Fractal sets in the field of p-adic analogue of the complex numbers

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Abstract The p-adic numbers field $\mathbb{Q}_p$ and the p-adic analogue of the complex numbers field $\mathbb{C}_p$ have a rich algebraic and geometric structure that in some ways rivals that of the corresponding objects for the real or complex fields. In this paper, we attempt to find and understand geometry structures of general sets in a p-adic setting. Several kinds of fractal measures and dimensions of sets in $\mathbb{C}_p$ are studied. Some typical fractal sets are constructed. It is worthwhile to note that, there exist some essential differences between p-adic case and classical case.

Keywords: **p-adic, algebraic extension, Hausdorff dimension, box-counting dimension, packing dimension, self-similar**.

1 Introduction

Fractal measures and dimensions of sets in Euclidean spaces are fundamental objects of geometry measure theory[1,2], such as Hausdorff measures and dimensions, box-counting dimensions, packing measures and dimensions. In this paper we investigate whether there are analogous notions in the field $\mathbb{Q}_p$ of p-adic numbers and the field $\mathbb{C}_p$ of p-adic analogue of the complex numbers[3−11]. The main aim is to study the geometric structure of general sets and measures in the p-adic case. It is worthwhile to note that, there

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exist some essential differences which come from the different topological and algebraic structures between classical case and p-adic case.

Let p be a prime number, recall that the field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if an arbitrary rational number $x \neq 0$ is represented as $x = p^r \frac{m}{n}$, where $r = \mathrm{ord}_p x \in \mathbb{Z}$, and m and n are not divisible by p, then $|x|_p = p^{-r}$. This norm in $\mathbb{Q}_p$ satisfies the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$.

Every element x of $\mathbb{Q}_p$ can be thought of as a unique formal series

$$\sum_{i=m}^{\infty} b_i p^i, 0 \leq b_i \leq p-1.$$

The set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is a subring of $\mathbb{Q}_p$ called the ring of p-adic integers. It is well known that $\mathbb{Q}_p$ is locally compact and $\mathbb{Z}_p$ is compact. There is a unique Haar measure μ on $\mathbb{Q}_p$, such that $\mu(\mathbb{Z}_p) = 1$, $\mu(p^n\mathbb{Z}_p) = p^{-n}$ for any $n \in \mathbb{Z}$.

The familiar construction of the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ from the rational numbers can be imitated in the p-adic context. This give rise to the field of p-adic numbers $\mathbb{Q}_p$ and after taking algebraic closure and then completions we get $\mathbb{C}_p$, the p-adic analogue of the complex numbers.

Let K be an algebraic extension of $\mathbb{Q}_p$ of degree n, then $n = e \cdot f$, where e is the index of ramification and f is the residue field degree. We say that the extension of K over $\mathbb{Q}_p$ is unramified if $e = 1$, ramified if $e > 1$, and totally ramified if $e = n$. Let $A_K = \{x \in K : |x|_p \leq 1\}$, $M_K = \{x \in K : |x|_p < 1\}$. Then A_K is a ring, which is the integral closure of $\mathbb{Z}_p$ in K. M_K is its unique maximal ideal. The field A_K/M_K is called the residue field of K. It's a field extension of $\mathbb{F}_p$ of degree f, where $\mathbb{F}_p$ is the finite field of integers modulo the prime p. An element $\pi \in K$ is called an uniformizer if $|\pi|_p = p^{-\frac{1}{e}}$. Every $x \in K$ can be written in one and only one way as

$$\sum_{i=m}^{+\infty} a_i \pi^i,$$

where each a_i satisfies $a_i^{p^f} = a_i$, i.e., the a_i's are Teichmüller digits. There is a unique Haar measure μ_K on K, such that $\mu(A_K) = 1$, $\mu(\pi^n A_K) = p^{-nf}$ for any $n \in \mathbb{Z}$.

Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic norm, where $\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. Let $A = \{x \in \mathbb{C}_p : |x|_p \leq 1\}$ be the valuation ring of $\mathbb{C}_p$, and $M = \{x \in \mathbb{C}_p : |x|_p < 1\}$ be its maximal ideal. The residue field A/M is the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Any nonzero element of $\mathbb{C}_p$ is a product of a fractional power of p, a root of unity, and an element in the open unit disc about 1. $\mathbb{C}_p$ is called the p-adic analogue of the complex numbers. The possible values of $|\cdot|_p$ on $\mathbb{C}_p$ is the set of all rational powers of p. $\mathbb{C}_p$ is not locally compact and A is not compact. There is no Haar measure on $\mathbb{C}_p$.

We define $D_a(r) = \{x \in \mathbb{C}_p : |x-a|_p \leq r\}$ be the disc of radius r about a point $a \in \mathbb{C}_p$. Let $D(r) = D_0(r)$ for brevity.

Now we start to study the fractal analysis in $\mathbb{C}_p$.

2 Hausdorff measures and dimensions

We define the diameter of any subset E of $\mathbb{C}_p$ as $|E|_p = \sup\{|x-y|_p : x, y \in E\}$. It is easy to see that $|E|_p$ takes value from $\{p^k : k \in \mathbb{Q}\}$.

If $\{U_j\}_{j=1}^{\infty}$, $U_j \subset \mathbb{C}_p$ is a collection of sets of diameters at most r that cover $E \subset \mathbb{C}_p$, i.e., $E \subset \bigcup_{j=1}^{+\infty} U_j$ with $|U_j|_p \leqslant r$, $j = 1, 2, ...$, then we say that $\{U_j\}$ is a r-cover of E. Here r is a rational power of p.

We now discuss the Hausdorff measures and dimensions in $\mathbb{C}_p$.

Definition 2.1. *Let $E \subset \mathbb{C}_p$ be any subset in $\mathbb{C}_p$, for $s \geq 0$ and $r > 0$, we call*

$$\mathcal{H}_r^s(E) = \inf\{\sum_{j=1}^{+\infty} |U_j|_p^s : \{U_j\} \textit{ is a } r\textit{-cover of } E\} \tag{1}$$

the s-dimensional approximate Hausdorff measure of E, and call the limit(obviously, it exists)

$$\lim_{r \to 0} \mathcal{H}_r^s(E) = \mathcal{H}^s(E) \tag{2}$$

the s-dimensional Hausdorff measure of E.

Similar to the classical case, we immediately get:

Proposition 2.1. *$\mathcal{H}_r^s$ and $\mathcal{H}^s$ are outer measures; and $\mathcal{H}^s$ is an ultra-metric outer measure.*

Theorem 2.1. *Hausdorff measures behave nicely under translations and dilations in $\mathbb{C}_p$: for $E \subset \mathbb{C}_p$, $\lambda \in \mathbb{C}_p$,*

$$\mathcal{H}^s(E+\lambda) = \mathcal{H}^s(E), \quad \mathcal{H}^s(\lambda E) = |\lambda|_p^s \mathcal{H}^s(E), \tag{3}$$

where $E + \lambda = \{x + \lambda : x \in E\}, \quad \lambda E = \{\lambda x : x \in E\}$.

Proof. For the first equality in (3). Let $\{U_j\}$ be a r-cover of E, then $\{U_j + \lambda\}$ is a r-cover of $E + \lambda$, which leads to $\mathcal{H}_r^s(E+\lambda) \leqslant \mathcal{H}_r^s(E)$. So

$$\mathcal{H}^s(E+\lambda) \leqslant \mathcal{H}^s(E). \tag{4}$$

Similarly, we can also get $\mathcal{H}^s(E-\lambda) \leqslant \mathcal{H}^s(E)$. Replacing E by $E+\lambda$, combining with (4), we get the first equality in (3).

The similar argument gives the second equality in (3).

Proposition 2.2. *Let $0 \leqslant s < t < +\infty$, then*

$$\mathcal{H}^s(E) < +\infty \Rightarrow \mathcal{H}^t(E) = 0; \qquad \mathcal{H}^t(E) > 0 \Rightarrow \mathcal{H}^s(E) = +\infty.$$

From this proposition, we can define the Hausdorff dimensions.

Definition 2.2. *The Hausdorff dimension of a set $E \subset \mathbb{C}_p$ is defined by*

$$\begin{aligned} \dim_H E &= \sup\{s : \mathcal{H}^s(E) > 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\} \\ &= \inf\{t : \mathcal{H}^t(E) < \infty\} = \inf\{t : \mathcal{H}^t(E) = 0\}. \end{aligned} \tag{5}$$

Proposition 2.3. *Let $A = \{x \in \mathbb{C}_p : |x|_p \leq 1\}$, then $\dim_H A = +\infty$.*

Proof. Let $\{U_j\}$ be any countable r-cover of A with $r < 1$. We suppose that each U_j is contained in A. With out lose of generality, the cover $\{U_j\}$ can be ordered so that

$$|U_1|_p \geq |U_2|_p \geq \cdots \geq |U_j|_p \geq \cdots .$$

Then there must exist a positive real number r_0, such that there are infinite many U_j satisfying $|U_j|_p = r_0$. Otherwise, there must be a positive real sequence $\{r_i\}$, with $1 > r_1 > r_2 \cdots$, and a positive integer sequence $\{n_i\}$, such that $|U_1|_p = \cdots |U_{n_1}|_p = r_1$, $|U_{n_1+1}|_p = \cdots |U_{n_2}|_p = r_2, \cdots$. From the topological structure of $\mathbb{C}_p$, each U_j is contained in a disc with diameter $|U_j|_p$. Then there must exist a disc $|V_1| \subset A$, $|V_1|_p = r_1$, but V_1 is disjoint with each $U_1, \cdots, U_{n_1}$. Moreover, in the disc V_1, there must exist a disc V_2, $|V_2|_p = r_2$, but V_2 is disjoint with each $U_{n_1+1}, \cdots, U_{n_2}$. Inductively, we get a set sequence $\{V_i\}$, satisfying $V_1 \supset V_2 \supset \cdots$. Here each V_i is disjoint with the set $\bigcup_1^{n_i} U_j$. Clearly, $A \supset \bigcap_{i=1}^{+\infty} V_i \neq \emptyset$, but for each point $x \in \bigcap_{i=1}^{+\infty} V_i$, x is not contained in any U_j, which is impossible. Hence, there exist a real number r_0, and there are infinite many U_j satisfying $|U_j|_p = r_0$, which easily implies that $\sum_{j=1}^{+\infty} |U_j|_p^s = +\infty$ for any $s \geq 0$. The proof is completed.

Remark 2.1. *This proposition is quite different with the $\mathbb{C}$ case at first sight. The reason is that $\mathbb{C}_p$ is not locally compact and A is not compact. However, we have $\dim_H \mathbb{Z}_p = 1$, which is similar as the $\mathbb{R}$ case.*

Theorem 2.2. *The Hausdorff dimensions of sets in $\mathbb{C}_p$ has the following properties:*

1) *monotony property: $E \subset F \subset \mathbb{C}_p \Rightarrow \dim_H E \leqslant \dim_H F$;*

2) *countable stabilization property: $\dim_H \bigcup_{k\geqslant 1} E_k = \sup_{k\geqslant 1}\{\dim_H E_k\}$;*

3) *For $E \subset \mathbb{C}_p$, then $0 \leq \dim_H E \leq +\infty$; if E contains a disc in $\mathbb{C}_p$, then $\dim_H E = +\infty$;*

4) $\dim_H E = \sup\{\dim_H F :$ *compact* $F \subset E\}$.

Remark 2.2. *Any open set in $\mathbb{C}_p$ has infinity Hausdorff dimension.*

Theorem 2.3. *Let K be a n-degree algebraic extension of $\mathbb{Q}_p$, $E \subset K$, $s \geq 0$, $r > 0$, then*

$$\mathcal{H}_r^s(E) = \inf\{\sum_{j=1}^{+\infty} |U_j|_p^s : \bigcup U_j \supset E, U_j \subset K, |U_j|_p \leq r\} \tag{6}$$

Proof. Denote $\inf\{\sum_{j=1}^{+\infty} |U_j|_p^s : \bigcup U_j \supset E, U_j \subset K, |U_j|_p \leq r\}$ by $\mathcal{H}_r^{s,K}(E)$. $\mathcal{H}_r^s(E) \leq \mathcal{H}_r^{s,K}(E)$ is obvious.

Take any r-cover $\{U_j\}$ of E in $\mathbb{C}_p$. Let $V_j = U_j \bigcap K$. Since $E \subset K$, we have $\bigcup V_j \supset E$, $|V_j|_p \leq |U_j|_p$. Hence, $\{V_j\}$ is a r-cover of E in K, and

$$\sum_j |V_j|_p^s \leq \sum_j |U_j|_p^s,$$

which leads to $\mathcal{H}_r^s(E) \geq \mathcal{H}_r^{s,K}(E)$. The proof is completed.

Remark 2.3. *Theorem 2.3 shows that the Hausdorff measures and dimensions of a set E do not depend on the algebraic extension fields where E lives in.*

Now we turn to study the Hausdorff net measures and dimensions in $\mathbb{C}_p$.

Definition 2.3. *Let $\mathcal{F}$ be a family of subsets in $\mathbb{C}_p$, if $\forall r > 0$, $\forall x \in \mathbb{C}_p$, there exists $A \in \mathcal{F}$, such that $x \in A$ and $|A|_p \leqslant r$, then $\mathcal{F}$ is called a net in $\mathbb{C}_p$.*

Denote by $\mathcal{N}(\mathbb{C}_p)$ the collection of all nets in $\mathbb{C}_p$, $\mathfrak{D}$ the net consisted of all discs, and $2^{\mathbb{C}_p}$ the net consisted of all subsets in $\mathbb{C}_p$.

Definition 2.4. *Let $s \geqslant 0$, $E \subset \mathbb{C}_p$, $\mathcal{F}$ a net in $\mathbb{C}_p$, then*

$$\mathcal{H}_{\mathcal{F}}^s(E) = \lim_{r\to 0} \inf\{\sum_{j=1}^{+\infty} |U_j|_p^s : \{U_j\} \subset \mathcal{F}, |U_j|_p \leq r, \bigcup_{j\geq 1} U_j \supset E\}$$

is the s-dimensional Hausdorff net measure of E about the net $\mathcal{F}$.

Let $\mathcal{F}_1$, $\mathcal{F}_2 \in \mathcal{N}(\mathbb{C}_p)$, if there exist positive constants c_1, $c_2 > 0$, such that $\forall E \subset \mathbb{C}_p$, $\forall s \geqslant 0$,

$$c_1 \mathcal{H}_{\mathcal{F}_1}^s(E) \leqslant \mathcal{H}_{\mathcal{F}_2}^s(E) \leqslant c_2 \mathcal{H}_{\mathcal{F}_1}^s(E), \tag{7}$$

then $\mathcal{F}_1$ and $\mathcal{F}_2$ are called equivalent, denoted by $\mathcal{F}_1 \backsimeq \mathcal{F}_2$, and the s-dimensional Hausdorff net measures $\mathcal{H}_{\mathcal{F}_1}^s$ and $\mathcal{H}_{\mathcal{F}_2}^s$ are equivalent, denoted by $\mathcal{H}_{\mathcal{F}_1}^s \backsimeq \mathcal{H}_{\mathcal{F}_2}^s$. Moreover, if

$$\mathcal{H}_{\mathcal{F}_1}^s(E) = \mathcal{H}_{\mathcal{F}_2}^s(E) \tag{8}$$

for all $E \subset \mathbb{C}_p$ and $s \geqslant 0$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are called strong equivalent, denoted by $\mathcal{F}_1 \equiv \mathcal{F}_2$.

Theorem 2.4. $2^{\mathbb{C}_p} \equiv \mathfrak{D}$.

Proof. Let $E \subset \mathbb{C}_p$, $s \geq 0$. Take any r-cover $\{U_j\}$(finite or countable) of E. We assume that $|U_j|_p = r_j$, where r_j is a rational power of p. Take $x_j \in U_j$, and let $V_j = x_j + D(r_j)$, then we get $V_j \supset U_j$, and $|V_j|_p = |U_j|_p = r_j$. So

$$\sum_j |U_j|_p^s = \sum_j |V_j|_p^s.$$

Since $\{V_j\} \subset \mathfrak{D}$, and the arbitrariness of $\{U_j\}$, we have

$$\inf(\sum |U_j|_p^s) = \inf(\sum |V_j|_p^s),$$

which leads to $\mathcal{H}^s(E) \geqslant \mathcal{H}^s_{\mathfrak{D}}(E)$. The opposite inequality is obvious. The proof is completed.

Theorem 2.4 tell us when we calculate the Hausdorff measure and dimension of some set in $\mathbb{C}_p$, we just need to consider the covers contained in the disc net. Moreover, combining with Theorem 2.3, if we know E is a subset of some algebraic extension field K first, we only need to consider the covers contained in the discs net of K.

3 Iterated function systems and self-similar sets

Let D be a closed subset of $\mathbb{C}_p$. A mapping $f : D \to \mathbb{C}_p$ is called a contraction if there is a number c with $0 < c < 1$ such that

$$|f(x) - f(y)|_p \leq c|x - y|_p$$

for all $x, y \in D$. If the equality holds, i.e. if $|f(x) - f(y)|_p = c|x - y|_p$, then we call f a contracting similarity.

A finite family of contractions $\{f_i\}_{i=1}^m$ with $m \geq 2$, is called an iterated function system or IFS. A non-empty compact subset E of D is called an attractor of the IFS if

$$E = \bigcup_{i=1}^m f_i(E).$$

The fundamental property of an iterated system is that it determines an unique attractor, which is usually a fractal. If each map of the IFS of E is a contracting similarity, then we call E a self-similar set. We say that the IFS satisfy the open set condition if there exists a non-empty bounded open set V such that

$$V \supset \bigcup_{i=1}^{m} f_i(V)$$

with the union disjoint. Then, as the classical case, we have

Theorem 3.1. *If the IFS $\{f_i\}_{i=1}^m$ of the self-similar set $E \subset \mathbb{C}_p$ satisfies the open set condition with the ratios $0 < c_i < 1$ for $1 \leq i \leq m$, then the Hausdorff dimension s of E satisfies the equation:*

$$\sum_{i=1}^{m} c_i^s = 1.$$

Example 3.1. *Cantor type sets in $\mathbb{C}_p$*

Let $2 \leq q \leq p$, An IFS $\{f_i\}_{i=0}^{q-1}$ is given by:

$$f_i(x) = px + i, \quad i = 0, 1, \cdots, q-1.$$

Then $\mathcal{C}_p^q = \{x \in \mathbb{Z}_p : x = \sum_{j=0}^{+\infty} x_i p^i, 0 \leq x_i < q\}$ is the attractor of the IFS. Obviously, the open set condition holds with open set $V = A$. Hence $\dim_H \mathcal{C}_p^q = \frac{\ln q}{\ln p}$.

In the special case $q = p$, the IFS becomes $\{f_i(x) = px + i\}_{i=0}^{p-1}$ and the self-similar set becomes $\mathbb{Z}_p$. Moreover, the IFS of $\mathbb{Z}_p$ is not unique, which can be changed in $\{f_i(x) = px + i\}_{i=1}^p$.

Example 3.2. *Valuation ring of a finite degree algebraic extension of $\mathbb{Q}_p$*

Let K be an algebraic extension of $\mathbb{Q}_p$ of degree n, then $n = e \cdot f$, where e is the index of ramification and f is the residue field degree. Let π be an uniformizer. The map $g(x) = \pi x + a, a \in \mathbb{C}_p$ is a contracting similarity with ratio $p^{-\frac{1}{e}}$.

Take an IFS as $\{g_i(x) = \pi x + a_i\}_{i=1}^{p^f}$ where each a_i satisfies $a_i^{p^f} = a_i$, i.e., the a_i's are Teichmüller digits. Then $|a_i|_p = 1$ for each i, and the attractor is $A_K = \{x \in K : |x|_p \leq 1\}$. The Hausdorff dimension s of A_K satisfies the equation $\sum_{i=1}^{p^f} (p^{-\frac{1}{e}})^s = 1$, which implies $\dim_H A_K = e \cdot f = n$.

Example 3.3. *Hausdorff dimensions of some basic sets in $\mathbb{C}_p$*

From the above two examples we have

$$\dim_H \mathbb{Z}_p = \dim_H \mathbb{Q}_p = 1,$$

and

$$\dim_H A_K = \dim_H K = n$$

if K is a n-degree algebraic extension of $\mathbb{Q}_p$.

Let $\mathbb{Q}_p^{unram}$ be the maximal unramified extension of $\mathbb{Q}_p$, then $\mathbb{Q}_p^{unram}$ is the union of all the finite unramified extensions of $\mathbb{Q}_p$, and

$$\mathbb{Q}_p^{unram} \subset \overline{\mathbb{Q}}_p \subset \mathbb{C}_p.$$

Hence

$$\dim_H \mathbb{Q}_p^{unram} = \dim_H \overline{\mathbb{Q}}_p = \dim_H \mathbb{C}_p = +\infty.$$

Proposition 3.1. *Let E be a self-similar set in $\mathbb{C}_p$ with the IFS $\{f_i(x) = a_i x + b_i\}_{i=1}^m$, then $E \subset \mathbb{Q}_p(a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_m)$.*

Lemma 3.1. *Let $f(x) = ax + b$, $g(x) = cx + d$ be two different contracting similarities, then the IFS $\{f, g\}$ satisfies the open set condition if and only if $|b-d|_p > \max\{|a|_p, |c|_p\}$.*

Proof. Without losing generality, we suppose $b, d \in A$. then $f(A) = D(|a|_p) + b \subset A$ and $g(A) = D(|c|_p) + d \subset A$, $f(A) \bigcap g(A) = \emptyset$ if and only if $|b - d|_p > \max\{|a|_p, |c|_p\}$. Moreover, $|b - d|_p \leq \max\{|a|_p, |c|_p\}$ implies $f(A) \subset g(A)$ or $g(A) \subset f(A)$. The proof is completed.

Due to the Lemma 3.1, the following Theorem holds immediately.

Theorem 3.2. *The IFS $\{f_i(x) = a_i x + b_i\}_{i=1}^m$ satisfies the open set condition if and only if $|b_i - b_j|_p > \max\{|a_i|_p, |a_j|_p\}$, $\forall i \neq j$.*

Corollary 3.1. *For the IFS $\mathbf{f} = \{f_i(x) = a_i x + b_i\}_{i=1}^m$, if $\max_i\{|a_i|_p\} < \min_{i\neq j}\{|b_i - b_j|_p\}$, then $\mathbf{f}$ satisfies the open set condition.*

Obviously, for Example 3.1 and Example 3.2, the condition in Corollary 3.1 are hold naturally.

4 Box-counting dimensions

First, we give some notations. Let $E \subset \mathbb{C}_p$, $r \in \mathbb{R}$, denote

$N_r(E)$, the smallest number of discs of diameters r that cover E;

$N_r^*(E)$, the smallest number of discs of diameters at most r that cover E;

$M_r(E)$, the largest number of disjoint discs of diameters r which have non-empty intersections with E.

Remark 4.1. *In the definitions of $N_r(E)$ and $N_r^*(E)$, the discs cover E are disjoint because of the topological structure of $\mathbb{C}_p$.*

On these three values, we have

Proposition 4.1. *Let $E \subset \mathbb{C}_p$, $r \in \mathbb{R}$, then $N_r(E) = N_r^*(E) = M_r(E)$.*

Proof. Firstly, we prove $N_r(E) = N_r^*(E)$.

$N_r^*(E) \leqslant N_r(E)$ is obvious.

To prove $N_r^*(E) \geq N_r(E)$, let $U_1, U_2, ..., U_{N_r^*(E)}$ be the discs with diameters at most r which cover E, $U_j \bigcap E \neq \emptyset$, $1 \leq j \leq N_r^*(E)$. Without losing generality, let

$$|U_1|_p = |U_2|_p = ... = |U_k|_p = r,$$

and

$$|U_j|_p < r,\ k < j \leqslant N_r^*(E),$$

where $0 \leqslant k \leqslant N_r^*(E)$.

Then for all $i \neq j$, $k < i, j \leq N_r^*(E)$, we have

$$d(U_i, U_j) > r.$$

In fact, if their exist some $i \neq j$, $k < i, j \leqslant N_r^*(E)$ such that $d(U_i, U_j) \leqslant r$, then since $|U_i|_p < r$ and $|U_j|_p < r$, we have

$$|U_i \bigcup U_j|_p \leqslant r.$$

Hence, there is a new disc U_0 such that $U_i \bigcup U_j \subset U_0$ and $|U_0|_p = r$, which is contradict to the fact that $N_r^*(E)$ is the smallest number of discs of diameters at most r that cover E. Thus, for all $k < i, j \leqslant N_r^*(E)$, $d(U_i, U_j) > r$.

For any i with $k < i \leqslant N_r^*(E)$, fix some $x_i \in U_i \bigcap E$. Then $\forall i, j$, $i \neq j$ with $k < i, j \leqslant N_r^*(E)$, we have $d(x_i + D(r), U_j) > 0$, so $d(x_i + D(r), x_j) > 0$. Hence, $x_j \notin x_i + D(r)$, and $(x_j + D(r)) \bigcap (x_i + D(r)) = \emptyset$, which results to $d(x_i + D(r), x_j + D(r)) > 0$ for all i, j with $k < i, j \leqslant N_r^*(E)$. Thus,

$$U_1, U_2, ..., U_k, x_{k+1} + D(r), x_{k+2} + D(r), ..., x_{N_r^*(E)} + D(r)$$

are disjoint discs with diameters r that cover E, which leads to $N_r(E) \leqslant N_r^*(E)$.

Secondly, we turn to prove $N_r(E) = M_r(E)$.

Let $U_1, U_2, ..., U_{N_r(E)}$ be the discs with diameters r which cover E, then each U_j has a non-empty intersection with E. Combining with the fact that any two discs in $\mathbb{C}_p$ are either disjoint or one containing another, we get $N_r(E) \leqslant M_r(E)$.

To prove $M_r(E) \leq N_r(E)$. Let $U_1, U_2, ..., U_{N_r(E)}$ be the discs with diameters r which cover E, $U_j \bigcap E \neq \emptyset$, $j = 1, 2, ..., N_r(E)$. Let $V_1, V_2, ..., V_{M_r(E)}$ be the disjoint discs of diameters r which have non-empty intersection with E. Then $\forall i$, $1 \leqslant i \leqslant M_r(E)$, their exists a point x_i with $x_i \in V_i \bigcap E$. Therefore, $\forall x_i$, there is some U_j such that $x_i \in U_j$, which leads to $V_i = U_j$. Moreover, $\forall 1 \leqslant i \leqslant M_r(E)$, we can always find some j, $1 \leqslant j \leqslant N_r(E)$, such that $V_i = U_j$. From the disjoint property of $\{U_i\}$ and $\{V_j\}$, we can conclude that $M_r(E) \leqslant N_r(E)$.

The proof is completed.

Remark 4.2. *In the classical case in $\mathbb{R}$ or $\mathbb{C}$, we have not the above result.*

Remark 4.3. *Let K be an algebraic extension of $\mathbb{Q}_p$ of degree n, $n = e \cdot f$. e is the index of ramification and f is the residue field degree. we may denote $N_r^K(E), N_r^{*,K}(E), M_r^K(E)$ in the similar way for discs all in K. But now r only takes values in $\{p^{\frac{m}{e}} : m \in \mathbb{Z}\}$. The equality still holds that $N_r^K(E) = N_r^{*,K}(E) = M_r^K(E)$.*

Definition 4.1. *Let E be a non-empty bounded subset in $\mathbb{C}_p$, then the upper and lower box-counting dimensions of E are respectively defined as*

$$\overline{\dim}_B E = \limsup_{r \to 0} \frac{\ln(N_r(E))}{-\ln r}, \tag{9}$$

$$\underline{\dim}_B E = \liminf_{r\to 0} \frac{\ln(N_r(E))}{-\ln r}. \tag{10}$$

If $\overline{\dim}_B E = \underline{\dim}_B E$, then the box-counting dimension of E exists, denoted by $\dim_B E$.

Theorem 4.1. *Let K be a n-degree algebraic extension of $\mathbb{Q}_p$, $E \subset K$ be a non-empty subset, then*

$$\overline{\dim}_B E = \limsup_{r\to 0} \frac{\ln(N_r^K(E))}{-\ln r}, \tag{11}$$

$$\underline{\dim}_B E = \liminf_{r\to 0} \frac{\ln(N_r^K(E))}{-\ln r}. \tag{12}$$

Proof. Let $r > 0$, there must exist a positive integer m, such that

$$p^{-\frac{m}{e}} \le r < p^{-\frac{m-1}{e}}.$$

Let $\{U_1, U_2, \cdots, U_{N_r(E)}\}$ be the discs in $\mathbb{C}_p$ with diameters r which cover E. Then for each U_i, $U_i \bigcap E \neq \emptyset$. Hence, there exists a point $a_i \in E \subset K$, such that $U_i = D_{a_i}(r)$. $U_i \bigcap K = \{x \in K : |x - a_i|_p \le r\} = \{x \in K : |x - a_i|_p \le p^{-\frac{m}{e}}\}$. So, $\{U_1 \bigcap K, U_2 \bigcap K, \cdots, U_{N_r(E)} \bigcap K\}$ is a discs cover of E with diameters $p^{-\frac{m}{e}}$. Hence we have

$$N_{p^{-\frac{m}{e}}}(E) \le N^K_{p^{-\frac{m}{e}}}(E) \le N_r(E).$$

The proof is completed.

Theorem 4.2. *Let K be a n-degree algebraic extension of $\mathbb{Q}_p$, μ_K be the Haar measure on K, $E \subset K$ be a non-empty bounded subset, then*

$$\overline{\dim}_B E = \limsup_{r\to 0} \left(n - \frac{\ln \mu(E(r))}{\ln r}\right), \tag{13}$$

$$\underline{\dim}_B E = \liminf_{r\to 0} \left(n - \frac{\ln \mu(E(r))}{\ln r}\right), \tag{14}$$

where $E(r) = \{x \in K : d(x, E) \le r\}$.

Proof. Using Theorem 4.1, we only need to verify

$$\limsup_{r\to 0} \frac{\ln(N_r^K(E))}{-\ln r} = \limsup_{r\to 0} \left(n - \frac{\ln \mu(E(r))}{\ln r}\right), \tag{15}$$

for the case of the lower box-counting dimension is similar.

Note that the Haar measure of any disc in K with diameter r is r^n. If $E \subset K$ can be covered by $N_r^K(E)$'s disjoint discs of diameters of r, then $E(r)$ can also be covered by them, and $\mu(E(r)) \leqslant N_r^K(E)r^n$. On the other hand, all the $N_r^K(E)$'s disjoint discs are covered by $E(r)$, and hence $\mu(E(r)) \geqslant N_r^K(E)r^n$. Therefore, $\mu(E(r)) = N_r^K(E)r^n$. The proof is completed.

Remark 4.4. *Theorem 4.1 and Theorem 4.2 give other definitions of the upper and lower box-counting dimensions of a bounded set E. Notice that these definitions do not depend on the choice of the algebraic extension field of $\mathbb{Q}_p$ where E lives in.*

Theorem 4.3. *Let E be a non-empty bounded subset in $\mathbb{C}_p$, then*
(1) $\dim_H E \leqslant \underline{\dim}_B E$;
(2) $\underline{\dim}_B(E)$, $\overline{\dim}_B(E)$ *are monotone;*
(3) *for $E \subset \mathbb{C}_p$, we have* $0 \leq \overline{\dim}_B E, \underline{\dim}_B E \leq +\infty$; *if E has open subset, then* $\dim_B E = +\infty$;
(4) $\overline{\dim}_B(E \bigcup F) = \max\{\overline{\dim}_B E, \overline{\dim}_B F\}$;
(5) $\overline{\dim}_B E = \overline{\dim}_B \overline{E}$, $\underline{\dim}_B E = \underline{\dim}_B \overline{E}$, *where $\overline{E}$ is the closure of E.*

Proof. We prove the above properties in order.

(1) Let $s < \dim_H E$, then $\mathcal{H}^s(E) = +\infty$. Therefore, there is an $r_0 \in \mathbb{R}$ such that $\mathcal{H}_r^s(E) > 1$ for all $r < r_0, r \in \mathbb{R}$.

Combining with $\mathcal{H}_r^s(E) \leqslant N_r(E)r^s$, we have

$$1 < N_r(E)r^s \tag{16}$$

for all $r < r_0, r \in \mathbb{R}$. Therefore, $\underline{\dim}_B E \geqslant s$, and hence $\underline{\dim}_B E \geqslant \dim_H E$ since s is arbitrary.

(2)

$$\text{“}E \subset F \Rightarrow N_r(E) \leq N_r(F),\ \forall r \in \mathbb{R}\text{”}$$

leads to the monotone properties of $\overline{\dim}_B E$ and $\underline{\dim}_B E$.

(3) From the definition, it is easy to get $\underline{\dim}_B A = \overline{\dim}_B A = +\infty$. If E has open subset, then E must contain a disc, which results to $\dim_B E = +\infty$.

(4) From the monotony, it is easy to get $\overline{\dim}_B(E \bigcup F) \geqslant \max\{\overline{\dim}_B E, \overline{\dim}_B F\}$.

For the opposite side of the above inequality, let $\alpha > \max\{\overline{\dim}_B E, \overline{\dim}_B F\}$, then for sufficiently small $r \in \mathbb{R}$, $N_r(E) \leqslant r^{-\alpha}$, $N_r(F) \leqslant r^{-\alpha}$,

$$N_r(E \bigcup F) \leqslant N_r(E) + N_r(F) \leqslant 2r^{-\alpha}.$$

Hence, $\overline{\dim}_B(E \bigcup F) \leqslant \alpha$, and so

$$\overline{\dim}_B(E \bigcup F) \leqslant \max\{\overline{\dim}_B E, \overline{\dim}_B F\}.$$

(5) Let $U_1, U_2, ..., U_k$ be a collection of discs of diameters r. If $\bigcup_{j=1}^k U_j \supset E$, then $\bigcup_{j=1}^k U_j \supset \overline{E}$ due to the fact that discs in $\mathbb{C}_p$ are both open and closed. Hence, $N_r(\overline{E}) \leqslant N_r(E)$ leads to the result.

In order to deeply study the box-counting dimension, we construct a compact set $\mathcal{C}_p^{q_1,q_2} \subset \mathbb{Z}_p$, such that $\underline{\dim}_B \mathcal{C}_p^{q_1,q_2} < \overline{\dim}_B \mathcal{C}_p^{q_1,q_2}$.

Example 4.1. *Cantor type set $\mathcal{C}_p^{q_1,q_2}$ in $\mathbb{Z}_p$*

Take q_1, q_2 with $2 \leqslant q_1 < q_2 \leqslant p-1$, and s, t such that $\frac{\ln q_1}{\ln p} < s < t < \frac{\ln q_2}{\ln p}$.

We start constructing a Cantor type set $\mathcal{C}_p^{q_1}$ in first step. Thus, we have q_1 discs $I_{1,1}, I_{1,2}, ..., I_{1,q_1}$ with diameters p^{-1}, which satisfy

$$q_1(p^{-1})^s \leqslant 1. \tag{17}$$

Then in each $I_{1,j}$ perform the similar construction of $\mathcal{C}_p^{q_2}$ k_1 times where k_1 is so large that

$$q_1 q_2^{k_1}(p^{-1-k_1})^t = q_1 p^{-t}(q_2 p^{-t})^{k_1} \geqslant 1. \tag{18}$$

After that, continue the construction of $\mathcal{C}_p^{q_1}$ k_2 times again, where k_2 is so large that

$$q_1 q_2^{k_1} q_1^{k_2}(p^{-1-k_1-k_2})^s = q_1 q_2^{k_1} p^{-(1+k_1)s}(q_1 p^{-s})^{k_2} \leqslant 1. \tag{19}$$

Continue this process without ending. We get a sequence of positive integers $\{k_j\}$ and a Cantor type set $\mathcal{C}_p^{q_1,q_2} \subset \mathbb{Z}_p$.

Theorem 4.4. *Let $2 \leqslant q_1 < q_2 \leqslant p-1$, and $\frac{\ln q_1}{\ln p} < s < t < \frac{\ln q_2}{\ln p}$, then the Cantor set $\mathcal{C}_p^{q_1,q_2} \subset \mathbb{Z}_p$ constructed in Example 4.1 has the following property:*

$$\underline{\dim}_B \mathcal{C}_p^{q_1,q_2} \leqslant s < t \leqslant \overline{\dim}_B \mathcal{C}_p^{q_1,q_2}.$$

Proof. From the above construction, it is easy to get

$$
\begin{aligned}
&N_{p^{-1}}(\mathcal{C}_p^{q_1,q_2}) = q_1,\\
&N_{p^{-(1+k_1)}}(\mathcal{C}_p^{q_1,q_2}) = q_1 q_2^{k_1},\\
&N_{p^{-(1+k_1+k_2)}}(\mathcal{C}_p^{q_1,q_2}) = q_1 q_2^{k_1} q_1^{k_2},\\
&\cdots\\
&N_{p^{-(1+k_1+k_2+\ldots+k_{2j-1})}}(\mathcal{C}_p^{q_1,q_2}) = q_1 q_2^{k_1} q_1^{k_2} \ldots q_2^{k_{2j-1}},\\
&N_{p^{-(1+k_1+k_2+\ldots+k_{2j})}}(\mathcal{C}_p^{q_1,q_2}) = q_1 q_2^{k_1} q_1^{k_2} \ldots q_1^{k_{2j}},\\
&\cdots.
\end{aligned}
$$

Hence,

$$
\underline{\dim}_B \mathcal{C}_p^{q_1,q_2} \leqslant \underline{\lim}_{j\to\infty} \frac{\ln N_{1+k_1+k_2+\ldots+k_{2j}}(\mathcal{C}_p^{q_1,q_2})}{(1+k_1+k_2+\ldots+k_{2j})\ln p} = \underline{\lim}_{j\to\infty} \frac{\ln q_1 q_2^{k_1} q_1^{k_2} \ldots q_1^{k_{2j}}}{(1+k_1+k_2+\ldots+k_{2j})\ln p} \leqslant s,
$$

$$
\overline{\dim}_B \mathcal{C}_p^{q_1,q_2} \geqslant \overline{\lim}_{j\to\infty} \frac{\ln N_{1+k_1+k_2+\ldots+k_{2j-1}}(\mathcal{C}_p^{q_1,q_2})}{(1+k_1+k_2+\ldots+k_{2j-1})\ln p} = \overline{\lim}_{j\to\infty} \frac{\ln q_1 q_2^{k_1} q_1^{k_2} \ldots q_2^{k_{2j-1}}}{(1+k_1+k_2+\ldots+k_{2j-1})\ln p} \geqslant t.
$$

The proof is completed.

5 Packing measures and dimensions

In this section, we discuss the packing measures and dimensions in $\mathbb{C}_p$.

Definition 5.1. *Let $E \subset \mathbb{C}_p$, we call a family of disjoint discs of diameters at most r a r-packing of E if each disc in this family has a non-empty intersection with E.*

Let E be a non-empty subset of $\mathbb{C}_p$, $s \geqslant 0$, $r \in \mathbb{R}$, we define

$$
P_r^s(E) = \sup\{\sum_{j=1}^{+\infty} |U_j|_p^s : \{U_j\} \text{ is } r\text{-packing of } E\}. \tag{20}
$$

Theorem 5.1. *Let K be a n-degree algebraic extension of $\mathbb{Q}_p$, $E \subset K$, we can define a r-packing of E in K in the similar way where all discs are in K, then*

$$
P_r^s(E) = \sup\{\sum_{j=1}^{+\infty} |U_j|_p^s : \{U_j\} \textit{ is } r\textit{-packing of } E \textit{ in } K\}. \tag{21}
$$

The proof is similar to previous two cases. Obviously, $P_r^s(E)$ decreases as $r \to 0$, and hence we can define the pre-packing measures as follows.

Definition 5.2. *Let E be a non-empty subset in $\mathbb{C}_p$, $s \geqslant 0$, $r \in \mathbb{R}$, then*

$$P^s(E) = \lim_{r \to 0} P_r^s(E) \tag{22}$$

is called the s-dimensional pre-packing measure of E.

It is easy to prove that for $0 \leqslant s < t < +\infty$,

$$P^s(E) < +\infty \Rightarrow P^t(E) = 0,$$

$$P^s(E) = +\infty \Leftarrow P^t(E) > 0,$$

so we can give the definition of the pre-packing dimensions.

Definition 5.3. *Let E be a non-empty subset in $\mathbb{C}_p$, then*

$$\Delta(E) = \sup\{s : P^s(E) = +\infty\} = \inf\{s : P^s(E) = 0\} \tag{23}$$

is called the pre-packing dimension of E.

For the pre-packing measures and dimensions, we have

Theorem 5.2. *Let $s \geqslant 0$, $E \subset \mathbb{C}_p$ be any subset in $\mathbb{C}_p$, then*

$$\mathcal{H}^s(E) \leqslant P^s(E). \tag{24}$$

Proof. Obviously, a r-disc cover of E is also a r-packing of E. So $\forall r \in \mathbb{R}$, $\mathcal{H}_r^s(E) \leqslant P_r^s(E)$, which result in $\mathcal{H}^s(E) \leqslant P^s(E)$.

Theorem 5.3. *Let $E \subset \mathbb{C}_p$ be any subset in $\mathbb{C}_p$, then*

(1) $\dim_H E \leqslant \Delta(E)$;

(2) $\overline{\dim}_B E = \Delta(E)$.

Proof. The conclusion (1) is the direct corollary of Theorem 5.2. We now prove (2).

Let $s \geqslant 0$, since $M_r(E)r^s \leqslant P_r^s(E)$, then

$$\limsup_{r\to 0} M_r(E)r^s \leqslant P^s(E).$$

If $s > \Delta(E)$, then $P^s(E) = 0$, so $\limsup_{r\to 0} M_r(E)r^s = 0$. Hence, $s \geqslant \overline{\dim}_B E$, thus $\overline{\dim}_B E \leqslant \Delta(E)$.

For the opposite inequality, we only need to prove it providing $\Delta(E) > 0$. Let $0 < \alpha < s < \Delta(E)$. We define a real numbers sequence r_j inductively.

First, since $s < \Delta(E)$, then $P^s(E) = +\infty$, we can choose r_0 such that

$$\sup\{\sum |U_j|_p^s : \{U_j\} \text{ is } r_0\text{-packing of } E\} > 1.$$

If r_j is defined, then we define r_{j+1} for $j \geqslant 0$. Choose an r_j-packing such that $\sum |U_j|_p^s \geqslant 1$. Let $n_k = \#\{U_i : p^{-k-1} < |U_i|_p \leq p^{-k}\}$, then

$$\sum_{k\geqslant 0} p^{-ks} n_k \geq \sum |U_j|_p^s \geqslant 1. \tag{25}$$

Hence, we can find k such that $p^{-ks}n_k \geqslant p^{-k\alpha}(1-p^{-\alpha})$, otherwise, $\sum_{k\geqslant 0} p^{-ks}n_k < 1$. we define $r_{j+1} = p^{-(k+1)}$.

From the above construction, for each r_j, we have a packing $\{V_i : 1 \leqslant i \leqslant n_k\}$, $|V_i|_p = p^{-k-2}$, and $n_k \geqslant p^{k(s-\alpha)}(1-p^{-\alpha})$, and so

$$M_{p^{-(k+2)}}(E) \geqslant n_k \geqslant p^{k(s-\alpha)}(1-p^{-\alpha}),$$

which results in $\overline{\dim}_B E \geqslant s - \alpha$, and $\overline{\dim}_B E \geqslant \Delta(E)$.

The proof is completed.

We give a example to show that P^s are not outer measures.

Example 5.1. *non-negative integers* $\mathbb{Z}^+$

It is easy to verify that $P^1(n) = 0$, $\forall n \in \mathbb{Z}^+$, but $P^1(\mathbb{Z}^+) = 1$ since $\mathbb{Z}^+$ is dense in $\mathbb{Z}_p$.

Definition 5.4. *Let* $E \subset \mathbb{C}_p$, $s \geqslant 0$, *then*

$$\mathcal{P}^s(E) = \inf\{\sum P^s(E_i) : E = \bigcup E_i\},$$

and

$$\dim_P E = \inf\{\sup\{\Delta(E_i) : E = \bigcup E_i\}\}$$

are called the s-dimensional packing measure and packing dimension of E, respectively.

Packing measures $\mathcal{P}^s$ are outer measures, with some similar properties as the Hausdorff measures. $\dim_P$ has also some properties similar to the Hausdorff dimensions, such as the countable stabilization property. We omit them here.

Theorem 5.4. *Let $s \geqslant 0$, $E \subset \mathbb{C}_p$ be any subset in $\mathbb{C}_p$. Then*

$$\mathcal{H}^s(E) \leqslant \mathcal{P}^s(E) \leqslant P^s(E), \tag{26}$$

and

$$\dim_H E \leqslant \dim_P E \leqslant \Delta(E). \tag{27}$$

Proof. The right sides of (26) and (27) are obvious from the definitions.

For the left side in (26), notice that Hausdorff measures are outer measures, we have

$$\mathcal{H}^s(E) = \inf\{\sum \mathcal{H}^s(E_i) : E = \bigcup E_i\}.$$

Combine with Theorem 5.2, we have

$$\mathcal{H}^s(E) = \inf\{\sum \mathcal{H}^s(E_i) : E \subset \bigcup E_i\} \leq \inf\{\sum P^s(E_i) : E \subset \bigcup E_i\} = \mathcal{P}^s(E).$$

Using the same method, we also get the left side of (27). The proof is completed.

6 Comparison of different dimensions

The relations between the three kinds of dimensions have been discussed above. They are

$$\dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E = \Delta(E), \tag{28}$$

$$\dim_H E \leq \dim_P E \leq \Delta(E). \tag{29}$$

Section 4 gives an example $\mathcal{C}_p^{q_1,q_2}$ for $\underline{\dim}_B \mathcal{C}_p^{q_1,q_2} < \overline{\dim}_B \mathcal{C}_p^{q_1,q_2}$. We have also an example for which the inequality $\dim_H E \leq \underline{\dim}_B E$ holds strictly.

Example 6.1. *non-negative integers* $\mathbb{Z}^+$

Since $\mathbb{Z}^+$ is a countable set, by Theorem 2.2, $\dim_H \mathbb{Z}^+ = 0$.

On the other hand, it is easy to see that $\mathbb{Z}^+$ is dense in $\mathbb{Z}_p$. Then by Theorem 4.3, we get that

$$\dim_B \mathbb{Z}^+ = \dim_B \mathbb{Z}_p = 1.$$

Then we have

$$\dim_H \mathbb{Z}^+ = 0 < 1 = \dim_B \mathbb{Z}^+. \tag{30}$$

This example also shows that the box-counting dimensions have no countable stabilization property.

For the self-similar set we discussed in Section 3. As in the classical case, we also have

Theorem 6.1. *If the IFS* $\{f_i\}_{i=1}^m$ *of the self-similar set* $E \subset \mathbb{C}_p$ *satisfies the open set condition with the ratios* $0 < c_i < 1$ *for* $1 \leq i \leq m$. *Let* s *satisfies the equation* $\sum_{i=1}^m c_i^s = 1$, *then*

$$\dim_H E = \dim_B E = \dim_P E.$$

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