\textbf{\textit{p}-adic Laplacian in Local Fields}

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\textbf{Abstract:} In this paper a family of multi-dimensional fractional differential operators $T^\alpha$ and their corresponding pseudo-differential equations over \textit{p}-adic fields are investigated. The test function class $\mathcal{D}(\mathbb{Q}_p^n)$ and distribution class $\mathcal{D}'(\mathbb{Q}_p^n)$ are invariant under the actions of these operators. The \textit{p}-adic Laplacian $\Delta_p$ and a fundamental solution of the Laplace equation are constructed. We study the spectral properties of the Laplacian $\Delta_p$, and obtain an orthonormal basis of the eigen-functions of this operator in $L^2(\mathbb{Q}_p^n)$. Furthermore, the Cauchy problems for the wave and heat equations on the \textit{p}-adic fields related to $\Delta_p$ are also studied.

\textbf{Keywords:} \textit{p}-adic fields, Laplacian, pseudo-differential operators, eigen-values, Cauchy problem.
1 Introduction

The main object of this paper is the $p$-adic Laplacian on $\mathbb{Q}_p^n$. To construct this operator, one need to consider the problem of how to define derivative operators on $\mathbb{Q}_p$, which is an important topic in the study of $p$-adic analysis\cite{1,2}. Many mathematicians, such as J.E. Gibbs\cite{3}, P.L. Butzer\cite{4}, C.W. Onneweer\cite{5}, W.X. Zheng\cite{6} and V. S. Vladimirov\cite{7} paid their great attention to this topic. However, the test function class $\mathcal{D}(\mathbb{Q}_p)$ are not invariant under the actions of their definitions of derivatives. In the 90’s, W.Y. Su\cite{8,9} has given a definition of derivatives and integrals, denoted by $T^s$, for general locally compact Vilenkin group $G$, using the pseudo-differential operators, including derivatives and integrals of fractional orders. The test function class $\mathcal{D}(\mathbb{Q}_p)$, together with its distribution class $\mathcal{D}'(\mathbb{Q}_p)$ are invariant under the actions of these fractional operators. For each $s \in \mathbb{R}$, $T^s$ is a pseudo-differential operator with the symbol $\langle \xi \rangle^s$ owing to the formula that

$$T^s f = (\langle \xi \rangle^s f)(\chi),$$

where $\langle \xi \rangle = \max\{1, |\xi|_p\}$. These operators can be used to study many interesting topics in harmonic analysis\cite{10,11}, approximation theory\cite{12-14}, fractal analysis\cite{15-18} and other scientific fields.

In \cite{19}, the convolution kernel $\kappa_s$ of the pseudo-differential operator $T^s$ is given and some important properties of $\kappa_s$ are obtained which play a key role in considerations related to fractional differential operators. A fundamental solution of the pseudo-differential equation

$$P(T^s)f = g, \quad g \in \mathcal{D}'(\mathbb{Q}_p), \quad s \in \mathbb{R},$$

with respect to an unknown distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ is obtained, where $P$ is a polynomial of finite order.

In this paper, firstly, we extend the definition of the fractional differential operators to the multi-dimensional space $\mathbb{Q}_p^n$. A family of multi-dimensional operators $T^s$ and their corresponding pseudo-differential equations are investigated. The test function class $\mathcal{D}(\mathbb{Q}_p^n)$ and distribution class $\mathcal{D}'(\mathbb{Q}_p^n)$ are invariant under these operators. Secondly, we give the definition of the $p$-adic Laplacian $\Delta_p$, analogous to that in the Euclidean case. A fundamental solution of the Laplace equation is constructed. Spectral properties of the Laplacian $\Delta_p$ are studied, and an orthonormal basis of eigen-functions of $\Delta_p$ in $L^2(\mathbb{Q}_p^n)$ is obtained. Finally, we investigate the Cauchy problems for the wave and heat equations on the $p$-adic fields related to $\Delta_p$, and obtain solutions of these equations.
2 A brief review of the $p$-adic analysis

In this section, we make a brief review of the $p$-adic analysis\[^{1-4}\]. Let $p$ be a prime number. Recall that the field $\mathbb{Q}_p$ of $p$-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean $p$-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if a nonzero rational number $x$ is represented as $x = p^r \frac{m}{n}$, where $r = \text{ord}_p x \in \mathbb{Z}$, and $m$ and $n$ are not divisible by $p$, then $|x|_p = p^{-r}$. This norm satisfies the \textbf{strong triangle inequality} that $|x + y|_p \leq \max(|x|_p, |y|_p)$ for any $x, y \in \mathbb{Q}_p$.

Every element $x$ in $\mathbb{Q}_p$ can be thought as a unique formal series
\[
\sum_{i=m}^{\infty} x_i p^i, \quad 0 \leq x_i \leq p - 1, \quad x_m \neq 0.
\]

The set $B_0 = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is a subring of $\mathbb{Q}_p$ called the \textbf{ring of $p$-adic integers}. It is well known that $\mathbb{Q}_p$ is locally compact and $B_0$ is compact. There is a Haar measure $dx$ on $\mathbb{Q}_p$, normalized that $\int_{B_0} dx = 1$. For any $r \in \mathbb{Z}$, denote by $B_r$ the disc of radius $p^r$ with center $0 \in \mathbb{Q}_p$ and by $S_r$ its boundary:
\[
B_r = \{x \in \mathbb{Q}_p : |x|_p \leq p^r\},
\]
\[
S_r = \{x \in \mathbb{Q}_p : |x|_p = p^r\}.
\]

It is clear that $\int_{B_r} dx = p^r$ and $\int_{S_r} dx = p^r (1 - \frac{1}{p})$.

The space $\mathbb{Q}_p^n$, consisting of points $x = (x_1, x_2, \cdots, x_n)$, where $x_j \in \mathbb{Q}_p$, is a locally compact metric measure space. The $p$-adic norm on $\mathbb{Q}_p^n$ is defined by
\[
|x|_p = \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n.
\]

Denote by $B^n_r = \{x \in \mathbb{Q}_p^n : |x|_p \leq p^r\}$ the ball of radius $p^r$ with the center $0 \in \mathbb{Q}_p$, $r \in \mathbb{Z}$. In fact,
\[
B^n_r = B_r \times B_r \times \cdots \times B_r.
\]

The Haar measure $dx$ on the field $\mathbb{Q}_p$ can be extended to a product measure $d^n x = dx_1 dx_2 \cdots dx_n$ on $\mathbb{Q}_p^n$ in the usual way.

A complex-valued function $\varphi$ defined on $\mathbb{Q}_p^n$ is called \textbf{locally-constant}, if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that
\[
\varphi(x + x') = \varphi(x), \quad \forall x' \in B^n_{l(x)}.
\]
We denote by $\mathcal{E}(\mathbb{Q}_p^n)$ the linear space of locally-constant functions, $\mathcal{D}(\mathbb{Q}_p^n)$ the linear space of locally-constant functions with compact supports, on $\mathbb{Q}_p^n$, respectively, and $\mathcal{D} = \mathcal{D}(\mathbb{Q}_p)$. $\mathcal{E} = \mathcal{E}(\mathbb{Q}_p)$ for short. Convergence in $\mathcal{E}(\mathbb{Q}_p^n)$ is defined in the following way: $\varphi_k \to 0$ in $\mathcal{E}(\mathbb{Q}_p^n)$ as $k \to \infty$ if and only if for any compact set $E \subset \mathbb{Q}_p^n$, $\varphi_k \to 0$ uniformly in $E$. Convergence in $\mathcal{D}(\mathbb{Q}_p^n)$ is defined that: $\varphi_k \to 0$ in $\mathcal{D}(\mathbb{Q}_p^n)$ as $k \to \infty$ if and only if all $\varphi_k$ assume constant values on cosets of a ball $B^n_l$ and are supported in a ball $B^n_N$, where $N, l$ are two numbers, not depending on $k$, and $\varphi_k \to 0$ uniformly. $\mathcal{D}(\mathbb{Q}_p^n)$ is called the test function class on $\mathbb{Q}_p^n$.

We denote by $\mathcal{D}'(\mathbb{Q}_p^n)$ the distribution space on $\mathcal{D}(\mathbb{Q}_p^n)$, $\mathcal{D}' = \mathcal{D}'(\mathbb{Q}_p)$. $\mathcal{D}'(\mathbb{Q}_p^n)$ is a complete topological space. Convergence in $\mathcal{D}'(\mathbb{Q}_p^n)$ is defined in the following way: $f_k \to 0$ as $k \to \infty$ in $\mathcal{D}'(\mathbb{Q}_p^n)$ if and only if $(f_k, \varphi) \to 0$ for any $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$.

For a compact set $E$, denote by $1_E$ the characteristic function of $E$. There is a canonical $\delta$-sequence $\delta^n_k = p^n k 1_{B^n_k}$, and a canonical 1-sequence $\Delta^n_k = 1_{B^n_k}$, $k \in \mathbb{Z}$, in $\mathcal{D}(\mathbb{Q}_p^n)$. It is easy to check $\delta^n_k \to \delta$ in $\mathcal{D}'(\mathbb{Q}_p^n)$ and $\Delta^n_k \to 1$ in $\mathcal{E}(\mathbb{Q}_p^n)$, as $k \to \infty$. Obviously, if we denote $\delta_k = \delta^n_1$ and $\Delta_k = \Delta^n_1$, then

$$\delta^n_k(x) = \delta_k(x_1) \delta_k(x_2) \cdots \delta_k(x_n), \quad x = (x_1, x_2, \cdots, x_n),$$

and

$$\Delta^n_k(x) = \Delta_k(x_1) \Delta_k(x_2) \cdots \Delta_k(x_n), \quad x = (x_1, x_2, \cdots, x_n).$$

The Fourier transform and inverse Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined by the formulæ

$$\varphi(\xi) = \int_{\mathbb{Q}_p^n} \varphi(x) \chi_p(-\xi \cdot x) d^n x, \quad \xi \in \mathbb{Q}_p^n;$$

$$\varphi^\vee(x) = \int_{\mathbb{Q}_p^n} \varphi(\xi) \chi_p(\xi \cdot x) d^n \xi, \quad x \in \mathbb{Q}_p^n,$$

where $\chi_p(\xi \cdot x) = \chi_p(\xi_1 x_1) \chi_p(\xi_2 x_2) \cdots \chi_p(\xi_n x_n) = e^{2\pi i \sum_{j=1}^n (\xi_j x_j)}$, $\xi \cdot x$ is the scalar product of $\xi$ and $x$, and the function $\chi_p(x)$ is a fixed non-trivial additive character on $\mathbb{Q}_p$ which is trivial on $B_0$. It is known that the Fourier transform and the inverse transform are linear isomorphisms from $\mathcal{D}(\mathbb{Q}_p^n)$ onto $\mathcal{D}(\mathbb{Q}_p^n)$. The transforms could be extended to distribution space. For each $f \in \mathcal{D}'(\mathbb{Q}_p^n)$, $f^\wedge$ and $f^\vee$ are defined by the relations

$$(f^\wedge, \varphi) = (f, \varphi^\wedge), \quad \forall \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

$$(f^\vee, \varphi) = (f, \varphi^\vee), \quad \forall \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

It is easy to see $\Delta^n_k \wedge = \delta^n_k$, $k \in \mathbb{Z}$.
For distributions \( f \in \mathcal{D}'(Q^n_p), g \in \mathcal{D}'(Q^m_p) \), the direct product of them is defined by
\[
(f(x) \times g(y), \varphi) = (f(x), (g(y), \varphi(x, y))), \quad \forall \varphi \in \mathcal{D}(Q^{n+m}_p),
\]
since any test function \( \varphi \in \mathcal{D}(Q^{n+m}_p) \) can be represented in a finite sum of the form
\[
\varphi(x, y) = \sum_k \varphi_k(x) \psi_k(y), \quad \varphi_k \in \mathcal{D}(Q^n_p), \quad \psi_k \in \mathcal{D}(Q^m_p).
\]
Thus \( f(x) \times g(y) \in \mathcal{D}'(Q^{n+m}_p) \). Moreover, the direct product is commutative, that is
\[
f(x) \times g(y) = g(y) \times f(x).
\]
Particularly, for \( g = 1 \), the above equality implies that
\[
(f(x), \int_{Q^n_p} \varphi(x, y)d^m y) = \int_{Q^m_p} (f(x), \varphi(x, y))d^m y, \quad \varphi \in \mathcal{D}(Q^{n+m}_p).
\]
The convolution \( f \ast g \) for distributions \( f, g \in \mathcal{D}'(Q^n_p) \) is defined\(^{[1,2]}\) that:
\[
(f \ast g, \varphi) = \lim_{k \to \infty} (f(x) \times g(y), \Delta_k(x)\varphi(x + y)) \quad \text{if the limit exists for all} \ \varphi \in \mathcal{D}(Q^n_p),
\]
where \( f(x) \times g(y) \) is the direct product of distributions \( f, g \). The formula
\[
(f \ast g)^\wedge = f^\wedge g^\wedge
\]
holds if the convolution \( f \ast g \) exists. If \( f, g \in \mathcal{D}'(Q^n_p) \) and \( \text{supp}g \subset B^N_N \) for some \( N \in \mathbb{Z} \), then the convolution \( f \ast g \) exists and
\[
(f \ast g, \varphi) = (f(x) \times g(y), \Delta_N^n(y)\varphi(x + y)), \quad \varphi \in \mathcal{D}(Q^n_p).
\]
Moreover, if \( g = \varphi \in \mathcal{D}(Q^n_p) \), then \( f \ast \varphi \in \mathcal{E}(Q^n_p) \) and \( f \ast \varphi \) takes the form
\[
(f \ast \varphi)(x) = (f(y), \varphi(x - y)), \quad x \in Q^n_p.
\]

3 \ n-dimensional pseudo-differential operator \( T^\alpha \)

In\(^{[8,9]}\), W.Y. Su made a definition of derivatives and integrals, of fractional orders, for general locally compact Vilenkin group \( G \), by using of pseudo-differential operators. The test function class \( \mathcal{D} \) and the distribution class \( \mathcal{D}' \) are invariant under these fractional operators.
For $\xi \in \mathbb{Q}_p$, denote $\langle \xi \rangle = \max \{1, |\xi|_p\}$. Obviously, $\langle \xi \rangle \in \mathcal{E}$. For $s \in \mathbb{R}$, $T^s$ is defined to be a pseudo-differential operator with the symbol $\langle \xi \rangle^s$ owing to the formula that

$$T^s \varphi = (\langle \xi \rangle^s \varphi^\wedge)^\vee, \quad \varphi \in \mathcal{D}.$$  

It is easy to check that $T^s \varphi$ exists in $\mathcal{D}$. The definition domain of $T^s$ can be extended to the distribution space $\mathcal{D}'$ by the relation

$$(T^s f, \varphi) = (f, T^s \varphi), \quad f \in \mathcal{D}', \quad \varphi \in \mathcal{D}.$$  

So for $f \in \mathcal{D}'$, we still have

$$T^s f = ((\langle \xi \rangle^s f^\wedge)^\vee.$$  

We call the operator $T^s$ the derivative operator on $\mathcal{D}'$ of order $s$ for $s > 0$, and the integral operator on $\mathcal{D}'$ of order $-s$ for $s < 0$. For $s = 0$, $T^0 f = f$ for all $f \in \mathcal{D}'$, $T^0$ is the identity operator.

In [19], the convolution kernel $\kappa_s$ of the pseudo-differential operator $T^s$ is given and some important properties of $\kappa_s$ are revealed which play a key role in problems related to fractional operator $T^s$.

$$\kappa_s = \left( \frac{1 - p^s}{1 - p^{-s-1}|x|_p^{-s-1}} + \frac{p^s - 1}{ps+1 - 1} \right) \Delta_0, \quad \text{for } s \neq 0, -1,$$

and $\kappa_0 = \delta$, $\kappa_{-1} = (1 - \frac{1}{p})(1 - \log_p |x|_p) \Delta_0$, where $|x|_p^{-s-1}$ is a distribution$^{[2,19]}$ in $\mathcal{D}'(\mathbb{Q}_p)$,

$$(|x|_p^{-s-1}, \varphi) = \int_{\mathbb{Q}_p} |x|_p^{-s-1}(\varphi(x) - \varphi(0))dx, \quad \varphi \in \mathcal{D}, \quad s \neq 0.$$  

The convolution kernel $\kappa_s$ has the following properties:

$$\kappa_s * \kappa_t = \kappa_{s+t}, \quad \forall s, t \in \mathbb{R}.$$  

Moreover, $\kappa_s$ is continuous on $s \in \mathbb{R}$.

We now consider the $n$-dimensional case.

Firstly, we give the definition of the partial differential operator $T^s_{x_j}$ for distributions in $\mathcal{D}'(\mathbb{Q}_p^n)$, $1 \leq j \leq n$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, it can be represented as a finite sum of the form

$$\varphi(x) = \sum_k \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\cdots \varphi_{k_n}(x_n), \quad \varphi_{k_j} \in \mathcal{D}.$$  

We define

$$T^s_{x_j} \varphi(x) = \sum_k \varphi_{k_1}(x_1) \cdots T^s \varphi_{k_j}(x_j) \cdots \varphi_{k_n}(x_n).$$
Obviously, the partial differential operator $T_{x_j}^*$ is well-defined and $T_{x_j}^* (\mathcal{D}(\mathbb{Q}_p^n)) = \mathcal{D}(\mathbb{Q}_p^n)$. We can extend the definition domain of the operator $T_{x_j}^*$ to $\mathcal{D}'(\mathbb{Q}_p^n)$ by the relation

$$(T_{x_j}^* f, \varphi) = (f, T_{x_j}^* \varphi), \quad f \in \mathcal{D}'(\mathbb{Q}_p^n), \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

And we have $T_{x_j}^* (\mathcal{D}'(\mathbb{Q}_p^n)) = \mathcal{D}'(\mathbb{Q}_p^n)$.

Secondly, we investigate the $n$-dimensional pseudo-differential operator $T^\alpha$ on $\mathcal{D}'(\mathbb{Q}_p^n)$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index, $\alpha_j \in \mathbb{R}$, with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. For $\alpha, \beta \in \mathbb{R}^n$, denote $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n)$. For $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^n$, denote $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. For example, for $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{Q}_p^n$, if we denote $\langle \xi \rangle = (\langle \xi_1 \rangle, \langle \xi_2 \rangle, \cdots, \langle \xi_n \rangle)$, then

$$\langle \xi \rangle^\alpha = \langle \xi_1 \rangle^{\alpha_1} \langle \xi_2 \rangle^{\alpha_2} \cdots \langle \xi_n \rangle^{\alpha_n}.$$

We write

$$\kappa_\alpha(x) = \kappa_{\alpha_1}(x_1) \times \kappa_{\alpha_2}(x_2) \times \cdots \times \kappa_{\alpha_n}(x_n),$$

where $\times$ is the direct product operation. In particular, for $\alpha = (0, 0, \cdots, 0)$,

$$\kappa_0(x) = \delta(x) = \delta(x_1) \times \delta(x_2) \times \cdots \times \delta(x_n).$$

We define the $n$-dimensional fractional operator $T^\alpha$ on the distribution class $\mathcal{D}'(\mathbb{Q}_p^n)$ by the following convolution form,

$$T^\alpha f = \kappa_\alpha \ast f,$$

and call $\kappa_\alpha$ the $n$-dimensional convolution kernel of $T^\alpha$. In particular, for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have

$$T^\alpha \varphi(x) = (\kappa_{\alpha_1}(y_1) \times \kappa_{\alpha_2}(y_2) \times \cdots \times \kappa_{\alpha_n}(y_n), \varphi(x - y)), \quad x \in \mathbb{Q}_p^n.$$

The following are some basic properties of the pseudo-differential operators $T^\alpha$ and their convolution kernels $\kappa_\alpha$.

**Proposition 3.1.** Let $\alpha, \beta \in \mathbb{R}^n$. Then

$$\kappa_\alpha \ast \kappa_\beta = \kappa_{\alpha + \beta}.$$

**Proof.**

$$\kappa_\alpha \ast \kappa_\beta = (\kappa_{\alpha_1} \ast \kappa_{\beta_1}) \times (\kappa_{\alpha_2} \ast \kappa_{\beta_2}) \times \cdots \times (\kappa_{\alpha_n} \ast \kappa_{\beta_n})$$

$$= \kappa_{\alpha_1 + \beta_1} \times \kappa_{\alpha_2 + \beta_2} \times \cdots \times \kappa_{\alpha_n + \beta_n} = \kappa_{\alpha + \beta}. \ ●$$
Proposition 3.2. Let $\alpha \in \mathbb{R}^n$. Then $\kappa^\wedge = \langle \xi \rangle^\alpha$.

Proof. 

\[ \kappa^\wedge = \kappa^\wedge_{\alpha_1} \times \kappa^\wedge_{\alpha_2} \cdots \kappa^\wedge_{\alpha_n} = \langle \xi \rangle^{\alpha_1} \langle \xi \rangle^{\alpha_2} \cdots \langle \xi \rangle^{\alpha_n} = \langle \xi \rangle^\alpha. \]

From the above two propositions, we obtain

Proposition 3.3. Let $\alpha \in \mathbb{R}^n$, $f \in \mathcal{D}'(Q_p^n)$. Then 

\[ T^\alpha f = \kappa_\alpha * f = (\langle \xi \rangle^\alpha f^\wedge)^\vee. \]

Proof. $(\kappa_\alpha * f)^\wedge = \kappa_\alpha^\wedge f^\wedge = \langle \xi \rangle^\alpha f^\wedge$. \#

Proposition 3.4. Let $\alpha, \beta \in \mathbb{R}^n$, $f \in \mathcal{D}'(Q_p^n)$. Then 

\[ T^{\alpha + \beta} f = T^\alpha T^\beta f = T^\beta T^\alpha f. \]

Proof. \[ T^{\alpha + \beta} f = \kappa_{\alpha + \beta} * f = \kappa_\alpha * \kappa_\beta * f = T^\alpha T^\beta f. \]

Proposition 3.5. Let $\alpha \in \mathbb{R}^n$, $f \in \mathcal{D}'(Q_p^n)$, $\varphi \in \mathcal{D}(Q_p^n)$. Then 

\[ (T^\alpha f, \varphi) = (f, T^\alpha \varphi). \]

Proof. Since $T^\alpha f = (\langle \xi \rangle^\alpha f^\wedge)^\vee$, we have 

\[ (T^\alpha f, \varphi) = (\langle \xi \rangle^\alpha f^\wedge, \varphi^\vee) = (f^\wedge, \langle \xi \rangle^\alpha \varphi^\vee) = (f, (\langle \xi \rangle^\alpha \varphi^\wedge)^\vee) = (f, T^\alpha \varphi). \]

Proposition 3.6. \[ \mathcal{D}(Q_p^n), \mathcal{E}(Q_p^n) \text{ and } \mathcal{D}'(Q_p^n) \text{ are invariant under the operators } T^\alpha. \]

Proof. We only prove the $\mathcal{D}(Q_p^n)$ case, since the others can be obtained by similar arguments. For $\varphi \in \mathcal{D}(Q_p^n)$, we have $\varphi^\wedge \in \mathcal{D}(Q_p^n)$, then $\langle \xi \rangle^\alpha \varphi^\wedge \in \mathcal{D}(Q_p^n)$, since $\langle \xi \rangle^\alpha \in \mathcal{E}(Q_p^n)$. Thus $T^\alpha \varphi = (\langle \xi \rangle^\alpha \varphi^\wedge)^\vee \in \mathcal{D}(Q_p^n)$. Hence $T^\alpha (\mathcal{D}(Q_p^n)) \subset \mathcal{D}(Q_p^n)$.

On the other hand, let $\psi \in \mathcal{D}(Q_p^n)$, consider the equation $T^\alpha \varphi = \psi$. Let $\varphi = T^{-\alpha} \psi = \kappa_{-\alpha} * \psi$, then from Proposition 3.1,

\[ T^\alpha \varphi = \kappa_{\alpha} * \varphi = \kappa_{\alpha} * \kappa_{-\alpha} * \psi = \delta * \psi = \psi. \]

Hence, $T^\alpha (\mathcal{D}(Q_p^n)) \supset \mathcal{D}(Q_p^n)$. \#

Theorem 3.1. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $f \in \mathcal{D}'(Q_p^n)$. Then 

\[ T^\alpha = T_{x_1}^{\alpha_1} \circ T_{x_2}^{\alpha_2} \circ \cdots \circ T_{x_n}^{\alpha_n}, \]

where $\circ$ denotes the composition operation. Moreover, the compositions are commutable.
Proof. Let \( \varphi \in D(\mathbb{Q}_p^n) \). It must have a finite sum form that
\[
\varphi(x) = \sum_k \varphi_k(x_1)\varphi_k(x_2)\cdots\varphi_k(x_n), \quad \varphi_k \in D.
\]

Then
\[
T^\alpha \varphi = \sum_k T^\alpha \varphi_k(x_1)\varphi_k(x_2)\cdots\varphi_k(x_n)
= \sum_k (\kappa_{\alpha_1} \ast \varphi_k_1)(\kappa_{\alpha_2} \ast \varphi_k_2)\cdots(\kappa_{\alpha_n} \ast \varphi_k_n)
= \sum_k T^\alpha_{x_1} \varphi_k T^\alpha_{x_2} \varphi_k_2 \cdots T^\alpha_{x_n} \varphi_k_n
= \sum_k T^\alpha_{x_1} \circ T^\alpha_{x_2} \circ \cdots \circ T^\alpha_{x_n} \varphi_k_1(x_1)\varphi_k_2(x_2)\cdots\varphi_k_n(x_n)
= T^\alpha_{x_1} \circ T^\alpha_{x_2} \circ \cdots \circ T^\alpha_{x_n} \varphi.
\]

Let \( f \in D'(\mathbb{Q}_p^n) \). Then for \( \varphi \in D(\mathbb{Q}_p^n) \), using Proposition 3.5, we have
\[
(T^\alpha f, \varphi) = (f, T^\alpha \varphi) = (f, T^\alpha_{x_1} \circ T^\alpha_{x_2} \circ \cdots \circ T^\alpha_{x_n} \varphi) = (T^\alpha_{x_1} \circ T^\alpha_{x_2} \circ \cdots \circ T^\alpha_{x_n} f, \varphi). \]

Taking \( \alpha = (0, \cdots, 0, \alpha_j, 0, \cdots, 0) \) in the above theorem, we immediately get

**Corollary 3.1.** Let \( 1 \leq j \leq n, \alpha_j \in \mathbb{R}, f \in D'(\mathbb{Q}_p^n) \). Then
\[
T^\alpha_{x_j} f = T^{(0, \cdots, 0, \alpha_j, 0, \cdots, 0)} f.
\]

Finally, we give some examples.

**Example 3.1.** \( T^\alpha 1 = 1 \).

*Proof.* Since \( 1^\alpha = \delta \), \( \forall \varphi \in D(\mathbb{Q}_p^n) \), we have
\[
(T^\alpha 1, \varphi) = ((\xi)^\alpha \delta, \varphi) = ((\xi)^\alpha \delta(\xi), \varphi^\vee(\xi)) = (\delta(\xi), (\xi)^\alpha \varphi^\vee(\xi)) = \varphi^\vee(0) = (1, \varphi). \]

**Example 3.2.** \( T^\alpha \delta = \kappa_\alpha \).

*Proof.* \( T^\alpha \delta = \kappa_\alpha \ast \delta = \kappa_\alpha. \)

**Example 3.3.** Let \( \alpha = (-1, -1, \cdots, -1) \), \( \varphi \in D(\mathbb{Q}_p^n) \). Then
\[
T^\alpha \varphi(x) = T^{-1, -1, \cdots, -1} \varphi(x) = (1 - \frac{1}{p})^n \int_{x + B_0^n} \prod_{j=1}^n (1 - \log_p |y_j - x_j|_p) \varphi(y) d^m y.
\]
Example 3.4. Let $\alpha \in \mathbb{R}^n$, $\eta \in \mathbb{Q}_p^n$. Then

$$T^\alpha \varphi = \kappa_\alpha \ast \varphi = \sum_k \kappa_{-1} \ast \varphi_{1}(x_1) \kappa_{-1} \ast \varphi_{2}(x_2) \cdots \kappa_{-1} \ast \varphi_{n}(x_n)$$

Proof. Since $\varphi$ can be represented as a finite sum of the form

$$\varphi = \sum_k \varphi_{k_1}(x_1) \varphi_{k_2}(x_2) \cdots \varphi_{k_n}(x_n), \quad \varphi_{k_j} \in \mathcal{D},$$

$$T^\alpha \varphi = \kappa_\alpha \ast \varphi = \sum_k \kappa_{-1} \ast \varphi_{1}(x_1) \kappa_{-1} \ast \varphi_{2}(x_2) \cdots \kappa_{-1} \ast \varphi_{n}(x_n)$$

$$= \sum_k \prod_{j=1}^n (1 - \frac{1}{p^n}) \int_{x_j + B_0^n} (1 - \log_p |y_j - x_j|_p) \varphi_{k_j}(y_j) dy_j$$

$$= (1 - \frac{1}{p^n}) \sum_k \prod_{j=1}^n (1 - \log_p |y_j - x_j|_p) \varphi_{k_1}(y_1) \varphi_{k_2}(y_2) \cdots \varphi_{k_n}(y_n) dy_1 dy_2 \cdots dy_n$$

$$= (1 - \frac{1}{p^n}) \sum_k \prod_{j=1}^n (1 - \log_p |y_j - x_j|_p) \varphi(y) d^n y. \quad \sharp$$

4 the Laplacian $\Delta_p$

Now we introduce the Laplacian $\Delta_p$ on $\mathbb{Q}_p^n$. $\Delta_p$ is an operator that

$$\Delta_p f(x) = \sum_{j=1}^n T_{x_j}^2 f(x), \quad f \in \mathcal{D}'(\mathbb{Q}_p^n).$$

If we denote by $e^j = (0, \cdots, 0, 1, 0, \cdots, 0)$, the $j$-th unit vector of $\mathbb{R}^n$, then

$$\Delta_p f(x) = \sum_{j=1}^n T^{2e^j} f(x).$$

Since $\sum_{j=1}^n T^{2e^j} f = (\sum_{j=1}^n \langle \xi_j \rangle^2 f^\wedge)^\wedge$, the Laplacian $\Delta_p$ is a pseudo-differential operator with the symbol $\sum_{j=1}^n \langle \xi_j \rangle^2$. 

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Example 4.1. Let $\psi(x) = \chi_p(p^{-1}e \cdot x)\Delta_0^n$, with $e = (1,1,\ldots,1)$. Then $\psi(x)$ is an eigen-function of the Laplacian $\Delta_p$,

$$\Delta_p\psi(x) = np^2\psi(x).$$

Proof. We can easily get that,

$$\psi^\wedge(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(p^{-1}e \cdot x)\Delta_0^n \chi_p(-\xi \cdot x) d^n x = \int_{B_0^n} \chi_p(x \cdot (p^{-1}e - \xi)) d^n x = 1_{p^{-1}e+B_0^n},$$

and

$$\sum_{j=1}^n \langle \xi \rangle^2 \psi^\wedge(\xi) = \sum_{j=1}^n \langle \xi \rangle^2 1_{p^{-1}e+B_0^n} = np^2 1_{p^{-1}e+B_0^n}.$$

Hence,

$$\Delta_p\psi(x) = (\sum_{j=1}^n \langle \xi \rangle^2 \psi^\wedge(\xi))^\wedge(x) = np^2 \int_{p^{-1}e+B_0^n} \chi_p(\xi \cdot x) d^n \xi$$

$$= np^2 \int_{B_0^n} \chi_p(p^{-1}e \cdot x)\chi_p(\xi \cdot x) d^n \xi = np^2 \chi_p(p^{-1}e \cdot x)\Delta_0^n = np^2\psi(x).$$

Example 4.2. Let $\psi(x) = \chi_p(p^{-1}e \cdot x)\Delta_0^n$, $a \in \mathbb{Q}_p, a \neq 0, b = (b_1,b_2,\ldots,b_n) \in \mathbb{Q}_p^n$. Then

$$\Delta_p\psi(ax+b) = \begin{cases} np^2|a|^2\psi(ax+b), & \text{for } |a|_p > p^{-1}; \\ np\psi(ax+b), & \text{for } |a|_p \leq p^{-1}. \end{cases}$$

Proof. The Fourier transform of $\psi(ax+b)$ is

$$(\psi(ax+b))^\wedge(\xi) = |a|^{-n}_{p}\chi_p(\frac{b \cdot \xi}{a})\psi^\wedge(\frac{\xi}{a}) = |a|^{-n}_{p}\chi_p(\frac{b \cdot \xi}{a})1_{a(p^{-1}e+B_0^n)}.$$

Hence,

$$\Delta_p\psi(ax+b) = (\sum_{j=1}^n \langle \xi \rangle^2 (\psi(ax+b))^\wedge(\xi))^\wedge(x)$$

$$= \int_{\mathbb{Q}_p^n} \sum_{j=1}^n \langle \xi \rangle^2 |a|^{-n}_{p}\chi_p(\frac{b \cdot \xi}{a})1_{a(p^{-1}e+B_0^n)}\chi_p(\xi \cdot x) d^n \xi$$

$$= \int_{a(p^{-1}e+B_0^n)} \sum_{j=1}^n \langle \xi \rangle^2 |a|^{-n}_{p}\chi_p((x + \frac{b}{a}) \cdot \xi) d^n \xi.$$
For $|a|_p \leq p^{-1}$, we have $a(p^{-1}e + B^n_0) \subset B^n_0$, then

$$
\Delta_p \psi(ax + b) = n \int_{a(p^{-1}e + B^n_0)} |a|_p^{-n} \chi_p((x + \frac{b}{a}) \cdot \xi)d^n\xi
$$

$$
= n \int_{B^n_0} \chi_p((x + \frac{b}{a}) \cdot (ap^{-1}e + a\xi))d^n\xi
$$

$$
= n \int_{B^n_0} \chi_p(p^{-1}e \cdot (ax + b))\chi_p(\xi \cdot (ax + b))d^n\xi
$$

$$
= n\chi_p(p^{-1}e \cdot (ax + b))\Delta^n_0(ax + b) = np\psi(ax + b).
$$

For $|a|_p > p^{-1}$, noticing that $\forall \xi \in a(p^{-1}e + B^n_0)$, $|\xi|_p = p|a|_p > 1$, we have

$$
\Delta_p \psi(ax + b) = \int_{a(p^{-1}e + B^n_0)} \sum_{j=1}^n |\xi_j|^2 |a|_p^{-n} \chi_p((x + \frac{b}{a}) \cdot \xi)d^n\xi
$$

$$
= np^2|a|_p^2 \int_{a(p^{-1}e + B^n_0)} |a|_p^{-n} \chi_p((x + \frac{b}{a}) \cdot \xi)d^n\xi = np^2|a|_p^2 \psi(ax + b).
$$

Let

$$
P(x) = \sum_r a_r x^r = \sum_{r_1, r_2, \ldots, r_n} a_{r_1, r_2, \ldots, r_n} x_{r_1} x_{r_2} \cdots x_{r_n}
$$

be a polynomial defined on $\mathbb{R}^n$, where $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ are multi-indexes and $a_r \in \mathbb{C}$ are constants. Let $\mathcal{P}$ be a pseudo-differential operator on $\mathcal{D}'(\mathbb{Q}^n_p)$, with the kernel $P(\xi)$, i.e.,

$$
\mathcal{P}f = (P(\xi)g^\Lambda)^\vee, \quad f \in \mathcal{D}'(\mathbb{Q}^n_p).
$$

In particular, if $P(\xi) = \sum_{j=1}^n |\xi|^2$, then $\mathcal{P} = \Delta_p$.

Let us consider the equation

$$
\mathcal{P}f = g, \quad g \in \mathcal{D}'(\mathbb{Q}^n_p). \tag{4.1}
$$

**Theorem 4.1.** If $P(x) \neq 0$ when all $x_i \geq 1$, then the equation (4.1) has a unique solution in $\mathcal{D}'(\mathbb{Q}^n_p)$ that

$$
f = (P^{-1}(\xi)g^\Lambda)^\vee.
$$

**Proof.** Since $P(x) \neq 0$ when all $x_i \geq 1$, the functions $P(\xi)$ and $P^{-1}(\xi)$ are both belong to $\mathcal{E}(\mathbb{Q}^n_p)$. Let $f = (P^{-1}(\xi)g^\Lambda)^\vee$. Then

$$
\mathcal{P}f = (P(\xi)f^\Lambda)^\vee = (P(\xi)P^{-1}(\xi)g^\Lambda)^\vee = g.
$$

For the uniqueness, we need to investigate solutions of the homogeneous equation

$$
\mathcal{P}f = 0. \tag{4.2}
$$
By applying to the equation (4.2) the Fourier transform, we get

\[ P(\langle \xi \rangle)f^\wedge = 0. \]

As \( P(x) \neq 0 \) when all \( x_i \geq 1 \), we have \( P(\langle \xi \rangle) \neq 0 \), then \( f^\wedge = 0 \), so \( f = 0 \). Thus the homogeneous equation (4.2) has only a trivial solution. \( \sharp \)

A fundamental solution of (4.1) is a distribution \( f \) such that \( Pf = \delta \).

**Theorem 4.2.** The equation (4.1) has a fundamental solution

\[ f_P(x) = (P^{-1}(\langle \xi \rangle))^\vee, \quad \text{i.e.,} \quad Pf = \delta. \]

**Proof.**

\[ Pf = (P(\langle \xi \rangle)P^{-1}(\langle \xi \rangle))^\vee = 1^\vee = \delta. \] \( \sharp \)

This theorem shows that the solution of equation (4.1) can be represented as

\[ f = f_P \ast g. \]

**Corollary 4.1.** If there exists a function of finite sum \( Q(x) = \sum_s b_s x^s \) defined on \( \mathbb{R}^n \), \( b_s \in \mathbb{C}, s \in \mathbb{R}^n \), such that \( Q(x) = P^{-1}(x) \), then the fundamental solution of (4.1) is

\[ f_P = \sum_s b_s \kappa_s. \]

**Proof.** Using Proposition 3.2, we obtain

\[ f_P = (Q(\langle \xi \rangle))^\vee = (\sum_s b_s (\langle \xi \rangle)^s)^\vee = \sum_s b_s (\langle \xi \rangle)^s)^\vee = \sum_s b_s \kappa_s. \] \( \sharp \)

**Corollary 4.2.** The Poisson equation \( \Delta_p f = g, g \in \mathcal{D}'(\mathbb{Q}^n_p) \) has a fundamental solution

\[ f_\Delta = (\frac{1}{(\xi_1^2 + (\xi_2^2 + \cdots + (\xi_n^2)^2)^\vee}, \text{ which is a distribution with support contained in } B^n_0. \]

**Proof.** Noticing that the function \( P(x) = x_1^2 + x_2^2 + \cdots + x_n^2 \), we have

\[ f_\Delta = (P^{-1}(\langle \xi \rangle))^\vee = \left( \frac{1}{(\xi_1^2 + (\xi_2^2 + \cdots + (\xi_n^2)^2)^\vee. \right. \]

\( \text{supp} f_\alpha \subset B_0 \) is a direct corollary of the fact that \( P^{-1}(\langle \xi \rangle) \in \mathcal{E} \), taking constant values on cosets of \( B^n_0 \). \( \sharp \)
5 Special properties of the Laplacian $\Delta_p$

The Laplacian $\Delta_p$ is a pseudo-differential operator with the symbol $\sum_{j=1}^{n} (\langle \xi_j \rangle)^2$,

$$\Delta \psi = \left( \sum_{j=1}^{n} (\langle \xi_j \rangle^2 \psi^*) \right)^\vee, \forall \psi.$$ 

It can be defined on those functions $\psi$ in the Hilbert space $L^2(Q^n_p)$, satisfying $\sum_{j=1}^{n} (\langle \xi_j \rangle)^2 \psi^* \in L^2(Q^n_p)$. We denote the collection of these functions by $D(\Delta_p)$, and call it the **domain of the Laplacian** $\Delta_p$ in $L^2(Q^n_p)$.

**Lemma 5.1.** $(\sum_{j=1}^{n} (\langle \xi_j \rangle)^2)^\rho \in L^2(Q^n_p)$ if and only if $\rho < -\frac{n}{4}$.

**Proof.** If $\rho < -\frac{n}{4}$, then

$$\int_{Q^n_p} \left( \sum_{j=1}^{n} (\langle \xi_j \rangle)^2 \right)^{2\rho} d^n \xi = \int_{B^n_0} n^{2\rho} d^n \xi + \int_{Q^n_p \setminus B^n_0} \left( \sum_{j=1}^{n} (\langle \xi_j \rangle)^2 \right)^{2\rho} d^n \xi$$

$$= n^{2\rho} + \sum_{r=1}^{\infty} \int_{|\xi|_p = p^r} \left( \sum_{j=1}^{n} (\langle \xi_j \rangle)^2 \right)^{2\rho} d^n \xi$$

$$\leq n^{2\rho} + \sum_{r=1}^{\infty} (p^{2r})^{2\rho} p^{rn} (1 - \frac{1}{p^n})$$

$$= n^{2\rho} + (1 - \frac{1}{p^n}) \sum_{r=1}^{\infty} p^{(n+4\rho)r}$$

$$< \infty.$$ 

If $\rho \geq -\frac{n}{4}$, then

$$\int_{Q^n_p} \left( \sum_{j=1}^{n} (\langle \xi_j \rangle)^2 \right)^{2\rho} d^n \xi = n^{2\rho} + \sum_{r=1}^{\infty} \int_{|\xi|_p = p^r} \left( \sum_{j=1}^{n} (\langle \xi_j \rangle)^2 \right)^{2\rho} d^n \xi$$

$$\geq n^{2\rho} + \min(n^{2\rho}, 1) \sum_{r=1}^{\infty} (p^{2r})^{2\rho} p^{rn} (1 - \frac{1}{p^n})$$

$$= n^{2\rho} + \min(n^{2\rho}, 1)(1 - \frac{1}{p^n}) \sum_{r=1}^{\infty} p^{(n+4\rho)r}$$

$$= \infty.$$ 

**Theorem 5.1.** $D(\Delta_p) \not\subset L^2(Q^n_p)$ and $\Delta_p(D(\Delta_p)) = L^2(Q^n_p)$. Furthermore, $D(\Delta_p)$ is dense in $L^2(Q^n_p)$. 

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Proof. By Lemma 5.1 and the fact that the Fourier transform is a unitary operator in \( L^2(Q^n_p) \), there exists a \( \psi \in L^2(Q^n_p) \), such that \( \psi^\wedge = (\sum_{j=1}^n (\xi_j)^2)^{-\frac{n}{2}} \in L^2(Q^n_p) \).

Still by Lemma 5.1, we have
\[
\sum_{j=1}^n (\xi_j)^2 \psi^\wedge = (\sum_{j=1}^n (\xi_j)^2)^{-\frac{n}{2}} \notin L^2(Q^n_p).
\]

Thus \( \psi \in L^2(Q^n_p) \), but \( \psi \notin \mathcal{D}(\Delta_p) \). So \( \mathcal{D}(\Delta_p) \subseteq L^2(Q^n_p) \).

Let \( \varphi \in L^2(Q^n_p) \). Consider the solution of the equation \( \Delta_p \psi = \varphi \), i.e.,
\[
\psi = \left( \left( \sum_{j=1}^n (\xi_j)^2 \right)^{-1} \varphi^\wedge \right)^\vee.
\]

Then \( \varphi \in L^2(Q^n_p) \) and \( \left| \left( \sum_{j=1}^n (\xi_j)^2 \right)^{-1} \right| \leq n^{-1} \) implies that \( \left( \sum_{j=1}^n (\xi_j)^2 \right)^{-1} \varphi^\wedge \in L^2(Q^n_p) \), so \( \psi \in L^2(Q^n_p) \). Hence, \( \Delta_p(\mathcal{D}(\Delta_p)) = L^2(Q^n_p) \).

Noticing the fact that \( \mathcal{D}(Q^n_p) \subset \mathcal{D}(\Delta_p) \) and \( \mathcal{D}(Q^n_p) \) is dense in \( L^2(Q^n_p) \), we get the density of \( \mathcal{D}(\Delta_p) \) in \( L^2(Q^n_p) \). ♯

**Theorem 5.2.** The Laplacian \( \Delta_p \) is a non-negative self-adjoint operator on \( L^2(Q^n_p) \).

Proof. Using the Parseval equality, we can easily get the following formula:
\[
(\Delta_p \psi, \varphi) = \int_{Q^n_p} \sum_{j=1}^n \langle \xi_j \rangle^2 \psi^\wedge(\xi) \overline{\varphi^\wedge(\xi)} d^n \xi = (\psi, \Delta_p \varphi), \quad \forall \psi, \varphi \in \mathcal{D}(\Delta_p),
\]

\[
||\Delta_p \psi||^2 = (\Delta_p \psi, \Delta_p \psi) = \int_{Q^n_p} \left( \sum_{j=1}^n \langle \xi_j \rangle^2 \right)^2 \psi^\wedge(\xi)^2 d^n \xi, \quad \psi \in \mathcal{D}(\Delta_p).
\]

Here \((\cdot, \cdot)\) is the scalar product in the Hilbert space \( L^2(Q^n_p) \), and \( || \cdot || \) is the \( L^2 \)-norm. Moreover,
\[
(\Delta_p \psi, \psi) = \int_{Q^n_p} \sum_{j=1}^n \langle \xi_j \rangle^2 |\psi^\wedge(\xi)|^2 d^n \xi > 0, \quad 0 \neq \psi \in \mathcal{D}(\Delta_p). \quad ♯
\]

There is a non-negative self-adjoint operator \([20] \Delta_p^{\frac{1}{2}} \) with the symbol \( (\sum_{j=1}^n \langle \xi_j \rangle^2)^{\frac{1}{2}} \), associated with \( \Delta_p \). The domain of \( \Delta_p^{\frac{1}{2}} \) is
\[
\mathcal{D}(\Delta_p^{\frac{1}{2}}) = \{ \psi \in L^2(Q^n_p) : (\sum_{j=1}^n \langle \xi_j \rangle^2)^{\frac{1}{2}} \psi^\wedge \in L^2(Q^n_p) \}.
\]
We have
$$D(\Delta_p) = \{ \psi : \psi \in D(\Delta_p^\frac{1}{2}) \text{ and } \Delta_{p,\alpha}^\frac{1}{2} \psi \in D(\Delta_p^\frac{1}{2}) \}.$$  

Furthermore, there is a non-negative quadratic form $Q(\cdot, \cdot)$ on $L^2(\mathbb{Q}_p^n)$ with domain $D(\Delta_p^\frac{1}{2}) \times D(\Delta_p^\frac{1}{2})$ such that
$$Q(\psi, \varphi) = (\Delta_p^\frac{1}{2} \psi, \Delta_p^\frac{1}{2} \varphi), \quad \forall \psi, \varphi \in D(\Delta_p^\frac{1}{2}),$$

If one define $Q_* (\psi, \varphi) = Q(\psi, \varphi) + (\psi, \varphi)$ for any $\psi, \varphi \in D(\Delta_p^\frac{1}{2})$, then $(D(\Delta_p^\frac{1}{2}), Q_*(\cdot, \cdot))$ is a Hilbert space.

**Proposition 5.1.** For any $\eta \in \mathbb{Q}_p^n$, the additive character $\chi_p(\eta \cdot x)$ is an eigen-function of the Laplacian $\Delta_p$ with respect to the eigen-value $\sum_{j=1}^n (\eta_j)^2$.

**Proof.** Using Example 3.4, we have
$$\Delta_p \chi_p(\eta \cdot x) = \sum_{j=1}^n T^{2\alpha_j} \chi_p(\eta \cdot x) = \sum_{j=1}^n (\eta_j)^2 \chi_p(\eta \cdot x) = \sum_{j=1}^n (\eta_j)^2 \chi_p(\eta \cdot x). \quad \#$$

Let us consider the eigen-value problem in $\mathbb{Q}_p^n$,
$$\Delta_p \psi = \lambda \psi, \quad \psi \in L^2(\mathbb{Q}_p^n). \quad (5.1)$$

From Theorem 5.2, the spectrum of the operator $\Delta_p$ consists of non-negative eigenvalues.

Let $\lambda = 0$. Then $\Delta_p \psi = 0$, which implies $\psi = 0$ from Theorem 5.1. Hence, $\lambda = 0$ is not an eigen-value of $\Delta_p$.

Let $\lambda > 0$. Applying to the equation (5.1) the Fourier transform, we get
$$(\sum_{j=1}^n (\xi_j)^2 - \lambda) \psi^\wedge(\xi) = 0.$$  

From here we conclude that the eigen-values of the Laplacian $\Delta_p$ have the form
$$\lambda_{N_1, N_2, \ldots, N_n} = \sum_{j=1}^n \beta^{2N_j}, \quad N_j \in \mathbb{Z}^+, \quad j = 1, 2, \ldots, n.$$  

Now we construct an orthonormal basis of eigen-functions of the Laplacian $\Delta_p$ in $L^2(\mathbb{Q}_p^n)$.

Recall that in the 1-dimensional case, an orthonormal basis of eigen-functions of $T^*$ in $L^2(\mathbb{Q}_p)$ is given in [19].
Lemma 5.2. \cite{Kozyrev19} Let \( n = 1, s \in \mathbb{R} \). The set of test functions \( \{\psi_{Nk\varepsilon}(x)\} \) is an orthonormal basis of eigen-functions of \( T^s \) in \( L^2(\mathbb{Q}_p) \), where

\[
\psi_{Nk\varepsilon}(x) = p^{-\frac{N}{2}} x_p(p^{N-1}kx) \Delta_0(p^N x - \varepsilon), \quad N \in \mathbb{Z}, \quad k = 1, 2, \cdots, p - 1, \quad \varepsilon \in \mathbb{Q}_p/B_0.
\]

Moreover,

\[
T^s \psi_{1-N,k\varepsilon}(x) = \begin{cases} 
   p^{Ns} \psi_{1-N,k\varepsilon}(x), & \text{for } N > 0, \\
   \psi_{1-N,k\varepsilon}(x), & \text{for } N \leq 0.
\end{cases}
\]

The orthonormal basis \( \{\psi_{Nk\varepsilon}(x)\} \) is a \( p \)-adic wavelet basis in \( L^2(\mathbb{Q}_p) \) constructed by S.V. Kozyrev\cite{Kozyrev21}.

For the Laplacian \( \Delta_p \), we have

**Theorem 5.3.** The set of test functions \( \{\prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j)\} \) is an orthonormal basis of eigen-functions of the Laplacian \( \Delta_p \) in \( L^2(\mathbb{Q}_p^n) \), where \( N_j \in \mathbb{Z}, k_j = 1, 2, \cdots, p - 1, \varepsilon_j \in \mathbb{Q}_p/B_0, j = 1, 2, \cdots, n \). Moreover,

\[
\Delta_p \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j) = \sum_{j=1}^n p^{2 \max\{0,N_j\}} \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j).
\]

Proof. Taking \( \psi(x) = \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j) \), using Lemma 5.2, we have

\[
\Delta_p \psi(x) = \Delta_p \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j) = \sum_{j=1}^n \prod_{j'=1}^n T^2_{x_j} \psi_{N_j k_j \varepsilon_j}(x_j)
\]

\[
= \sum_{j=1}^n \prod_{1 \leq j' \leq n, j' \neq j} \psi_{N_j k_j \varepsilon_j}(x_j) T^2_{x_j} \psi_{N_j k_j \varepsilon_j}(x_j)
\]

\[
= \sum_{j=1}^n \prod_{1 \leq j' \leq n, j' \neq j} \psi_{N_j k_j \varepsilon_j}(x_j) p^{2 \max\{0,1-N_j\}} \psi_{N_j k_j \varepsilon_j}(x_j)
\]

\[
= \sum_{j=1}^n p^{2 \max\{0,1-N_j\}} \psi(x).
\]

For the orthogonality of \( \{\prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j)\} \), consider the scalar product \( \langle \psi, \varphi \rangle \) in \( L^2(\mathbb{Q}_p^n) \), where \( \psi(x) = \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j) \) and \( \varphi(x) = \prod_{j=1}^n \psi_{N'_j k'_j \varepsilon'_j}(x_j) \).

\[
\langle \psi(x), \varphi(x) \rangle = \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j), \prod_{j=1}^n \psi_{N'_j k'_j \varepsilon'_j}(x_j)
\]

\[
= \prod_{j=1}^n \psi_{N_j k_j \varepsilon_j}(x_j), \psi_{N'_j k'_j \varepsilon'_j}(x_j) = \prod_{j=1}^n \delta_{N_j N'_j} \delta_{k_j k'_j} \delta_{\varepsilon_j \varepsilon'_j}.
\]
For the completeness of \( \{ \prod_{j=1}^{n} \psi_{N_{j}k_{j}e_{j}}(x_{j}) \} \), consider the Fourier coefficient of \( \Delta_{0}^{a} \).

\[
(\Delta_{0}^{a}, \prod_{j=1}^{n} \psi_{N_{j}k_{j}e_{j}}) = \prod_{j=1}^{n} p^{-N_{j}} \int_{B_{0}^{p} - N_{j}k_{j}} \chi_{p}(-p^{N_{j} - 1}k_{j}x_{j}) dx_{j} = \prod_{j=1}^{n} p^{-N_{j}} \delta_{e_{j},B_{0}} \gamma(N_{j}),
\]

where \( \gamma \) is a function defined as \( \gamma(t) = 0 \) if \( t \leq 0 \), \( \gamma(t) = 1 \) if \( t \geq 1 \).

Hence,

\[
\sum_{j=1}^{n} |(\Delta_{0}^{a}, \prod_{j=1}^{n} \psi_{N_{j}k_{j}e_{j}})|^2 = \sum_{j=1}^{n} \prod_{j=1}^{n} p^{-N_{j}} \delta_{e_{j},B_{0}} \gamma(N_{j})
\]

\[
= \sum_{1 \leq N_{j} < +\infty, j=1,2,\ldots,n} p^{-N_{j}} = (p - 1)^{n} \prod_{j=1}^{n} p^{-N_{j}} = 1 = \| \Delta_{0}^{a} \|^2.
\]

Thus the Parseval equality of \( \Delta_{0}^{a} \) holds, which proves the completeness of \( \{ \prod_{j=1}^{n} \psi_{N_{j}k_{j}e_{j}}(x_{j}) \} \). #

### 6 Cauchy problem for wave equations on \( Q_{p}^{n} \)

In this section, we consider the initial value problem

\[
\begin{align*}
\frac{\partial^{2}u}{\partial t^{2}} - a\Delta_{p}^{s}u &= f(x,t), \quad x \in Q_{p}^{n}, \quad 0 < t \leq T, \\
u(x,0) &= \varphi(x), \quad x \in Q_{p}^{n}, \\
u_{t}(x,0) &= \psi(x), \quad x \in Q_{p}^{n},
\end{align*}
\]

(6.1)

where \( a \neq 0 \), \( s \in \mathbb{R}, \quad T > 0 \), the function \( f \) and the initial function \( \varphi \) and \( \psi \) are complex valued.

**Theorem 6.1.** The homogeneous equation

\[
\begin{align*}
\frac{\partial^{2}u}{\partial t^{2}} - a\Delta_{p}^{s}u &= 0, \quad x \in Q_{p}^{n}, \quad 0 < t \leq T, \\
u(x,0) &= 0, \quad x \in Q_{p}^{n}, \\
u_{t}(x,0) &= \psi(x), \quad x \in Q_{p}^{n},
\end{align*}
\]

(6.2)

has a fundamental solution

\[
E(x,t) = \begin{cases} 
\left( \frac{e^{\pi(\sum_{j=1}^{n}(\xi_{j})^{2})^{1/2}t - e^{-\pi(\sum_{j=1}^{n}(\xi_{j})^{2})^{1/2}t}}}{2\pi(\sum_{j=1}^{n}(\xi_{j})^{2})^{1/2}} \right)^{\nu}(x), & \text{for } a > 0, \\
\left( \frac{\sin(\sqrt{-a}(\sum_{j=1}^{n}(\xi_{j})^{2})^{1/2}t)}{\sqrt{-a}(\sum_{j=1}^{n}(\xi_{j})^{2})^{1/2}} \right)^{\nu}(x), & \text{for } a < 0,
\end{cases}
\]

where \( E(x,t) \in \mathcal{D}'(Q_{p}^{n}) \) has a compact support in \( B_{0}^{n} \) for any \( t \in [0,T] \). Moreover, for \( \psi \in \mathcal{D}'(Q_{p}^{n}) \) the equation (6.2) has a solution

\[
u(x,t) = E(x,t) * \psi.
\]
Proof. Let \( \psi = \delta \), denote by \( E(x,t) \) the fundamental solution of (6.2). Applying to (6.2) the Fourier transform, we get

\[
\frac{\partial^2 E(\xi,t)}{\partial t^2} = a\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right) E(\xi,t),
\]

\[ E(\xi,0) = 0, \quad E^\wedge_t(\xi,0) = 1. \]

If \( a > 0 \), then

\[
E^\wedge(\xi,t) = C_1 e^{-\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t} + C_2 e^{\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t},
\]

where \( C_1 \) and \( C_2 \) are two constants satisfying

\[ C_1 + C_2 = 0, \]

and

\[-C_1 \sqrt{a} \left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} + C_2 \sqrt{a} \left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} = 1.\]

So

\[-C_1 = C_2 = \frac{1}{2\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}}}.\]

Hence,

\[ E^\wedge(\xi,t) = \frac{e^{-\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t} - e^{\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t}}{2\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}}},\]

and \forall t \in [0, T], \( E^\wedge(\xi,t) \in \mathcal{E}(\mathbb{Q}_p^n) \) assumes constant values on cosets of \( B_0^n \). So

\[ E(x,t) = \left(\frac{e^{-\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t} - e^{\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t}}{2\sqrt{a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}}}\right)^\wedge(x),\]

and \forall t \in [0, T], \( E(x,t) \in \mathcal{D}'(\mathbb{Q}_p^n) \) with \( \text{supp} E(x,t) \subset B_0^n \).

If \( a < 0 \), then

\[ E^\wedge(\xi,t) = \frac{\sin(\sqrt{-a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t)}{\sqrt{-a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}}},\]

and also \forall t \in [0, T], \( E^\wedge(\xi,t) \in \mathcal{E}(\mathbb{Q}_p^n) \) assumes constant values on cosets of \( B_0^n \). So

\[ E(x,t) = \left(\frac{\sin(\sqrt{-a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}} t)}{\sqrt{-a}\left(\sum_{j=1}^{n} \langle \xi_j^2 \rangle \right)^{\frac{1}{2}}}\right)^\wedge(x),\]

and we have \forall t \in [0, T], \( E(x,t) \in \mathcal{D}'(\mathbb{Q}_p^n) \) with \( \text{supp} E(x,t) \subset B_0^n \).
Hence ∀ \in [0, T], E(x, t) * \psi(x) exists. Let u(x, t) = E(x, t) * \psi(x). Then we have

\[
\left( \frac{\partial^2}{\partial t} - a \Delta_p \right) u(x, t) = \left( \frac{\partial^2}{\partial t} - a \Delta_p \right) (E(x, t) * \psi(x)) \\
= \left( \frac{\partial}{\partial t} - a \Delta_p \right) E(x, t) * \psi(x) \\
= 0 * \psi(x) = 0,
\]

and

\[u_t(x, 0) = E_t(x, 0) * \psi(x) = \delta(x) * \psi(x) = \psi(x). \]

Hence, \( u(x, t) = E(x, t) * \psi(x) \) is a solution of (6.2).

For the function \( f(x, t) \) defined on \( Q^n_p \times [0, T] \), we say that \( f \in \mathcal{E}(Q^n_p) \) uniformly with respect to \( t \), if its exponent of local constancy do not depend on \( t \).

**Lemma 6.1.** Let \( \omega(x, t) \in \mathcal{E}(Q^n_p) \) uniformly with respect to \( t \), and \( \omega \) is continuous on \( t \). Then

\[
\Delta_p^* \int_0^t \omega(x, \tau) d\tau = \int_0^t \Delta_p^* \omega(x, \tau) d\tau.
\]

**Proof.** It is easy to check that

\[
\int_0^t \omega(x, \tau) d\tau \in \mathcal{E}(Q^n_p) \text{ and } \int_0^t \Delta_p^* \omega(x, \tau) d\tau \in \mathcal{E}(Q^n_p).
\]

Then for any \( \phi \in D(Q^n_p) \), we have

\[
\left( \Delta_p^* \int_0^t \omega(x, \tau) d\tau, \phi(x) \right) = \left( \int_0^t \omega(x, \tau) d\tau, \Delta_p^* \phi(x) \right) \\
= \int_{Q^n_p} d^n x \int_0^t \omega(x, \tau) \Delta_p^* \phi(x) d\tau.
\]

Using Fubini Theorem, we get

\[
\left( \Delta_p^* \int_0^t \omega(x, \tau) d\tau, \phi(x) \right) = \int_0^t d\tau \int_{Q^n_p} \omega(x, \tau) \Delta_p^* \phi(x) d^n x \\
= \int_0^t d\tau \int_{Q^n_p} \Delta_p^* \omega(x, \tau) \phi(x) d^n x \\
= \int_{Q^n_p} d^n x \int_0^t \Delta_p^* \omega(x, \tau) \phi(x) d\tau \\
= \left( \int_0^t \Delta_p^* \omega(x, \tau) d\tau, \phi(x) \right).
\]
Hence,
\[ \Delta_p^s \int_0^t \omega(x, \tau) d\tau = \int_0^t \Delta_p^s \omega(x, \tau) d\tau. \]

**Theorem 6.2.** Denote by \( M_\psi \) the solution of the homogenous equation (6.2), \( \varphi, \psi \in \mathcal{E}(\mathbb{Q}_p^n) \), \( f(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \in [0, T] \), \( f(x, t) \in C[0, T] \). Then the inhomogeneous equation (6.1) has a solution \( u(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \), uniformly with respect to \( t \in [0, T] \), \( u(x, t) \in C^2[0, T] \), with
\[
\frac{\partial}{\partial t} M_\psi(x, t) + \int_0^t M_f(x, t - \tau) d\tau.
\]

Proof. A solution of (6.1) is given by
\[
u = u_1 + u_2 + u_3,
\]
where \( u_2 \) is the solution of (6.2), and \( u_1, u_3 \) are solutions of the following two equations, respectively.
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - a \Delta_p^s u &= 0, \quad x \in \mathbb{Q}_p^n, \quad 0 < t \leq T, \\
u(x, 0) &= \varphi(x), \quad x \in \mathbb{Q}_p^n, \\
u_t(x, 0) &= 0, \quad x \in \mathbb{Q}_p^n,
\end{align*}
\]
(6.3)

and
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - a \Delta_p^s u &= f(x, t), \quad x \in \mathbb{Q}_p^n, \quad 0 < t \leq T, \\
u(x, 0) &= 0, \quad x \in \mathbb{Q}_p^n, \\
u_t(x, 0) &= 0, \quad x \in \mathbb{Q}_p^n.
\end{align*}
\]
(6.4)

Let \( u_1 = \frac{\partial}{\partial t} M_\varphi \). Then
\[
\frac{\partial^2 u_1}{\partial t^2} - a \Delta_p^s u_1 = \frac{\partial}{\partial t} \left( \frac{\partial^2 M_\varphi}{\partial t^2} - a \Delta_p^s M_\varphi \right) = 0,
\]

\[
u_1(x, 0) = \frac{\partial}{\partial t} M_\varphi(x, t)|_{t=0} = \varphi(x),
\]

\[
u_{1t}(x, 0) = \frac{\partial^2}{\partial t^2} M_\varphi(x, t)|_{t=0} = a \Delta_p^s M_\varphi(x, t)|_{t=0} = 0.
\]

So \( u_1 = \frac{\partial}{\partial t} M_\varphi \) solves the equation (6.3).

Let \( f_\tau = f(x, \tau) \), \( u_3 = \int_0^t M_{f_\tau}(x, t - \tau) d\tau \). It is easy to check that \( u_3(x, 0) = 0 \).
Since \( f(x,t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \in [0,T] \) and \( f(x,t) \in C[0,T] \), we have that \( \forall t \in [0,T] \), \( M_{f_\tau}(x,t-\tau) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( \tau \) and is continuous on \( \tau \). Hence, \( u_3(x,t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \). So we have

\[
\frac{\partial u_3}{\partial t} = M_{f_\tau}(x,t-\tau)|_{\tau=t} + \int_0^t \frac{\partial M_{f_\tau}(x,t-\tau)}{\partial t} d\tau = \int_0^t \frac{\partial M_{f_\tau}(x,t-\tau)}{\partial t} d\tau.
\]

Then

\[
u_3(x,0) = \frac{\partial u_3}{\partial t}|_{t=0} = 0.
\]

Using Lemma 6.1, we get

\[
\frac{\partial^2 u_3}{\partial t^2} = \frac{\partial M_{f_\tau}(x,t-\tau)}{\partial t}|_{\tau=t} + \int_0^t \frac{\partial^2 M_{f_\tau}(x,t-\tau)}{\partial t^2} d\tau
\]

\[
= f(x,t) + a \int_0^t \Delta_p^s M_{f_\tau}(x,t-\tau) d\tau
\]

\[
= f(x,t) + a \Delta_p^s u_3.
\]

So \( u_3 = \int_0^t M_{f_\tau}(x,t-\tau) d\tau \) is a solution of (6.4).

Since \( \varphi, \psi \in \mathcal{E}(\mathbb{Q}_p^n) \), it is obvious that \( u_1, u_2 \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \). Hence, \( u(x,t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \in [0,T] \). ♯

**Lemma 6.2.** If \( a < 0, s > n \), then the fundamental solution \( E(x,t) \) is a continuous function supported in \( B_0^n \) for any \( 0 < t \leq T \).

**Proof.** If \( a < 0, s > n \), then for any \( 0 < t \leq T \),

\[
\int_{\mathbb{Q}_p^n} |E^\Lambda(\xi,t)| d^n\xi = \int_{\mathbb{Q}_p^n} \frac{\sin(-a(\sum_{j=1}^n \langle \xi_j \rangle^2) \frac{\tau}{2} t)}{\sqrt{-a(\sum_{j=1}^n \langle \xi_j \rangle^2) \frac{\tau}{2}}} |d^n\xi
\]

\[
\leq \frac{1}{\sqrt{-an}} \int_{\mathbb{Q}_p^n} \frac{1}{(\sum_{j=1}^n \langle \xi_j \rangle^2) \frac{\tau}{2}} |d^n\xi
\]

\[
= \frac{1}{\sqrt{-an}} + \frac{1}{\sqrt{-a}} \sum_{r=1}^{+\infty} \int_{|\xi|=p^r} \frac{1}{(\sum_{j=1}^n \langle \xi_j \rangle^2) \frac{\tau}{2}} |d^n\xi
\]

\[
\leq \frac{1}{\sqrt{-an}} + \frac{1}{\sqrt{-a}} (1 - \frac{1}{p^n}) \sum_{r=1}^{+\infty} p^{(n-s)r}
\]

\[
< \infty.
\]

So \( \forall t \in (0,T] \), \( E^\Lambda(\xi,t) \in L^1(\mathbb{Q}_p^n) \). Hence \( E(x,t) \) is a continuous function supported in \( B_0^n \) for any \( 0 < t \leq T \). ♯
Theorem 6.3. If \( a < 0, s > n, \varphi, \psi \in \mathcal{E}(Q^n_p), f(x,t) \in \mathcal{E}(Q^n_p) \) uniformly with respect to \( t \in [0, T], \) \( f(x,t) \in C[0, T], \) then the equation (6.1) has a solution \( u(x,t) \in \mathcal{E}(Q^n_p) \) uniformly with respect to \( t \in [0, T], \) \( u(x,t) \in C^2[0, T], \) with

\[
\begin{align*}
\varphi(x) &= \int_{Q^n_p} E(x-\eta,t)\psi(\eta)d^n\eta + \int_0^t \frac{\partial}{\partial \tau} E(x-\eta,t-\tau)f(\eta,\tau)d^n\eta.
\end{align*}
\]

Proof. Using Theorem 6.2 and Lemma 6.2, we have

\[
\begin{align*}
u(x,t) &= M_{\psi} + \frac{\partial}{\partial t} M_{\psi}(x,t) + \int_0^t M_{f}(x,t-\tau)d\tau \\
 &= E(x,t) \ast \psi + \frac{\partial}{\partial t} E(x,t) \ast \varphi + \int_0^t E(\cdot,t) \ast f_r(x,t-\tau)d\tau \\
 &= \int_{Q^n_p} E(x-\eta,t)\psi(\eta)d^n\eta + \int_{Q^n_p} \frac{\partial}{\partial t} E(x-\eta,t)\varphi(\eta)d^n\eta \\
 &\quad + \int_0^t d\tau \int_{Q^n_p} E(x-\eta,t-\tau)f(\eta,\tau)d^n\eta. \quad \# 
\end{align*}
\]

7 Cauchy problem for heat equations on \( Q^n_p \)

In this section, we consider another initial value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - a\Delta_s^p u &= f(x,t), \quad x \in Q^n_p, \quad 0 < t \leq T, \\
u(x,0) &= \varphi(x), \quad x \in Q^n_p,
\end{align*}
\]

\[(7.1)\]

where \( a \neq 0, s \in \mathbb{R}, T > 0, \) the function \( f \) and the initial function \( \varphi \) are complex valued.

Theorem 7.1. The homogeneous equation

\[
\begin{align*}
\frac{\partial u}{\partial t} - a\Delta_s^p u &= 0, \quad x \in Q^n_p, \quad 0 < t \leq T, \\
u(x,0) &= \varphi(x), \quad x \in Q^n_p,
\end{align*}
\]

\[(7.2)\]

has a fundamental solution

\[
F(x,t) = (e^{a(\sum_{j=1}^{n} (\xi_j)^2)t})^\varphi(x),
\]

where \( F(x,t) \in \mathcal{D}'(Q^n_p) \) has a compact support in \( B^n_0, \) for any \( t \in [0, T]. \) Moreover, for \( \varphi \in \mathcal{D}'(Q^n_p) \) the equation (7.2) has a solution

\[
u(x,t) = F(x,t) \ast \varphi.
\]

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Proof. Let $\varphi = \delta$, denote by $F(x, t)$ the fundamental solution of (7.2). Applying to (7.2) the Fourier transform, we get

$$\frac{\partial F^\wedge(\xi, t)}{\partial t} = a(\sum_{j=1}^{n}(\xi_j)^2)^{\ast} F^\wedge(\xi, t), \quad F^\wedge(\xi, 0) = 1.$$ 

Thus,

$$F^\wedge(\xi, t) = e^{a(\sum_{j=1}^{n}(\xi_j)^2)^{\ast} t},$$

and $\forall t \in [0, T], F^\wedge(\xi, t) \in \mathcal{E}(\mathbb{Q}_p^n)$ assumes constant values on cosets of $B_0^n$. So

$$F(x, t) = (e^{a(\sum_{j=1}^{n}(\xi_j)^2)^{\ast} t})^\wedge(x),$$

and $\forall t \in [0, T], F(x, t) \in \mathcal{D}'(\mathbb{Q}_p^n)$ with supp$F(x, t) \subset B_0^n$.

Hence $\forall t \in [0, T], F(x, t) \ast \varphi(x)$ exists. Let $u(x, t) = F(x, t) \ast \varphi(x)$. Then

$$\left(\frac{\partial}{\partial t} - a\Delta_p^s\right)u(x, t) = \left(\frac{\partial}{\partial t} - a\Delta_p^s\right)(F(x, t) \ast \varphi(x))$$

$$= \left((\frac{\partial}{\partial t} - a\Delta_p^s)F(x, t)\right) \ast \varphi(x)$$

$$= 0 \ast \varphi(x) = 0,$$

and

$$u(x, 0) = F(x, 0) \ast \varphi(x) = \delta(x) \ast \varphi(x) = \varphi(x).$$

Hence, $u(x, t) = F(x, t) \ast \varphi(x)$ is a solution of (7.2).

\[ \sharp \]

**Theorem 7.2.** If we denote by $W_{\varphi}$ the solution of the homogenous equation (7.2), $\varphi \in \mathcal{E}(\mathbb{Q}_p^n)$, $f(x, t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0, T]$, $f(x, t) \in C[0, T]$, then the inhomogeneous equation (7.1) has a solution $u(x, t) \in \mathcal{E}(\mathbb{Q}_p^n)$ uniformly with respect to $t \in [0, T]$, $u(x, t) \in C^1[0, T]$, with

$$u = W_{\varphi} + \int_0^t W_{f_\tau}(x, t - \tau) d\tau.$$

**Proof.** A solution of (7.1) is given by

$$u = u_1 + u_2,$$

where $u_1$ is the solution of (7.2), and $u_2$ is the solution of the following equation.

$$\left(\frac{\partial}{\partial t} - a\Delta_p^s\right)u = f(x, t), \quad x \in \mathbb{Q}_p^n, \quad 0 < t \leq T,$$

$$u(x, 0) = 0, \quad x \in \mathbb{Q}_p^n.$$  \(7.3\)
Let \( f_s = f(x, \tau), u_2 = \int_0^t W_{f_s}(x, t - \tau) d\tau \). It is easy to get \( u_2(x, 0) = 0 \).

Since \( f(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \in [0, T] \) and \( f(x, t) \in C[0, T] \), we have \( \forall t \in [0, T], W_{f_s}(x, t - \tau) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( \tau \) and is continuous on \( \tau \). Hence, \( u_2(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \). Using Lemma 6.1, we have

\[
\frac{\partial u_2}{\partial t} = W_{f_s}(x, t - \tau)|_{\tau=t} + \int_0^t \frac{\partial W_{f_s}(x, t - \tau)}{\partial t} d\tau
\]

\[
= f(x, t) + \int_0^t a\Delta_p W_{f_s}(x, t - \tau)d\tau
\]

\[
= f(x, t) + a\Delta_p u_2.
\]

So \( u_2 = \int_0^t W_{f_s}(x, t - \tau)d\tau \) solves the equation (6.3).

Since \( \varphi \in \mathcal{E}(\mathbb{Q}_p^n) \) and \( \text{supp} F(x, t) \in \mathbb{B}_0^n \), we get that \( u_1(x, t) = F(x, t) * \varphi(x) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \). Hence, \( u(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \in [0, T] \).

**Lemma 7.1.** If \( a < 0, s > 0 \), then the fundamental solution \( F(x, t) \) is a non-negative continuous function supported in \( \mathbb{B}_0^n \) for any \( 0 < t \leq T \).

**Proof.** If \( a < 0, s > 0 \), then for any \( 0 < t \leq T \),

\[
\int_{\mathbb{Q}_p^n} F^*(\xi, t) d\mu_\xi = \int_{\mathbb{Q}_p^n} e^{a(\sum_{j=1}^n \xi_j^2)^s t} d\mu_\xi
\]

\[
= \int_{\mathbb{B}_0^n} e^{a s t} d\mu_\xi + \int_{\mathbb{Q}_p^n \setminus \mathbb{B}_0^n} e^{a(\sum_{j=1}^n \xi_j^2)^s t} d\mu_\xi
\]

\[
= e^{a s t} + \sum_{r=1}^{+\infty} \int_{[|\xi|=p^r]} e^{a(\sum_{j=1}^n \xi_j^2)^s t} d\mu_\xi
\]

\[
\leq e^{a s t} + \sum_{r=1}^{+\infty} \int_{[|\xi|=p^r]} e^{a p^{2s} t} d\mu_\xi
\]

\[
= e^{a s t} + (1 - \frac{1}{p^n}) \sum_{r=1}^{+\infty} e^{a p^{2s} t} p^{nr}
\]

\[
< \infty.
\]

So \( \forall t \in (0, T], F^*(\xi, t) \in L^1(\mathbb{Q}_p^n) \), and hence \( F(x, t) \) is a continuous function supported in \( \mathbb{B}_0^n \) for any \( 0 < t \leq T \). For the non-negative property of \( F(x, t) \), one can verify it by a direct calculation.

**Theorem 7.3.** If \( a < 0, s > 0, \varphi \in \mathcal{E}(\mathbb{Q}_p^n), f(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \) uniformly with respect to \( t \in [0, T], f(x, t) \in C[0, T], \) then the equation (7.1) has a solution \( u(x, t) \in \mathcal{E}(\mathbb{Q}_p^n) \)
uniformly with respect to \( t \in [0,T] \), \( u(x,t) \in C^1[0,T] \), with

\[
u(x,t) = \int_{\mathbb{Q}_p} F(x-\eta,t)\varphi(\eta)d\eta + \int_0^t d\tau \int_{\mathbb{Q}_p} F(x-\eta,t-\tau)f(\eta,\tau)d\eta.
\]

Proof. From Theorem 7.2 and Lemma 7.2, we have

\[
u(x,t) = W_\varphi + \int_0^t W_{f_\tau}(x,t-\tau)d\tau = F(x,t) * \varphi + \int_0^t F(\cdot, t) * f_\tau(x,t-\tau)d\tau
\]

\[
= \int_{\mathbb{Q}_p} F(x-\eta,t)\varphi(\eta)d\eta + \int_0^t d\tau \int_{\mathbb{Q}_p} F(x-\eta,t-\tau)f(\eta,\tau)d\eta.
\]

References


