained in [13]). Usually we shall not expect the existence of a translation invariant Dirichlet form for any degenerate fixed point of a given resistance renormalization map of certain p.c.f. fractal. In [10], the first two authors constructed an example called the eyebolted Vicsek cross which possesses one class of such translation invariant Dirichlet forms which can not be self-similar, but they also constructed an example called the Sierpinski sickle which can not possess a translation invariant form at a degenerate fixed point. The knowledge of this type of Dirichlet forms on self-similar sets is much less than the well-studied self-similar Dirichlet forms, and even we do not have any further examples in this direction and the study of the associated Laplacians is still far from complete.

The purpose of this paper is two-folds: first is to give a scheme describing the translation invariant Dirichlet forms discussed above, and study their spectral asymptotics; second is to construct more examples of such Dirichlet forms. As supplement to the work [14], we confirm the existence of the forms for the class of generalized Sierpinski gaskets by computing the explicit conductance growth ratios (or equivalently, reciprocals of the resistance decay ratios) and confirming that they are strictly greater than 1 (or satisfy [14, Assumption 4.3]). We also show that the construction of such form fails on all the Vicsek checkerboards by showing that they do not satisfy Assumption 4.3 in [14]. We also construct a new example which is a class of asymptotically 2-dimensional diffusions depending on two parameters on the 3-dimensional SG. For all these examples, we study their spectral asymptotic behavior by using the first result.

We organize the paper as follows. In Section 2, we give some preliminaries about self-similar sets and Dirichlet forms (or resistance forms in particular). In Section 3, we give a scheme for constructing the translation invariant Dirichlet forms, and prove our main result about the spectral asymptotics. In Section 4, we study in detail the examples constructed in previous works and construct a new example of asymptotically 2-dimensional diffusion with 3 different types of conductance on 3-dimensional SG. As a corollary, we give the spectral asymptotics of all the examples.

2. **Preliminaries**

Let $\{F_i\}_{i=1}^N$ be an iterated function system (IFS) on $\mathbb{R}^d$ such that

$$F_i(x) = \varrho(x - b_i) + b_i, \quad 1 \le i \le N, \tag{2.1}$$

where $0 < \varrho < 1$ and $b_i \in \mathbb{R}^d$. Let $K = \bigcup_{i=1}^N F_i(K)$ be the corresponding self-similar set, and let μ be the self-similar measure on K defined by $\mu = \frac{1}{N}\sum_{i=1}^N \mu \circ F_i^{-1}$. If the IFS satisfies the open set condition (OSC), i.e., there is a nonempty bounded open set O such that $F_i(O) \subset O$ and $F_i(O) \cap F_j(O) = \emptyset$ for $i \neq j$, then the Hausdorff dimension of K is $\dim_H(K) = \alpha = \frac{\log N}{|\log \varrho|}$, and μ is the α-dimensional Hausdorff measure normalized on K, which is α-regular in the sense that

$$\mu(B(x,r)) \asymp r^\alpha,$$

for $0 < r < \operatorname{diam}(K)$ and $x \in K$ with $B(x,r) := \{y \in K : \ |x-y| < r\}$. (Note that we use $f \asymp g$ to mean $C^{-1}f \le g \le Cf$ for some $C > 0$.)

We always assume that K is connected. We define the symbolic space of K as usual. Let $\Sigma = \{1, \cdots N\}$ be the alphabet, Σ_n be the set of all the words with length n, and Σ_∞ be the set of infinite words $\omega = \omega_1\omega_2\cdots$; let $\pi : \Sigma_\infty \to K$ be defined by $\{x\} = \{\pi(\omega)\} = \bigcap_{n\ge1} K_{\omega_1\cdots\omega_n}$, a symbolic representation of $x \in K$ by ω, where $K_{\omega_1\cdots\omega_n} = F_{\omega_1} \circ \cdots \circ F_{\omega_n}(K)$.

Following Kigami [24], we define the *critical set* C and the *post-critical set* $\mathcal{P}$ for K by

$$C = \pi^{-1}\Big(\bigcup\nolimits_{1\le i<j\le N}(K_i \cap K_j)\Big), \quad \mathcal{P} = \bigcup\nolimits_{m\ge1} \tau^m(C),$$

where $K_i = F_i(K)$, $\tau : \Sigma_\infty \to \Sigma_\infty$ is the left shift by one index. If $\mathcal{P}$ is a finite set, we call $\{F_i\}_{i=1}^N$ a *post-critically finite* (p.c.f.) IFS, and K a p.c.f. self-similar set. The *boundary* of K is defined to be $V_0 = \pi(\mathcal{P})$. (We always assume $\#(V_0) \ge 2$ to avoid triviality.) We also define

$$V_n = \bigcup_{i\in\{1,\ldots,N\}} F_i(V_{n-1}), \quad V_* = \bigcup_{n\ge1} V_n.$$

It is clear that $\{V_n\}_{n=0}^\infty$ is an increasing sequence of sets, and K is the closure of V_*. For any $\omega \in \Sigma_n$, we call $K_\omega := F_\omega(K)$ a *cell* of K and denote by $V_\omega := F_\omega(V_0)$ the boundary of K_ω, where $F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}$.

It is known that a p.c.f. IFS in (2.1) satisfies the OSC [6]. (More generally, this is true if the associate similar matrices A_i of F_i (instead of the ϱ in (2.1)) are commensurable, i.e., there exists a matrix A and integers n_i such that $A_i = A^{n_i}$; but it is not true without this assumption [35].) Hence the p.c.f. self-similar set K has dimension α, and is associated with a self-similar measure μ that is α-regular.

It is also known in [10] that a p.c.f. IFS in (2.1) satisfies the following separation property:

(H): *there exists* $C_0 > 0$ *such that for any integer* $n \ge 1$ *and any two words* ω *and* τ *with length* n *and* $K_\omega \cap K_\tau = \emptyset$, $\operatorname{dist}(K_\omega, K_\tau) \ge C_0\varrho^n$.

2.1. **Resistance forms.**

([25, Chapter 3]) Let X be a set and $\ell(X)$ be the space of functions on X. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions.

(RF1). $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constant functions and $\mathcal{E}$ is a nonnegative symmetric bilinear form on $\mathcal{F}$. In addition, $\mathcal{E}(u) := \mathcal{E}(u,u) = 0$ if and only if u is constant.
(RF2). Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by that $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(RF3). If $x \neq y$, then there is $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.
(RF4). For any $p, q \in X$,

$$\sup\left\{\frac{|u(p)-u(q)|^2}{\mathcal{E}(u)}:\ u \in \mathcal{F}, \mathcal{E}(u,u) > 0\right\} \tag{2.2}$$

is finite.
(RF5). $\bar{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u}) \leq \mathcal{E}(u)$ for any $u \in \mathcal{F}$, where $\bar{u} = (u \vee 0) \wedge 1$.

For a resistance form $(\mathcal{E}, \mathcal{F})$, if we denote by

$$R(p,q)^{-1} := \inf\{\mathcal{E}(u):\ u \in \mathcal{F}, u(p) = 1, u(q) = 0\}$$

for any distinct $p, q \in X$, and set $R(p,p) = 0$ for any $p \in X$, we have by [24] that $R(\cdot,\cdot)$ is a metric on X, which is call the *effective resistance metric*. It is known [24] that a regular harmonic structure on a p.c.f. self-similar set gives a resistance form on the fractal.

Kigami and Lapidus [26] obtained the spectral asymptotics of Laplacians defined on general p.c.f. self-similar sets admitting a regular harmonic structure under self-similar measures. To be precise, let $(\{F_i\}_{i=0}^N, K, V_0)$ be a p.c.f. self-similar set with boundary V_0. Assume that there exists a regular harmonic structure which gives a resistance form $(\mathcal{E}, \mathcal{F})$ on K satisfying the self-similar property, i.e., for any $u \in \mathcal{F}$ and $1 \leq i \leq N$, $u \circ F_i \in \mathcal{F}$ and

$$\mathcal{E}(u) = \sum_{i=1}^N \frac{1}{r_i}\mathcal{E}(u \circ F_i), \tag{2.3}$$

where $0 < r_i < 1$ for $1 \leq i \leq N$. Let μ be a self-similar measure on K with weights $\{\mu_i\}_{i=1}^N$. Let $\rho(t)$ be the eigenvalue counting function of the corresponding Laplacian on K, with Dirichlet or Neumann boundary conditions. Then

$$\rho(t) \asymp t^{\frac{d_S}{2}}$$

for sufficiently large t, where d_S is the spectral dimension defined in (1.1).

In this paper, we consider a class of not necessarily self-similar resistance forms on p.c.f. self-similar sets defined by (2.1), which can be viewed as degenerate fixed points of the associated resistance renormalizing maps. They have the translation invariant property and satisfy another type of identity instead of (2.3), which may allow different conductance growth ratio on different "directions". We will discuss the details in the next section.

3. **Spectral asymptotics**

In this section, we consider a certain kind of resistance forms on a p.c.f. self-similar set $(K, \{F_i\}_{i=1}^N, V_0)$ satisfying (2.1). We consider the form $(\mathcal{E}, \mathcal{F})$ defined on V_* as following:

$$\mathcal{E}(u) = \lim_{n\to\infty} E_n(u), \quad \forall u \in \ell(V_*),$$

where

$$E_n(u) = \sum_{\omega\in\Sigma_n}\sum_{p,q\in V_0} c(F_\omega(p), F_\omega(q))\Big(u(F_\omega(p)) - u(F_\omega(q))\Big)^2$$

are *compatible* discrete resistance forms on V_n, with $c(F_\omega(p), F_\omega(q)) \geq 0$, the *conductance* between points $F_\omega(p)$ and $F_\omega(q)$. Here "compatible" means that for any $n \geq 1$, $u \in \ell(V_{n-1})$, it always holds that

$$E_{n-1}(u) = \min\{E_n(v) : v \in \ell(V_n), v|_{V_{n-1}} = u\}.$$

In addition, we call the function v which attains the minimal energy the *harmonic extension* of u in V_n. A function $h \in \ell(V_*)$ is called *harmonic* if it minimizes the energy at each level of V_n. The domain $\mathcal{F}$ is defined to be

$$\mathcal{F} = \{u \in \ell(V_*) : \mathcal{E}(u) < \infty\}.$$

We consider the forms with the following assumptions:
(A1): *Assume that for any $n \geq 0$, and any $p, q \in V_0$, $c(F_\omega(p), F_\omega(q))$ is independent of $\omega \in \Sigma_n$, that is*

$$c(F_\omega(p), F_\omega(q)) = c_n(p, q).$$

(A2): *Assume that for any distinct $p, q \in V_0$, either $c_n(p, q) = 0$ for all $n \geq 0$ or there exists a constant $\kappa(p, q) > 1$ such that $c_n(p, q) \asymp \kappa(p, q)^n$. In addition, assume that $(V_0, \kappa(p, q))$ is irreducible, that is, for any pair $p, q \in V_0$, there is a chain $p = p_1, p_2, \cdots, p_{m-1}, p_m = q$ on V_0 such that $\kappa(p_i, p_{i+1}) \neq 0$ for $i = 1, 2, \cdots, m-1$.*

We call a resistance form $(\mathcal{E}, \mathcal{F})$ satisfying **(A1)** and **(A2)** a *homogeneous form*. It is natural that the form can be extended to the completion of V_* under $R(\cdot, \cdot)$. In our situation, the completion is the same with K, provided property **(H)**, see Proposition 3.2. Intersecting the self-similar forms, we have a typical class of homogeneous forms, which are the forms with exactly one essential conductance growth ratio, including the Lindstrøm's construction on nested fractals [29], of which the asymptotic ratios of eigenvalue counting functions are known [26] since they belong to the self-similar forms. Besides this case, we here want to focus more on the non-self-similar case. Known examples include the so-called asymptotically one-dimensional diffusions on SG [18] and their generalizations [14]. In [12], the authors characterized all homogeneous forms on SG. It is proved that there is a dichotomy situation: either $(\mathcal{E}, \mathcal{F})$ is asymptotically one-dimensional as in [18], or $(\mathcal{E}, \mathcal{F})$ is the standard self-similar form established by Kigami in [23]. It is seen that such a form always behaves invariant under translation but not necessarily invariant under the scaling maps F_i. The forms display local anisotropy, i.e., there are some preferred directions of motion which dominate at small scales.

We denote by

$$\kappa_{\max} := \max\{\kappa(p, q) : p, q \in V_0\},$$

and let

$$\kappa_0 = \max\left\{ s : \begin{array}{c} \forall p \neq q \text{ in } V_0, \exists \text{ a chain } p = p_1, p_2, \cdots, p_m = q \text{ in } V_0 \\ \ni \kappa(p_i, p_{i+1}) \geq s, 1 \leq i \leq m-1 \end{array} \right\}.$$

We write V_0 into equivalent classes. For $p, q \in V_0$, set $p \sim q$ if $p = q$ or there is a chain $p = p_1, p_2, \cdots, p_m = q$ in V_0 such that $\kappa(p_i, p_{i+1}) > \kappa_0$ for $1 \leq i \leq m-1$. It is clear that " $\sim$ " gives an equivalent relation on V_0. If there is no p, q such that $\kappa(p, q) > \kappa_0$, then set " $\sim$ " to be the trivial relation that each equivalent class consists of only one singleton in V_0.

For $n \geq 1$, the equivalence relation "$\sim$" on V_0 induces an equivalent relation "$\sim_n$" on V_n: Let x, y be two distinct points in V_n, we say $x \sim_n y$ if and only if there is a chain $x = x_1, x_2, \cdots, x_m = y$ such that for each $1 \leq i \leq m-1$, there is a word $\omega_i \in \Sigma_n$ such that $x_i = F_{\omega_i}(p_i)$ and $x_{i+1} = F_{\omega_i}(q_i)$ for some $p_i, q_i \in V_0$ with $p_i \sim q_i$. For $p \in F_\omega(V_0)$ with some $|\omega| = n$, denoted by $[p]_n$ the equivalent class of p under $\sim_n$.

We will need an additional assumption on $(\mathcal{E}, \mathcal{F})$ to estimate the eigenvalue asymptotic ratio of its associated Laplacian with respect to the measure μ.

(A3): *Assume that there exists some $\omega \in \Sigma_{n_0}$ for some $n_0 \geq 1$ such that for all $p \in F_\omega(V_0)$, we have*

$$[p]_{n_0} \cap V_0 = \emptyset.$$

Proposition 3.1. *Assume* **(A1)**, *for $k \geq 0$, let $\mathcal{E}^{(k)}(u)$ be the form with conductances $c_{n+k}(p,q)$ on V_n, $n \geq 0$. Then $\mathcal{E}(u) = \mathcal{E}^{(0)}(u)$ satisfies the following identity*

$$\mathcal{E}(u) = \sum_{i=1}^{N} \mathcal{E}^{(1)}(u \circ F_i),$$

and consequently for any $n \geq 1$,

$$\mathcal{E}(u) = \sum_{\omega \in \Sigma_n} \mathcal{E}^{(n)}(u \circ F_\omega). \tag{3.1}$$

Let μ be the normalized Hausdorff measure on K, i.e. μ satisfies $\mu = \frac{1}{N}\sum_{i=1}^{N} \mu \circ F_i^{-1}$.

Proposition 3.2. *Let $(\mathcal{E}, \mathcal{F})$ be the form satisfying the assumption* **(A1)** *and* **(A2)**. *Then the resistance metric $R(\cdot,\cdot)$ on V_* satisfies the estimate*

$$C^{-1}|x-y|^{-\frac{\log \kappa_{\max}}{\log \varrho}} \leq R(x,y) \leq C|x-y|^{-\frac{\log \kappa_0}{\log \varrho}}, \quad \forall x, y \in V_*. \tag{3.2}$$

Moreover, $R(\cdot,\cdot)$ is a bounded on V_* with

$$R(x,y) \leq C\left(\sum_{p,q \in V_0 : \kappa(p,q) = \kappa_0} \frac{1}{c_0(p,q)}\right), \quad \forall x, y \in V_*. \tag{3.3}$$

Consequently, $R(\cdot,\cdot)$ can be continuously extended to K, and $(\mathcal{E}, \mathcal{F})$ turns out to be a local regular Dirichlet form on $L^2(K, \mu)$.

Proof. For distinct points x, y in V_*, let n be the integer such that $C_0\varrho^{n+1} \leq |x-y| < C_0\varrho^n$, where C_0 is a positive constant has the same value as that in property **(H)**. We may assume that $n \geq 0$. Let ω and τ be two words with length n such that $x \in K_\omega$ and $y \in K_\tau$, then by **(H)**, we have that

$$K_\omega \cap K_\tau \neq \emptyset.$$

For the upper bound of (3.2), pick $z \in K_\omega \cap K_\tau = V_\omega \cap V_\tau$, we will estimate $R(x,z)$. We can find a sequence of decreasing cells $\{K_{\omega_k}\}_{k \geq 0}$ such that $\omega_0 = \omega$, $\omega_k \in \Sigma_{n+k}$, $k \geq 0$, and $x \in \bigcap_{k \geq 0} K_{\omega_k}$. Choose an infinite chain $z = z_0, z_1, \cdots$ such that $\lim_{k\to\infty} z_k = x$ and z_k and z_{k+1} are contained in $F_{\omega_k}(V_0)$, $\forall k \geq 0$. Then for any $u \in \mathcal{F}$ with $\mathcal{E}(u) \neq 0$, we have

$$|u(x) - u(z)| \leq \sum_{k \geq 0} |u(z_k) - u(z_{k+1})| \leq C \sum_{k \geq 0} \sum_{p,q \in V_0 :\, \kappa(p,q) = \kappa_0} \sqrt{\frac{\mathcal{E}(u)}{c_{n+k}(p,q)}}$$

$$\leq C' \sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} \sqrt{\frac{\mathcal{E}(u)}{\kappa_0^n c_0(p,q)}}.$$

So we have

$$R(x,z) \leq \kappa_0^{-n}\left(C' \sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} \sqrt{\frac{1}{c_0(p,q)}}\right)^2. \tag{3.4}$$

Similarly, we have the same bound for $R(y,z)$. Hence

$$R(x,y) \leq R(x,z) + R(y,z) \leq C\kappa_0^{-n} \leq C|x-y|^{-\frac{\log\kappa_0}{\log\varrho}}.$$

For the lower bound of (3.2), we may find a positive integer m_0 such that, the $n+m_0$ cell $K_{\widetilde{\omega}}$ containing x does not intersect any $n+m_0$ cell which contains y. Then let u be the $(n+m_0)$-piecewise harmonic function with values 1 on $K_{\widetilde{\omega}}$ and 0 on any other points in V_{n+m_0}. Hence $u=0$ on y, and we have

$$\mathcal{E}(u) \leq C \max_{p,q\in V_0} c_{n+m_0}(p,q) \leq C'\kappa_{\max}^n,$$

where C and C' are constants independent of n and x, y. Thus

$$R(x,y) \geq \mathcal{E}(u)^{-1} \geq C'^{-1}\kappa_{\max}^{-n} \geq C^{-1}|x-y|^{-\frac{\log\kappa_{\max}}{\log\varrho}}.$$

A slight modification of identity (3.4) gives the estimate (3.3).

The regularity of $(\mathcal{E},\mathcal{F})$ can be seen by using piecewise harmonic functions to approximate any $u\in C(K)$, and the locality follows from identity (3.1). □

Let Δ be the Laplacian (infinitesimal generator) of the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$, then $\mathcal{F}$ is compactly imbedded in $C(K)$, and hence the Δ has a discrete spectrum which has the only limit point $+\infty$. Let $\rho(x)$ and $\rho_{\mathcal{N}}(x)$ be the Dirichlet and Neumann eigenvalue counting functions of the Laplacian Δ defined by $(\mathcal{E},\mathcal{F})$ under μ.

Proposition 3.3. *Let K be a compact connected set and ν be a Borel measure on K with full support, and let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(K,\nu)$ with $\mathcal{F}\subset C(K)$. Denote by $\{P_t\}_{t\geq 0}$ the associated semigroup of heat operators of $(\mathcal{E},\mathcal{F})$. Suppose $\mathcal{L}\subset\mathcal{F}$ is a closed linear sublattice of $L^2(K,\nu)$, and there exists $C>0$ such that*

$$P_t u \leq Cu, \qquad \forall\ t>0,\ \ u\geq 0,\ \ u\in\mathcal{L}. \tag{3.5}$$

Then $\mathcal{L}$ has dimension at most one.

Proof. The essentially idea of the proof comes from [5, Theorems 7.2, 7.3]. Suppose $\mathcal{L}$ is nontrivial, let $u\geq 0$ be any non-zero element in $\mathcal{L}$, then $u\in C(K)$. Let $U=\{x\in K: u(x)\neq 0\}$. We claim that $U=K$, modulo a ν-null set. If $\upsilon\in C(K)$ and $|\upsilon|\leq\alpha u$ for some $\alpha\geq 0$, then by the Markovian property of $\{P_t\}_{t>0}$ and (3.5), we have

$$|P_t\upsilon| \leq P_t|\upsilon| \leq \alpha P_t u \leq \alpha C u.$$

Hence for

$$\mathcal{G} = \{\upsilon\in C(K): |\upsilon|\leq\alpha u \ \text{ for some } \alpha\geq 0\},$$

$P_t(\mathcal{G})\subseteq\mathcal{G}$ for all $t\geq 0$. As U is an open set by definition, $\mathcal{G}$ contains all the continuous functions that are compactly supported in U. The L^2-closure of $\mathcal{G}$ is the set of all $\upsilon\in L^2(K,\nu)$ with $\upsilon=0$ on $K\setminus U$. So U is an *invariant* set of the semigroup $\{P_t\}_{t>0}$. (A ν-measurable set $B\subset K$ is said to be P_t-invariant if $P_t(1_B f)=1_B P_t f$ ν-a.e. for any $f\in L^2$

and $t > 0$.) Hence by [8, Theorem 1.6.1], $1_U \in \mathcal{F}$. However, as K is connected, this holds if and only if $U = K$ or $U = \emptyset$. Since u is nonzero, we conclude that $U = K$, and the claim follows.

Now, if $u \in \mathcal{L}$, then u^+ and u^- are in $\mathcal{L}$ and have disjoint supports. It follows from the claim that one of them must vanish. Hence $u \in \mathcal{L}$ implies $u \geq 0$ or $(-u) \geq 0$. If u, v are two distinct positive elements of $\mathcal{L}$, then $u + \eta v$ is either positive or negative for all $\eta \in \mathbb{R}$. But the sum must change sign as η increases through $\mathbb{R}$. Hence there is η such that $u + \eta v = 0$. This is a contradiction, and hence $\mathcal{L}$ is one dimensional. □

Lemma 3.4. *Let K be a p.c.f. self-similar set satisfying (2.1) and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form satisfying* **(A1)** *and* **(A2)**. *Let Λ_1 be the eigenfunction space of λ_1, the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. Then Λ_1 is of dimension one.*

Proof. We make use of the Rayleigh quotient for the first eigenvalue:

$$\lambda_1 = \inf_{u \in \mathcal{F}_0, u \neq 0} \frac{\mathcal{E}(u)}{||u||_2^2}, \tag{3.6}$$

where $\mathcal{F}_0 := \{u \in \mathcal{F} : u|_{V_0} = 0\}$. There exists a function $u \in \mathcal{F}$ attains the infimum, and all such functions must be eigenfunctions with eigenvalue λ_1. Therefore by the Markovian property of the Dirichlet form, we see that Λ_1 is a closed *sublattice*, hence also u^+, u^- are contained in Λ_1 (here $u^\pm$ means the positive and negative parts of u). For any $u \in \Lambda_1$, we have

$$P_t u = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Delta^n u = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-\lambda_1)^n u = e^{-t\lambda_1} u \leq u.$$

By using Proposition 3.3 with $\mathcal{L} = \Lambda_1$, we see that Λ_1 is of dimension at most one, and thus Λ_1 is one dimensional since Λ_1 is nontrivial. □

Lemma 3.5. *Let K be a p.c.f. self-similar set satisfying (2.1) and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(K, \mu)$ satisfying* **(A1)**, **(A2)** *and* **(A3)**. *There exists $C > 0$ such that for any initial conductance $\{c_0(p,q)\}_{p,q \in V_0}$, we have*

$$C^{-1} \left(\sum_{p,q \in V_0:\ \kappa(p,q) = \kappa_0} \frac{1}{c_0(p,q)} \right)^{-1} \leq \lambda_1 \leq C \sum_{p,q \in V_0:\ \kappa(p,q) = \kappa_0} c_0(p,q), \tag{3.7}$$

where λ_1 is the first eigenvalues of $-\Delta$ with the Dirichlet boundary condition.

Proof. We will make use of the Rayleigh quotient in (3.6) again. We will choose a piecewise harmonic function $u_0 \in \mathcal{F}_0$. By **(A3)**, we can find an n_0-level cell K_ω such that for any $p \in F_\omega(V_0)$, $[p]_{n_0} \cap V_0 = \emptyset$. Set u_0 be 1 on $\bigcup_{p \in F_\omega(V_0)} [p]_{n_0}$, 0 on other points in V_{n_0}, and harmonic in each n_0 level cell. Then

$$||u_0||_2^2 \geq \int_{F_\omega(K)} u_0^2 d\mu = \mu(K_\omega) = N^{-n_0}.$$

Also observe that $\mathcal{E}(u_0) \le C \sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} c_{n_0}(p,q)$ for some $C > 0$. Therefore

$$\lambda_1 \le \frac{\mathcal{E}(u_0)}{\|u_0\|_2^2} \le C \sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} c_{n_0}(p,q)\cdot N^{n_0} \le C' \sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} c_0(p,q)$$

for some $C' > 0$.

To estimate the lower bound, we let $u \in \mathcal{F}$, then

$$|u(x) - u(y)|^2 \le R(x,y)\mathcal{E}(u), \quad x, y \in K.$$

It follows that for $u \in \mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$, $u \ne 0$, by choosing y to be some $p \in V_0$, we have

$$|u(x)|^2 \le R(x,p)\mathcal{E}(u), \quad \forall x \in K.$$

Integrating both sides with respect to μ, we obtain

$$\|u\|_2^2 \le \int_K R(x,p)d\mu(x)\cdot\mathcal{E}(u).$$

Using (3.3), we have $C_1 > 0$ such that

$$C_1\left(\sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} \frac{1}{c_0(p,q)}\right)^{-1} \le \frac{\mathcal{E}(u)}{\|u\|_2^2}.$$

Since u is arbitrary, this implies that $C_1\left(\sum_{p,q\in V_0:\ \kappa(p,q)=\kappa_0} \frac{1}{c_0(p,q)}\right)^{-1} \le \lambda_1$. This completes the proof of the lemma. □

Lemma 3.6. *For all $t \ge 0$ and $n \ge 0$,*

$$N^n\rho^{(n)}\left(\frac{t}{N^n}\right) \le \rho^{(0)}(t), \quad \text{and} \quad \rho_N^{(0)}(t) \le N^n\rho_N^{(n)}\left(\frac{t}{N^n}\right). \tag{3.8}$$

Here $\rho^{(n)}(t)$ ($\rho_N^{(n)}(t)$) is the eigenvalue counting function of the Dirichlet (Neumann) Laplacian associated with $\mathcal{E}^{(n)}$ with respect to the Hausdorff measure μ. We refer to the similar proof in [26, Propsitions 6.2, 6.3]. The technique is that first we restrict $\mathcal{E}^{(0)}$ on the sub-domain $\mathcal{F}_1 := \{u \in \mathcal{F} : u|_{V_1} = 0\}$. Denote by $\rho\left(t; \mathcal{E}^{(0)}, \mathcal{F}_1\right)$ the corresponding eigenvalue counting function, then by making use of the identity (3.1) we have the relation

$$\rho\left(t; \mathcal{E}^{(0)}, \mathcal{F}_1\right) = N\rho^{(1)}\left(\frac{t}{N}\right),$$

where $\frac{1}{N}$ in the bracket is the scaling factor of μ. Using this repeatedly and that $\rho\left(t; \mathcal{E}_0^{(n)}, \mathcal{F}_1\right) \le \rho^{(n)}(t)$, we obtain the first inequality in (3.8). The second inequality can be shown by constructing another Dirichlet form which has domain $\mathcal{F}_2 := \{u \in \ell(K \setminus V_1) : u \circ F_i = f_i \text{ on } K \setminus V_0 \text{ for some } f_i \in \mathcal{F}, i = 1, 2, \cdots, N\}$ and using a similar argument.

Theorem 3.7. *Assume that* **(A1)**, **(A2)** *and* **(A3)** *hold, then for $t_0 = \inf\{t : \rho(t) > 0\}$,*

$$\rho(t) \asymp t^{\frac{\log N}{\log(N\kappa_0)}}, \qquad t > t_0.$$

Similarly, the same inequality holds when $\rho(t)$ is replaced by $\rho_N(t)$ and for any $t_0 > 0$.

Proof. By Lemma 3.4, we see that if we denote by $\lambda_1^{(n)}$ the first eigenvalue of the Dirichlet Laplacian associated with $\mathcal{E}^{(n)}$, then we have $\rho^{(n)}\left(\lambda_1^{(n)}\right) = 1$, and $\rho_N^{(n)}\left(\lambda_1^{(n)}\right) \leq \rho^{(n)}\left(\lambda_1^{(n)}\right)+N = N+1$. Then by Lemma 3.6, we have

$$\rho_N^{(0)}\left(N^n\lambda_1^{(n)}\right) \leq C \cdot N^n.$$

Letting $t = N^n\lambda_1^{(n)}$, by Lemma 3.5, we have $t \asymp N^n\kappa_0^n$ and $N^n \asymp t^{\log N/\log(N\kappa_0)}$. It follows that

$$\rho(t) = \rho^{(0)}(t) \leq \rho_N^{(0)}(t) \leq Ct^{\frac{\log N}{\log(N\kappa_0)}}$$

for some $C > 0$. The same argument yields the other inequality. □

4. **Examples**

In this section, we present several examples to illustrate Theorem 3.7. The first class of examples is the one-parameter family of diffusions on a class of generalized Sierpinski gaskets (on SG is constructed in [18] and on generalized SGs is considered in [14]). We then give two more examples, one is the translation invariant forms on the eyebolted Vicsek cross [11]; the other is a new class of forms with 3 different types of conductance on the 3-dimensional SG.

The *asymptotically one-dimensional diffusion* on the 2-dimensional SG was firstly constructed by Hattori, Hattori and Watanabe [18] using a probabilistic method. Then Hambly and Kumagai [14] extended this construction of Dirichlet forms to some nested fractals and obtained their heat kernel estimates. It was proved that under an assumption ([14, Assumption 4.3]) on the explicit conductance growth ratio, their construction will successfully yield a one-parameter family of local regular Dirichlet forms on those fractals. There are two classes of nested fractals considered in [14], one is the generalized SGs, the other is the Vicsek checkerboards. We will prove that this assumption does **HOLD** for all the generalized SGs, but it does **NOT HOLD** for all the Vicsek checkerboards. Hambly and Jones [13] obtained the heat kernel estimates and spectral asymptotics (as a byproduct) for the standard SG, and recently the authors improved their lower bound in [12] by using a purely analytic method.

Then we will recall another example, the eyebolted Vicsek cross (constructed by two of the authors in [11]), on which they proved that there exist two kinds of resistance forms. One of them is self-similar; the other satisfies the conditions **(A1)**-**(A3)** in Section 3.

In the next, we will give a construction of a resistance form on the 3-dimensional SG, whose conductances (resistances) are of 3 different types. Note that in [14], it is mentioned without detail that the multi-parameter diffusions would exist.

4.1. **Example: one-parameter diffusions on the generalized Sierpinski gaskets.**

We consider a class of nested fractals which is generalized from the SG, see Figure 1. Take a d-dimensional level-ℓ Sierpinski gasket as K for $d, \ell \geq 2$. The set V_0 consists of $d+1$ points denoted by $p_1, p_2, \ldots, p_{d+1}$ which are vertices of a d-dimensional tetrahedron.

In [14], it is shown that the effective conductivity map $\alpha : [0, w^*] \to [0, w^*]$ is a strictly increasing map with two fixed points 0 and $1(= w^*)$, where the case $w = w^* = 1$ is corresponding to the standard resistance form on K, and the existence of such form was proved by Lindstrøm [29]. Following [14], for $w \in (0, 1)$, for each $n \geq 0$, we denote by

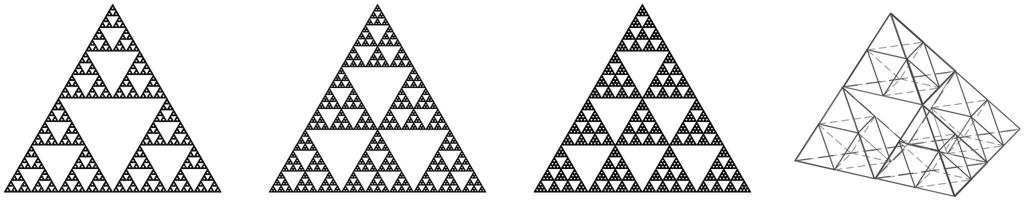

Figure 1. The d-dimensional level-ℓ Sierpinski gaskets with $(d,\ell) = (2,2),(2,3),(2,4),(3,2)$.

$a_n := R_n(w)$ and $b_n := R_n(w)\alpha^{-n}(w)$ the two kinds of conductances on V_n, where a_n is assigned to be the conductance of each of the edges of a $(d-1)$-dimensional face (call it the bottom face of K) and its parallel hyperplanes, while other edges are of conductance b_n. Then by [14, Lemmas 3.9 and 3.11] we have

$$a_n \asymp R_G^n, \qquad b_n \asymp (R_G\beta)^n,$$

where $R_G(>1)$ is the reciprocal of the standard resistance renormalization factor of the $(d-1)$-dimensional level-ℓ SG, and $\beta \in (0,1)$. By [14, Theorem 4.4], if the condition $R_G\beta > 1$ holds (See [14, Assumption 4.3]), then the compatible conductance sequence $\{(a_n,b_n)\}_{n\geq 0}$ will give a regular local Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$, where μ is the normalized Hausdorff measure on K.

As a supplement to [14], for all the generalized SGs, we compute the explicit value of $R_G\beta$ and show that indeed $R_G\beta > 1$. As a result, the construction of asymptotically one-dimensional diffusions truly exist on all the generalized SGs.

To this end, we fist give some notations. Let $d \geq 2$, $i \geq 1$ be integers, define $H(d,i)$ inductively:

$$\begin{aligned} H(1,i) &= 1 \text{ for all } i \geq 1; \\ H(d,1) &= 1 \text{ for all } d \geq 2; \\ H(d,i) &= \sum_{j=1}^{i} H(d-1,j) \text{ for all } d \geq 2 \text{ and } i \geq 1. \end{aligned}$$

Let $d \geq 2$ and $\ell \geq 2$, then for a d-dimensional level-ℓ SG, denoted by K, for $1 \leq i \leq \ell$, $H(d,i)$ is the number of i-th layer level-1 cells of K that parallel to the bottom face. In particular, $H(d,\ell)$ is the number of level-1 cells in a $(d-1)$-dimensional level-ℓ SG.

Proposition 4.1. *Let K be a d-dimensional level-ℓ SG, let $0 < w < 1$ and $(a_0,b_0) = (1,w)$ be the conductances on V_0. Then*

$$a_n \asymp R_G^n, \qquad b_n \asymp \left(\sum_{i=1}^{\ell} \frac{1}{H(d,i)}\right)^n.$$

Consequently, Assumption 4.3 in [14] *holds in this case.*

Proof. The asymptotic of a_n that $a_n \asymp R_G^n$ is shown in [14], and $R_G > 1$ is also provided therein. We will study the asymptotic of b_n.

On the one hand, we consider V_0 with conductance (a_n,b_n) and V_1 with conductance (a_{n+1},b_{n+1}). Note that they are compatible. The effective conductance between the top vertex P and the union set of d points in V_0 contained in the bottom face is clearly

$D_n := db_n$ by the parallel law. By the monotonicity law [7], D_n becomes larger if we replace each conductance a_{n+1} by ∞. Denote by D'_n the effective conductance with a_{n+1} replaced by ∞, then by computation using the series law and parallel law, it gives that $D'_n = db_{n+1}\left(\sum_{i=1}^{\ell}\frac{1}{H(d,i)}\right)^{-1}$. Hence by $D_n \le D'_n$, we have

$$db_{n+1}\left(\sum_{i=1}^{\ell}\frac{1}{H(d,i)}\right)^{-1} \ge db_n,$$

which implies that for all $n \ge 0$,

$$\frac{b_{n+1}}{b_n} \ge \sum_{i=1}^{\ell}\frac{1}{H(d,i)}. \tag{4.1}$$

On the other hand, we still let V_1 be equipped with conductance (a_{n+1}, b_{n+1}), consider the harmonic function u on V_1 with values 1 at the top vertex P and 0 at the d points in V_0 contained in the bottom face. Then the graph energy $E(u) = db_n$ by the compatibility of $\{(a_n, b_n)\}_{n=0}^{\infty}$. Consider each $(d-1)$-dimensional layer of level-1 edges, denoted by H, we then have $a_{n+1}|u(x)-u(y)|^2 \le E(u)$ for $x, y \in H$ connected by a edge, and hence $|u(x)-u(y)|^2 \le db_n/a_{n+1}$. As a consequence, $|u(x)-u(y)|^2$ is of order at most $O(\beta^n)$ for any $x, y \in H$. (In particular, on the bottom face, u is of order at most $O(\beta^{n/2})$.)

Write $E(u)$ into the summation of horizontal energies $E_i^{(h)}(u)$ on each $(d-1)$-dimensional layer H_i for $i = 1, \ldots, \ell$, and the vertical energies $E_i^{(v)}(u)$ between two neighboring layers H_{i-1} and H_i for $i = 1, \ldots, \ell$, then

$$\begin{aligned} E(u) &= \sum_{i=1}^{\ell} E_i^{(h)}(u) + \sum_{i=1}^{\ell} E_i^{(v)}(u) \\ &\ge \sum_{i=1}^{\ell} E_i^{(v)}(u) = \sum_{i=1}^{\ell} \sum_{\substack{x\in H_{i-1}, y\in H_i \\ x\sim y}} b_{n+1}(u(x)-u(y))^2, \end{aligned} \tag{4.2}$$

where $x \sim y$ means that x and y are connected by conductance b_n.

Fix some $x_i \in H_i$, we obtain $E(u)$ is no less than

$$\sum_{i=1}^{\ell} \sum_{\substack{x\in H_{i-1}, y\in H_i \\ x\sim y}} b_{n+1}(u(x_{i-1})-u(x_i)+u(x)-u(x_{i-1})+u(x_i)-u(y))^2. \tag{4.3}$$

By counting the numbers of the pairs $x \sim y$ for $x \in H_{i-1}$ and $y \in H_i$, and using the estimate $|u(x)-u(y)|^2$ for x, y belonging to the same layer H_i, we have

$$\begin{aligned} E(u) &\ge \sum_{i=1}^{\ell} dH(d,i)b_{n+1}\left((u(x_{i-1})-u(x_i))^2 - |u(x_{i-1})-u(x_i)|\cdot O(\beta^{n/2})\right) - b_{n+1}\cdot O(\beta^n) \\ &\ge \sum_{i=1}^{\ell} dH(d,i)b_{n+1}(u(x_{i-1})-u(x_i))^2 - b_{n+1}\cdot O(\beta^{n/2}) \\ &\ge db_{n+1}\left\{\sum_{i=1}^{\ell}\frac{1}{H(d,i)}\right\}^{-1} - b_{n+1}\cdot O(\beta^{n/2}). \end{aligned} \tag{4.4}$$

By $E(u) = db_n$ and (4.4), we see that

$$\frac{b_n}{b_{n+1}} \geq \left\{ \sum_{i=1}^{\ell} \frac{1}{H(d,i)} \right\}^{-1} - O(\beta^{n/2}). \tag{4.5}$$

Combining (4.1), (4.5) and $\beta \in (0,1)$, we have that

$$b_n \asymp \left\{ \sum_{i=1}^{\ell} \frac{1}{H(d,i)} \right\}^{n}.$$

As a result, since $\sum_{i=1}^{\ell} \frac{1}{H(d,i)} > \frac{1}{H(d,1)} = 1$, we have that $R_G\beta > 1$, proving the Assumption 4.3 in [14]. □

We then apply Theorem 3.7 to obtain the spectral asymptotics for this class of Dirichlet forms as following.

Corollary 4.2. *Let $d \geq 2$ and $\ell \geq 2$ be two integers, K be a d-dimensional level-ℓ SG, and μ be the normalized Hausdorff measure on K. Let the Dirichlet form $(\mathcal{E}, \mathcal{F})$ be defined as above on $L^2(K,\mu)$. Let Δ be the generator of the form and $\rho(t)$ be its eigenvalue counting function, then for t large enough,*

$$\rho(t) \asymp t^{\frac{\log H(d+1,\ell)}{\log(H(d+1,\ell)S(d,\ell))}},$$

where $S(d,\ell) = \sum_{i=1}^{\ell} \frac{1}{H(d,i)}$.

Proof. Given the conductances $\{(a_n, b_n)\}_{n\geq 0}$ for $(\mathcal{E}, \mathcal{F})$ on K, it is easy to check that conditions **(A1)**, **(A2)** and **(A3)** are all satisfied. By Theorem 3.7 with $N = H(d+1,\ell)$ and $\kappa_0 = S(d,\ell)$, we have

$$\rho(t) \asymp t^{\frac{\log H(d+1,\ell)}{\log(H(d+1,\ell)\cdot S(d,\ell))}}.$$

□

4.2. **Example: Vicsek checkerboards (fails).** Let $k \geq 1$ be an integer. The family of $(2k+1)$-Vicsek checkerboards was also considered in [14], see Figure 2. Let $V_0 =$

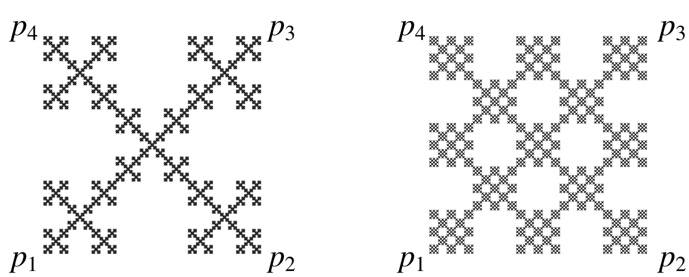

FIGURE 2. The $(2k+1)$-Vicsek checkerboard with $k = 1, 2$.

$\{p_1, p_2, p_3, p_4\}$ be the set of 4 vertices of a unit square. As in [14], let a_0 be the conductance on the two diagonals on V_0, and b_0 be the conductance on each side on V_0. Put $a_0 = 1$ and $b_0 = w$ such that $w \in (0, w^*)$ where $w^* > 0$ is the fixed point for the conductivity map α. Let $\{(a_n, b_n)\}_{n\geq 0}$ be the compatible sequence on the approximating graphs of K. Then it is known in [14] that $a_n \asymp R_G^n = (2k+1)^n$ and $b_n \asymp (R_G\beta)^n$ with some $\beta \in (0,1)$. We will show that $R_G\beta \leq 1$ for all the $(2k+1)$-Vicsek checkerboards. Thus in this case, the Assumption 4.3 in [14] does not hold.

Proposition 4.3. *For the* $(2k+1)$*-Vicsek checkerboards, Assumption 4.3 in* [14] *fails.*

Proof. On V_1 we put conductance (a_{n+1}, b_{n+1}), and by the compatibility, its trace on V_0 is (a_n, b_n). Let u be the harmonic function on V_1 such that $u(p_1) = u(p_3) = 0$ and $u(p_2) = u(p_4) = 1$, then it is clear that $E(u) = 4b_n$. By the monotonicity law, if we remove the cells that are not located on the two diagonals, then the new harmonic function, denoted by υ, satisfies $E(\upsilon) \leq E(u)$. By computation, we have that

$$E(\upsilon) = \frac{4(a_{n+1} + b_{n+1})b_{n+1}}{(a_{n+1} + b_{n+1}) + 4kb_{n+1}}. \tag{4.6}$$

By (4.6) and that $E(\upsilon) \leq E(u) = 4b_n$, we obtain

$$\frac{b_{n+1}}{b_n} \leq 1 + \frac{4kb_{n+1}}{a_{n+1} + b_{n+1}} = 1 + O(\beta^n).$$

This shows that $R_G\beta \leq 1$, and hence [14, Assumption 4.3] fails for the Vicsek checkerboards. □

4.3. **Example: translation invariant forms on the eyebolted Vicsek cross.** In $\mathbb{R}^2$, let $\{p_1, p_2, p_3, p_4\}$ be the four vertices of the unit square S, and let p_0 be the center of S, that is, $p_0 = (0,0)$ and $p_1 = (-1/2, -1/2)$, $p_2 = (1/2, -1/2)$, $p_3 = (1/2, 1/2)$, $p_4 = (-1/2, 1/2)$. Divide S into a mesh of sub-squares of size $1/9$, and pick 21 sub-squares as shown in Figure 3.

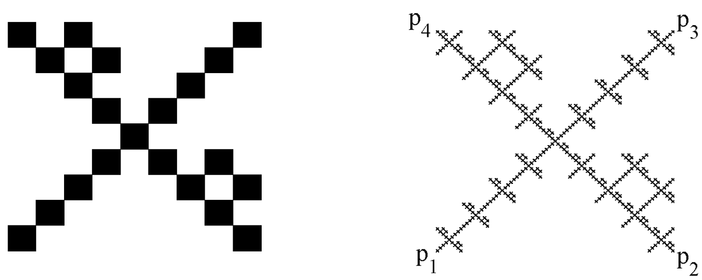

FIGURE 3. The eyebolted Vicsek cross.

For each sub-square Q, let $F_Q : S \to S$ be given by

$$F_Q(x) = x/9 + p_Q$$

where p_Q is chosen so that $F_Q(S) = Q$. Renumber the maps F_Q by $\{F_i\}_{i=1}^{21}$. Let K be the unique nonempty compact set such that $K = \bigcup_{i=1}^{21} F_i(K)$. Then $(K, \{F_i\}_{i=1}^{21})$ is a p.c.f. self-similar set with boundary $V_0 = \{p_1, p_2, p_3, p_4\}$. We call this modified Vicsek cross an *eyebolted Vicsek cross*. The Hausdorff dimension of K is $\alpha = \log 21/\log 9$, and the self-similar measure with the natural weight is the normalized α-dimensional Hausdorff measure μ on K.

It is shown in [10, Theorem 5.1] that on the eyebolted Vicsek cross, there are two local regular Dirichlet forms that can be constructed. One satisfies the energy self-similar identity (2.3), the other one is from a reverse recursive construction and does not satisfy (2.3) but satisfies (3.1). We now consider the second construction. The details can be found in

[10].

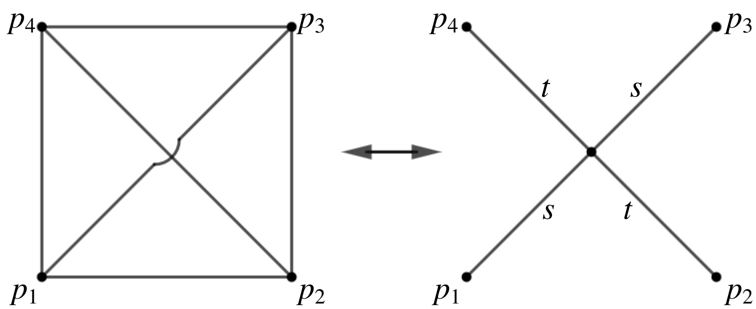

FIGURE 4. the $\boxtimes - X$ transform.

Let $\{s_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ be two positive sequences such that $s_n = 9^{-n}$ and $t_{n-1} = 9t_n - \frac{t_n^2}{9^{-n}+t_n}$ for $n \geq 1$ with $t_0 = 1$, and let the conductances $c_n(x,y)$ on V_n be given by

$$c_n(F_\omega(p_i), F_\omega(p_{i+1}))^{-1} = 2(s_n + t_n),\ i = 1,2,3,4,\ (p_5 = p_1)$$

$$c_n(F_\omega(p_1), F_\omega(p_3))^{-1} = 2(s_n + \frac{s_n^2}{t_n}),$$

$$c_n(F_\omega(p_2), F_\omega(p_4))^{-1} = 2(t_n + \frac{t_n^2}{s_n}),$$

for any $\omega \in \Sigma_n$, using the $\boxtimes - X$ transform illustrated in Figure 4. For $u \in C(K)$ and $n \geq 0$, let

$$E_n(u) = \sum_{\omega\in\Sigma_n}\sum_{p,q\in V_0} c_n(F_\omega(p), F_\omega(q))\Big(u(F_\omega(p)) - u(F_\omega(q))\Big)^2.$$

By the compatibility of E_n and E_{n-1}, we see that $\{E_n(u)\}_{n\geq 0}$ is an increasing sequence on n, and define

$$\mathcal{E}(u) = \lim_{n\to\infty} E_n(u), \quad \mathcal{F} = \{u \in C(K) : \mathcal{E}(u) < \infty\}.$$

Note that $\mathcal{F}$ is dense in $C(K)$ by approximating $u \in C(K)$ through the piecewise harmonic functions applied to the subcells. Hence it is not hard to see that $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on K.

Lemma 4.4. *Let $t_{n-1} = 9t_n - \frac{t_n^2}{9^{-n}+t_n}$ with $t_0 = 1$ be a positive sequence as above, then*

$$t_n \asymp 8^{-n}.$$

Proof. Let $x_n = 8^n t_n$, then we have

$$x_{n-1} = x_n + \frac{1}{8} \cdot \frac{x_n}{1 + (9/8)^n x_n}. \tag{4.7}$$

Thus $\{x_n\}$ is a non-increasing sequence and bounded from below, hence $\lim\limits_{n\to\infty} x_n$ exists. On the other hand, let $y_n = 9^n t_n$, then we have

$$y_{n-1} = y_n - \frac{1}{9} \cdot \frac{y_n^2}{1 + y_n}. \tag{4.8}$$

Hence y_n is a non-decreasing sequence and we will show that $\lim_{n\to\infty} y_n = \infty$. Indeed, if $\lim_{n\to\infty} y_n = y$ is finite, then letting $n \to \infty$ in (4.8), we should have $y = y - \frac{1}{9} \cdot \frac{y^2}{1+y}$, this gives $y = 0$, a contradiction because $y_0 = 1$ and y_n is non-decreasing.

Pick n_0 large enough, and for $n \geq n_0$, we see by (4.7) that

$$\begin{aligned} x_n &= x_{n_0} + \sum_{k=n_0}^{n-1} (x_{k+1} - x_k) \\ &= x_{n_0} - \frac{1}{8} \cdot \sum_{k=n_0}^{n-1} \frac{x_{k+1}}{1 + y_{k+1}} \\ &\geq x_{n_0} \left(1 - \frac{1}{8} \cdot \sum_{k=n_0}^{\infty} \frac{1}{y_{k+1}} \right). \end{aligned} \tag{4.9}$$

By (4.8), we have

$$\frac{y_n}{y_{n-1}} = \left(1 - \frac{1}{9} \cdot \frac{y_n}{1 + y_n} \right)^{-1} > \left(1 - \frac{1}{9} \cdot \frac{1}{2} \right)^{-1} = \frac{18}{17},$$

and hence $y_n \geq \left(\frac{18}{17}\right)^n$. Substituting this into (4.9), we have for some n_0 large that

$$x_n \geq \frac{1}{2} x_{n_0},$$

proving that $t_n \asymp 8^{-n}$. □

Corollary 4.5. *Let K be the eyebolted Vicsek cross, μ be the normalized Hausdorff measure on K. Let the Dirichlet form $(\mathcal{E}, \mathcal{F})$ be defined as above on $L^2(K, \mu)$. Let Δ be the generator of the form and $\rho(t)$ be its eigenvalue counting function, then for t large enough,*

$$\rho(t) \asymp t^{\frac{\log 21}{\log 168}}.$$

Proof. By Lemma 4.4, we have

$$\begin{aligned} &c_n(F_\omega(p_i), F_\omega(p_{i+1})) \asymp 8^n, \qquad i = 1, 2, 3, 4.\ (p_5 = p_1) \\ &c_n(F_\omega(p_1), F_\omega(p_3)) \asymp 9^n, \\ &c_n(F_\omega(p_2), F_\omega(p_4)) \asymp \left(\frac{64}{9}\right)^n, \end{aligned}$$

for $\omega \in \Sigma_n$. Then it is easy to check that conditions **(A1)**, **(A2)** and **(A3)** are all satisfied. By Theorem 3.7 with $N = 21$ and $\kappa_0 = 8$, we have

$$\rho(t) \asymp t^{\frac{\log 21}{\log 168}}.$$

□

4.4. **Example: a new class of diffusions on the 3-dimensional SG.** Let p_1, p_2, p_3, p_4 be the four vertices of a unit tetrahedron in $\mathbb{R}^3$. We define the 3-dimensional Sierpinski gasket K to be the unique nonempty compact set in $\mathbb{R}^3$ with the contractions $\{F_i\}_{i=1}^4$ on $\mathbb{R}^3$ such that $F_i(x) = \frac{1}{2}(x-p_i)+p_i$, $1 \le i \le 4$. It is not hard to see that K is a p.c.f. self-similar set in the sense of Kigami[24] with the boundary points $V_0 = \{p_1, p_2, p_3, p_4\}$. For $n \ge 1$, let $V_n = \bigcup_{i=1}^4 F_i(V_{n-1})$. Denote by G_n the graph with vertices V_n and edges between each pair of two distinct points $p, q \in V_n$ such that $p, q \in F_\omega(V_0)$ for some $|\omega| = n$.

Assign conductance on the edges of G_n in the following way. For any $n \ge 0$, let a_n, b_n, c_n be positive real numbers. For each word ω with $|\omega| = n$, set conductance a_n on the edge $\overline{F_\omega(p_2)F_\omega(p_3)}$ and c_n on the edge $\overline{F_\omega(p_1)F_\omega(p_4)}$, then set conductance b_n on the rest, i.e. the four edges $\overline{F_\omega(p_1)F_\omega(p_2)}$, $\overline{F_\omega(p_1)F_\omega(p_3)}$, $\overline{F_\omega(p_4)F_\omega(p_2)}$ and $\overline{F_\omega(p_4)F_\omega(p_3)}$. See Figure 5.

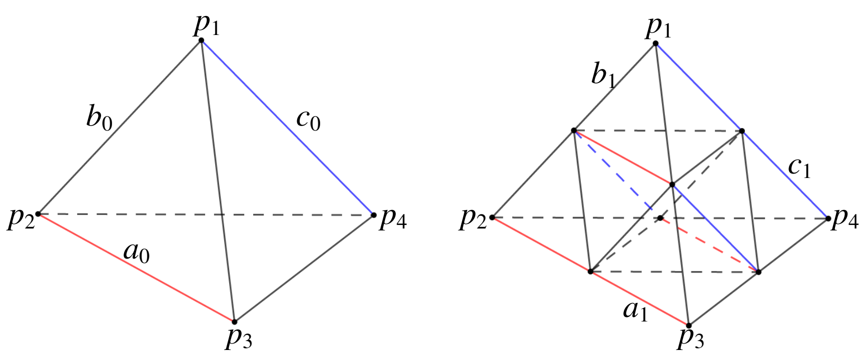

FIGURE 5. The conductances on G_0 and G_1.

In order that $\{G_n\}_{n\ge0}$ are compatible with conductances $\{a_n, b_n, c_n\}_{n\ge0}$, we need computations to obtain the conditions on $\{a_n, b_n, c_n\}_{n\ge0}$. By [24], assume on G_0, the conductance is $\{A, B, C\}$, and the associated Laplacian matrix is

$$H_0 = \begin{pmatrix} -2B-C & B & B & C \\ B & -A-2B & A & B \\ B & A & -A-2B & B \\ C & B & B & -2B-C \end{pmatrix},$$

then on G_1, we assume conductance to be $\{a, b, c\}$, and its trace on G_0 can be written as $H_1|_{V_0} = T - J^t X^{-1} J$, where

$$T = \begin{pmatrix} -2b-c & & & \\ & -a-2b & & \\ & & -a-2b & \\ & & & -2b-c \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & a & a & 0 \\ b & 0 & b & 0 \\ b & b & 0 & 0 \\ c & 0 & 0 & c \\ 0 & b & 0 & b \\ 0 & 0 & b & b \end{pmatrix},$$

and

$$X = \begin{pmatrix} -2a-4b & b & b & 0 & b & b \\ b & -a-4b-c & a_n & b & 0 & c \\ b & a & -a-4b-c & b & c & 0 \\ 0 & b & b & -4b-2c & b & b \\ b & 0 & c & b & -a-4b-c & a \\ b & c & 0 & b & a & -a-4b-c \end{pmatrix}.$$

By solving $H_1|_{V_0} = H_0$, we have that

$$A = \frac{(a+b)(b^2(4b+c)+a^2(3b+2c)+ab(9b+5c))}{2(a+2b)(3ab+4b^2+2ac+3bc)}, \tag{4.10}$$

$$B = \frac{2b(a+b)(b+c)}{3ab+4b^2+2ac+3bc}, \tag{4.11}$$

$$C = \frac{(b+c)(a(b^2+5bc+2c^2)+b(4b^2+9bc+3c^2))}{2(2b+c)(3ab+4b^2+2ac+3bc)}. \tag{4.12}$$

For all $n \geq 1$, by viewing $(a_{n-1}, b_{n-1}, c_{n-1}) = (A, B, C)$, $(a_n, b_n, c_n) = (a, b, c)$, and setting $v_n = \frac{a_n}{c_n}$ and $w_n = \frac{b_n}{c_n}$, we obtain

$$(v_{n-1}, w_{n-1}) = T(v_n, w_n),$$

where $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ such that $T = T(v, w) = \big(T_1(v, w), T_2(v, w)\big)$ with T_1, T_2 given by

$$T_1(v,w) := \frac{(1+2w)(v^3(2+3w)+w^3(1+4w)+v^2w(7+12w)+vw^2(6+13w))}{(1+w)(v+2w)(v(2+5w+w^2)+w(3+9w+4w^2))},$$

$$T_2(v,w) := \frac{4w(v+w)(1+2w)}{v(2+5w+w^2)+w(3+9w+4w^2)}.$$

We list some basic properties of the map T in the following.

Lemma 4.6. *The map T has the following properties:*
(i). $T(v, 0) = (v, 0)$ for $v \in [1, \infty)$;
(ii). $T(\{1\} \times [0, 1]) \subseteq \{1\} \times [0, 1]$, in particular, $T(1, 1) = (1, 1)$;
(iii). if $w > 0$ and $v > 1$, then $T_1(v, w) < v$.

Proof. The proof of (i),(ii) and (iii) are direct calculations. For example, (iii) is by the fact that given $w > 0$, $v > 1$,

$$T_1(v,w) - v = \frac{(1-v)w^3(1+v^2+6w+8w^2+v(2+6w))}{(1+w)(v+2w)\big(v(2+5w+w^2)+w(3+9w+4w^2)\big)} < 0.$$

□

We will give a criteria for the existence of the positive sequence $\{a_n, b_n, c_n\}_{n\geq 0}$ by first studying the dynamical behavior of mapping T on the following three different cases.

Proposition 4.7. *There exists a nondecreasing function f mapping $[1, +\infty)$ onto $\Big[1, \frac{3+\sqrt{17}}{2}\Big)$ such that for $v_0 \geq 1$,*
(i). if $0 < w_0 < f(v_0)$, then there exists a unique sequence of positive pairs $\{(v_n, w_n)\}_{n\geq 0}$ such that for all $n \geq 1$, $(v_{n-1}, w_{n-1}) = T(v_n, w_n)$ and $w_n < f(v_n)$. Moreover $\lim\limits_{n\to\infty} v_n$ exists and is finite;
(ii). if $w_0 = f(v_0)$, then there exists a unique sequence of positive pairs $\{(v_n, w_n)\}_{n\geq 0}$ such

that for all $n \geq 1$, $(v_{n-1}, w_{n-1}) = T(v_n, w_n)$ and $v_n \geq 1$, $w_n = f(v_n)$;
(iii). if $w_0 > f(v_0)$, then there does not exist any sequence of positive pairs $\{(v_n, w_n)\}_{n\geq 0}$ such that for all $n \geq 1$, $(v_{n-1}, w_{n-1}) = T(v_n, w_n)$.

Proof. Our proof is based on studying the dynamical property of map T. We will first give some claims with proofs. In the following, we will always set $\theta = \frac{1}{2}(3 + \sqrt{17})$.

Claim 1: T is a one-to-one map from $[1, \infty) \times [0, \theta]$ to its image.

Proof of Claim 1: Denote by $T = (T_1, T_2)$ and $\frac{\partial T_i}{\partial v}$, $\frac{\partial T_i}{\partial w}$, $i = 1, 2$ the partial derivatives. Then we have

$$\begin{aligned}\frac{\partial T_1}{\partial v} =&(1 + 2w)(v^4(4 + 16w + 17w^2 + 3w^3) + 2v^3w(14 + 59w + 69w^2 + 18w^3)\\ &+ w^4(29 + 139w + 200w^2 + 80w^3) + 2vw^3(40 + 185w + 251w^2 + 92w^3)\\ &+ v^2w^2(73 + 323w + 409w^2 + 131w^3))/\\ &((1 + w)(v + 2w)^2(v(2 + 5w + w^2) + w(3 + 9w + 4w^2))^2),\end{aligned}$$

$$\begin{aligned}\frac{\partial T_1}{\partial w} =&2(1 - v)w^2(3v^2 + 6v^3 + 3v^4 + 7vw + 45v^2w + 45v^3w + 7v^4w + 3w^2 + 82vw^2\\ &+ 198v^2w^2 + 82v^3w^2 + 3v^4w^2 + 36w^3 + 304vw^3 + 304v^2w^3\\ &+ 36v^3w^3 + 131w^4 + 430vw^4 + 131v^2w^4 + 184w^5 + 185vw^5 + 80w^6)/\\ &((1 + w)^2(v + 2w)^2(2v + 3w + 5vw + 9w^2 + vw^2 + 4w^3)^2),\end{aligned}$$

and

$$\frac{\partial T_2}{\partial v} = \frac{4w^2(1 + 2w)(1 + 4w + 3w^2)}{(v(2 + 5w + w^2) + w(3 + 9w + 4w^2))^2},$$

$$\frac{\partial T_2}{\partial w} = \frac{4(v^2(2 + 8w + 9w^2) + w^2(3 + 12w + 14w^2) + 2vw(2 + 7w + 6w^2 - 3w^3))}{(v(2 + 5w + w^2) + w(3 + 9w + 4w^2))^2}.$$

It is not hard to check that on $[1, \infty)\times[0, \theta]$, we have $\frac{\partial T_1}{\partial v} > 0$, $\frac{\partial T_1}{\partial w} \leq 0$ and $\frac{\partial T_2}{\partial v} > 0$, $\frac{\partial T_2}{\partial w} > 0$. Suppose that there are two points (v, w) and $(\widetilde{v}, \widetilde{w})$ in $[1, \infty) \times [0, \theta]$ such that $T(v, w) = T(\widetilde{v}, \widetilde{w})$. Without loss of generality, assume that $v \leq \widetilde{v}$. Let $\eta : [0, 1] \to [1, \infty) \times [0, \theta]$ be the straight line connecting these two points, that is $\eta(s) = (s\widetilde{v} + (1 - s)v, s\widetilde{w} + (1 - s)w)$. Then there exists $s_1, s_2 \in (0, 1)$ such that $\frac{dT_1(\eta(s))}{ds}|_{s=s_1} = 0$ and $\frac{dT_2(\eta(s))}{ds}|_{s=s_2} = 0$. We then show that we must have $(v, w) = (\widetilde{v}, \widetilde{w})$ case by case.
Case 1: $v = \widetilde{v}$ or $w = \widetilde{w}$. By $\frac{dT_2(\eta(s))}{ds}(s_2) = 0$ and the chain rule, we have

$$\frac{\partial T_2}{\partial v}(\eta(s_2))(\widetilde{v} - v) + \frac{\partial T_2}{\partial w}(\eta(s_2))(\widetilde{w} - w) = 0.$$

By using the fact that $\frac{\partial T_2}{\partial v} > 0$ and $\frac{\partial T_2}{\partial w} > 0$, we must have $v = \widetilde{v}$ and $w = \widetilde{w}$.
Case 2: $v < \widetilde{v}$ and $w < \widetilde{w}$. Similar to Case 1, we have

$$\frac{\partial T_2}{\partial v}(\eta(s_2))(\widetilde{v} - v) + \frac{\partial T_2}{\partial w}(\eta(s_2))(\widetilde{w} - w) = 0,$$

which is impossible since $\frac{\partial T_2}{\partial v} > 0$ and $\frac{\partial T_2}{\partial w} > 0$.

Case 3: $v < \widetilde{v}$ and $w > \widetilde{w}$. By $\frac{dT_1(\eta(s))}{ds}(s_1) = 0$ and the chain rule, we have

$$\frac{\partial T_1}{\partial v}(\eta(s_1))(\widetilde{v} - v) + \frac{\partial T_1}{\partial w}(\eta(s_1))(\widetilde{w} - w) = 0,$$

which is impossible since $\frac{\partial T_1}{\partial v} > 0$ and $\frac{\partial T_1}{\partial w} \leq 0$.

Above all, we conclude that T is a one-to-one map between $[1, \infty)\times[0, \theta]$ and its image.

Let $S_0 = [1, +\infty) \times [0, 1]$ and $S_1 = T(S_0)$ be the image of S_0 under T. For $n \geq 1$, let $S_n = T(S_{n-1})$. Let $S_\infty = \bigcup\limits_{n\geq 0} S_n$.

Claim 2: $S_0 \subsetneqq S_1$. Moreover, T is homeomorphic between S_∞ and S_∞.

Proof of Claim 2: Firstly, by Lemma 4.6(i), (ii), T maps the line segment $\{1\} \times [0, 1]$ into $\{1\} \times [0, 1]$ and $T(1, 1) = (1, 1)$, $T(1, 0) = (1, 0)$, and hence by the continuity of T we see that T maps $\{1\} \times [0, 1]$ onto itself. Secondly, each point in $[1, \infty) \times \{0\}$ is fixed under the map T. Thirdly, let $\gamma_0(t) = (t, 1)$, $t \in [1, \infty)$ be a straight horizontal line in $\mathbb{R}^2$. Let $\gamma_1 = T(\gamma_0)$ be the image of the curve $\gamma_0(t)$, then we have by T that

$$\gamma_1(t) = \left(\frac{3(5 + 19t + 19t^2 + 5t^3)}{16(2 + t)^2}, \frac{3 + 3t}{4 + 2t}\right),\ t \in [1, \infty).$$

The curve γ_1 is located strictly over the curve γ_0 for $t > 1$ by that the second coordinate $\frac{3+3t}{4+2t} > 1$. Also for $(v, w) \in S_0$,

$$\begin{aligned} T_1(v, w) &= \frac{(1 + 2w)(v^3(2 + 3w) + w^3(1 + 4w) + v^2w(7 + 12w) + vw^2(6 + 13w))}{(1 + w)(v + 2w)(v(2 + 5w + w^2) + w(3 + 9w + 4w^2))} \\ &\geq \frac{v^3}{8(2 + v)^2} \to \infty \text{ as } v \to \infty. \end{aligned}$$

By Claim 1 and using the invariance of domain theorem, we conclude from above that $S_0 \subsetneqq T(S_0)$, and also S_0 and S_1 are homeomorphic under the map T. Consequently, $T(S_\infty) = S_\infty$ and T is a homeomorphism between S_∞ and S_∞.

Claim 3: $S_\infty \subseteqq [1, \infty) \times [0, \theta)$.

Proof of Claim 3: Let $0 < w_0 < \theta$, $v_0 > 1$. For $n \geq 1$, let $(v_{-n}, w_{-n}) = T(v_{-(n-1)}, w_{-(n-1)})$. We see by computation that $v_{-1} > 1$. Let $h(x) = \frac{4x(1+2x)}{2+5x+x^2}$ be a function on $[0, \infty)$. Note that θ is a fixed point of h and $h(t) < \theta$ for $0 < t < \theta$. Then

$$w_{-1} = \frac{4w_0(v_0 + w_0)(1 + 2w_0)}{v_0(2 + 5w_0 + w_0^2) + w_0(3 + 9w_0 + 4w_0^2)} < \frac{4w_0(1 + 2w_0)}{2 + 5w_0 + w_0^2} = h(w_0) < \theta.$$

Therefore by induction, $v_{-n} > 1$ and $w_{-n} < \theta$ and the claim holds.

For $n \geq 1$, let $\gamma_n(t) = T(\gamma_{n-1}(t))$ and write $\gamma_n(t) = (\alpha_n(t), \beta_n(t))$, $t \in [1, \infty)$.

Claim 4: $\alpha_n'(t) > 0, \beta_n'(t) > 0$ and $\left(\frac{\alpha_n}{\beta_n}\right)'(t) > 0$ for any $t \in (1, \infty)$.

Proof of Claim 4: We show this by induction. First we have

$$\alpha_1'(t) = \left(\frac{3(5 + 19t + 19t^2 + 5t^3)}{16(2 + t)^2}\right)'(t) = \frac{3(28 + 57t + 30t^2 + 5t^3)}{16(2 + t)^3} > 0.$$

and $\beta_1'(t) = \left(\frac{3+3t}{4+2t}\right)' = \frac{3}{2(2+t)^2} > 0$, and moreover we have

$$\left(\frac{\alpha_1}{\beta_1}\right)'(t) = \left(\frac{5+14t+5t^2}{8(2+t)}\right)'(t) = \frac{23+20t+5t^2}{8(2+t)^2} > 0.$$

Assume inductively that for some $k \geq 1$, $\beta_k'(t) > 0$ and $\left(\frac{\alpha_k}{\beta_k}\right)'(t) > 0$. we will show that $\beta_{k+1}'(t) > 0$ and $\left(\frac{\alpha_{k+1}}{\beta_{k+1}}\right)'(t) > 0$.
Since by product rule, $\alpha_k' = \left(\beta_k \cdot \frac{\alpha_k}{\beta_k}\right)' = \beta_k' \cdot \frac{\alpha_k}{\beta_k} + \beta_k \cdot \left(\frac{\alpha_k}{\beta_k}\right)' > 0$. Then by chain rule we have

$$\beta_{k+1}' = (T_2(\alpha_k, \beta_k))' = \frac{\partial T_2}{\partial v} \cdot \alpha_k' + \frac{\partial T_2}{\partial w} \cdot \beta_k' > 0.$$

Next we show that $\left(\frac{\alpha_{k+1}}{\beta_{k+1}}\right)' = \left(\frac{T_1(\alpha_k,\beta_k)}{T_2(\alpha_k,\beta_k)}\right)' > 0$, which is equivalent to

$$\left(\frac{2\alpha_k^2 + 5\alpha_k\beta_k + 3\alpha_k^2\beta_k + \beta_k^2 + 9\alpha_k\beta_k^2 + 4\beta_k^3}{4\beta_k(1+\beta_k)(\alpha_k + 2\beta_k)}\right)' > 0. \tag{4.13}$$

By rearranging and using product rule, (4.13) is equivalent to

$$\begin{aligned}&(\beta_k\alpha_k' - \alpha_k\beta_k')(2\alpha_k^2 + 8\alpha_k\beta_k + 4\alpha_k^2\beta_k + 9\beta_k^2 + 14\alpha_k\beta_k^2 + 3\alpha_k^2\beta_k^2 + 12\beta_k^3 + 12\alpha_k\beta_k^3 + 14\beta_k^4)\\ &+ (\alpha_k^2\beta_k^2 + 6\alpha_k\beta_k^3 + 11\beta_k^4)\alpha_k' + 6\beta_k^4\beta_k' > 0.\end{aligned}$$

In fact this is true by $\alpha_k > 0, \beta_k > 0, \alpha_k' > 0, \beta_k' > 0$, and $\beta_k\alpha_k' - \alpha_k\beta_k' > 0$ from $\left(\frac{\alpha_k}{\beta_k}\right)' > 0$ which is by the inductive assumption.

Thus $\beta_n'(t) > 0$ and $\left(\frac{\alpha_n}{\beta_n}\right)'(t) > 0$ for all $n \geq 1$ and $t \in (1, \infty)$, and hence $\alpha_n'(t) > 0$ by product rule. This completes the proof of Claim 4.

For $n \geq 1$, let α_n^{-1} be the inverse of α_n, and define $f_n(v) = \beta_n(\alpha_n^{-1}(v))$, then f_n is a strictly increasing bounded continuous function on $[1, \infty)$. Also by Claim 4, we have that

$$(\beta_n - \delta\alpha_n)' = \left(\beta_n\left(1 - \delta \cdot \frac{\alpha_n}{\beta_n}\right)\right)' = \left(1 - \delta \cdot \frac{\alpha_n}{\beta_n}\right) \cdot \beta_n' - \delta \cdot \beta_n \cdot \left(\frac{\alpha_n}{\beta_n}\right)' \leq 0,$$

if we chose δ such that $\delta \geq \frac{\beta_n}{\alpha_n}$, and hence we have

$$0 < \frac{df_n(v)}{dv} = \frac{\beta_n'(\alpha_n^{-1}(v))}{\alpha_n'(\alpha_n^{-1}(v))} \leq \delta. \tag{4.14}$$

Since $\alpha_n \geq 1$ and $\beta_n < \theta$, we have $f_n'(v)$ is uniformly bounded for v and n.

For fixed $v > 1$, $f_n(v)$ is increasing on n by the fact that $S_n \subsetneq S_{n+1}$. Let

$$f(v) = \lim_{n\to\infty} f_n(v). \tag{4.15}$$

Since $\{f_n\}_{n\geq 1}$ are uniformly bounded by Claim 3 and equi-continuous by (4.14), $f(v)$ is continuous and nondecreasing for $v \in [1, \infty)$.

Let $\Gamma := (v, f(v))$, $v \in [1, \infty)$ be the graph of f.

Claim 5: T is a bijection between Γ and Γ.

Proof of Claim 5: Denote by $\overline{S_\infty}$ the closure of S_∞. Since by Claim 2, T is a homeomorphism on S_∞, and hence $T(\overline{S_\infty}) = \overline{S_\infty}$. That is to say $T(\partial\overline{S_\infty}) = \partial\overline{S_\infty}$. Since

$\partial\overline{S_\infty} = (\{1\} \times [0,1]) \cup ([1,\infty) \times \{0\}) \cup \Gamma$ and T is invariant on the other two subsets by Lemma 4.6, we should have $T(\Gamma) = \Gamma$. Since $\Gamma \subseteq [1,\infty) \times [0,\theta)$, T is one-to-one on Γ.

Claim 6: For $(v_0, w_0) \in S_\infty$ with $(v_0, w_0) \neq (1,1)$, we have $\{(v_n, w_n)\}_{n\geq 0}$ exists and $w_n \asymp 2^{-n}$, $\lim\limits_{n\to\infty} v_n$ exists and is finite.

Proof of Claim 6: The existence follows from Claim 2. Since $S_\infty = \bigcup\limits_{n\geq 0} S_n$, we may assume that $(v_0, w_0) \in S_0$ by that T is bijective on S_∞. Suppose $v_0 > 1$, then v_n is strictly increasing by Lemma 4.6. Assume that $0 < w_0 \leq 1$, then $w_n \leq 1$ by Claim 2. We first show that $\{v_n\}_{n\geq 0}$ is uniformly bounded. If it is not true, that is $v_n \to \infty$ as $n \to \infty$. By using map T, for any small $\varepsilon > 0$, we see that for all n large enough,

$$w_n > \frac{(4-\varepsilon)w_{n+1}(1+2w_{n+1})}{2+5w_{n+1}+w_{n+1}^2}.$$

Then we get $w_{n+1} < \frac{8}{12-3\varepsilon} w_n$, and we have $w_n \to 0$ as $n \to \infty$ if we pick small ε. By using map T, we have for n large enough,

$$\begin{aligned}
\frac{v_n}{v_{n-1}} &= \frac{v_n(1+w_n)(v_n+2w_n)\big(v_n(2+5w_n+w_n^2)+w_n(3+9w_n+4w_n^2)\big)}{(1+2w_n)\big(v_n^3(2+3w_n)+w_n^3(1+4w_n)+v_n^2w_n(7+12w_n)+v_nw_n^2(6+13w_n)\big)} \\
&\leq \frac{v_n(1+w_n)(v_n+2w_n)(v_n+4w_n)(2+5w_n+w_n^2)}{v_n^3(1+2w_n)(2+3w_n)} \\
&\leq (1+w_n)\left(1+2\frac{w_n}{v_n}\right)\left(1+4\frac{w_n}{v_n}\right)\cdot\frac{(2+5w_n+w_n^2)}{(1+2w_n)(2+3w_n)} \\
&\leq (1+4w_n)^3,
\end{aligned}$$

and thus $\{v_n\}_{n\geq 0}$ is uniformly bounded since $\{w_n\}_{n\geq 0}$ has exponential decay, a contradiction. This proves that $\lim\limits_{n\to\infty} v_n$ exists and is finite.

Next we will show that $\lim\limits_{n\to\infty} w_n = 0$.

If $\lim\limits_{n\to\infty} w_n = 0$ is not true, then there exists $\varepsilon > 0$ such that there is a subsequence $\{n_k\}_{k\geq 0}$ such that $w_{n_k} \geq \varepsilon$. By using T again, we have for $n = n_k$,

$$\begin{aligned}
v_n - v_{n-1} &= \frac{(v_n-1)w_n^3(1+v_n^2+6w_n+8w_n^2+2v_n+6v_nw_n)}{(1+w_n)(v_n+2w_n)\big(v_n(2+5w_n+2w_n^2)+w_n(3+9w_n+4w_n^2)\big)} \\
&\geq C_0(v_n-1),
\end{aligned}$$

where C_0 depends only on ε. We then have $\lim\limits_{n\to\infty} v_n \geq \sum_k C_0(v_{n_k}-1) \to \infty$. This contradicts the fact that $\lim\limits_{n\to\infty} v_n$ is bounded. Hence $w_n \to 0$ as $n \to \infty$.

Thus, by substituting $w_n \to 0$ and $v_n \asymp 1$ as $n \to \infty$ into the expression of T, we easily see that there is $\delta \in (0,1)$ such that $w_n \lesssim \delta^n$ for large n. As a consequence,

$$\frac{w_{n-1}}{w_n} = \frac{4(v_n+w_n)(1+2w_n)}{v_n(2+5w_n+w_n^2)+w_n(3+9w_n+4w_n^2)}, \tag{4.16}$$

gives that $2 - C\delta^n \leq w_{n-1}/w_n \leq 2 + C\delta^n$. Therefore we have $w_n \asymp 2^{-n}$.

Claim 7: For $(v_0, w_0) \in \Gamma$, if $v_0 = 1$, then $(v_n, w_n) = (1,1)$; if $v_0 > 1$, then $\lim\limits_{n\to\infty} w_n = \theta$, $v_n \asymp \left(\frac{4+4\theta}{2+3\theta}\right)^n = \left(\frac{7-\sqrt{17}}{2}\right)^n$.

Proof of Claim 7: The $v_0 = 1$ case is trivial. So we assume $v_0 > 1$. We first show that $\lim_{n\to\infty} w_n = \theta$. By that v_n is increasing and f is nondecreasing, we see that w_n is nondecreasing, and by Claim 5, $w_n \le \theta$, thus $\lim_{n\to\infty} w_n = \theta_0 \le \theta$ exists. If $\theta_0 < \theta$, we see from (4.16) that for v_n large enough,

$$\frac{w_{n-1}}{w_n} \asymp \frac{4(1+2w_n)}{2+5w_n+w_n^2} > \delta_0 > 1,$$

where δ_0 only depends on θ_0. We then have $w_n \le C\delta_0^{-n} \to 0$, a contradiction. Thus we must have $\lim_{n\to\infty} w_n = \theta$. By this, we have from

$$\frac{v_n}{v_{n-1}} \asymp \frac{(1+w_n)(2+5w_n+w_n^2)}{(1+2w_n)(2+3w_n)} > 1$$

that there is some $\delta > 1$ such that for n large enough,

$$v_n \ge C^{-1}\delta^n. \tag{4.17}$$

From (4.17), we then show that $\theta - w_n \le C\delta^{-n}$ for n large enough.

Indeed, By (4.17) and the fact that w_n is bounded above by θ, we have $w_n/v_n \le C\delta^{-n}$. Substituting this into

$$1 \ge \frac{w_{n-1}}{w_n} = \frac{4(v_n+w_n)(1+2w_n)}{v_n(2+5w_n+w_n^2)+w_n(3+9w_n+4w_n^2)}, \tag{4.18}$$

we have

$$\frac{4(1+2w_n)}{2+5w_n+w_n^2} \le 1 + C'\delta^{-n}, \tag{4.19}$$

which implies that

$$\theta - w_n \le C\delta^{-n}. \tag{4.20}$$

For n large enough, from $v_n/w_n \ge C\delta^n$ and (4.20), we have

$$\frac{v_n}{v_{n-1}} = \frac{(1+w_n)(2+5w_n+w_n^2)}{(1+2w_n)(2+3w_n)} + O(\delta^{-n}) = \frac{4+4\theta}{2+3\theta} + O(\delta^{-n}),$$

which implies that $v_n \asymp \left(\frac{4+4\theta}{2+3\theta}\right)^n = \left(\frac{7-\sqrt{17}}{2}\right)^n$. This complete the proof of Claim 7.

Claim 8: For (v_0, w_0) such that $v_0 \ge 1$, $w_0 > f(v_0)$, there is no $\{(v_n, w_n)\}_{n\ge 0}$ as a solution.

Proof of Claim 8: Assume that (v_0, w_0) has a solution $\{(v_n, w_n)\}_{n\ge0}$ with $f(v_0) < w_0$. Then $w_0 \le h^{(n)}(w_n) \to \theta$ where h is defined in Claim 3, and thus $w_n \le \theta$ for all $n \ge 1$ by the same reason. Thus we must have $f(v_n) < w_n \le \theta$ for all $n \ge 0$. Since $w_n \ge 1$, by the mapping T, $v_n \to \infty$ as $n \to \infty$, we may assume that v_0 is large enough.

Now we define a sequence of positive numbers $d_n := |w_n - f(v_n)| = w_n - f(v_n)$ for $n \ge 0$. We want to show that there is $\delta \in (0, 1)$ such that

$$d_{n+1} > \delta^{-1} d_n. \tag{4.21}$$

Denote by $p = (v_n, w_n)$, $q = (v_n, f(v_n))$. Consider $T(p)$ and $T(q)$, we have $T(p) = (v_{n-1}, w_{n-1})$ by definition. By Claim 5, $T(q)$ is on the curve $w = f(v)$. Then on one hand, we have

$$T_2(v_n, w_n) - T_2(v_n, f(v_n)) = \int_{f(v_n)}^{w_n} \frac{\partial T_2}{\partial w} dw \le \sup_{w_n \le w \le f(v_n)} \left|\frac{\partial T_2}{\partial w}(v_n, w)\right| (w_n - f(v_n)). \tag{4.22}$$

We observe that as $v \to \infty$ and $w \to \theta$, we have

$$\frac{\partial T_2}{\partial w} \to \frac{4(2+8w+9w^2)}{(2+5w+w^2)^2} \to \frac{20+35\theta}{4(1+2\theta)^2}. \tag{4.23}$$

On the other hand, by $\frac{\partial T_1}{\partial w} \le 0$,

$$T_1(v_n, w_n) - T_1(v_n, f(v_n)) = \int_{f(v_n)}^{w_n} \frac{\partial T_1}{\partial w} dw \le 0, \tag{4.24}$$

and hence $T_1(v_n, f(v_n)) \ge v_{n-1}$. We then turn to estimate $f(T_1(v_n, f(v_n))) - f(v_{n-1})$. We see that for m large enough,

$$\begin{aligned}
f_m(T_1(v_n, f(v_n))) - f_m(v_{n-1}) &= \int_{T_1(p)}^{T_1(q)} f_m'(x)dx \\
&\le \left(\sup_{x \ge v_{n-1}} f_m'(x)\right) \cdot (T_1(q) - T_1(p)) \\
&\le \frac{\theta}{v_{n-1}} \int_{f_m(v_n)}^{w_n} \left(-\frac{\partial T_1}{\partial w}\right) dw \quad \text{(using (4.14))} \\
&\le \frac{\theta}{v_{n-1}} \sup_{w \ge f_m(v_n), v = v_n} \left|\frac{\partial T_1}{\partial w}\right| \cdot (w_n - f_m(v_n)) \\
&\le \frac{\theta}{v_{n-1}} \cdot v_n \cdot \sup_{w \ge f_m(v_n)} \frac{2(3+7w+3w^2)w^2}{(1+w)^2(2+5w+w^2)^2} \cdot (w_n - f_m(v_n)).
\end{aligned}$$

Observe that as $n \to \infty$ and $w \to \theta$, we have $\frac{v_n}{v_{n-1}} \to \frac{4+4\theta}{2+3\theta}$, and thus

$$\frac{\theta}{v_{n-1}} \cdot v_n \cdot \frac{2(3+7w+3w^2)w^2}{(1+w)^2(2+5w+w^2)^2} \to \frac{32+57\theta}{2(1+\theta)(1+2\theta)^2}. \tag{4.25}$$

By using that $f_m(x) \to f(x)$ uniformly as $m \to \infty$, we see that for n large enough,

$$\begin{aligned}
T_2(v_n, f(v_n)) - f(v_{n-1}) &= f(T_1(v_n, f(v_n))) - f(v_{n-1}) \\
&\le \left(\frac{32+57\theta}{2(1+\theta)(1+2\theta)^2} + o(1)\right)(w_n - f(v_n)).
\end{aligned} \tag{4.26}$$

Now since

$$\frac{20+35\theta}{4(1+2\theta)^2} + \frac{32+57\theta}{2(1+\theta)(1+2\theta)^2} = \frac{137\theta+77}{146\theta+82} < 1, \tag{4.27}$$

and by using (4.22), (4.23), and (4.26), we see that there is $\delta \in (0,1)$ such that for n large enough,

$$w_{n-1} - f(v_{n-1}) = T_2(v_n, w_n) - T_2(v_n, f(v_n)) + T_2(v_n, f(v_n)) - f(v_{n-1}) < \delta(w_n - f(v_n)).$$

Thus (4.21) holds and this contradicts the fact that w_n is bounded from above by θ. Hence Claim 8 is true.

Above all, (i) follows from Claim 6; (ii) follows from Claims 5 and 7; (iii) follows from Claim 8. The proof is complete. □

Corollary 4.8. *Suppose $a_0 \ge c_0 > 0$, Then*
(i). if $0 < \frac{b_0}{c_0} < f\left(\frac{a_0}{c_0}\right)$, then there exists a unique positive solution $\{a_n, b_n, c_n\}_{n \ge 0}$ satisfying $a_n \ge c_n$, $\frac{b_n}{c_n} < f\left(\frac{a_n}{c_n}\right)$ and $a_n \asymp c_n \asymp 2^n$, $b_n \asymp 1$;

(ii). if $\frac{b_0}{c_0} = f\left(\frac{a_0}{c_0}\right)$, then there exists a unique positive solution $\{a_n, b_n, c_n\}_{n\geq 0}$ satisfying $a_n \geq c_n$, $\frac{b_n}{c_n} = f(\frac{a_n}{c_n})$; moreover, if $a_0 = c_0$, then $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n \cdot a_0$, otherwise if $a_0 > c_0$, then $a_n \asymp 2^n$, $b_n \asymp c_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$;
(iii). if $\frac{b_0}{c_0} > f\left(\frac{a_0}{c_0}\right)$, then there does not exist any positive solution.

Proof. (i). By Proposition 4.7 (i) and (4.11), we have

$$\frac{b_{n-1}}{b_n} = \frac{2(v_n + w_n)(1 + w_n)}{3w_nv_n + 4w_n^2 + 2v_n + 3w_n} = 1 + O(2^{-n}),$$

which implies that $b_n \asymp 1$ and hence $c_n = b_n/w_n \asymp 2^n$ and $a_n = v_nc_n \asymp 2^n$.

(ii). In the case $a_0 = b_0 = c_0$, we have $v_n = w_n = 1$ and hence $\frac{b_{n-1}}{b_n} = \frac{2}{3}$. Hence $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n \cdot a_0$.

We then assume $a_0 > c_0$. By Proposition 4.7 (ii) and (4.11), we have

$$\frac{b_{n-1}}{b_n} = \frac{2(v_n + w_n)(1 + w_n)}{3w_nv_n + 4w_n^2 + 2v_n + 3w_n} = \frac{2 + 2w_n}{2 + 3w_n} + o(\delta^n) = \frac{2 + 2\theta}{2 + 3\theta} + o(\delta^n),$$

for some $\delta \in (0, 1)$, thus we obtain $b_n \asymp \left(\frac{2+3\theta}{2+2\theta}\right)^n$, where $\theta = \frac{1}{2}(3 + \sqrt{17})$. Hence $c_n = b_n/w_n \asymp b_n \asymp \left(\frac{2+3\theta}{2+2\theta}\right)^n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$ and $a_n = v_nc_n \asymp \left(\frac{4+4\theta}{2+3\theta}\right)^n \cdot \left(\frac{2+3\theta}{2+2\theta}\right)^n \asymp 2^n$.

(iii). This is direct from Proposition 4.7 (iii). □

Corollary 4.9. *Let K be the 3-dimensional Sierpinski gasket, μ be the normalized Hausdorff measure on K. Let (a_0, b_0, c_0) be positive real numbers with $a_0 \geq c_0$ such that $\frac{b_0}{c_0} = f\left(\frac{a_0}{c_0}\right)$, where f is defined in Proposition 4.7. Then the compatible sequence $\{(a_n, b_n, c_n)\}_{n\geq 0}$ defines a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$. Let Δ be the generator of the form and $\rho(t)$ be its eigenvalue counting function, then for t large enough, we have*

(i). if $a_0 = c_0$, then $\rho(t) \asymp t^{\frac{\log 4}{\log 6}}$;

(ii). if $a_0 > c_0$, then $\rho(t) \asymp t^{\frac{\log 4}{\log\left(\frac{7+\sqrt{17}}{2}\right)}}$.

Proof. (i). By Corollary 4.8, if $a_0 = c_0$, then $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n \cdot a_0$, and thus $\kappa_0 = \frac{3}{2}$, and the relation $\sim$ is the trivial relation that each point in V_0 is a single equivalent class. We see that conditions **(A1)-(A3)** hold. By Theorem 3.7, we obtain with $N = 4$ and $\kappa_0 = \frac{3}{2}$ that

$$\rho(t) \asymp t^{\frac{\log 4}{\log 6}}.$$

We note that for this case, we can apply the result in [26] to get a more delicate estimate

$$\rho(t) = \left(G\left(\frac{\log t}{2}\right) + o(1)\right) t^{\frac{\log 4}{\log 6}},$$

where $G(x)$ is a periodic bounded function with period $T = \frac{1}{2}\log 6$.

(ii). If $a_0 > c_0$, then by Corollary 4.8, $a_n \asymp 2^n$, $b_n \asymp c_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$, and thus $\kappa_0 = \frac{7+\sqrt{17}}{8}$, and the relation $\sim$ is given by $V_0 = \{p_1\} \cup \{p_2, p_3\} \cup \{p_4\}$. We can check that conditions **(A1)-(A3)** hold. By Theorem 3.7, we obtain with $N = 4$ and $\kappa_0 = \frac{7+\sqrt{17}}{8}$ that

$$\rho(t) \asymp t^{\frac{\log 4}{\log\left(\frac{7+\sqrt{17}}{2}\right)}}.$$

□

References

[1] M. Barlow, *Diffusions on fractals*, vol. **1690** of Lect. Notes Math., Springer, 1998, pp. 1–121.

[2] M. Barlow and R. Bass, *Transition densities for Brownian motion on the Sierpiński carpet*, Probab. Theory Related Fields, **91** (1992), pp. 307–330.

[3] M. Bodin, *Discrete characterisations of Lipschitz spaces on fractals*, Math. Nachr., **282** (2009), pp. 26–43.

[4] Z. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stochastic Process. Appl., **108** (2003), pp. 27–62.

[5] E.B. Davies, *One-parameter semigroups*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London Mathematical Society Monographs, **15**, London-New York, 1980.

[6] Q.-R. Deng and K.-S. Lau, *Open set condition and post-critically finite self-similar sets*, Nonlinearity, **21** (2008), pp. 1227–1232.

[7] P. Doyle and J. Snell, *Random walks and electric networks*, Carus Mathematical Monographs Vol. **22**, Mathematical Association of America, Washington, DC 1984.

[8] M. Fukushima and Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter, Studies in Mathematics, **19**, Berlin, Second revised and extended edition ed., 2011.

[9] A. Grigor'yan, J. Hu, and K.-S. Lau, *Heat kernels on metric-measure spaces and an application to semilinear elliptic equations*, Trans. Amer. Math. Soc., **355** (2003), pp. 2065–2095.

[10] Q. Gu and K.-S. Lau, *Dirichlet forms and convergence of Besov norms on self-similar sets*, Ann. Acad. Sci. Fenn. Math., (to appear).

[11] Q. Gu and K.-S. Lau, *Dirichlet forms and critical exponents on fractals*, Trans. Amer. Math. Soc., (to appear).

[12] Q. Gu, K.-S. Lau and H. Qiu, *On a recursive construction of Dirichlet form on the Sierpiński gasket*, J. Math. Anal. Appl., **474** (2019), pp. 674–692.

[13] B. Hambly and O. Jones, *Asymptotically one-dimensional diffusion on the Sierpiński gasket and multi-type branching processes with varying environment*, J. Theoret. Probab., **15** (2002), pp. 285–322.

[14] B. Hambly and T. Kumagai, *Heat kernel estimates and homogenization for asymptotically lower-dimensional processes on some nested fractals*, Potential Anal., **8** (1998), pp. 359–397.

[15] B. Hambly and T. Kumagai, Transition density estimates for diffusion processes on post critically finite self-similar fractals, Proc. London Math. Soc. (3), **78** (1999), pp. 431–458.

[16] B. Hambly, V. Metz, and A. Teplyaev, *Self-similar energies on post-critically finite self-similar fractals*, J. London Math. Soc. (2), **74** (2006), pp. 93–112.

[17] B. Hambly and W. Yang, *Degenerate limits for one-parameter families of non-fixed-point diffusions on fractals*, J. Fractal Geom., **6** (2019), pp. 1–51.

[18] K. Hattori, T. Hattori and H. Watanabe, *Asymptotically one-dimensional diffusions on the Sierpiński gasket and the abc-gaskets*, Probab. Theory Related Fields, **100** (1994), pp. 85–116.

[19] J. Hu and X.-S. Wang, *Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals*, Studia Math., **177** (2006), pp. 153–172.

[20] A. Jonsson, *Brownian motion on fractals and function spaces*, Math. Zeit., **222** (1996), pp. 495–504.

[21] A. Jonsson, *triangulation of close sets and bases in function spaces*, Ann. Acad. Sci. Fenn. Math., **29**(2004), pp. 43-58

[22] A. Jonsson and H. Wallin, *Function spaces on subsets of R^n*, Math. Reports Vol. **2**, Acad. Publ., Harwood, 1984.

[23] J. Kigami, *A harmonic calculus on the Sierpinski spaces*, Japan J. Appl. Math., **6**, (1989), pp. 259–290.

[24] J. Kigami, *Analysis on fractals*, Cambridge Univ. Press, 2001.

[25] J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*, Mem. Amer. Math. Soc., **216** (2012), no. 1015.

[26] J. Kigami and L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Comm. Math. Phys., **158**,1993, pp. 93–125.

[27] S.-L. Kong and K.-S. Lau, *Critical exponents of induced Dirichlet forms on self-similar sets*, arXiv:1612.01708.

[28] S.-L. Kong, K.-S. Lau and T.-K. Wong, *Random walks and induced Dirichlet forms on self-similar sets*, Adv. Math., **320**(2017), 1099-1134.
[29] T. Lindstrøm, *Brownian motion on nested fractals*, Mem. Amer. Math. Soc., **83** (1990), no. 420.
[30] V. Metz, *Hilbert's projective metric on cones of Dirichlet forms*, J. Funct. Anal., **127** (1995), pp. 438–455.
[31] R. Peirone, *Existence of self-similar energies on finitely ramified fractals*, J. Anal. Math., **123** (2014), pp. 35–94.
[32] K. Pietruska-Pałuba, *Some function spaces related to the Brownian motion on simple nested fractals*, Stochastics and Stochastics Reports, **67** (1999), pp. 267–285.
[33] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals*, Ann. Sci. École Norm. Sup. (4), **30** (1997), pp. 605–673.
[34] R. Strichartz, *Differential equations on fractals: a tutorial*, Princeton University Press, 2006.
[35] A. Tetenov, K. Kamalutdinov, D. Vaulin, *Self-similar Jordan arcs which do not satisfy OSC*, arXiv:1512.00290.

Department of Mathematics, Memorial University of Newfoundland, NL A1C5S7, Canada
Email address: `001gqs@163.com`

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China, & Department of Mathematics, University of Pittsburgh, Pittsburgh, Pa. 15217, U.S.A.
Email address: `kslau@math.cuhk.edu.hk`

Department of Mathematics, Nanjing University, Nanjing, 210093, China
Email address: `huaqiu@nju.edu.cn`