BOUNDARY VALUE PROBLEMS FOR A FAMILY OF DOMAINS IN THE SIERPINSKI GASKET

ZIJIAN GUO, RACHEL KOGAN, HUA QIU, AND ROBERT S. STRICHARTZ

Abstract. For a family of domains in the Sierpinski gasket, we study harmonic functions of finite energy, characterizing them in terms of their boundary values, and study their normal derivatives on the boundary. We characterize those domains for which there is an extension operator for functions of finite energy. We give an explicit construction of the Green's function for these domains.

1. Introduction

Consider the domain Ω_x in the Sierpinski Gasket ($\mathcal{SG}$) consisting of all points above the horizontal line L_x at the distance x from the top vertex q_0, for $0 < x \leq 1$.

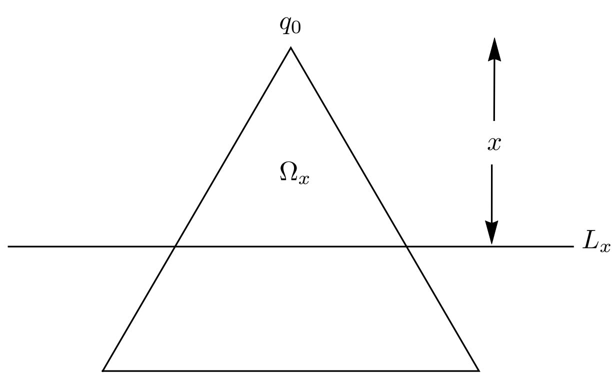

Figure 1.1.

Let $S(x) = \mathcal{SG} \cap L_x$. For x not a dyadic rational, this is a Cantor set. The boundary of Ω_x consists of $S(x)$ together with q_0. By general principles, harmonic functions on Ω_x are determined by their boundary values, where harmonic functions are defined to be solutions of $\triangle h = 0$ on the interior of Ω_x, where $\triangle$ is the Kigami

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Laplacian on $\mathcal{SG}$. The study of such harmonic functions was initiated in [S1], and continued in [OS] for the special case $x = 1$. In this paper, we extend the results in [OS] to the general case. In Section 2, we give an explicit description of the analog of the Poisson kernel to recover the harmonic function from its boundary values, in terms of the Haar series expansion of the boundary values on $S(x)$, and we characterize the boundary values that correspond to harmonic functions of finite energy. In Section 3, we define normal derivatives on the boundary and give a description of the Dirichlet-to-Neumann map as a multiplier transform on the Haar series expansion.

In Section 4, we study the extension problem for functions of finite energy on Ω_x to functions of finite energy on $\mathcal{SG}$. We are able to characterize the values of x for which such extensions are possible. In particular, the value $x = 1$ studied in [OS] does not admit such extensions. This may be regarded as the first of a family of Sobolev extension problems, based on Sobolev spaces on $\mathcal{SG}$ discussed in [S2]. We leave these as open problems for future research. Related problems are studied in [LS] and [LRSU].

In Section 5, we give a construction of a Green's function on Ω_x to solve the Dirichlet problem $-\triangle u = F$ on Ω_x, $u|_{\partial\Omega_x} = 0$ via an integral transform of F. The construction of the Green's function is analogous to Kigami's construction on $\mathcal{SG}$.

The reader is referred to the books [Ki] and [S3] for a description of the theory of the Laplacian on $\mathcal{SG}$, and related fractals. It would be interesting to extend the results of this paper to other domains in $\mathcal{SG}$, and to domains in other fractals. In this regard, we offer the following cautionary tale. Consider the fractal $\mathcal{SG}_3$, defined similarly to $\mathcal{SG}$ but by subdivisions of the sides of triangles into three rather than two pieces(see Figure 1.2).

$\mathcal{SG}_3$

Figure 1.2.

We may consider domains Ω_x defined as before, with the boundary $S(x)$ modeled as a Cantor set with divisions into three pieces. There is a natural analog of Haar functions on $S(x)$, with two generators as shown in Figure 1.3.

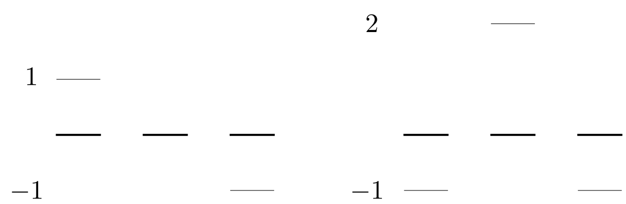

Figure 1.3. Haar generators.

Because the second generator is symmetric rather than skew-symmetric, we cannot glue to zero at the top, so the analog of Lemma 2.3 does not hold. It is not clear how to overcome this difficulty.

2. Harmonic Functions on Ω_x

For $0 < x \leq 1$, there is a unique representation

$$x = \sum_{k=1}^{\infty} 2^{-n_k} \tag{2.1}$$

for a sequence

$$0 < n_1 < n_2 < \cdots \tag{2.2}$$

of increasing positive integers. We will approximate Ω_x by the increasing sequence of domains $\Omega_x^{(m)}$ where each $\Omega_x^{(m)}$ is the closure of $\Omega_{x_{[m]}}$ where

$$x_{[m]} = \sum_{k=1}^{m} 2^{-n_k} \tag{2.3}$$

is the partial sum of (2.1). (Note that (2.3) is not the representation of $x_{[m]}$ of the form (2.1) since it is a finite binary representation.) The domain $\Omega_x^{(m)}$ is a finite union of cells, specifically 1 n_1-cell, 2 n_2-cells, 4 n_3-cells, $\cdots$, 2^{m-1} n_m-cells. Figure 2.1 illustrates $\Omega_x^{(m)}$ for $m = 1, 2, 3$ for two choices of x. The boundary of $\Omega_x^{(m)}$ consists of the top vertex q_0 together with the 2^m bottom vertices of the n_m-cells.

Following [S1] we define

$$Rx = \sum_{k=2}^{\infty} 2^{-n_k} = x - 2^{-n_1} \tag{2.4}$$

and the function $\alpha_0(x)$ by the identity

$$\alpha_0(x) = \frac{1}{1 + 2(\frac{5}{3})^{n_2 - n_1}(1 - \alpha_0(Rx))} \tag{2.5}$$

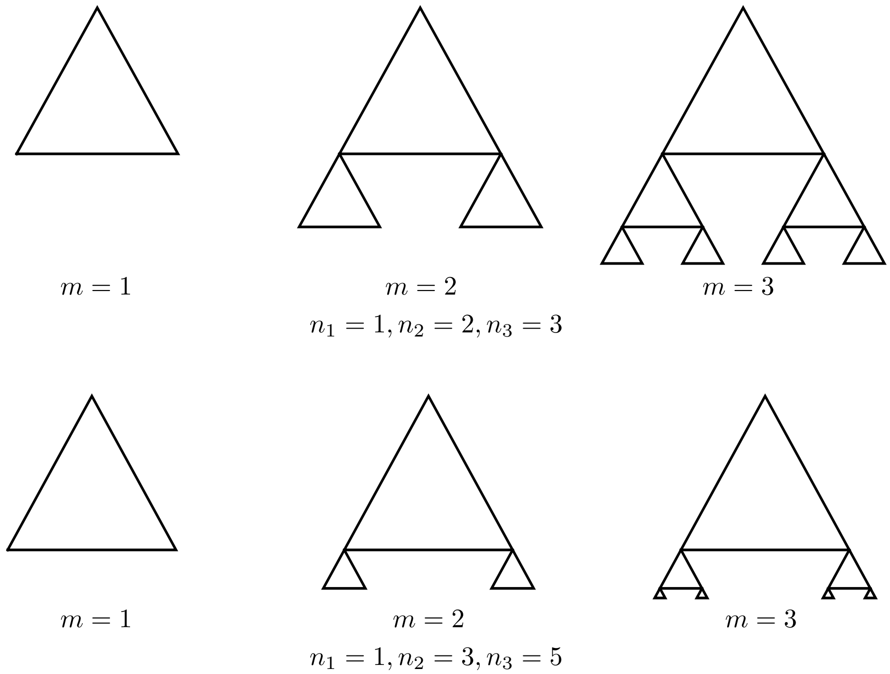

Figure 2.1. Some examples of $\Omega_x^{(m)}$ for $m = 1, 2, 3$.

which is easily solved to obtain a variant of a continued fraction representation

$$\alpha_0(x) = \lim_{k\to\infty} \alpha_0^{(k)}(x) \tag{2.6}$$

for

$$\alpha_0^{(k)} = \cfrac{1}{1 + 2\left(\frac{5}{3}\right)^{n_2-n_1}(1 - \cfrac{1}{1 + 2\left(\frac{5}{3}\right)^{n_3-n_2}(1 - \cfrac{1}{\ddots\, \cfrac{1}{1 + 2\left(\frac{5}{3}\right)^{n_k-n_{k-1}}}}}}. \tag{2.7}$$

See Figure 2.2 for the graph of $\alpha_0(x)$ on $(0, 1]$.

We also define

$$\alpha_1(x) = \frac{1 - \alpha_0(x)^2}{2\alpha_0(x) + 1}, \quad \alpha_2(x) = \frac{\alpha_0(x) - \alpha_0(x)^2}{2\alpha_0(x) + 1}. \tag{2.8}$$

Note that

$$\alpha_0(x) + \alpha_1(x) + \alpha_2(x) = 1. \tag{2.9}$$

These functions enable us to describe harmonic functions in Ω_x. The boundary of Ω_x consists of the top vertex q_0 and $S(x) = L_x \cap \mathcal{SG}$. If x is not a dyadic rational then $S(x)$ is a Cantor set. We will assume this holds. Then a harmonic function is determined by the value $h(q_0)$ and the expansion of $h|_{\mathcal{SG}}$ in a Haar basis.

Definition 2.1. The harmonic function h_0 satisfies

$$h_0(q_0) = 1, \quad h_0|_{S(x)} = 0. \tag{2.10}$$

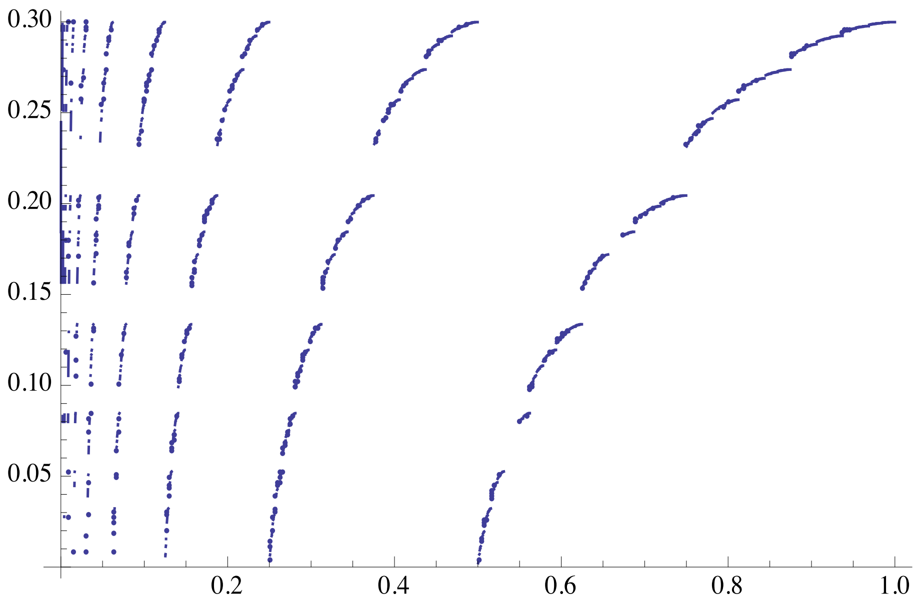

Figure 2.2. The graph of $\alpha_0(x)$.

The harmonic function h_1 satisfies

$$h_1(q_0) = 0, \quad h_1|_{S(x)\cap F_0^{n_1-1}F_1(\mathcal{SG})} = 1, \quad h_1|_{S(x)\cap F_0^{n_1-1}F_2(\mathcal{SG})} = -1. \tag{2.11}$$

We write h_0^x and h_1^x when we need to explicitly show the dependence on x.

Note that $1 - h_0$ satisfies

$$(1-h_0)(q_0) = 0, \quad (1-h_0)|_{S(x)} = 1, \tag{2.12}$$

so that $1 - h_0$ and h_1 vanish at q_0 and give the first two Haar functions when restricted to $S(x)$. Also it is shown in [S1] that

$$h_0(F_0^{n_1-1}F_1q_0) = h_0(F_0^{n_1-1}F_2q_0) = \alpha_0(x) \tag{2.13}$$

and

$$h_1(F_0^{n_1-1}F_1q_0) = -h_1(F_0^{n_1-1}F_2q_0) = \alpha_1(x) - \alpha_2(x). \tag{2.14}$$

Lemma 2.2. *Let $y = 2^{n_1}Rx$. Then*

$$h_0^x \circ (F_0^{n_1-1}F_1) = h_0^x \circ (F_0^{n_1-1}F_2) = \alpha_0(x)h_0^y \tag{2.15}$$

and

$$h_1^x \circ (F_0^{n_1-1}F_1) = -h_1^x(F_0^{n_1-1}F_2) = 1 + (\alpha_1(x) - \alpha_2(x) - 1)h_0^y. \tag{2.16}$$

Proof. The function $\alpha_0(x)h_0^y$ is a harmonic function on Ω_y with boundary values $\alpha_0(x)$ at q_0 and zero on $S(y)$. Note that $F_0^{n_1-1}F_1(S(y)) = S(x)$, so $h_0^x \circ (F_0^{n_1-1}F_1)$ is also a harmonic function on Ω_y vanishing on $S(y)$, and it assume the value $\alpha_0(x)$ at q_0 by (2.13). Thus (2.15) holds. A similar argument shows that (2.14) implies (2.16).□

Next we consider the general Haar basis functions on $L^2(S(x))$. Let $\omega = (\omega_1, \cdots, \omega_m)$ be a word of length $|\omega| = m$, with each $\omega_j = 1$ or 2. Then

$$S_\omega(x) = S(x) \cap F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \cdots F_0^{n_m-n_{m-1}-1} F_{\omega_m}(\mathcal{SG}) \tag{2.17}$$

describe the dyadic pieces of $S(x)$. In particular,

$$S(x) = \bigcup_{|\omega|=m} S_\omega(x). \tag{2.18}$$

The Cantor measure μ on $S(x)$ assigns measure 2^{-m} to each piece $S_\omega(x)$. The Haar function ψ_ω is supported on $S_\omega(x)$ and satisfies

$$\psi_\omega|_{S_{\omega 1}(x)} = 2^{m/2} \text{ and } \psi_\omega|_{S_{\omega 2}(x)} = -2^{m/2}. \tag{2.19}$$

Then $1 \cup \{\psi_\omega\}$ is an orthonormal basis for $L^2(S(x), d\mu)$. We define h_ω^x to be the harmonic function on Ω_x with boundary values $h_\omega^x(q_0) = 0$ and $h_\omega^x|_{S(x)} = \psi_\omega$.

Lemma 2.3. *Let $y_m = 2^{n_m} R^m x$. Then h_ω^x is supported in*

$$\Omega_x \cap F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \cdots F_0^{n_m-n_{m-1}-1} F_{\omega_m}(\mathcal{SG})$$

and

$$h_\omega^x \circ (F_0^{n_1-1} F_{\omega_1} F_0^{n_2-n_1-1} F_{\omega_2} \cdots F_0^{n_m-n_{m-1}-1} F_{\omega_m}) = 2^{m/2} h_1^{y_m}. \tag{2.20}$$

Proof. The key observation is that, because of skew-symmetry, the function h_1 not only vanishes at q_0 but also has normal derivative vanishing at q_0. Thus we may glue the function defined by (2.20) to zero outside this cell and still have a harmonic function. This function clearly has the required boundary values for h_ω^x.□

Theorem 2.4. *The energies are given by*

$$\mathcal{E}(h_0^x) = (1 - \alpha_0(x))^2 \sum_{j=1}^{\infty} 2^{2-j} \left(\frac{5}{3}\right)^{2n_1 - n_j}, \tag{2.21}$$

$$\mathcal{E}(h_1^x) = 6 \left(\frac{1 - \alpha_0(x)}{2\alpha_0(x) + 1}\right)^2 \left(\frac{5}{3}\right)^{n_1} + 2 \left(\frac{3\alpha_0(x)}{2\alpha_0(x) + 1}\right)^2 \left(\frac{5}{3}\right)^{n_1} \mathcal{E}(h_0^y), \tag{2.22}$$

and

$$\mathcal{E}(h_\omega^x) = 2^m \left(\frac{5}{3}\right)^{n_m} \mathcal{E}(h_1^{y_m}) \tag{2.23}$$

where $m = |\omega|$. Moreover, there exist positive constants C_1 and C_2, independent of x, such that

$$C_1 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \leq \mathcal{E}(h_\omega^x) \leq C_2 2^m \left(\frac{5}{3}\right)^{n_{m+1}}. \tag{2.24}$$

Proof. We compute the energy of h_0^x on the top cell $F_0^{n_1}(\mathcal{SG})$ using (2.13) to be $\left(\frac{5}{3}\right)^{n_1} 2(\alpha_0(x) - 1)^2$, since there are two edges where the difference of h_0^x is $\alpha_0(x) - 1$. On the remaining cells $F_0^{n_1-1} F_1(\mathcal{SG})$ and $F_0^{n_1-1} F_2(\mathcal{SG})$ the function h_0^x is equal to $\alpha_0(x) h_0^y \circ (F_0^{n_1-1} F_1)^{-1}$ and $\alpha_0(x) h_0^y \circ (F_0^{n_1-1} F_2)^{-1}$ by (2.15). These each have energy $\alpha_0(x)^2 \left(\frac{5}{3}\right)^{n_1} \mathcal{E}(h_0^y)$, so

$$\mathcal{E}(h_0^x) = 2 \left(\frac{5}{3}\right)^{n_1} ((\alpha_0(x) - 1)^2 + \alpha_0(x)^2 \mathcal{E}(h_0^y)). \tag{2.25}$$

Before iterating this identity we observe that

$$(1-\alpha_0(y))\alpha_0(x) = \frac{1}{2}\left(\frac{5}{3}\right)^{n_1-n_2}(1-\alpha_0(x)). \tag{2.26}$$

This follows from (2.5) and the observation, from (2.7), that $\alpha_0(x)$ depends only on the sequence of differences $n_k - n_{k-1}$ and therefore $\alpha_0(y) = \alpha_0(2^{n_1}Rx) = \alpha_0(Rx)$. Thus

$$\mathcal{E}(h_0^x) = (1-\alpha_0(x))^2\left(2\left(\frac{5}{3}\right)^{n_1} + \left(\frac{5}{3}\right)^{2n_1-n_2}\right) + 4\left(\frac{5}{3}\right)^{n_2}\alpha_0(x)^2\alpha_0(y)^2\mathcal{E}(h_0^{y_2})$$

and by iterating we obtain (2.21).

Similarly, we use (2.14) to compute the energy of h_1^x on the top cell $F_0^{n_1}(\mathcal{SG})$ to be $\left(\frac{5}{3}\right)^{n_1} 6(\alpha_2(x)-\alpha_1(x))^2 = \left(\frac{5}{3}\right)^{n_1} 6\left(\frac{1-\alpha_0(x)}{2\alpha_0(x)+1}\right)^2$ by (2.8). Then by using (2.16) we compute the energy in each of the other cells to be $\left(\frac{5}{3}\right)^{n_1}(\alpha_1(x)-\alpha_2(x)-1)^2\mathcal{E}(h_0^y)$ $= \left(\frac{5}{3}\right)^{n_1}\left(\frac{3\alpha_0(x)}{2\alpha_0(x)+1}\right)^2\mathcal{E}(h_0^y)$, and by adding we obtain (2.22). Then (2.23) follows by Lemma 2.3.

To obtain the estimate (2.24) we observe that since $0 \le \alpha_0(x) \le \frac{3}{10}$ it follows from (2.7) that $\alpha_0(x)$ is bounded above and below by multiples of $\left(\frac{5}{3}\right)^{n_1-n_2}$. It follows from (2.21) that $\mathcal{E}(h_0^x)$ is bounded above and below by multiples of $\left(\frac{5}{3}\right)^{n_1}$ since the infinite series is dominated by its first term. We get the same estimate for $\mathcal{E}(h_1^x)$ using (2.22) since the second summand is bounded by a multiple of $\left(\frac{5}{3}\right)^{2(n_1-n_2)}\left(\frac{5}{3}\right)^{n_1}\left(\frac{5}{3}\right)^{n_2-n_1}$. Then (2.24) follows from this estimate and (2.23). $\square$

Corollary 2.5. *Let h be the harmonic function on Ω_x with boundary values $h(q_0) = a$ and $h|_{S(x)} = f$, where*

$$f = b + \sum_\omega c_\omega \psi_\omega \tag{2.27}$$

for

$$c_\omega = \int_{S(x)} f\psi_\omega d\mu. \tag{2.28}$$

Then $\mathcal{E}(h)$ is bounded above and below by multiples of

$$\left(\frac{5}{3}\right)^{n_1}(a-b)^2 + \sum_{m=0}^{\infty}\sum_{|\omega|=m} 2^m\left(\frac{5}{3}\right)^{n_{m+1}}|c_\omega|^2. \tag{2.29}$$

In particular, h has finite energy if and only if (2.29) *is finite.*

Proof. By subtracting a constant we may assume without loss of generality that $a = 0$ (this does not change c_ω). Then from (2.27) we have

$$h = b(1-h_0) + \sum_{m=0}^{\infty}\sum_{|\omega|=m} c_\omega h_\omega, \tag{2.30}$$

and the functions $h_0 \cup \{h_\omega\}$ are orthogonal in energy by symmetry considerations. Thus

$$\mathcal{E}(h) = b^2\mathcal{E}(1-h_0) + \sum_{m=0}^{\infty}\sum_{|\omega|=m}|c_\omega|^2\mathcal{E}(h_\omega) \tag{2.31}$$

and the result follows by the estimates (2.24). □

We are also interested in the corresponding result for the L^2 norm of h. Using similar reasoning we can show that $\| h \|_2^2$ is bounded above and below by multiples of

$$\left(\frac{1}{3}\right)^{n_1}(a^2+b^2)+\sum_{m=0}^{\infty}\sum_{|\omega|=m}2^m\left(\frac{1}{3}\right)^{n_{m+1}}|c_\omega|^2. \tag{2.32}$$

Of course this allows the coefficients to grow so that $\sum_\omega |c_\omega|^2$ is infinite, meaning that the boundary values f on $S(x)$ may not be in $L^2(S(x))$.

3. Normal Deriatives

We follow the general outline from [OS] to define a normal derivative on $S(x)$. We define

$$\partial_n u|_{S(x)} = \lim_{m\to\infty} 2^m \sum_{|\omega|=m} (-\partial_n u(\tilde{F}_\omega q_0))\chi_{S(x)\cap \tilde{F}_\omega(\mathcal{SG})} \tag{3.1}$$

if the limit exists, where

$$\tilde{F}_\omega = F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2}\cdots F_0^{n_m-n_{m-1}-1}F_{\omega_m}. \tag{3.2}$$

The cells $\tilde{F}_\omega(\mathcal{SG})$ for $|\omega| = m$ cover $S(x)$, and $\tilde{F}_\omega q_0$ is the top vertex. Since $\partial_n u(\tilde{F}_\omega q_0)$ is outer directed, upward, we insert the minus sign to get an outer directed normal across $S(x)$.

Lemma 3.1. *$\partial_n h_0^x$ is the constant function on $S(x)$ with value $-2\left(\frac{5}{3}\right)^{n_1}(1-\alpha_0(x))$.*

Proof. We compute $\partial_n h_0^x(q_0) = 2\left(\frac{5}{3}\right)^{n_1}(1-\alpha_0(x))$ from the cell $F_0^{n_1}(\mathcal{SG})$. Next consider the cell $F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1}(\mathcal{SG})$. The top vertex is $F_0^{n_1-1}F_{\omega_1}q_0$, and by symmetry (on the cell $F_0^{n_1}(\mathcal{SG})$), $\partial_n h_0^x(F_0^{n_1-1}F_{\omega_1}q_0) = \frac{1}{2}\partial_n h_0^x(q_0)$ for $\omega_1 = 1, 2$. Thus $2\sum_{|\omega|=1}(-\partial_n h_0^x(\tilde{F}_\omega q_0))\chi_{S(x)\cap\tilde{F}_\omega(\mathcal{SG})} = -\partial_n h_0^x(q_0)\chi_{S(x)}$. By similar reasoning there is no change on the right side of (3.1) as m increases. □

Lemma 3.2. $\partial_n h_\omega^x = 6\cdot 2^m\left(\frac{5}{3}\right)^{n_{m+1}}\left(\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1}\right)\psi_\omega$.

Proof. On the cell $F_0^{n_1}(\mathcal{SG})$ we compute (using (2.14))

$$\partial_n h_1^x(F_0^{n_1-1}F_1q_0) = -\partial_n h_1^x(F_0^{n_1-1}F_2q_0) = 3\left(\frac{5}{3}\right)^{n_1}(\alpha_1(x)-\alpha_2(x)),$$

so we have

$$\sum_{|\omega|=1}2(-\partial_n h_1^x(\tilde{F}_\omega q_0))\chi_{S(x)\cap\tilde{F}_\omega(\mathcal{SG})} = 6\left(\frac{5}{3}\right)^{n_1}(\alpha_1(x)-\alpha_2(x))\psi_\emptyset,$$

and by the same reasoning as in Lemma 3.1, this does not change if we instead sum over $|\omega| = m$ for any $m \geq 2$. So this gives the correct result for $\omega = \emptyset$. We then use Lemma 2.3 to scale the result for general ω. □

Theorem 3.3. *Suppose h and f are given as in Corollary 2.5. Then $\partial_n h$ is given by*

$$2(b-a)\left(\frac{5}{3}\right)^{n_1}(1-\alpha_0(x))+\sum_{m=0}^{\infty}\sum_{|\omega|=m}6\cdot 2^m\left(\frac{5}{3}\right)^{n_{m+1}}\left(\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1}\right)c_\omega\psi_\omega \tag{3.3}$$

provided the series converges. In other words, the Dirichlet-to-Neumann map $f \to \partial_n h$ *is a Haar series multiplier map with multiplier* $6 \cdot 2^m \left(\frac{5}{3}\right)^{n_{m+1}} \left(\frac{1-\alpha_0(y_m)}{2\alpha_0(y_m)+1}\right)$.

Corollary 3.4. *Suppose f satisfies*

$$\sum_{m=0}^{\infty} \sum_{|\omega|=m} 2^{2m} \left(\frac{5}{3}\right)^{2n_{m+1}} |c_\omega|^2 < \infty. \tag{3.4}$$

Then $\partial_n h$ is well-defined in $L^2(S(x))$ and $\| \partial_n h \|_2^2$ is bounded above and below by a multiple of (3.4).

Proof. The Theorem follows from Lemma 3.2, and the Corollary follows from the fact that $\frac{1-\alpha_0(x)}{2\alpha_0(x)+1}$ is uniformly bounded above and below independent of x. □

Note that the finiteness of (3.4) is a stronger condition than the finiteness of (2.29), so harmonic functions of finite energy do not necessarily satisfy (3.4), but functions h satisfying the conditions of Corollary 3.4 automatically have finite energy.

Corollary 3.5. *Suppose h is as in Corollary 2.5 with coefficients that satisfy* (3.4)*, and v is any function of finite energy of Ω_x, then the following Gauss-Green formula holds:*

$$\mathcal{E}(h, v) = v(q_0)\partial_n h(q_0) + \int_{S(x)} v \partial_n h d\mu. \tag{3.5}$$

Proof. v is continuous since v is of finite energy, hence has a well-defined restriction to $S(x)$ that is bounded and thus in $L^2(\mu)$. Apply the standard Gauss-Green formula on the domain $\bigcup_{|\omega|\le m} \tilde{F}_\omega(\mathcal{SG})$ and take the limit as $m \to \infty$. □

4. Extending Functions of Finite Energy

In this section we will write Ω_x^+ for the region above $L(x)$ that was previously denoted Ω_x, and Ω_x^- for the region below $L(x)$. Under the assumption that x is not a dyadic rational, $S(x)$ is the common boundary of Ω_x^+ and Ω_x^-. For functions $u^\pm$ defined on $\Omega_x^\pm$, we use $\mathcal{E}_{\Omega_x^\pm}(u^\pm)$ to denote the energies of $u^\pm$, which are naturally defined by taking the graph energy sum with edges restricted to lie in $\Omega_x^\pm$, and then computing the usual renormalized limit. Let $dom\mathcal{E}_{\Omega_x^\pm}$ to denote the collections of functions of finite energy on $\Omega_x^\pm$, respectively.

The first issue that we address is under what conditions can we glue together functions $u^\pm$ of finite energy on $\Omega_x^\pm$ to obtain a function of finite energy on $\mathcal{SG}$. Since functions of finite energy are continuous, $u^\pm$ must have boundary values on $S(x)$ that agree. It turns out that this is the only condition that we need to impose. This is not surprising since the same is true for gluing functions of finite energy on domains that intersect at a finite set of points.

Theorem 4.1. *Let $u^\pm \in dom\mathcal{E}_{\Omega_x^\pm}$, and suppose*

$$u^+|_{S(x)} = u^-|_{S(x)}, \tag{4.1}$$

the values being defined by continuity. Then

$$u = \begin{cases} u^+ \ \text{on } \overline{\Omega}_x^+, \\ u^- \ \text{on } \overline{\Omega}_x^-, \end{cases} \tag{4.2}$$

belongs to dom$\mathcal{E}$ in $\mathcal{SG}$ and

$$\mathcal{E}(u) = \mathcal{E}_{\Omega_x^+}(u^+) + \mathcal{E}_{\Omega_x^-}(u^-). \tag{4.3}$$

Proof. Let S_m denote the strip of 2^m cells of order n_m containing $S(x)$, and let $B_m^\pm$ denote the unions of the cells of order n_m contained in $\Omega_x^\pm$. Then

$$\mathcal{E}^{(n_m)}(u) = \mathcal{E}^{(n_m)}_{B_m^+}(u) + \mathcal{E}^{(n_m)}_{B_m^-}(u) + \mathcal{E}^{(n_m)}_{S_m}(u).$$

Since $\mathcal{E}^{(n_m)}_{B_m^\pm}(u) \to \mathcal{E}_{\Omega_x^\pm}(u^\pm)$ as $m \to \infty$, it suffices to show

$$\mathcal{E}^{(n_m)}_{S_m}(u) \to 0. \tag{4.4}$$

Let C denote one of the n_m-cells in S_m with boundary points $x_m \in \Omega_x^+$ and $y_m, z_m \in \Omega_x^-$. We need to estimate

$$\left(\frac{5}{3}\right)^{n_m} [(u^+(x_m) - u^-(y_m))^2 + (u^+(x_m) - u^-(z_m))^2 + (u^-(y_m) - u^-(z_m))^2]. \tag{4.5}$$

It suffices to estimate the first two terms in (4.5) since $u^-(y_m) - u^-(z_m) = (u^+(x_m) - u^-(z_m)) - (u^+(x_m) - u^-(y_m))$, and by symmetry it suffices to estimate the first term. Let $S_m^\pm$ be the portion of S_m above or below $S(x)$. There will be an infinite sequence of points $\{x_m, x_{m+1}, \cdots\}$ in S_m^+ and $\{y_m, y_{m+1}, \cdots\}$ in S_m^-, both converging to the same point $p \in S(x)$. Since $u^+(p) = u^-(p)$ by (4.1), we may write

$$u^+(x_m) - u^-(y_m) = \sum_{j=m}^{\infty} (u^+(x_j) - u^+(x_{j+1})) - \sum_{j=m}^{\infty} (u^-(y_j) - u^-(y_{j+1})). \tag{4.6}$$

Now each pair (x_j, x_{j+1}) are vertices of a cell C_j of order n_{j+1} in Ω_x^+. Note that all these cells are essentially disjoint, and $C = \bigcup_j C_j$.

So we have the estimate

$$|u^+(x_j) - u^+(x_{j+1})| \leq \left(\frac{3}{5}\right)^{n_{j+1}/2} \mathcal{E}_{C_j}(u^+)^{1/2}. \tag{4.7}$$

By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \sum_{j=m}^{\infty} |u^+(x_j) - u^+(x_{j+1})| &\leq \left(\sum_{j=m}^{\infty} \left(\frac{3}{5}\right)^{n_{j+1}}\right)^{1/2} \left(\sum_{j=m}^{\infty} \mathcal{E}_{C_j}(u^+)\right)^{1/2} \\ &\leq c \left(\frac{3}{5}\right)^{n_m/2} \mathcal{E}_{C \cap S_m^+}(u^+)^{1/2}. \end{aligned} \tag{4.8}$$

By similar reasoning we obtain the same estimate with $|u^-(y_j) - u^-(y_{j+1})|$, so by (4.6) we have

$$\left(\frac{5}{3}\right)^{n_m} |u^+(x_m) - u^-(y_m)|^2 \leq c\mathcal{E}_{C \cap S_m^+}(u^+) + c\mathcal{E}_{C \cap S_m^-}(u^-). \tag{4.9}$$

Summing (4.9) over all the 2^m cells C yields

$$\mathcal{E}^{(n_m)}_{S_m}(u) \leq c\mathcal{E}_{S_m^+}(u^+) + c\mathcal{E}_{S_m^-}(u^-) \tag{4.10}$$

and $\mathcal{E}_{S_m^\pm}(u^\pm) \to 0$ because $\bigcap_m S_m^\pm = S(x)$ and $S(x)$ has measure zero in the Kusuoka measure, (this follows easily from Theorem 5.1 of ([AHS]).□

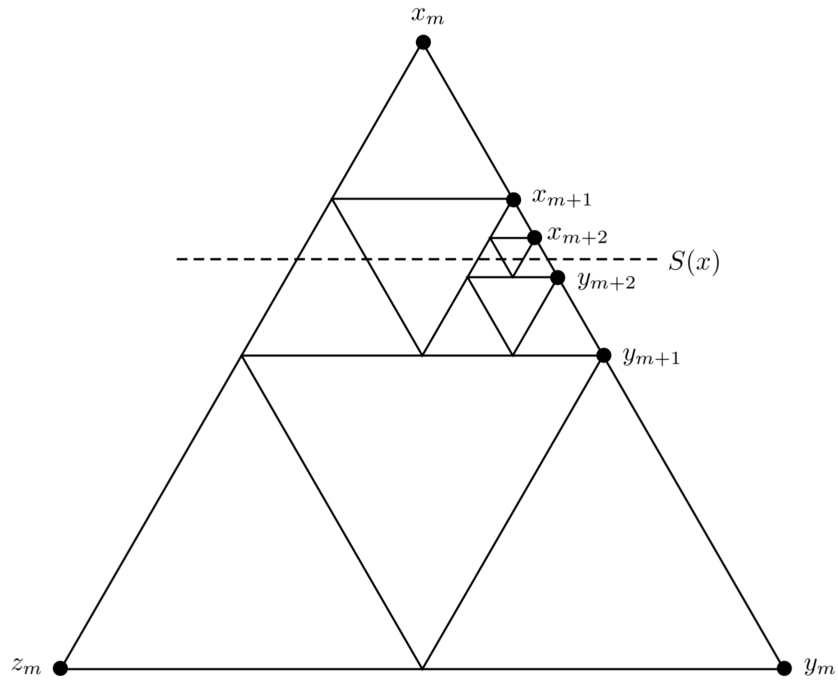

Figure 4.1.

It is easy to characterize the restrictions to $S(x)$ of functions of finite energy on Ω_x^+.

Theorem 4.2. *A function f on $S(x)$ is the restriction to $S(x)$ of a function u^+ of finite energy on Ω_x^+ if and only if f has a Haar series expansion* (2.27) *with* (2.29) *finite (here a=0), and* (2.29) *is bounded by a multiple of $\mathcal{E}_{\Omega_x^+}(u^+)$.*

Proof. Let h be the harmonic function on Ω_x^+ with the same boundary values f. Since harmonic functions minimize energy, $\mathcal{E}_{\Omega_x^+}(h) \leq \mathcal{E}_{\Omega_x^+}(u^+)$, and the result follows from Corollary 2.5. □

However, there is no such simple result for Ω_x^-. We pose the following extension problem.

Problem 4.3. *Does there exist a bounded linear extension operator (meaning $Tu|_{\Omega_x^+} = u$) $T : dom_{\Omega_x^+}(\mathcal{E}) \to dom_{\mathcal{SG}}(\mathcal{E})$?*

There is a simple obstruction to solving this problem.

Definition 4.4. x satisfies the *nonconsecutive condition with bound N* if there are no N consecutive integers in the sequence $\{n_m\}$. If there is some N for which this holds then x is said to satisfy the *nonconsecutive condition.*

Note that a generic value of x will not satisfy this condition. However there are uncountably many (of Hausdorff dimension 1) values of x that do satisfy the condition. Perhaps the simplest choice has $n_m = 2m - 1$, with $N = 2$.

Theorem 4.5. *Let E denote the collection of x satisfying the nonconsecutive condition. Then the Hausdorff dimension of E is* 1.

Proof. Let E_N denote the collection of x satisfying the nonconsecutive condition with bound N. Then $E = \bigcup_{N\geq 2} E_N$ and

$$E_2 \subset E_3 \subset \cdots \subset E_N \subset \cdots . \tag{4.11}$$

We will first prove that the Hausdorff dimension of E_N is the unique positive root of the equation

$$2 - 2^s - 2^{-Ns} = 0. \tag{4.12}$$

Consider the set E_N. We divide it into the disjoint union $E_N = \bigcup_{k\geq 1} E_{N,k}$ where $E_{N,k}$ is the set of x in E_N whose n_1-digit is k. Obviously, for each k, $E_{N,k}$ is a similar copy of $E_{N,1}$ with contraction ratio 2^{1-k}. Since the Hausdorff dimension is stable for countable unions, we just need to compute the dimension of $E_{N,1}$. For this set, by the nonconsecutive condition, we can write

$$E_{N,1} = \left(\bigcup_{j\geq 3}(2^{-1} + E_{N,j})\right)\cup\cdots\cup\left(\bigcup_{j\geq N+1}(2^{-1} + \cdots + 2^{-(N-1)} + E_{N,j})\right). \tag{4.13}$$

Since $|E_{N,j}| \leq 1/2^j$, it is easy to check that the above union is disjoint. Moreover, (4.13) is essentially a self-similar identity for the set $E_{N,1}$ with contraction ratios,

$$2^{-2}, 2^{-3}, \cdots ; 2^{-3}, 2^{-4}, \cdots ; 2^{-N}, 2^{-(N+1)}, \cdots ,$$

satisfying the open set condition (with the open set $(2^{-1}, 1)$). (See [M] for the theory of infinitely generated self-similar sets.) Hence the Hausdorff dimension of $E_{N,1}$ is the solution of the equation

$$1 = \sum_{k=2}^{N}\sum_{j\geq k}(2^{-s})^j = \sum_{k=2}^{N}\frac{(2^{-s})^k}{1-2^{-s}} = \frac{2^{-2s} - 2^{-s(N+1)}}{(1-2^{-s})^2}, \tag{4.14}$$

which simplifies to (4.12). So we get the Hausdorff dimension of E_N.

Using (4.11), an easy calculation will show that the Hausdorff dimension of E is 1.□

If x fails to satisfy the nonconsecutive condition, then there are pairs of points in Ω_x^+ that are much closer to each other in $\mathcal{SG}$ than in Ω_x^+. For example, if $n_j = j$ for $j \leq N$ then the points $F_1F_2^{N-1}q_0$ and $F_2F_1^{N-1}q_0$ in Ω_x^+ are distance on the order of $\left(\frac{3}{5}\right)^N$ apart in the resistance metric on $\mathcal{SG}$, but are far apart in Ω_x^+. Note that $h_1^x(F_1F_2^{N-1}q_0) - h_1^x(F_2F_1^{N-1}q_0) = 2h_1^x(F_1F_0^{N-1}q_0)$ and $\mathcal{E}(h_1^x)$ is bounded. The estimate analogous to (4.7) shows

$$c \leq \left(\frac{3}{5}\right)^{N/2}\mathcal{E}(u)^{1/2}$$

for any extension u of h_1^x to $\mathcal{SG}$, hence $\mathcal{E}(u) \geq c\left(\frac{5}{3}\right)^N$. This means that the bound on the operator T, if it exists, would be bounded below by a multiple of $\left(\frac{5}{3}\right)^{N/2}$.

The same reasoning applies locally if $\{n_m\}$ has a consecutive string of N integers. Thus if such strings exist for all N then T cannot be bounded. On the other hand it is easy to see that if the nonconsecutive condition holds for x then distances in Ω_x^+ and $\mathcal{SG}$ are comparable. Note that this is very reminiscent of the type of condition that appears in the work of Peter Jones in the Sobolev extension problem in domains in Euclidean space ([J], [R]).

Theorem 4.6. *The extension Problem 4.3 has a positive solution if and only if x satisfies the nonconsecutive condition, in which case the bound on T is $O(\left(\frac{10}{3}\right)^{N/2})$.*

Proof. We need to construct an extension operator T under the assumption that x satisfies the nonconsecutive condition. In view of our previous results, it suffices

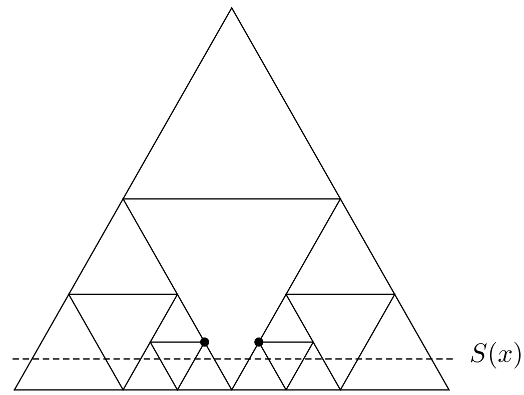

Figure 4.2.

to solve the extension problem for the functions h_ω (and also $1-h_0$), say $Th_\omega = \tilde{h}_\omega$ where the functions $\tilde{h}_\omega$ are orthogonal in energy and

$$\mathcal{E}(\tilde{h}_\omega) \leq C(N)2^m \left(\frac{5}{3}\right)^{n_{m+1}}. \tag{4.15}$$

Suppose first that $N = 2$. Consider first $1 - h_0$ and $h_\emptyset$. Assume for simplicity that $n_1 = 1$. Then $n_2 \geq 3$. Then $S(x)$ passes through the cells $F_1F_0(\mathcal{SG})$ and $F_2F_0(\mathcal{SG})$. We will extend $1 - h_0$ to be identically 1 on the bottom portions of these cells, and $h_\emptyset$ to be 1 on $F_1F_0(\mathcal{SG})$ and -1 on $F_2F_0(\mathcal{SG})$. On the remaining four cells of level 2 we make the extension harmonic with boundary values 0 on the bottom vertices (see Figure 4.3).

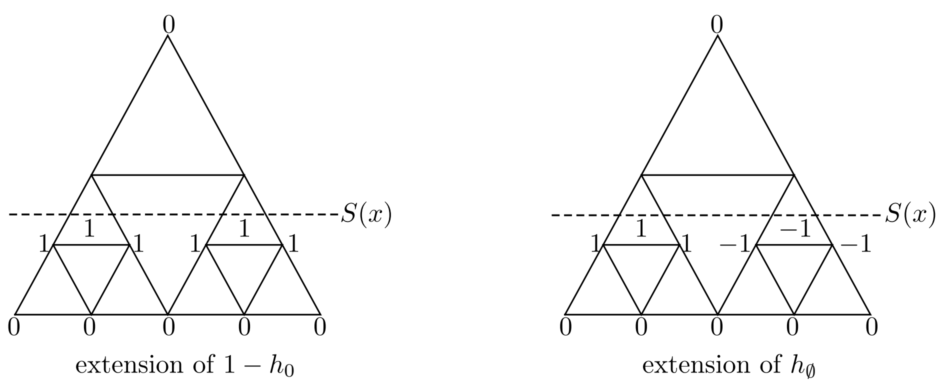

Figure 4.3.

Note that the added energy of these extensions is exactly $8\left(\frac{5}{3}\right)^2$. Also, since one extension is symmetric and one is skew-symmetric with respect to the vertical reflection, they are orthogonal in energy. If $n_1 > 1$ we may repeat the same process on $F_0^{n_1-1}(\mathcal{SG})$ and then continue the extension to be identically zero on the complement of $F_0^{n_1-1}(\mathcal{SG})$. The added energy is exactly $8\left(\frac{5}{3}\right)^{n_1+1}$, but the energy of the original functions was also a multiple of $\left(\frac{5}{3}\right)^{n_1}$, so this is consistent with (4.15) with $m = 0$ and gives a uniform bound on the extension operator.

For the extension of h_ω we just have to repeat the same procedure miniaturized. If $|\omega| = m$ then h_ω is supported on a cell of order $n_{m+1} - 1$ and since $n_{m+2} \geq n_{m+1} + 2$ the right side of Figure 4.3 describes h_ω and its extension (except for a factor of $2^{m/2}$) to that cell, and then we may glue this to zero in the complement of the cell. Thus we get an extension with the same energy bound. For words ω with $|\omega| = m$, the extended function have disjoint support, so the energies are orthogonal. Comparing extensions for words of different length with overlapping support, we again have a symmetry/skew-symmetry dichotomy with respect to the local reflection in the vertical axis of the smaller supporting cell (this is the overlap of the supports) and so we again have energy orthogonality. This completes the proof for $N = 2$.

For general N the argument is simlar. In Figure 4.4 we show the extension of $h_\emptyset$ when $N = 3$ and $n_1 = 1, n_2 = 2, n_3 \geq 4$.

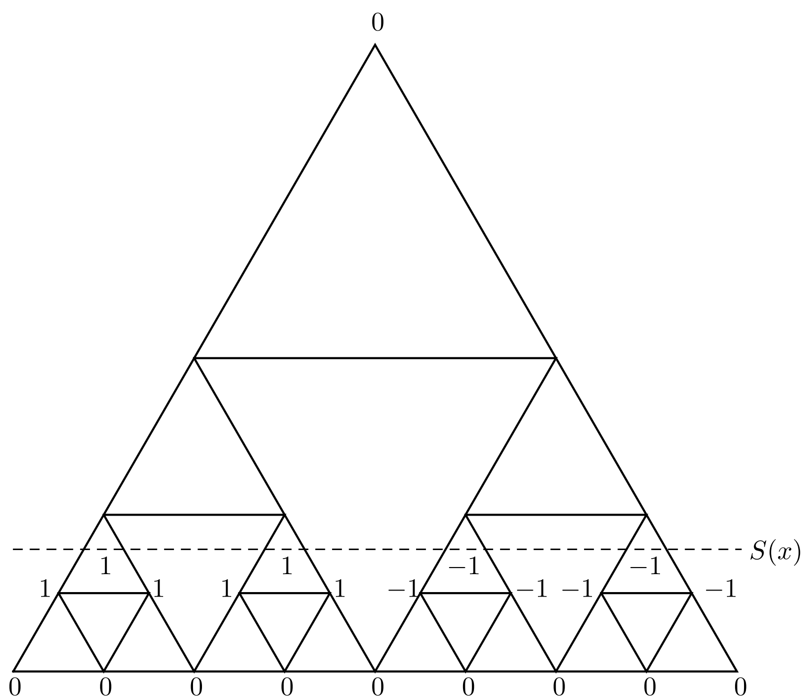

Figure 4.4.

Here we have 2^N cells of order N contributing to the energy, and this multiplies the energy by $O(\left(\frac{10}{3}\right)^N)$. Since the norm of the extension is measured in terms of the square root of the energy, we obtain the $O(\left(\frac{10}{3}\right)^{N/2})$ bound. $\square$

The optimal extension operator would produce functions that are harmonic on Ω_x^-. In particular, it would be interesting to have an explicit description of the functions h_ω^- that are harmonic on Ω_x^- and are equal to ψ_ω on $S(x)$, again under the nonconsecutive condition.

We may regard Theorem 4.2 as a trace theorem and Theorem 4.6 as an extension theorem for $dom\mathcal{E}$ regarded as a Sobolev space, and then we should ask if there are analogous results for other Sobolev spaces. In [S2] the spaces $dom_{L^2}(\Delta^k)$ on $\mathcal{SG}$ are considered as Sobolev spaces ($dom_{L^2}(\Delta^k) = \{u \in L^2(\mathcal{SG}) : \Delta^j u \in L^2(\mathcal{SG})$ for all $j \leq k\}$). Similarly for the space $\{u \in dom_{L^2}(\Delta^k) : \mathcal{E}(\Delta^k u) < \infty\}$. These spaces are easily characterized in terms of expansions in eigenfunctions of the Laplacian. A complete theory of the eigenspaces of the Laplacian on Ω_1 is given in [Q].

Problem 4.7. *For each of these Sobolev spaces, characterize the space of traces on $S(x)$ and restrictions to Ω_x^+, for x satisfying the nonconsecutive condition.*

It seems plausible that the trace problem may have a solution with a condition similar to (2.29) for the Haar expansion (2.27) with different multiples of $|c_\omega|^2$ depending on the Sobolev space. The restriction problem is likely to be more challenging. It is clear that restrictions of functions in $dom_{L^2}(\Delta^k)$ must satisfy $\Delta^j u \in L^2(\Omega_x^+)$ for $j \leq k$, but that is not sufficient because all harmonic functions automatically have $\Delta^j u = 0$. It would seem that the characterization of restriction Sobolev spaces would also have to involve conditions on traces on $S(x)$. Related problems are discussed in [LS] and [LRSU].

5. Green's Function

For a given k, let V_k denote the set of vertices on the k-level graph approximation of $\mathcal{SG}$. For a point $z \in V_k \setminus V_0$, let ϕ_z^k denote the piecewise harmonic spline of level k satisfying $\phi_z^k(t) = \delta_{zt}$ for $t \in V_k$ and extended harmonically on $\mathcal{SG}$. Notice that $\phi_z^k \in dom_0\mathcal{E}$ because $z \notin V_0$, and it is supported in the two k-cells meeting at z. Recall that in the standard theory (see the books [Ki] and [S3]), the Green's function $G(s,t)$ to solve the Dirichlet problem $-\Delta u = F$ on $\mathcal{SG}$, subject to the boundary condition $u|_{V_0} = 0$ via an integral transform $\int_{\mathcal{SG}} G(s,t)F(t)dt$, has the following explicit formula,

$$G(s,t) = \lim_{M\to\infty} G^M(s,t) \text{ (uniform limit)} \tag{5.1}$$

with

$$G^M(s,t) = \sum_{k=1}^{M} \sum_{z,z' \in V_k \setminus V_{k-1}} g(z,z')\phi_z^k(s)\phi_{z'}^k(t), \tag{5.2}$$

where

(5.3)
$$g(z,z') = \begin{cases} \frac{3}{10}\left(\frac{3}{5}\right)^k & \text{for } z = z' \in V_k \setminus V_{k-1}, \\ \frac{1}{10}\left(\frac{3}{5}\right)^k & \text{for } z \neq z' \in V_k \setminus V_{k-1}, \text{ contained in the same } (k-1)\text{-cell}, \\ 0, & \text{otherwise.} \end{cases}$$

To get an analogous Green's function on Ω_x, we should first modify the definition of those piecewise harmonic splines ϕ_z^k whose support intersects the boundary $S(x)$ of the domain Ω_x. More specially, let ω be a word of symbols $\{1,2\}$ with $|\omega| = m$ and $z = \tilde{F}_\omega(q_0)$. We redefine $\phi_z^{n_m}$ to be the piecewise harmonic spline with value 1 on z, 0 on $V_{n_m} \cap \Omega_x$ and $S(x)$, and extended harmonically on Ω_x. Obviously the support of $\phi_z^{n_m}$ is contained in two n_m-cells meeting at z, with $\phi_z^{n_m} = h_0^{y_m} \circ \tilde{F}_\omega^{-1}$ on the cell $\tilde{F}_\omega(\mathcal{SG})$ and with values unchanged on the other cell, denoted by $\tilde{\tilde{F}}_\omega(\mathcal{SG})$, where

$$\tilde{\tilde{F}}_\omega = \begin{cases} F_0^{n_1-1}F_{\omega_1}F_0^{n_2-n_1-1}F_{\omega_2}\cdots F_0^{n_{m-1}-n_{m-2}-1}F_{\omega_{m-1}}F_0^{n_m-n_{m-1}} & \text{for } m\geq 2,\\ F_0^{n_1} & \text{for } m=1.\end{cases} \tag{5.4}$$

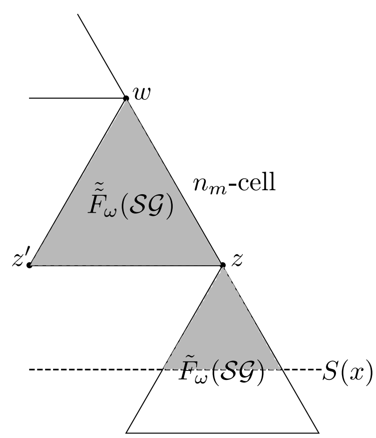

Figure 5.1. The support of $\phi_z^{n_m}$.

Lemma 5.1. *Let $z=\tilde{F}_\omega(q_0)$, then*

$$\mathcal{E}_{\Omega_x}(\phi_z^{n_m},v)=\left(\frac{5}{3}\right)^{n_m}\left(\frac{1+\alpha_0(y_{m-1})}{\alpha_0(y_{m-1})}v(z)-v(z')-v(w)\right) \tag{5.5}$$

for any $v\in dom_0\mathcal{E}_{\Omega_x}$, where $z'=\tilde{F}_{\omega_1\cdots\omega_{m-1}(3-\omega_m)}(q_0)$ and $w=\tilde{F}_{\omega_1\cdots\omega_{m-1}}(q_0)$ are the two n_m-neighbors of z(See Figure 5.1.).

Proof. On the cell $\tilde{F}_\omega(\mathcal{SG})$, by using the localized Gauss-Green formula (see (3.5)),

$$\mathcal{E}_{\Omega_x\cap\tilde{F}_\omega(\mathcal{SG})}(\phi_z^{n_m},v)=v(z)\partial_n\phi_z^{n_m}(z)=2\left(\frac{5}{3}\right)^{n_{m+1}}(1-\alpha_0(y_m))v(z). \tag{5.6}$$

The last equality follows from the same argument as the proof of Lemma 3.1 with suitable scaling.

On the other cell $\tilde{\tilde{F}}_\omega(\mathcal{SG})$, by using the standard theory,

$$\mathcal{E}_{\Omega_x\cap\tilde{\tilde{F}}_\omega(\mathcal{SG})}(\phi_z^{n_m},v)=\left(\frac{5}{3}\right)^{n_m}(2v(z)-v(z')-v(w)). \tag{5.7}$$

Summing the energies on the two cells, we get the desired result by using (2.5).□

Let T_x^m be the set of vertices in $V_{n_m}\cap\Omega_x$ which can be expressed as $\tilde{F}_\omega(q_0)$ for some word $\omega=\omega_1,\cdots,\omega_m$ of symbols $\{1,2\}$, and $T_x=\bigcup_{m\geq 1}T_x^m$.

Definition 5.2. For fixed m, let

$$G_{\Omega_x}^m(s,t)=\sum_{k=1}^{n_m}\sum_{z,z'\in(V_k\setminus V_{k-1})\cap\Omega_x}g_x(z,z')\phi_z^k(s)\phi_{z'}^k(t), \tag{5.8}$$

with
(5.9)

$$g_x(z,z') = \begin{cases} \frac{\alpha_0(y_{l-1})+\alpha_0(y_{l-1})^2}{2\alpha_0(y_{l-1})+1}\left(\frac{3}{5}\right)^{n_l} & \text{for } z=z'\in T_x^l \text{ with } l\le m, \\ \frac{\alpha_0(y_{l-1})^2}{2\alpha_0(y_{l-1})+1}\left(\frac{3}{5}\right)^{n_l} & \text{for } z\neq z'\in T_x^l,\ \text{being } n_l\text{-neighbors, with } l\le m, \\ g(z,z') & \text{for } z,z'\in V_k\setminus V_{k-1} \text{ contained in} \\ & \text{a } (k-1)\text{-cell in } \Omega_x, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is obvious that $G^m_{\Omega_x}(s,t)$ converges uniformly to a function $G_{\Omega_x}(s,t)$ as m goes to infinity.

Theorem 5.3. *G_{Ω_x} is the Green's function for Ω_x, namely*

$$u(s) = \int_{\Omega_x} G_{\Omega_x}(s,t)F(t)dt \tag{5.10}$$

solves the Dirichlet problem $-\Delta u = F$ on Ω_x with $u|_{\partial\Omega_x}=0$, for any continuous F.

Proof. Similar to the $\mathcal{SG}$ case, suppose we could prove

$$\mathcal{E}_{\Omega_x}(G^m_{\Omega_x}(\cdot,t),v) = \sum_{z\in V_{n_m}\cap\Omega_x} v(z)\phi_z^{n_m}(t) \tag{5.11}$$

for any $v\in dom_0\mathcal{E}_{\Omega_x}$.

Then just multiply (5.11) by $F(t)$ and integrate, using the standard arguments to interchange the energy and integral, to obtain

$$\mathcal{E}_{\Omega_x}(u_m,v) = \int_{\Omega_x} F(t) \sum_{z\in V_{n_m}\cap\Omega_x} v(z)\phi_z^{n_m}(t)dt \tag{5.12}$$

for

$$u_m(s) = \int_{\Omega_x} G^m_{\Omega_x}(s,t)F(t)dt. \tag{5.13}$$

Since

$$\sum_{z\in V_{n_m}\cap\Omega_x} v(z)\phi_z^{n_m}(t) \to v(t) \tag{5.14}$$

uniformly as $m\to\infty$, the right side of (5.12) converges to $\int_{\Omega_x} F(t)v(t)dt$, and the left side converges to $\mathcal{E}_{\Omega_x}(u,v)$ as m goes to ∞. Thus we have

$$\mathcal{E}_{\Omega_x}(u,v) = \int_{\Omega_x} Fvdt \tag{5.15}$$

for any $v\in dom_0\mathcal{E}_{\Omega_x}$, which yields that $-\Delta u = F$ with $u|_{\partial\Omega_x}=0$.

Hence our goal is to prove (5.11). The function $G^m_{\Omega_x}(s,t)$, which we regard as a function of the single variable s, could be viewed as a linear combination of terms $\phi_z^k(s)$. Then it is clear that $\mathcal{E}_{\Omega_x}(G^m_{\Omega_x}(\cdot,t),v)$ is a linear combination of $v(z)$ for $z\in V_{n_m}\cap\Omega_x$. So we need to compute the combination coefficient of $v(z)$ for each z.

Let $z_0\in V_{n_m}\cap\Omega_x$. If $z_0\notin T_x$, it is easy to observe that there exists an n_m-cell containing z_0 as an interior point. The terms in $G^m_{\Omega_x}$ that contribute to the coefficient of $v(z_0)$ all have supports away from $S(x)$. Thus the standard argument for the $\mathcal{SG}$ case shows that the coefficient of $v(z_0)$ should be $\phi_{z_0}^{n_m}(t)$.

Hence we only need to consider the case that $z_0 \in T_x$. We first do this when $z_0 \in T_x^m$. Let z_0' denote the unique n_m-neighbor of z_0 in the same level. Then the only terms in $G^m_{\Omega_x}$ that contribute to the coefficient of $v(z_0)$ are

$$g_x(z_0, z_0)\phi^{n_m}_{z_0}(s)\phi^{n_m}_{z_0}(t), g_x(z_0, z_0')\phi^{n_m}_{z_0}(s)\phi^{n_m}_{z_0'}(t),$$

$$g_x(z_0', z_0)\phi^{n_m}_{z_0'}(s)\phi^{n_m}_{z_0}(t), g_x(z_0', z_0')\phi^{n_m}_{z_0'}(s)\phi^{n_m}_{z_0'}(t).$$

By Lemma 5.1, the total contribution is

$$\begin{aligned}(5.16)\qquad &\left(\frac{5}{3}\right)^{n_m}\left(\frac{1+\alpha_0(y_{m-1})}{\alpha_0(y_{m-1})}g_x(z_0,z_0) - g_x(z_0',z_0)\right)\phi^{n_m}_{z_0}(t)\\ &+\left(\frac{5}{3}\right)^{n_m}\left(\frac{1+\alpha_0(y_{m-1})}{\alpha_0(y_{m-1})}g_x(z_0,z_0') - g_x(z_0',z_0')\right)\phi^{n_m}_{z_0'}(t).\end{aligned}$$

By substituting the value of $g_x(z_0, z_0) = g_x(z_0', z_0') = \frac{\alpha_0(y_{m-1})+\alpha_0(y_{m-1})^2}{2\alpha_0(y_{m-1})+1}\left(\frac{3}{5}\right)^{n_m}$ and $g_x(z_0, z_0') = g_x(z_0', z_0) = \frac{\alpha_0(y_{m-1})^2}{2\alpha_0(y_{m-1})+1}\left(\frac{3}{5}\right)^{n_m}$ into (5.16), it is easy to verify that

$$(5.17)\qquad \left(\frac{5}{3}\right)^{n_m}\left(\frac{1+\alpha_0(y_{m-1})}{\alpha_0(y_{m-1})}g_x(z_0,z_0) - g_x(z_0',z_0)\right) = 1,$$

and

$$(5.18)\qquad \left(\frac{5}{3}\right)^{n_m}\left(\frac{1+\alpha_0(y_{m-1})}{\alpha_0(y_{m-1})}g_x(z_0,z_0') - g_x(z_0',z_0')\right) = 0.$$

So the coefficient of $v(z_0)$ is $\phi^{n_m}_{z_0}(t)$.

Next we consider the general case. Suppose $z_0 = \tilde{F}_\omega(q_0) \in T_x^l$ with $1 \leq l < m$. We need to compute the coefficient of $v(z_0)$. The previous discussion immediately shows that the contribution of terms in $G^l_{\Omega_x}$ to $v(z_0)$ is $\phi^{n_l}_{z_0}(t)$. Now we consider the terms in $G^m_{\Omega_x} - G^l_{\Omega_x}$. Let $z_1 = \tilde{F}_{\omega 1}(q_0)$ and $z_2 = \tilde{F}_{\omega 2}(q_0)$. Notice that in all the terms in $G^m_{\Omega_x} - G^l_{\Omega_x}$ that contribute to $v(z_0)$, only those which contain $\phi^{n_{l+1}}_{z_1}(s)$ or $\phi^{n_{l+1}}_{z_2}(s)$ have supports intersecting the boundary $S(x)$. Moreover, in calculating the energy $\mathcal{E}_{\Omega_x}(\phi^{n_{l+1}}_{z_i}, v)$, only the part $\phi^{n_{l+1}}_{z_i}|_{\tilde{\tilde{F}}_{\omega i}(\mathcal{SG})}$ is involved in contributing to the coefficient of $v(z_0)$, for i=1,2. Comparing to the standard $\mathcal{SG}$ case, the function $\phi^{n_{l+1}}_{z_i}(s)$ has been redefined, but the restriction of it to $\tilde{\tilde{F}}_{\omega i}(\mathcal{SG})$ is unchanged. So the total contribution of $G^m_{\Omega_x} - G^l_{\Omega_x}$ to $v(z_0)$ is as same as the standard case, namely $\phi^{n_m}_{z_0}(t) - \phi^{n_l}_{z_0}(t)$. Thus we get that in $\mathcal{E}_{\Omega_x}(G^m_{\Omega_x}, v)$, the coefficient of $v(z_0)$ is $\phi^{n_m}_{z_0}(t)$, as required.

Thus we have proved (5.11). □

Theorem 5.4. *For continuous F, the normal derivative of the solution u given by* (5.10) *is continuous on $S(x)$.*

Proof. From Theorem 5.3,

$$(5.19)\qquad \partial_n u|_{S(x)} = \sum_{m\geq 1}\sum_{z,z'\in T_x^m} g_x(z,z')\partial_n\phi^{n_m}_z|_{S(x)}\int_{\Omega_x}\phi^{n_m}_{z'}(t)F(t)dt,$$

since only those terms containing ϕ^k_z whose supports intersect $S(x)$ contribute to the value of $\partial_n u|_{S(x)}$.

For fixed m, let $z = \tilde{F}_\omega(q_0) \in T_x^m$. Note that on the cell $\tilde{F}_\omega(\mathcal{SG})$, $\phi_z^{n_m} = h_0^{y_m} \circ \tilde{F}_\omega^{-1}$. By Lemma 3.1, we have

$$\partial_n \phi_z^{n_m}|_{S(x)} = -2\left(\frac{5}{3}\right)^{n_{m+1}} (1-\alpha_0(y_m))2^m \chi_{S_\omega(x)}. \tag{5.20}$$

On the other hand, for $z, z' \in T_x^m$, $g_x(z,z')$ is bounded above by a multiple of $\alpha_0(y_{m-1})\left(\frac{3}{5}\right)^{n_m}$, hence by a multiple of $\left(\frac{3}{5}\right)^{n_{m+1}}$ using (2.7). It is also easy to see that $\int_{\Omega_x} \phi_{z'}^{n_m}(t)F(t)dt$ is bounded above by a multiple of $\frac{1}{3^{n_m}} \parallel F \parallel_\infty$. Combing these estimates with (5.20), we conclude that $|\partial_n u||_{S(x)}$ is bounded above by a multiple of

$$\sum_{m\geq 1}\sum_{|\omega|=m} \frac{2^m}{3^{n_m}} \parallel F \parallel_\infty \chi_{S_\omega(x)}. \tag{5.21}$$

From (5.21), one can easily verify that $\partial_n u$ is continuous on $S(x)$.□

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