HARMONIC MEASURES OF DOMAINS IN NESTED FRACTALS

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Abstract. It is known that on each nested fractal, there exists a unique Brownian motion, which is associated with a self-similar local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a unique energy renormalization factor $\rho \in (0, 1)$. We prove, for any given open connected subset Ω of a nested fractal K, if Ω has a graph-directed self-similar boundary, the harmonic measure on the boundary $\partial\Omega$ with respect to any point in Ω is generated by a finite number of matrices which determine the partition of the flows given by certain Green's functions with one variable properly chosen. We also obtain the two-sided estimates of the energies of harmonic functions on Ω by a quadratic form of their boundary value functions. Our result widely extends the existing results studying the boundary value problems on domains in Sierpiński gasket type fractals.

1. Introduction

In classical analysis, let Ω be a smooth domain in $\mathbb{R}^d$, let $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator, the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has a unique solution u for some function f on the boundary. Moreover, the function u near the boundary has the same regularity (i.e., Hölder continuity or L^p boundedness) as f. In particular, if Ω is the open unit ball $B = \{x \in \mathbb{R}^d : |x| < 1\}$, then u has an expression of Poisson integral

$$u(x) = \int_{|\zeta|=1} f(\zeta)P(x, \zeta)dS(\zeta), \tag{1.2}$$

where dS is the normalized surface measure on the unit sphere, $P(x, \zeta) = \frac{1-|x|^2}{|x-\zeta|^d}$ is called the Poisson kernel, and the measure $P(x, \zeta)dS(\zeta)$ is known as the harmonic measure.

From probabilistic point of view, the harmonic measure is the hitting distribution of the Brownian motion on the boundary of a smooth domain. More precisely, let $\{X_t\}_{t\geq 0}$ be a Brownian motion in a smooth domain Ω killed at the boundary $\partial\Omega$ with $X_0 = x \in \Omega$, let τ be the first exit time of X_t from Ω, then the harmonic measure $d\mu_x$ for $x \in \Omega$ on the boundary $\partial\Omega$ satisfies that

$$\mathrm{P}_x(X_\tau \in A) = \mu_x(A) \qquad \text{for any } A \subseteq \partial\Omega. \tag{1.3}$$

In the fractal context, a local regular Dirichlet form plays the role of the Dirichlet integral $\int_\Omega |\nabla u|^2 dx$ and we denote the generator of the Dirichlet form by Δ, which is called the Laplacian. There is a large literature on the topic based on Kigami's analytic approach on the *post critically finite* (p.c.f.) self-similar sets, and the probabilistic approach of

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Lindstrøm on the nested fractals as well as Barlow and Bass on the Sierpiński carpet (see [1, 2, 3, 10, 11, 12, 13, 14, 17, 19, 21, 22, 23] and the references therein).

Let K be a self-similar set generated by an iterated function system $\{F_i\}_{i=1}^N$. Most of the previous studies are about Dirichlet form $(\mathcal{E}, \mathcal{F})$ having the self-similar property, which means that there exist N positive real numbers $\{\rho_i : i = 1, \ldots, N\}$ called *energy renormalization factors* such that for any function $u \in \mathcal{F}$, it holds that $u \circ F_i \in \mathcal{F}$ for any $i = 1, \ldots, N$, and

$$\mathcal{E}[u] = \sum_{i=1}^{N} \frac{1}{\rho_i} \mathcal{E}[u \circ F_i], \tag{1.4}$$

where $\mathcal{E}[u] := \mathcal{E}(u, u)$. Such Dirichlet forms which have a uniform energy renormalization factor (i.e., all the ρ_i's are equal) are known to exist on some classes of self-similar sets possessing certain strongly symmetric properties, for example, a nested fractal [17] or a generalized Sierpiński carpet [2], see also a very recent work [7] for two new classes of non-p.c.f. self-similar sets.

We are concerned with the harmonic measures of domains (nonempty open connected subsets) in self-similar sets possessing a self-similar local regular Dirichlet form. For such a fractal domain Ω with boundary $\partial\Omega$ (usually preserves certain self-similarity), we mainly focus on two problems motivated from the classical analysis: one is to find an exact description of the harmonic measures of points in Ω; the other is to estimate the energies of harmonic functions generated by given values on $\partial\Omega$. The study of such problems was initiated in [20, 9] for typical domains in the Sierpiński gasket. See also [16, 5] for extensions in higher level Sierpiński gaskets. However, the methods strongly depend on the delicate structure of Sierpiński gaskets, and the technique of the energy estimate depends on a rigorous requirement of the geometry of the domain.

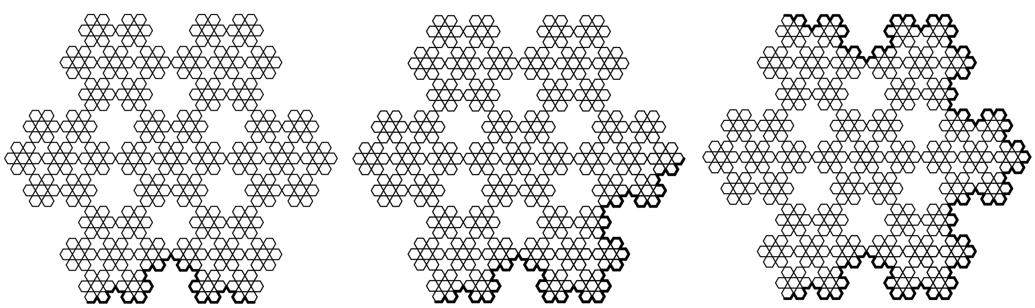

FIGURE 1. domains with graph-directed boundary in the Lindstrøm's snowflake

In this paper, we revisit this study and setup a general framework that includes far more flexible examples, see Figure 1 for typical domains in the Lindstrøm's snowflake, with Koch curve boundaries denoted by thick curves. We will focus on the nested fractals for brevity, and the techniques in this paper do not allow us to deal with the infinite ramified self-similar sets like the Sierpiński carpet. We assume the domains to have a "graph-directed boundary condition", and introduce a finite number of matrices, which we call flux distribution matrices, to determine the harmonic measures of points in Ω. These matrices are essentially determined by the graph-directed structures of both the domain and the boundary, together with the flows of certain Green's functions along $\partial\Omega$ with one variable being delicately chosen. As is known, in the classical theory, for a given regular domain Ω in $\mathbb{R}^d$, the harmonic measure of some point $x \in \Omega$ on the boundary $\partial\Omega$ is the normal derivative (or flow) of the Green's function $G_\Omega(x, \cdot)$. We will see that it is the same in the fractal case after we give meaning to the "normal derivative". For the energy estimate problem, we observe that the harmonic measures of any pair of points in Ω are equivalent due to the elliptic Harnack inequality, and thus by allowing involving harmonic measures of various points, we will provide an equivalent estimate of the energies of harmonic functions on Ω from the integrals of their values on $\partial\Omega$ against these harmonic measures.

We organize the paper as following. In Section 2, we give some preliminaries for the definition and basic properties of nested fractals, and then we introduce the basic theory about the electric networks and the formulas of Gauss-Green on finite graphs, which play central roles in the study of harmonic measures of domains in nested fractals with graph-directed boundaries. Then we state and prove our first main result about the harmonic measures of domains in nested fractals in Section 3. In Section 4, we illustrate our result by several examples. Finally, we provide an energy estimate of harmonic functions via their boundary values in Section 5.

2. **Preliminaries**

We will study the harmonic measures on domains of a class of self-similar sets in $\mathbb{R}^d$ called *nested fractals* introduced by Lindstrøm [17]. Let K be the self-similar set defined by an iterated function system (IFS) $\{F_i\}_{i=1}^N$ of the form $F_i(x) = \varrho A_i x + b_i$, where $N \geq 2$, $0 < \varrho < 1$, and for each $1 \leq i \leq N$, A_i is a $d \times d$ orthogonal matrix and $b_i \in \mathbb{R}^d$. Let P be the set of all fixed points of $\{F_i\}_{i=1}^N$. Call $p \in P$ an *essential fixed point* if there exist distinct $i, j \in \{1, \ldots, N\}$, and $q \in P$ such that $F_i(p) = F_j(q)$, and denote by P_0 the set of all essential fixed points. For any distinct points $x, y \in \mathbb{R}^d$, denote the bisecting hyperplane $H_{x,y} = \{z \in \mathbb{R}^d : |x - z| = |y - z|\}$ and write $R_{x,y}$ the orthogonal reflection with respect to $H_{x,y}$; let $\mathcal{G}$ denote the group generated by all the reflections $R_{x,y}$ for $x, y \in P_0$.

Definition 2.1 (**nested fractals**). *Let $\{F_i\}_{i=1}^N$ and K be as the above. We call K a* nested fractal *if it satisfies the following conditions:*

*(*OSC*) $\{F_i\}_{i=1}^N$ satisfies the open set condition;*
*(*Connectivity*) K is connected;*
*(*Symmetry*) K is invariant under $\mathcal{G}$;*
*(*Nesting*) for any $i, j \in \{1, \ldots, N\}$ with $i \neq j$, $F_i(K) \cap F_j(K) = F_i(P_0) \cap F_j(P_0)$.*

The typical nested fractals include the Sierpiński gasket (SG) and its higher level and higher dimensional generalizations, the Vicsek sets, pentagasket, Lindstrøm's snowflake, etc. It is known that on each nested fractal, there exists a unique Brownian motion (see [17] for the existence and [22] for the uniqueness). The corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the following self-similar identity: for any $u \in \mathcal{F}$, $u \circ F_i \in \mathcal{F}$ for $1 \leq i \leq N$ and

$$\mathcal{E}[u] = \frac{1}{\rho} \sum_{i=1}^{N} \mathcal{E}[u \circ F_i],$$

where $\rho \in (0, 1)$ is the energy renormalization factor.

On SG, Barlow and Perkins [4] proved that the transition probability $p_t(x,y)$ of the associated diffusion satisfies the following Li-Yau type sub-Gaussian estimate: for any $x,y\in \mathrm{SG}$ and $t>0$,

$$p_t(x,y)\asymp \frac{C}{V\left(x,t^{1/\beta}\right)}\exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right),$$

where $d(\cdot,\cdot)$ is the Euclidean metric on SG, $V(x,t^{1/\beta}):=\mu(B(x,t^{1/\beta}))\asymp t^{\alpha/\beta}$ is the volume of metric balls, $\alpha=\log 3/\log 2$ is the Hausdorff dimension of SG, and $\beta=\log 5/\log 2$ is called the walk dimension of SG. Later by Kumagai [15], the two sided sub-Gaussian estimate of heat kernel was proved to be true on the class of nested fractals with different parameters α and β under a new constructed shortest path metric.

The *effective resistance* between two disjoint non-empty closed subsets A,B of K is defined as:

$$R(A,B)^{-1}:=\inf\{\mathcal{E}[u]:\ u\in\mathcal{F}, u|_A=0, u|_B=1\}.$$

We write $R(x,A):=R(\{x\},A)$ and $R(x,y):=R(\{x\},\{y\})$ when x,y are two points. When we only consider points, then by setting $R(x,x)=0$ for all $x\in K$, it is known that $R(\cdot,\cdot)$ is indeed a metric on K, which is called the *effective resistance metric*.

2.1. **Graph-directed self-similar sets.** We first give the definition of graph-directed self-similar sets, see also [18]. Let $(\mathcal{A},\Gamma)$ be a directed graph with vertices $\mathcal{A}=\{1,\ldots,\ell\}$ and the set of directed edges Γ. For any $\gamma\in\Gamma$, if γ is a directed edge from i to j for some $i,j\in\mathcal{A}$, we denote by $I(\gamma)=i$ and $T(\gamma)=j$ the *initial vertex* and the *terminal vertex* separately. For $i,j\in\mathcal{A}$, let $\Gamma_i=\{\gamma\in\Gamma:\ I(\gamma)=i\}$ and let $\Gamma_{i,j}=\{\gamma\in\Gamma:\ I(\gamma)=i, T(\gamma)=j\}$. We assume each edge γ is associated with a contractive map Φ_γ on $\mathbb{R}^d$. Then there exists a vector of nonempty compact sets $\{D_i\}_{i=1}^{\ell}$ in $\mathbb{R}^d$, called *graph-directed self-similar sets*, such that

$$D_i=\bigcup_{j=1}^{\ell}\bigcup_{\gamma\in\Gamma_{i,j}}\Phi_\gamma(D_j),\qquad 1\le i\le \ell. \tag{2.1}$$

Let $m\ge 1$, a finite word $\boldsymbol{\gamma}=\gamma_1\gamma_2\cdots\gamma_m$ with $\gamma_i\in\Gamma$ for $i=1,\ldots,m$ is called *admissible* if $T(\gamma_i)=I(\gamma_{i+1})$ for any $i=1,\ldots,m-1$. Let $m\ge 0$, we denote by Γ_m^* the set of admissible words with length m, where we make convention that $\Gamma_0^*=\{\emptyset\}$, which contains only the empty word. We denote by $\Gamma^*=\bigcup_{m=0}^{\infty}\Gamma_m^*$ the set of all finite admissible words. For any $\boldsymbol{\gamma}=\gamma_1\gamma_2\cdots\gamma_m\in\Gamma^*$, we set $I(\boldsymbol{\gamma})=I(\gamma_1)$ and $T(\boldsymbol{\gamma})=T(\gamma_m)$ and define $\Phi_{\boldsymbol{\gamma}}=\Phi_{\gamma_1}\circ\cdots\circ\Phi_{\gamma_m}$.

Keep this in mind, we then turn to a particular situation, that is, domains in nested fractals. Let $(K,\{F_i\}_{i=1}^N)$ be a nested fractal. For $\ell\ge 1$, let $\{\Omega_1,\Omega_2,\ldots,\Omega_\ell\}$ be a vector of sets such that for $1\le i\le\ell$, each Ω_i is a connected open subset of K and the boundary of Ω_i in K is nonempty, which we denote by $D_i:=\partial\Omega_i$. We assume that $\{(\Omega_i,D_i)\}_{1\le i\le\ell}$ satisfy the following *graph-directed boundary condition*:

for $1\le i\le\ell$ and $1\le k\le N$, if $D_i\cap F_k(K)\ne\emptyset$, then there exist $\sigma\in\mathcal{G}$ and $1\le j\le\ell$ such that

$$\Omega_i\cap F_k(K)=F_k\circ\sigma(\Omega_j),\qquad D_i\cap F_k(K)=F_k\circ\sigma(D_j). \tag{2.2}$$

According to the configuration of K and $\{\Omega_i\}_{i=1}^{\ell}$, we define the directed graph on $\mathcal{A}=\{1,\ldots,\ell\}$ as following. For each above situation, we set γ to be a directed edge from i to j with the contractive map $\Phi_\gamma=F_k\circ\sigma$. Denote by Γ the set of all directed edges γ

between some indices $i, j \in \mathcal{A}$. In this way, we have a directed graph $(\mathcal{A}, \Gamma)$ and a set of similitudes $\{\Phi_\gamma\}_{\gamma\in\Gamma}$ such that for each γ, there is some $k \in \{1, \ldots, N\}$ and $\sigma \in \mathcal{G}$ such that $\Phi_\gamma = F_k \circ \sigma$. Furthermore, $\{D_i\}_{1\le i\le \ell}$ satisfy the equations (2.1) with these $\{\Phi_\gamma\}_{\gamma\in\Gamma}$, and hence, $\{D_i\}_{1\le i\le\ell}$ are graph-directed self-similar sets.

2.2. **Electric networks.** We will introduce some basic results for electric networks. One may find more details in [6]. Let G be a finite set, and let $g : G\times G \to \mathbb{R}$ be a nonnegative function such that

$$g(p, q) = g(q, p),\ g(p, q) \ge 0,\ g(p, p) = 0, \quad p, q \in G.$$

We call the pair (G, g) an *electric network*, and define the energy on (G, g) to be

$$\mathcal{E}[u] := \mathcal{E}(u, u) = \sum_{p,q\in G} g(p, q)(u(p) - u(q))^2,$$

for $u \in \mathcal{F}$, the domain of $\mathcal{E}$, which is the collection of all the real functions defined on G.

For $p, q \in G$, we write $p \sim q$ if $g(p, q) > 0$ and say that (G, g) is *connected* if for any $p, q \in G$ there is a path $p = p_0 \sim p_1 \sim \cdots \sim p_n = q$. In the rest of this section, we always assume that (G, g) is connected. We also define the graph Laplacian of a function u on G as

$$\Delta u(p) = \sum_{q\in G,\ p\sim q} g(p, q)(u(q) - u(p)), \qquad \forall p \in G.$$

u is said to be *harmonic* at p if

$$\Delta u(p) = 0.$$

By an elementary computation, we see that for any u on G,

$$\mathcal{E}[u] = -\sum_{p\in G} u(p)\Delta u(p). \tag{2.3}$$

Similar to the fractal case discussed previously, we define the *effective resistance* $R(A, B)$ of two disjoint nonempty subsets A and B of G as

$$R(A, B)^{-1} := \inf\{\mathcal{E}[u] : u = 0 \text{ on } A, u = 1 \text{ on } B\}.$$

The infimum can be attained by a unique function u_0 which is harmonic at any point in $G \setminus (A \cup B)$. We call u_0 the *realization* of $R(A, B)$ for the *potential problem pot*$(G; A, B)$. For $A = \{p\}, B = \{q\},\ p, q \in G$, $R(p, q)$ defines a metric on G by letting $R(p, p) = 0$ for any $p \in G$, and is called the *effective resistance metric* [14].

Let $G_0 \subset G$, for a function u on G_0, there is a unique function $\tilde{u}$ (usually we call *harmonic extension* of u) on G such that

$$\mathcal{E}[\tilde{u}] = \inf_{v|_{G_0}=u} \mathcal{E}[v],$$

and [14, Theorem 2.1.6]

$$\mathcal{E}[\tilde{u}] = \sum_{p,q\in G_0, p\ne q} g_0(p, q)(u(p) - u(q))^2, \tag{2.4}$$

with $g_0(p, q) \ge 0$ independent of u. The electric network (G_0, g_0) is called the *trace* of (G, g) to G_0.

To define the current in the network (G, g), we let

$$W(G) := \{(p, q) \in G \times G :\ g(p, q) > 0\}.$$

Set $N(p) = \{q : (p,q) \in W(G)\}$. A *flow* from A to B on G is a function $I : W(G) \to \mathbb{R}$ such that

$$I(p,q) = -I(q,p), \quad \sum_{q \in N(p)} I(p,q) = 0 \text{ for } p \in G \setminus (A \cup B).$$

The *total flux* of I is given by

$$T(I; A, B) := \sum_{p \in A} \sum_{q \in N(p)} I(p,q) = -\sum_{p \in B} \sum_{q \in N(p)} I(p,q).$$

Define the energy of I as

$$E[I] := E(I, I) = \sum_{(p,q) \in W(G)} g(p,q)^{-1} I^2(p,q).$$

Then we have (cf. [6])

$$R(A,B) = \inf\{E[I] : I \text{ is a flow from } A \text{ to } B \text{ on } G \text{ with total flux } 1\}. \tag{2.5}$$

We call a flow I from A to B on G with total flux 1 attaining the infimum a *feasible flow* for the *current problem* $\mathrm{cur}(G; A, B)$. For a function $u : G \to \mathbb{R}$, define $\nabla u : W(G) \to \mathbb{R}$ by

$$\nabla u(p,q) = g(p,q)(u(q) - u(p)).$$

It is shown [6] that the current problem $\mathrm{cur}(G; A, B)$ has the unique solution which is $R(A,B) \cdot \nabla u_0$ where u_0 is the unique solution of the potential problem $\mathrm{pot}(G; A, B)$.

2.3. **Discrete Gauss-Green's formulas.** We provide a discrete version of *Gauss-Green's formulas* on electric networks analogous to the classical result on $\mathbb{R}^d$. We recall that for a smooth bounded domain $\Omega \subset \mathbb{R}^d$, denote by $\overline{\Omega}$ the closure of Ω, let u be a function defined on $\overline{\Omega}$ having continuous second order derivatives. Then the first Gauss-Green's formula holds:

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} ds,$$

where ds is the $(n-1)$-dimensional area element on $\partial\Omega$ and $\mathbf{n}$ is the unit outer normal vector on $\partial\Omega$.

Let v be another function on Ω, by using the first formula, the second Gauss-Green's formula is written as the following

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}} \right) ds.$$

There is a discrete version of Gauss-Green's formulas on electric networks which will play an important role in the later study of harmonic measures on fractal domains in the next section.

Let (G, g) be a connected electric network as above. We say that Ω is a domain of G if Ω is a nonempty subset of G such that Ω is connected. We define the interior of Ω by

$$\Omega^\circ := \{p \in \Omega : \text{ if } q \in G \text{ such that } p \sim q, \text{ then } q \in \Omega\},$$

and let $\partial\Omega = \Omega \setminus \Omega^\circ$ be the boundary of Ω in G.

Assume Ω° is nonempty to avoid triviality. Then by definition of the discrete Laplacian Δ, we have that for any function u on Ω, the first Gauss-Green's formula holds:

$$\sum_{p \in \Omega^\circ} \Delta u(p) = \sum_{p \in \Omega^\circ} \sum_{q \sim p} g(p,q)(u(q) - u(p)) = \sum_{q \in \partial\Omega} \sum_{p \in \Omega,\ p \sim q} g(p,q)(u(q) - u(p)) =: \sum_{q \in \partial\Omega} \frac{\partial u}{\partial \mathbf{n}}(q),$$

where $\frac{\partial u}{\partial \mathbf{n}}(q) = \sum_{p\in\Omega,\ p\sim q} g(p,q)(u(q) - u(p))$ plays the same role as the outer normal derivative of u at $q \in \partial\Omega$.

Let v be another function on Ω, then we have

$$\begin{aligned}\sum_{p\in\Omega^\circ} v(p)\Delta u(p) &= \sum_{p\in\Omega^\circ}\sum_{q\sim p} g(p,q)v(p)(u(q)-u(p))\\ &= \sum_{p\in\Omega^\circ}\left(\sum_{q\in\Omega^\circ} + \sum_{q\in\partial\Omega}\right) g(p,q)v(p)(u(q)-u(p))\\ &= \frac{1}{2}\sum_{p,q\in\Omega^\circ} g(p,q)(u(q)-u(p))(v(p)-v(q)) + \sum_{p\in\Omega^\circ}\sum_{q\in\partial\Omega} g(p,q)v(p)(u(q)-u(p)).\end{aligned}$$

Changing the role of u and v in the above, we have

$$\sum_{p\in\Omega^\circ} u(p)\Delta v(p) = \frac{1}{2}\sum_{p,q\in\Omega^\circ} g(p,q)(u(q)-u(p))(v(p)-v(q)) + \sum_{p\in\Omega^\circ}\sum_{q\in\partial\Omega} g(p,q)u(p)(v(q)-v(p)).$$

Combine the above two equations with slight rearrangement, we obtain the discrete version of the second Gauss-Green's formula

$$\sum_{p\in\Omega^\circ} (v(p)\Delta u(p) - u(p)\Delta v(p)) = \sum_{q\in\partial\Omega}\left(v(q)\frac{\partial u}{\partial \mathbf{n}}(q) - u(q)\frac{\partial v}{\partial \mathbf{n}}(q)\right). \tag{2.6}$$

For a function u on Ω, ∇u on the edges defines a flow in Ω°, and the flux flowing outside Ω through $q \in \partial\Omega$ is exactly $\frac{\partial u}{\partial \mathbf{n}}(q)$.

We then turn to consider a domain Ω in a nested fractal K equipped with the standard Dirichlet form $(\mathcal{E}, \mathcal{F})$. For a harmonic function on Ω which takes finitely many boundary values, by tracing the form to the finite boundaries, we may view the harmonic function as to be defined on a finite electric network, and hence we can apply the discrete Gauss-Green's formulas obtained above to this effective electric network.

Let K be a nested fractal with a standard Dirichlet form $(\mathcal{E}, \mathcal{F})$, let Ω be a connected nonempty open subset of K such that its boundary D is a finite set. Let u be a harmonic function on Ω, we write the energy of u in Ω as $\mathcal{E}_\Omega[u]$, then $\mathcal{E}_\Omega[u]$ can be written as

$$\mathcal{E}_\Omega[u] = \sum_{p,q\in D} g(p,q)(u(p)-u(q))^2, \tag{2.7}$$

where (D, g) is the *trace* of $(\mathcal{E}, \mathcal{F})$ on Ω to D.

We define the flux of the harmonic function u (or the flow ∇u) flowing outside Ω through $p \in D$ as

$$\frac{\partial u}{\partial \mathbf{n}}(p) = \sum_{q\in D,\ q\neq p} g(p,q)(u(p)-u(q)).$$

Then we simply have

$$\sum_{p\in D}\frac{\partial u}{\partial \mathbf{n}}(p) = 0,$$

which also implies that for a harmonic function u in a domain Ω, the total flux of u flowing outside through its boundary D is 0.

Note that by the harmonicity of u and the compatibility of the sequence of electric networks approximating the fractal, we see that $\frac{\partial u}{\partial \mathbf{n}}(p)$ can be computed by the values of u

at arbitrarily small neighborhood of p in Ω. For the same reason, by viewing the domain with a finite boundary set (including the case that we view a subset of the boundary as one point by shorting the set if u is constant on it) as a finite electric network according to (2.7), the first and second discrete Gauss-Green's formulas are applicable for harmonic functions u and v on domains in self-similar sets.

3. Harmonic measures

Let $(K, \{F_i\}_{i=1}^N)$ be a nested fractal with essential fixed point set P_0. Let $\{\Omega_i\}_{i=1}^\ell$ be a collection of open domains in K with the boundaries $\{D_i\}_{i=1}^\ell$ such that they satisfy the graph-directed boundary condition (2.2). Denote by $\mathcal{A} = \{1, \ldots, \ell\}$ and for each $i \in \mathcal{A}$, we denote by $\mathcal{P}_i = \Omega_i \cap P_0$ and call $\mathcal{P}_i$ the *free vertices* since they are not contained in the boundary D_i. Then we denote the number of free vertices by $s_i = \#\mathcal{P}_i$. In order to state our result in a simpler way without losing generality, we assume the following.

- $(\mathcal{A}, \Gamma)$ is a connected graph.
- $s_i \geq 1$ for each $i \in \mathcal{A}$.

The first assumption is to make sure that $\{\Omega_i\}_{i=1}^\ell$ have relations with each other, otherwise we may separately deal with each connected component (sub-graphs) of $(\mathcal{A}, \Gamma)$.

The second assumption is also reasonable. Otherwise, there would be some Ω_i such that $\Omega_i \cap P_0 = \emptyset$, then by the connectedness of Ω_i and the nesting property of the fractal K, we see that in the graph $(\mathcal{A}, \Gamma)$, there should be no edge having i as a terminal vertex and all the edges with initial vertex i should point to some j such that $\mathcal{P}_j \neq \emptyset$. Therefore, in the second generation of the domains after one step iteration by $\{\Phi_\gamma\}_{\gamma\in\Gamma}$, all should contain points in P_0, and thus the problem with possibly $s_i = 0$ can be easily transferred into the problem with $s_i \geq 1$ for all i, thus satisfying the second assumption.

For each $\gamma \in \Gamma$ with $I(\gamma) = i$ and $T(\gamma) = j$, we associate γ with an $s_i \times s_j$ matrix M_γ, which is defined in the following way.

Let $p \in \mathcal{P}_i$, consider the effective resistance

$$R(D_i, p)^{-1} := \inf\{\mathcal{E}(u) : \ u \in \mathcal{F}, u = 0 \text{ on } D_i, u(p) = 1\}.$$

By the connectedness of Ω_i and using a standard PDE argument, the infimum in the definition of $R(D_i, p)$ can be attained by a unique function u, which is harmonic in $\Omega_i \setminus \{p\}$. The function $\upsilon := R(D_i, p)u$ coincides with the function $G_{\Omega_i}(p, \cdot)$ on Ω_i where $G_{\Omega_i}(\cdot, \cdot)$ is the Green's function on $\Omega_i \times \Omega_i$. Also υ gives a flow with total flux 1 from D_i to p.

Since $\gamma \in \Gamma_{i,j}$, we see that $\Phi_\gamma(\Omega_j) \subset \Omega_i$. Now consider the restriction of the function υ on $\Phi_\gamma(\Omega_j)$, and denote its pull-back through Φ_γ^{-1} by $\Phi_\gamma^{-1}(\upsilon)$, then $\Phi_\gamma^{-1}(\upsilon)$ is a function on Ω_j, which is harmonic in $\Omega_j \setminus \mathcal{P}_j$. For each $q \in \mathcal{P}_j$, we denote by $w(p, q)$ the flux of the harmonic function υ flowing outside $\Phi_\gamma(\Omega_j)$ through the vetex $\Phi_\gamma(q)$.

It is seen that $w(p, q)$ is some real number which might be negative. By varying $q \in \mathcal{P}_j$, we obtain a row vector $\mathbf{W}_\gamma(p) := \{w(p, q)\}_{q\in\mathcal{P}_j}$ of dimension s_j which records the fluxes of υ flowing outside $\Phi_\gamma(\Omega_j)$ through each vertex in $\{\Phi_\gamma(q) : \ q \in \mathcal{P}_j\}$ ($w(p, q)$ might be zero for some q). By varying $p \in \Omega_i$, we write together the row vectors $\mathbf{W}_\gamma(p)$ into an $s_i \times s_j$ matrix, and denote it by M_γ, each of whose entry denotes the flux of the unit flow from D_i to some vertex $p \in \mathcal{P}_i$ through some vertex $\Phi_\gamma(q)$ outside $\Phi_\gamma(\Omega_j)$. For each Ω_i, since the harmonic function υ has zero divergence in the interior $\Omega_i \setminus \{p\}$, the total flux of υ which is 1 should be equal to the summation of all the fluxes flowing outside $\cup_{\gamma\in\Gamma_i}\Phi_\gamma(\Omega_{T(\gamma)})$ through those finite vertices in $\cup_{\gamma\in\Gamma_i}\{\Phi_\gamma(q) : q \in \mathcal{P}_{T(\gamma)}\}$. Hence if we denote by $\mathcal{W}_\gamma(p)$

the summation of all entries in $\mathbf{W}_\gamma(p)$, then

$$\sum_{\gamma\in\Gamma_i} \mathcal{W}_\gamma(p) = 1, \qquad \text{for all } p \in \mathcal{P}_i. \tag{3.8}$$

For this reason, we call $\{M_\gamma\}_{\gamma\in\Gamma_i}$ the *flux distribution matrices* associated with Ω_i.

We then use the matrices $\{M_\gamma\}_{\gamma\in\Gamma}$ to construct a class of positive measures $\{\mu_{i,p} : i \in \mathcal{A}, p \in \mathcal{P}_i\}$ on D_i. Note that by (2.1), we have

$$D_i = \bigcup_{\boldsymbol{\gamma}\in\Gamma_m^*, I(\boldsymbol{\gamma})=i} \Phi_{\boldsymbol{\gamma}}(D_{T(\boldsymbol{\gamma})}), \qquad \text{for all } m \geq 1. \tag{3.9}$$

Denote by $D_{\boldsymbol{\gamma}} = \Phi_{\boldsymbol{\gamma}}(D_{T(\boldsymbol{\gamma})})$ for $\boldsymbol{\gamma} = \gamma_1 \cdots \gamma_m \in \Gamma_m^*$. Note that each of the matrix M_γ has row number $s_{I(\gamma)}$ and column number $s_{T(\gamma)}$, and by that $\boldsymbol{\gamma}$ is admissible, i.e., $T(\gamma_j) = I(\gamma_{j+1})$ for $j = 1, \ldots, m-1$, we see that the matrix product $M_{\gamma_1} \cdots M_{\gamma_m} := M_{\boldsymbol{\gamma}}$ is well defined, we then define

$$\mu_{i,p}(D_{\boldsymbol{\gamma}}) = \mathbf{e}_p M_{\boldsymbol{\gamma}} \mathbf{1}^t_{T(\boldsymbol{\gamma})}, \tag{3.10}$$

where $M_{\boldsymbol{\gamma}} = M_{\gamma_1} \cdots M_{\gamma_m}$, $\mathbf{e}_p = (0, \cdots, 1, \cdots, 0)$ is the s_i-dimensional unit row vector representing p in $\mathcal{P}_i$, and $\mathbf{1}^t_{T(\boldsymbol{\gamma})}$ is the $s_{T(\boldsymbol{\gamma})}$-dimensional column vector with all entries equal to 1. It is clear that $\mu_{i,p}(D_{\boldsymbol{\gamma}})$ equals the flux of υ flowing into Ω_i through the portion $\Phi_{\boldsymbol{\gamma}}(D_{T(\boldsymbol{\gamma})})$ on the boundary, and is strictly positive.

Then by (3.8) and (3.9), we see that $\mu_{i,p}$ is a probability measure on D_i.

We then turn to show that the measures $\{\mu_{i,p} : i \in \mathcal{A}, p \in \mathcal{P}_i\}$ are exactly harmonic measures, see Theorem 3.2, which is our main result in this section.

Before that, we first show that for a domain Ω, the harmonic measures of any two points $x, y \in \Omega$ on the boundary D are equivalent to each other. Let $R(\cdot,\cdot)$ be the effective resistance metric on a nested fractal K. We say that the *elliptic Harnack inequality* holds if there is $C > 0$ such that for any ball $B_R(x, r)$ under the resistance metric with center $x \in K$ and $r \in (0, 1)$, and for any nonnegative harmonic function u on $B_R(x, r)$, it holds that

$$\sup_{B_R(x,r/2)} u \leq C \inf_{B_R(x,r/2)} u.$$

Since the sub-Gaussian heat kernel estimates hold for the standard Dirichlet form on each nested fractal under a quasisymmetric modification of the resistance metric [15], the elliptic Harnack inequality holds for the standard Brownian motions on nested fractals under the resistance metric (see for example [8]). By using the elliptic Harnack inequality, we have the following.

Proposition 3.1. *Let $\{\Omega_i\}_{i=1}^\ell$ be a vector of domains with boundaries $\{D_i\}_{i=1}^\ell$. For each $i \in \{1, \ldots, \ell\}$, and any two two points x and y in Ω_i, denote by $\mu_{i,x}$ and $\mu_{i,y}$ the two harmonic measures of x and y on D_i respectively. Then there exists a constant $C > 0$ depending on x, y such that for any measurable set $A \subseteq D_i$,*

$$C^{-1}\mu_{i,y}(A) \leq \mu_{i,x}(A) \leq C\mu_{i,y}(A).$$

Proof. Let $f = 1_A$ be the indicator function of a subset A contained in D_i, and let u be the harmonic extension of f on Ω_i, then $u \geq 0$ in Ω_i and $u(x) = \mu_{i,x}(A)$ for $x \in \Omega$. Let $r = \min\{\mathrm{dist}_R(x, D_i), \mathrm{dist}_R(y, D_i)\}$, then $r > 0$. Using finite number of balls with radius $r/2$ chaining x and y, then the Harnack inequality implies that there exists $C > 0$ such that

$$C^{-1}u(y) \leq u(x) \leq Cu(y),$$

proving the conclusion. $\square$

Theorem 3.2. *Let K be a nested fractal. Let $\{(\Omega_i, D_i)\}_{i\in\mathcal{A}}$ be a vector of domains with boundaries in K satisfying the graph-directed boundary condition* (2.2). *Let $(\mathcal{A}, \Gamma)$ be its directed graph and $\{M_\gamma\}_{\gamma\in\Gamma}$ be the collection of the associated flux distribution matrices. Let $i \in \mathcal{A}$, $p \in \mathcal{P}_i$, then the measure $\mu_{i,p}$ defined in* (3.10) *is the harmonic measure of p on the boundary D_i with respect to Ω_i, i.e., for any $f \in L^1(D_i, \mu_{i,p})$, the unique harmonic function u on Ω_i generated by f satisfies*

$$u(p) = \int_{D_i} f(x) d\mu_{i,p}(x). \tag{3.11}$$

Proof. We first prove the result when f is a simple function on D_i, and the general case is by approximating f via a sequence of simple functions. Let $m \geq 1$ be an integer, assume that f is of the form

$$f = \sum_{|\gamma|=m} f_\gamma 1_{D_\gamma}$$

with $f_\gamma \in \mathbb{R}$. Let u be the unique harmonic extension of f on Ω_i. Let $\upsilon = G_{\Omega_i}(p, \cdot)$, then $\frac{\partial \upsilon}{\partial \mathbf{n}}(p) = 1$ and υ is harmonic in $\Omega_i \setminus \{p\}$. Since u and υ both are harmonic in $\Omega_i \setminus \{p\}$ and u, υ take finitely many different values on the boundary (for u, we can do shorting for the set D_γ for each $|\gamma| = m$, i.e., identifying each of the set D_γ as one singleton), it is equivalent to consider a finite electric network instead of the domain $\Omega_i \setminus \{p\}$ so that on this equivalent network, we can apply the second Gauss-Green's formula (2.6) with these u and υ. Hence we obtain that

$$\begin{aligned} 0 = \int_{\Omega_i \setminus \{p\}} u\Delta\upsilon - \upsilon\Delta u &= \int_{D_i \cup \{p\}} u\frac{\partial \upsilon}{\partial \mathbf{n}} - \upsilon\frac{\partial u}{\partial \mathbf{n}} \\ &= \int_{D_i \cup \{p\}} u\frac{\partial \upsilon}{\partial \mathbf{n}}, \end{aligned}$$

where in the last equality we use that $\upsilon = 0$ on D_i and $\frac{\partial u}{\partial \mathbf{n}}(p) = 0$ by the harmonicity of u at p. Then by $\frac{\partial \upsilon}{\partial \mathbf{n}}(p) = 1$, we obtain from above that

$$0 = \sum_{|\gamma|=m} f_\gamma \int_{D_\gamma} \frac{\partial \upsilon}{\partial \mathbf{n}} + u(p),$$

which by using the definition of $\mu_{i,p}$ is equivalent to

$$u(p) = \sum_{|\gamma|=m} f_\gamma \mu_{i,p}(D_\gamma),$$

proving that (3.11) holds for any simple function f.

For a general $f \in L^1(D_i, \mu_{i,p})$, we know that the harmonic measures of any two points $x, y \in \Omega$ are equivalent by Proposition 3.1. Thus the extension u is well defined at every point in Ω. To see that u is harmonic, we use a sequence of simple function f_n to approximate f in L^1, then the harmonic functions u_n generated by f_n are well defined. Then since the sequence u_n convergence uniformly to u in any compact subsets of Ω, we see that u is harmonic in Ω. $\square$

4. Examples

In this section, we illustrate Theorem 3.2 by several examples. Due to the complexity of the computation for the flux distribution matrices, we only give explicit formulas of harmonic measures for some simple examples. The examples include domains in the Sierpiński gasket appeared in several previous works [5, 9, 16, 20]. We will also give an example on the Vicsek set.

4.1. **Example: Sierpiński gasket and the bottom line.** Let K be the Sierpiński gasket, generated by the IFS $\{F_i\}_{i=1}^3$ with $F_i(x) = \frac{1}{2}(x - p_i) + p_i$ and $\{p_1, p_2, p_3\}$ are the three vertices of an equilateral triangle. Obviously, $P_0 = \{p_1, p_2, p_3\}$, and the standard Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K satisfies the self-similar property [12]

$$\mathcal{E}(u) = \frac{5}{3}\sum_{i=1}^{3}\mathcal{E}(u \circ F_i), \quad \forall u \in \mathcal{F}.$$

Let us consider the only one domain $\Omega = K \setminus \{\overline{p_1 p_2}\}$ with boundary $D = \overline{p_1 p_2}$, where $\overline{p_1 p_2}$ is the line segment connecting vertices p_1 and p_2, see Figure 2. This is an example

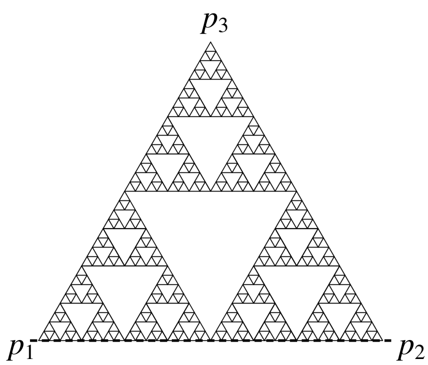

Figure 2. A domain in the Sierpiński gasket with a bottom line boundary

from the pioneering work by Owen and Strichartz [20], who proved that for this domain Ω, the harmonic measure of p_3 is the normalized Lebesgue measure on D. See [5, 9] for further discussions. In our situation, D can be viewed as a self-similar set generated by the IFS $\{F_1, F_2\}$, and this example obviously satisfies the graph-directed boundary condition (2.2), which has the directed graph $(\mathcal{A}, \Gamma)$ with only one vertex $\mathcal{A} = \{1\}$ and two edges $\Gamma = \{\gamma_1, \gamma_2\}$, each of which is from 1 to itself, where γ_i corresponds to the contractive map F_i for $i = 1, 2$ respectively. It is easy to calculate that the two associated matrices of γ_1 and γ_2 are $M_{\gamma_1} = M_{\gamma_2} = \begin{pmatrix} 1/2 \end{pmatrix}$, and thus the harmonic measure of p_3, which is μ_{1,p_3} in our setting, is the $(1/2, 1/2)$-self-similar measure on D.

Remark. A similar domain can be considered if one replace K by its level-3 extension. Recall that the level-3 Sierpiński gasket is generated by an IFS consisting of 6 contractive maps with ratio $1/3$ in a similar manner as K. Still consider the domain with the bottom line boundary. One can calculate similarly that the harmonic measure of the top point p_3 is a $(\frac{6+\eta}{18+4\eta}, \frac{3+\eta}{9+2\eta}, \frac{6+\eta}{18+4\eta})$-self-similar measure on the bottom line boundary D, where $\eta = \frac{\sqrt{2353}-15}{14}$, see [5] for a same result under a different consideration.

4.2. **Example: Sierpiński gasket, the half domain.** Still assume K to be the Sierpiński gasket, and $\{\Omega_1, \Omega_2\}$ are the following. Ω_1 is the left half of K, with boundary $D_1 = L \cap K$, and L is the bisecting line of p_1 and p_2, while Ω_2 is $K \setminus \{p_2\}$ with boundary $D_2 = \{p_2\}$, see Figure 3. The domain Ω_1 is studied by Li and Strichartz [16], see also [5] for extensions

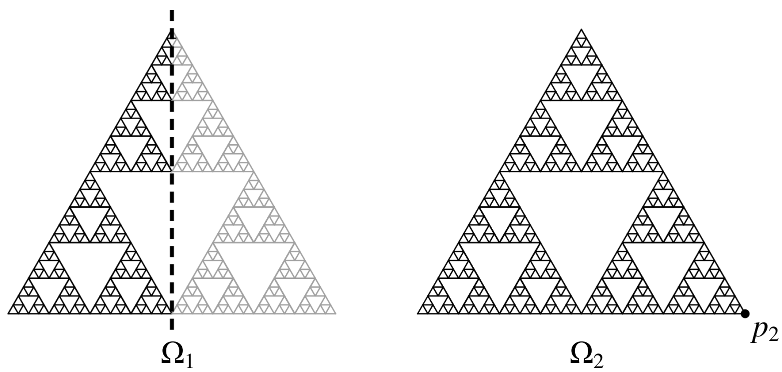

FIGURE 3. The left half domain in the Sierpiński gasket

by Cao and Qiu. It is easy to check that $\{(\Omega_i, D_i)\}_{i=1}^2$ satisfies (2.2), with the directed graph $(\mathcal{A}, \Gamma)$ as following. $\mathcal{A} = \{1, 2\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, where edge γ_1 is from 1 to 1 corresponding to F_3, edge γ_2 is from 1 to 2 corresponding to F_1, and edge γ_3 is from 2 to 2 corresponding to F_2, see Figure 4. It is easy to calculate that the associated matrices

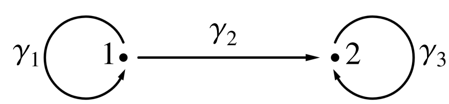

FIGURE 4. The directed graph $(\mathcal{A}, \Gamma)$ in Example 4.2

are $M_{\gamma_1} = \begin{pmatrix} 1/3 \end{pmatrix}$, $M_{\gamma_2} = \begin{pmatrix} 1 & -1/3 \end{pmatrix}$ and $M_{\gamma_3} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$. Using this, we calculate that the harmonic measure of p_1 with respect to Ω_1 is $\mu_{1,p_1} = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} \delta_{F_3^n F_1(p_2)}$ where δ_x is the Dirac measure at x.

4.3. **Example: domains in the Vicsek set.** Recall that the Vicsek set, denoted by K, is a typical nested fractal generated by the IFS $\{F_i\}_{i=1}^5$, with $F_i(x) = \frac{1}{3}(x - p_i) + p_i$, where $\{p_i\}_{i=1}^4$ are the four corner vertices of a unit square and p_5 is the center. The standard Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K satisfies [23]

$$\mathcal{E}(u) = 3 \sum_{i=1}^{5} \mathcal{E}(u \circ F_i), \quad \forall u \in \mathcal{F}.$$

Let $L_1 = \overline{p_1 p_2} \cap K$ and $L_2 = \overline{p_2 p_3} \cap K$. Then both of L_1 and L_2 are middle-third Cantor sets. We define two domains $\Omega_1 = K \setminus L_1$, $\Omega_2 = K \setminus (L_1 \cup L_2)$, with boundaries $D_1 = L_1$, $D_2 = L_1 \cup L_2$ respectively, see Figure 5. Then $\{(\Omega_i, D_i)\}_{i=1}^2$ satisfies the condition (2.2) with a directed graph $(\mathcal{A}, \Gamma)$ defined by $\mathcal{A} = \{1, 2\}$ and $\Gamma = \{\gamma_i\}_{i=1}^5$ as illustrated in Figure 6. The corresponding contractive maps of $\{\gamma_i\}_{i=1}^5$ are F_1, F_2, F_1, $F_3 \circ \sigma$ and F_2, where σ is the counterclockwise rotation by $\frac{\pi}{2}$ around p_5. Then a computation gives that the

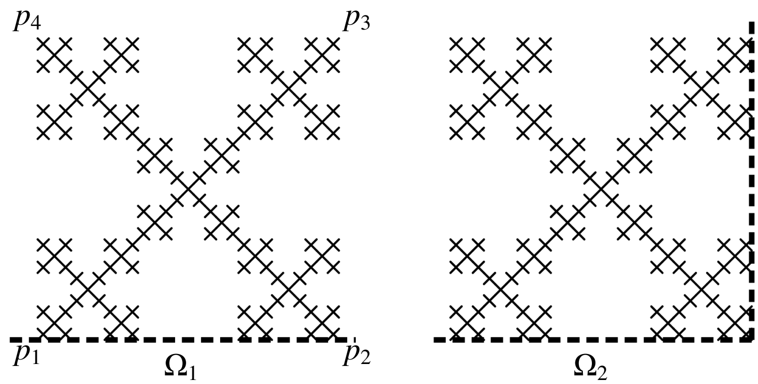

FIGURE 5. A couple of domains in the Vicsek set

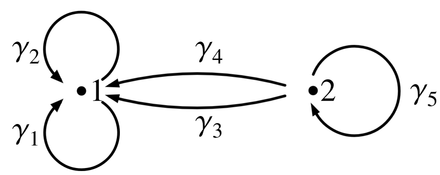

FIGURE 6. The directed graph $(\mathcal{A}, \Gamma)$ in Example 4.3

associated matrices are $M_{\gamma_1} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}$, $M_{\gamma_2} = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}$, $M_{\gamma_3} = \begin{pmatrix} 0 & \frac{13+t}{26+14t} \end{pmatrix}$, $M_{\gamma_4} = \begin{pmatrix} \frac{13+t}{26+14t} & 0 \end{pmatrix}$, $M_{\gamma_5} = \begin{pmatrix} \frac{6t}{13+7t} \end{pmatrix}$, where $t = \frac{\sqrt{69}-2}{5}$.

5. ENERGY ESTIMATES

In this section, we still consider domains $\{(\Omega_i, D_i)\}_{i=1}^{\ell}$ in a nested fractal $(K, \{F_i\}_{i=1}^N)$ satisfying the graph-directed boundary condition (2.2). For each $1 \leq i \leq \ell$, we aim to provide the energy estimate of a harmonic function u in Ω_i via the integrals of its boundary value f on D_i against harmonic measures. Throughout this section, we always use the notation $f \lesssim g$ if there is a constant $C > 0$ such that $f \leq Cg$, and write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

Let P_0 be the collection of essential fixed points of $(K, \{F_i\}_{i=1}^N)$, and $(\mathcal{E}, \mathcal{F})$ be the standard Dirichlet form on K with an energy renormalization factor ρ. Before proceeding, we first recall that for a harmonic function u on K with boundary values $\{u(p) : \ p \in P_0\}$, assuming the flux of u flowing out K through each point $p \in P_0$ is $\frac{\partial u}{\partial \mathbf{n}}(p)$, we simply have

$$\mathcal{E}[u] \asymp \sum_{p,q \in P_0} (u(p) - u(q))^2 \asymp \sum_{p \in P_0} \left| \frac{\partial u}{\partial \mathbf{n}}(p) \right|^2.$$

Consequently, we can pass the estimates to any m-level cell $K_w := F_w(K)$ with adjustment by ρ^{-m} (and ρ^m for flows separately), where for $w = w_1 w_2 \cdots w_m$, we write $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ for short. Denote by $\mathcal{E}_w[u]$ the energy of a harmonic function on K_w, then

$$\mathcal{E}_w[u] \asymp \rho^{-m} \sum_{p,q \in F_w(P_0)} (u(p) - u(q))^2 \asymp \rho^m \sum_{p \in F_w(P_0)} \left| \frac{\partial u}{\partial \mathbf{n}}(p) \right|^2. \tag{5.12}$$

We still use $(\mathcal{A}, \Gamma)$ to denote the directed graph of $\{(\Omega_i, D_i)\}_{i=1}^{\ell}$. In the rest of this section, for an arbitrary domain Ω_i, $i \in \mathcal{A}$, we omit the subscript i for brevity, just denote it by Ω with its boundary D. We also denote the free vertices of Ω as $\mathcal{P}$. For some $p \in \mathcal{P}$, we denote the harmonic measure $\mu_{i,p}$ obtained in Section 3 by μ_p. Still from Section 3, we know that there exists a finite number of flux distribution matrices which generate the harmonic measures μ_p on the boundary D, such that for a given function f on the boundary D, the values of the harmonic extension u at p is determined by the integral of f against μ_p. Denote by $\mathcal{E}_\Omega[u]$ the energy of u in Ω. Our purpose is to estimate $\mathcal{E}_\Omega[u]$ from above and below by means of the integrals of f on cells of D.

For two words $\boldsymbol{\gamma}, \boldsymbol{\eta} \in \Gamma_m^*$ with $m \geq 1$, we write $\boldsymbol{\gamma} \sim \boldsymbol{\eta}$ if two m-cells $\Omega_{\boldsymbol{\gamma}}$ and $\Omega_{\boldsymbol{\eta}}$ are contained in the same $(m-1)$-cell, or equivalently $\boldsymbol{\gamma}^- = \boldsymbol{\eta}^-$, where we denote $\Omega_{\boldsymbol{\gamma}} := \Phi_{\boldsymbol{\gamma}}(\Omega_{T(\boldsymbol{\gamma})})$ with $\Phi_{\boldsymbol{\gamma}} = \Phi_{\gamma_1} \circ \Phi_{\gamma_2} \circ \cdots \circ \Phi_{\gamma_m}$. Similarly, we denote $D_{\boldsymbol{\gamma}} := \Phi_{\boldsymbol{\gamma}}(D_{T(\boldsymbol{\gamma})})$ for short.

Our main result in this section is the following.

Theorem 5.1. *Let K be a nested fractal equipped with a standard self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a renormalization factor $\rho \in (0,1)$. Let Ω be a connected domain in K with a graph-directed boundary D. Let $f \in L^1(D, \mu_x)$ for some $x \in \Omega$ and u be the harmonic extension in Ω of f, then we have*

$$\mathcal{E}_\Omega[u] \asymp \sum_{m=1}^{\infty} \rho^{-m} \sum_{\substack{\boldsymbol{\gamma} \sim \boldsymbol{\eta}, \\ |\boldsymbol{\gamma}| = |\boldsymbol{\eta}| = m}} \sum_{p \in \mathcal{P}_{T(\boldsymbol{\gamma})},\, q \in \mathcal{P}_{T(\boldsymbol{\eta})}} \left(\int_{D_{T(\boldsymbol{\gamma})}} f \circ \Phi_{\boldsymbol{\gamma}} d\mu_p - \int_{D_{T(\boldsymbol{\eta})}} f \circ \Phi_{\boldsymbol{\eta}} d\mu_q \right)^2. \tag{5.13}$$

Proof. By the graph-directed boundary condition (2.2), for each cell $F_k K$ such that $F_k K \cap \Omega \neq \emptyset$, it is either contained in Ω or $F_k K \cap \Omega$ is an image of some Ω_j under the map $F_k \circ \sigma$ for some $\sigma \in \mathcal{G}$.

We first show the "$\lesssim$" in (5.13). We construct a sequence of piecewise harmonic functions $\{v_m\}_{m=1}^\infty$ on Ω satisfying the boundary value f step by step.

For v_1, we let v_1 be harmonic in each cell $F_k(K)$ contained in Ω, taking the same value as u at each of the free vertices in $\mathcal{P}$; and on each $\Omega_{\boldsymbol{\gamma}}$ contained in Ω with $|\boldsymbol{\gamma}| = 1$, taking the same value as the $\Phi_{\boldsymbol{\gamma}}$-image of the harmonic function on $\Omega_{T(\boldsymbol{\gamma})}$ generated by $f \circ \Phi_{\boldsymbol{\gamma}}$ on $D_{T(\boldsymbol{\gamma})}$, i.e., view $\Omega_{\boldsymbol{\gamma}}$ as a separated domain with value f restricted on $D_{\boldsymbol{\gamma}}$ and generate the function on $\Omega_{\boldsymbol{\gamma}}$. Obviously, v_1 satisfies the boundary value f on D. We denote by $\mathcal{E}_{\Omega,1}[v_1]$ the energy of v_1 on the union of 1-cells away from the boundary D.

By the harmonicity of v_1 in $\Omega \setminus \cup_{|\boldsymbol{\gamma}|=1} \Omega_{\boldsymbol{\gamma}}$, we have

$$\mathcal{E}_{\Omega,1}[v_1] \lesssim \rho^{-1} \sum_{p,q \in \Lambda_1} (v_1(p) - v_1(q))^2,$$

where Λ_1 is the collection of free vertices of Ω together with the $\Phi_{\boldsymbol{\gamma}}$-images of free vertices of those $\Omega_{T(\boldsymbol{\gamma})}$, and the inequality follows by (5.12). By using the formula in Theorem 3.2, we can write those $v_1(p)$'s into integrals of f against harmonic measures, that is for a vertex $p \in \mathcal{P}$, we have $v_1(p) = \int_D f d\mu_p$; and for a vertex being a $\Phi_{\boldsymbol{\gamma}}$-image of a free vertex q of $\Omega_{T(\boldsymbol{\gamma})}$, the value of $v_1(\Phi_{\boldsymbol{\gamma}}(q))$ equals $\int_{D_{T(\boldsymbol{\gamma})}} f \circ \Phi_{\boldsymbol{\gamma}} d\mu_q$. We then have

$$\begin{aligned}
&\sum_{p,q \in \Lambda_1} (v_1(p) - v_1(q))^2 \\
&= \sum_{p,q \in \mathcal{P}} \left(\int_D f d\mu_p - \int_D f d\mu_q \right)^2 + \sum_{|\boldsymbol{\gamma}|=1,\ p,q \in \mathcal{P}_{T(\boldsymbol{\gamma})}} \left(\int_{D_{T(\boldsymbol{\gamma})}} f \circ \Phi_{\boldsymbol{\gamma}} d\mu_p - \int_{D_{T(\boldsymbol{\gamma})}} f \circ \Phi_{\boldsymbol{\gamma}} d\mu_q \right)^2
\end{aligned}$$

$$+\sum_{\substack{\boldsymbol{\gamma}\neq\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=1}}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2$$
$$+\sum_{p\in\mathcal{P}}\sum_{|\boldsymbol{\gamma}|=1,\ q\in\mathcal{P}_{T(\boldsymbol{\gamma})}}\left(\int_D fd\mu_p-\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_q\right)^2. \tag{5.14}$$

For a vertex $p\in\mathcal{P}$, and a $\boldsymbol{\gamma}\in\Gamma_1^*$ with $\Omega_{\boldsymbol{\gamma}}\subset\Omega$, observe that the normalized restricted measure μ_p on $D_{\boldsymbol{\gamma}}$ is a linear combination with total weight 1 of the $\Phi_{\boldsymbol{\gamma}}$-images of harmonic measures $\{\mu_q:\ q\in\mathcal{P}_{T(\boldsymbol{\gamma})}\}$. So the first term on the right side of (5.14) satisfies

$$\sum_{p,q\in\mathcal{P}}\left(\int_D fd\mu_p-\int_D fd\mu_q\right)^2\lesssim\sum_{|\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=1}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2; \tag{5.15}$$

the second term on the right side of (5.14) satisfies

$$\sum_{|\boldsymbol{\gamma}|=1,\ p,q\in\mathcal{P}_{T(\boldsymbol{\gamma})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_q\right)^2$$
$$\lesssim\sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=2}}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2; \tag{5.16}$$

and the forth term on the right side of (5.14) is bounded by the second and the third term. Thus we obtain

$$\mathcal{E}_{\Omega,1}[\upsilon_1]\lesssim\rho^{-1}\sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=1}}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2$$
$$+\rho^{-1}\sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=2}}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2. \tag{5.17}$$

Next, for υ_2, we let $\upsilon_2=\upsilon_1$ in $\Omega\setminus\cup_{|\boldsymbol{\gamma}|=1}\Omega_{\boldsymbol{\gamma}}$, and define υ_2 on $\Omega_{\boldsymbol{\gamma}}$'s in a same way as we did for υ_1, that is, for each $\Omega_{\boldsymbol{\gamma}}$, pull back υ_1 to $\Omega_{T(\boldsymbol{\gamma})}$ and construct a piecewise harmonic function $\upsilon_2\circ\Phi_{\boldsymbol{\gamma}}$ as above. Then obviously υ_2 satisfies the boundary value f on D, and by denoting $\mathcal{E}_{\Omega,2}[\upsilon_2]$ the energy of υ_2 on $\cup_{|\boldsymbol{\gamma}|=1}\Omega_{\boldsymbol{\gamma}}\setminus\cup_{|\boldsymbol{\gamma}|=2}\Omega_{\boldsymbol{\gamma}}$, we obtain similarly that

$$\mathcal{E}_{\Omega,2}[\upsilon_2]\lesssim\rho^{-2}\sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=2}}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2$$
$$+\rho^{-2}\sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=3}}\sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}}d\mu_p-\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}}d\mu_q\right)^2. \tag{5.18}$$

Repeatedly continuing this process, we obtain a sequence of piecewise harmonic functions $\{\upsilon_m\}_{m=1}^{\infty}$ on Ω such that $\upsilon_m = \upsilon_{m-1}$ in $\Omega \setminus \cup_{|\boldsymbol{\gamma}|=m-1}\Omega_{\boldsymbol{\gamma}}$, and consequently,

$$\mathcal{E}_\Omega[u] \le \sum_{m=1}^{\infty} \mathcal{E}_{\Omega,m}[\upsilon_m] \lesssim \sum_{m=1}^{\infty} \rho^{-m} \sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=m}} \sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}} \left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}} d\mu_p - \int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}} d\mu_q\right)^2.$$

This proves "$\lesssim$" of (5.13).

We then prove the "$\gtrsim$" of (5.13). Pick any admissible $\boldsymbol{\gamma}$ with $|\boldsymbol{\gamma}| = 1$. We use $\Phi_{\boldsymbol{\gamma}}^{-1}$ to pull back the restriction of u on $\Omega_{\boldsymbol{\gamma}}$ into Ω_j, where $j = T(\boldsymbol{\gamma})$ and denote this function on Ω_j by $u_{\boldsymbol{\gamma}}$. Pick $p \in \mathcal{P}_j$, and denote $v = G_{\Omega_j}(p,\cdot)$. We apply the Gauss-Green's formula (2.6) for $u_{\boldsymbol{\gamma}}$ and v on $\Omega_j \setminus \mathcal{P}_j$ to obtain

$$\begin{aligned} 0 &= \int_{\Omega_j\setminus\mathcal{P}_j} v\Delta u_{\boldsymbol{\gamma}} - u_{\boldsymbol{\gamma}}\Delta v = \int_{D_j\cup\mathcal{P}_j} v\frac{\partial u_{\boldsymbol{\gamma}}}{\partial\mathbf{n}} - u_{\boldsymbol{\gamma}}\frac{\partial v}{\partial\mathbf{n}} \\ &= \sum_{q\in\mathcal{P}_j}\left(v(q)\frac{\partial u_{\boldsymbol{\gamma}}}{\partial\mathbf{n}}(q) - u_{\boldsymbol{\gamma}}(q)\frac{\partial v}{\partial\mathbf{n}}(q)\right) - \int_{D_j} u_{\boldsymbol{\gamma}}\frac{\partial v}{\partial\mathbf{n}} \\ &= \sum_{q\in\mathcal{P}_j} v(q)\frac{\partial u_{\boldsymbol{\gamma}}}{\partial\mathbf{n}}(q) - u_{\boldsymbol{\gamma}}(p) + \int_{D_j} f\circ\Phi_{\boldsymbol{\gamma}} d\mu_p. \end{aligned}$$

Hence we have

$$\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}} d\mu_p = u_{\boldsymbol{\gamma}}(p) - \sum_{q\in\mathcal{P}_{T(\boldsymbol{\gamma})}} v(q)\frac{\partial u_{\boldsymbol{\gamma}}}{\partial\mathbf{n}}(q), \tag{5.19}$$

and similarly for another admissible $\boldsymbol{\eta}$ such that $\boldsymbol{\eta}\sim\boldsymbol{\gamma}$ and $p' \in \mathcal{P}_{T(\boldsymbol{\eta})}$, we have

$$\int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}} d\mu_{p'} = u_{\boldsymbol{\eta}}(p') - \sum_{q\in\mathcal{P}_{T(\boldsymbol{\eta})}} v'(q)\frac{\partial u_{\boldsymbol{\eta}}}{\partial\mathbf{n}}(q), \tag{5.20}$$

with $v' = G_{\Omega_{T(\boldsymbol{\eta})}}(p',\cdot)$. By subtracting the above two equalities, using that $0 \le v(q) \le R(p, D_{T(\boldsymbol{\gamma})})$(or $0 \le v'(q) \le R(p', D_{T(\boldsymbol{\eta})})$) which is bounded by a universal constant, we obtain

$$\begin{aligned} &\left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}} d\mu_p - \int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}} d\mu_{p'}\right)^2 \\ &\lesssim \sum_{q\in\mathcal{P}_{T(\boldsymbol{\gamma})}}\left|\frac{\partial u_{\boldsymbol{\gamma}}}{\partial\mathbf{n}}(q)\right|^2 + \sum_{q\in\mathcal{P}_{T(\boldsymbol{\eta})}}\left|\frac{\partial u_{\boldsymbol{\eta}}}{\partial\mathbf{n}}(q)\right|^2 + (u_{\boldsymbol{\gamma}}(p) - u_{\boldsymbol{\eta}}(p'))^2 \lesssim \rho\cdot\mathcal{E}_{\Omega,1}[u], \end{aligned} \tag{5.21}$$

where we use (5.12) in the last estimate. Summing up (5.21) with all $\boldsymbol{\gamma}\sim\boldsymbol{\eta}$ in Γ_1^* and all possible p, p', we get

$$\rho^{-1} \sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=1}} \sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}} \left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}} d\mu_p - \int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}} d\mu_q\right)^2 \lesssim \mathcal{E}_{\Omega,1}[u]. \tag{5.22}$$

Similarly we have for $m \ge 2$ that

$$\rho^{-m} \sum_{\substack{\boldsymbol{\gamma}\sim\boldsymbol{\eta},\\ |\boldsymbol{\gamma}|=|\boldsymbol{\eta}|=m}} \sum_{p\in\mathcal{P}_{T(\boldsymbol{\gamma})},\ q\in\mathcal{P}_{T(\boldsymbol{\eta})}} \left(\int_{D_{T(\boldsymbol{\gamma})}} f\circ\Phi_{\boldsymbol{\gamma}} d\mu_p - \int_{D_{T(\boldsymbol{\eta})}} f\circ\Phi_{\boldsymbol{\eta}} d\mu_q\right)^2 \lesssim \mathcal{E}_{\Omega,m}[u]. \tag{5.23}$$

We then sum up the inequalities over m to obtain

$$\sum_{m=1}^{\infty}\rho^{-m}\sum_{\substack{\gamma\sim\eta,\\|\gamma|=|\eta|=m}}\sum_{p\in\mathcal{P}_{T(\gamma)},\ q\in\mathcal{P}_{T(\eta)}}\left(\int_{D_{T(\gamma)}} f\circ\Phi_\gamma d\mu_p-\int_{D_{T(\eta)}} f\circ\Phi_\eta d\mu_q\right)^2\lesssim\mathcal{E}_\Omega[u],\qquad(5.24)$$

proving the "$\gtrsim$" in (5.13).

□

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