Continuity of packing measure function of self-similar iterated function systems

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Abstract: In this paper, we focus on the packing measure of self-similar sets. Let K be a self-similar set whose Hausdorff dimension and packing dimension equal s, we state that if K satisfies the strong open set condition with an open set $\mathcal{O}$, then

$$\mathcal{P}^s(K \cap B(x,r)) \geq (2r)^s$$

for each open ball $B(x,r) \subset \mathcal{O}$ centered in K, where $\mathcal{P}^s$ denotes the s-dimensional packing measure. We use this inequality to obtain some precise density theorems for packing measure of self-similar sets, which can be applied to compute the exact value of the s-dimensional packing measure $\mathcal{P}^s(K)$ of K. Moreover, by using the above results, we show the continuity of the packing measure function of the attractors on the space of self-similar iterated function systems satisfying the strong separation condition. This result gives a complete answer to a question posed by L. Olsen in [14].

Keywords: self-similar set, iterated function system, packing measure, open set condition, strong separation condition.

1 Introduction

In this paper we will analysis the behaviour of the packing measure of self-similar sets with open set condition or strong separation condition. Recall the definition of packing measure, introduced by Tricot [17], Taylor and Tricot [16], which requires two limiting procedures. For $E \subset \mathbb{R}^d$ and $\delta > 0$, a δ-packing of E is a countable family of disjoint open balls of radii at most δ and with centers in E. For $s \geq 0$, the s-dimensional *packing premeasure* of E is defined as

$$P^s(E) = \inf_{\delta>0}\{P^s_\delta(E)\},$$

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where $P^s_\delta(E) = \sup\{\sum_{i=1}^\infty \mathrm{diam}(B_i)^s\}$ with the supremum taken over all δ-packing of E. Here $\mathrm{diam}(B_i)$ denotes the diameter of B_i. The s-dimensional *packing measure* of E is defined as

$$\mathcal{P}^s(E) = \inf\{\sum_{i=1}^{\infty} P^s(E_i) | E \subset \bigcup_{i=1}^{\infty} E_i\}.$$

The *packing dimension* of E is defined as

$$\dim_P(E) = \inf\{s \geq 0 | \mathcal{P}^s(E) = 0\} = \sup\{s \geq 0 | \mathcal{P}^s(E) = \infty\}.$$

The packing measure and packing dimension play an important role in the study of fractal geometry in a manner dual to the Hausdorff measure and Hausdorff dimension (See [3] and [9] for further properties of the above measures and dimensions).

Let $N \geq 2$ be an integer. Let $\mathbf{f} = \{f_1, f_2, \cdots, f_N\}$ be an *iterated function system* (*IFS*) on $\mathbb{R}^d$ of contractive similitudes. The corresponding *self-similar set* for $\mathbf{f}$ is the unique non-empty compact set $K \subset \mathbb{R}^d$ which is invariant under the action of the elements of $\mathbf{f}$:

$$K = \bigcup_{i=1}^{N} f_i(K).$$

It is well-known that if $\mathbf{f}$ satisfies the *open set condition*(*OSC*), i.e., there exists a nonempty bounded open set $\mathcal{O} \subset \mathbb{R}^d$ such that $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $1 \leq i \leq N$ and $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset$ for all $i \neq j$, then the Hausdorff dimension $\dim_H(K)$ and the packing dimension $\dim_P(K)$ of K coincide, and the common value $s = \dim_H(K) = \dim_P(K)$ is given by the following formula

$$\sum_{i=1}^{N} r_i^s = 1, \tag{1.1}$$

where r_i denotes the contraction ratio of f_i for $1 \leq i \leq N$. Moreover, the Hausdorff measure and packing measure of K are finite and positive. This was proved by Moran [11] in 1946 and rediscovered by Hutchinson [6] in the 1980s. Since the intersection of $\mathcal{O}$ and K may be empty, the OSC is in general too weak to imply results. One can strengthen the definition as follows: *The strong open set condition (SOSC)* holds if and only if furthermore $\mathcal{O} \cap K \neq \emptyset$. Schief proved that SOSC is equivalent to OSC in the Euclidean case, see [15]. There is another separation condition called the *strong separation condition (SSC)* which is satisfied if $f_i(K) \cap f_j(K) = \emptyset$ for all i, j with $i \neq j$. Obviously, SSC implies SOSC and the implication may not be inverted. In this paper, we will frequently assume these two conditions.

We shall need some standard notation from symbolic dynamics. For each positive integer k, let

$$W_k = \{1, 2, \cdots, N\}^k = \{\mathbf{i} = i_1 i_2 \cdots, i_k : i_j \in \{1, 2, \cdots, N\}\},$$

denote the space of words of length k with symbols $\{1, 2, \cdots, N\}$. Also, for $k = 0$, we define $W_0 = \{\emptyset\}$ and call $\emptyset$ the empty word. Moreover, set $W = \bigcup_{k \geq 0} W_k$ and denote the length of $\mathbf{i} \in W$ by $|\mathbf{i}|$. Assume now that $\mathbf{f} = \{f_1, f_2 \cdots, f_N\}$ is an *IFS* with invariant set K. Let $f_\emptyset = id, r_\emptyset = 1, K_\emptyset = K$. For each non-empty finite word $\mathbf{i} = i_1 i_2 \cdots i_k$ let $f_\mathbf{i} = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$, $r_\mathbf{i} = r_{i_1} r_{i_2} \cdots r_{i_k}$ and $K_\mathbf{i} = f_\mathbf{i}(K)$. Then $K = \bigcup_{\mathbf{i} \in W_k} K_\mathbf{i}$ for each k.

For a Borel measure μ on $\mathbb{R}^d$ and a Borel set E, we let $\mu|_E$ denote the restriction of μ to E. Let λ denote the self-similar measure satisfying

$$\lambda = \sum_i r_i^s \lambda \circ f_i^{-1}.$$

It is well known that under the assumption of OSC, $\lambda = \frac{\mathcal{H}^s|_K}{\mathcal{H}^s(K)} = \frac{\mathcal{P}^s|_K}{\mathcal{P}^s(K)}$, and $\lambda(K_\mathbf{i}) = r_\mathbf{i}^s$ for each $\mathbf{i} \in W$.

We always assume that K is in general position, i.e., not contained in a hyperplane. In [10] it is proved that, under this assumption and the OSC, the intersection of K with any l-dimensional C^1 submanifold of $\mathbb{R}^d$ with $0 < l < d$ is an $\mathcal{H}^s$- null set, and therefore a $\mathcal{P}^s$- null set.

Since the definitions of Hausdorff and packing measures are sometimes awkward to work with, there are only very few non-trivial examples of sets in $E \subset \mathbb{R}^d$ for which the exact Hausdorff measure $\mathcal{H}^{\dim_H(E)}(E)$ or packing measure $\mathcal{P}^{\dim_P(E)}(E)$ of E is known. For example, one can see papers [1], [5]. [18] is a recent review of relevant open questions in this field. In particular, there is no formula similar to (1.1) for the Hausdorff measure or packing measure of a self-similar set. In view of this, it is natural to ask if the Hausdorff measure and packing measure vary continuously with the IFS.

To make the above question precise we introduce the following notations.

Let N be a positive integer with $N \geq 2$ and let $X \subset \mathbb{R}^d$ be a compact set. Let $\mathbf{f} = (f_1, f_2, \cdots, f_N)$ be a IFS on X satisfying the OSC. In order to emphasize the relation between the corresponding fractal characteristics with $\mathbf{f}$, we denote the self-similar set of $\mathbf{f}$ as $K^\mathbf{f}$ and write the contraction ratio of f_i as $r_i^\mathbf{f}$ for each $1 \leq i \leq N$. Similarly, denote $r_*^\mathbf{f} = \min_{1 \leq i \leq N} r_i^\mathbf{f}$, the corresponding self-similar measure as $\lambda^\mathbf{f}$ and the Hausdorff dimension and packing dimension as $s^\mathbf{f}$.

Write

$$M_{OSC} = \{\mathbf{f}|\ \mathbf{f} = (f_1, f_2, \cdots, f_N) \text{ is a self-similar IFS on } X \text{ satisfying the OSC}\},$$

$$M_{SSC} = \{\mathbf{f}|\ \mathbf{f} = (f_1, f_2, \cdots, f_N) \text{ is a self-similar IFS on } X \text{ satisfying the SSC}\}.$$

It is obvious that $M_{SSC} \subset M_{OSC}$. We equip M_{OSC} and M_{SSC} with the metric induced by

$$D(\mathbf{f}, \mathbf{g}) = \max_{1 \le i \le N} \{\| f_i - g_i \|_\infty\},$$

for $\mathbf{f}, \mathbf{g} \in M_{OSC}$. It is not difficult to see that M_{SSC} is an open subset of M_{OSC}. Below $d(\cdot, \cdot)$ denotes the Euclidean metric between two points or two sets. For $\Delta > 0$ we write

$$M_\Delta = \{\mathbf{f}|\ \mathbf{f} \in M_{SSC}, \text{ and } d(K_i^{\mathbf{f}}, K_j^{\mathbf{f}}) > \Delta \text{ for all } i \neq j\}.$$

Also, one can easily prove that $M_{SSC} = \bigcup_{\Delta > 0} M_\Delta$ and each M_Δ is an open subset in M_{SSC}, see [14]. The metric spaces M_{OSC} and M_{SSC} provide a natural setting for investigating to what extent fractal characteristics of $K^{\mathbf{f}}$ vary continuously with $\mathbf{f}$. For example, let $\mathcal{K}$ denote the family of non-empty compact subsets of X equipped with the Hausdorff metric, then the map

$$\mathbf{f} \to K^{\mathbf{f}}$$

from M_{OSC} into $\mathcal{K}$ is continuous, see [2]. It also follows immediately from (1.1) that the dimension map

$$\mathbf{f} \to s^{\mathbf{f}}$$

from M_{OSC} into $\mathbb{R}$ is continuous. Following this line of investigation it is natural to ask if the measure maps

$$\mathbf{f} \to \mathcal{H}^{s^{\mathbf{f}}}(K^{\mathbf{f}}), \tag{1.2}$$

$$\mathbf{f} \to \mathcal{P}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) \tag{1.3}$$

from M_{OSC} into $\mathbb{R}$ are continuous.

For the Hausdorff measure map, Ayer and Strichartz [1] showed that in the special case of linear Cantor sets they found a point $\mathbf{f} \in M_{OSC} \setminus M_{SSC}$, at which the map (1.2) fails to be continuous. In [14] Olsen altered the space M_{OSC} to M_{SSC}, then he proved the continuity, i.e., the map

$$\mathbf{f} \to \mathcal{H}^{s^{\mathbf{f}}}(K^{\mathbf{f}})$$

from M_{SSC} into $\mathbb{R}$ is continuous.

Recall that in order to prove this continuity theorem, Olsen used a so-called explicit formula for the Hausdorff measure of self-similar sets which was established in [13]. Indeed, he showed that the Hausdorff measure coincides with the infimum of the reciprocal densities. Let $\mathbf{f} \in M_{OSC}$ and $K^{\mathbf{f}}$ be the corresponding self-similar set. Let $s^{\mathbf{f}}$ be the Hausdorff dimension and $\lambda^{\mathbf{f}}$ be the self-similar measure. Then

$$\mathcal{H}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) = \inf\{\frac{\operatorname{diam}(U)^{s^{\mathbf{f}}}}{\lambda^{\mathbf{f}}(U)} | U \text{ is open and convex }, U \cap K^{\mathbf{f}} \neq \emptyset\}.$$

Moreover, if furthermore $\mathbf{f}$ satisfies the SSC, i.e., there exists $\Delta > 0$ such that $\mathbf{f} \in M_{\Delta}$, then

$$\mathcal{H}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) = \inf\{\frac{\operatorname{diam}(U)^{s^{\mathbf{f}}}}{\lambda^{\mathbf{f}}(U)} | U \text{ is open and convex }, U \cap K^{\mathbf{f}} \neq \emptyset, \ \operatorname{diam}(U) \geq \Delta\}. \quad (1.4)$$

As pointed in [13], the above formulae are implicit in earlier work by Marion and Ayer & Strichartz, see [7],[8] and [1]. They used these formulae to compute the exact value of the s-dimensional Hausdorff measure $\mathcal{H}^s(\mathcal{C})$ of certain linear Cantor subsets $\mathcal{C}$ of $\mathbb{R}$, where s denotes the Hausdorff dimension of $\mathcal{C}$.

By the fact that $M_{SSC} = \bigcup_{\Delta>0} M_{\Delta}$ and each M_{Δ} is an open subset in M_{SSC}, the proof of Hausdorff measure continuity theorem can be simplified to prove that map (1.2) from M_{Δ} into $\mathbb{R}$ is continuous for each $\Delta > 0$. Moreover, in the proof of this continuity, it should be pointed that the restrictive condition $\operatorname{diam}(U) \geq \Delta$ for the family of U in (1.4) holds the key to the result. Actually, A detailed comparison of the self-similar measures of arbitrary elements in those families between one IFS and its nearby IFS is needed. The condition $\operatorname{diam}(U) \geq \Delta$ ensures the comparison successfully.

However, for the packing measure map, it is not clear if the map (1.3) is continuous. Let $\mathcal{A}$ denote the family of analytic subsets of M_{SSC} and $\sigma(\mathcal{A})$ denote the σ-algebra generated by $\mathcal{A}$. Olsen proved that the map (1.3) from M_{SSC} into $\mathbb{R}$ is $\sigma(\mathcal{A})$-measurable. However, this result is very weak compared to the continuity, see [14]. The proof also needs an explicit formula for the packing measure. Actually, a similar formula for packing measure in the SSC case was also proved in [13], i.e., for each $\mathbf{f} \in M_{\Delta}$ with $\Delta > 0$, the following formula holds.

$$\mathcal{P}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) = \sup\{\frac{(2r)^{s^{\mathbf{f}}}}{\lambda^{\mathbf{f}}(B(x,r))}, x \in K^{\mathbf{f}}, 0 < r \leq \frac{1}{2}\Delta\}. \quad (1.5)$$

However, in this formula, the family contains balls centered in $K^{\mathbf{f}}$ whose radii could be arbitrary small.

In that paper Olsen posed the following open question.

Question. *Is the packing measure function in* (1.3) *from* M_{SSC} *into* $\mathbb{R}$ *continuous? If it is not continuous, is it of Baire class* n *for some positive integer* n*? If it is not of Baire class* n *for some positive integer* n*, is it Borel measurable?*

In this paper, we will show that this map is indeed continuous, which gives a complete answer to this question. This leads to our main result.

Theorem 1.1. *The map* $\boldsymbol{f} \to \mathcal{P}^{s^{\boldsymbol{f}}}(K^{\boldsymbol{f}})$ *from* M_{SSC} *into* $\mathbb{R}$ *is continuous.*

In the proof of Theorem 1.1, the following revised version of (1.5) will play a key role. Indeed, we will show that if $\mathbf{f} \in M_\Delta$, then

$$\mathcal{P}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) = \sup\{\frac{(2r)^{s^{\mathbf{f}}}}{\lambda^{\mathbf{f}}(B(x,r))}, x \in K^{\mathbf{f}}, \frac{1}{2}r_*^{\mathbf{f}}\Delta \leq r \leq \frac{1}{2}\Delta\}, \tag{1.6}$$

where $r_*^{\mathbf{f}} = \min_{1\leq i\leq N} r_i^{\mathbf{f}}$. Comparing with (1.5), this formula rules out balls with small radii from the family. This formula is a direct corollary of the following theorem which is another main result in this paper.

Theorem 1.2. *Let* $K \subset \mathbb{R}^d$ *be the self-similar set associated with a IFS* $\boldsymbol{f} = \{f_1, f_2, \cdots, f_N\}$ *satisfying the SOSC for an open set* $\mathcal{O}$*, and with packing and Hausdorff dimension* s*. Then*

$$\mathcal{P}^s(K \cap B(x,r)) \geq (2r)^s \tag{1.7}$$

for any ball $B(x,r) \subset \mathcal{O}$ *centered in* K*.*

We should point that a similar result in the SSC case was proved by Olsen in [13], and from which the explicit formula (1.5) was obtained.

It is not known whether the packing measure continuity theorem still holds from M_{OSC} into $\mathbb{R}$. However, in the special setting of linear Cantor sets of real line $\mathbb{R}$. Feng [5] discussed the exact value of packing measure $\mathcal{P}^{\dim_P(\mathcal{C})}$ of self-similar Cantor sets $\mathcal{C}$ satisfying the OSC(where the open set is an interval). Moreover, his result implies that $\mathcal{P}^{\dim_P(\mathcal{C})}$ depends continuously on the IFS in M_{OSC}. In view of this, we guess that Theorem 1 could be generalized to the following setting.

Conjecture. *The packing measure function in* (1.3) *from* M_{OSC} *into* $\mathbb{R}$ *is continuous.*

However, we are not able to prove this.

This paper is organized as follows. In Section 2, we deal with the density theorem for packing measure of self-similar sets with SOSC. Firstly, we give the proof of Theorem 1.2 which plays an important role in giving the explicit formula of packing measure in SOSC case. Secondly, we prove the formula (1.6) by using the so-called blow-up principle in the SSC case. Section 3 is devoted to the proof of Theorem 1.1 by using the explicit formula (1.6).

2 Density theorems for packing measure of self-similar sets

We analyze the local behaviour of the packing measure of self-similar sets in this section. Let $N \geq 2$ be an integer. $\mathbf{f} = \{f_1, f_2, \cdots, f_N\}$ be a IFS on $\mathbb{R}^d$ of contractive similitudes. In this section, for the sake of simplicity, we always denote the self-similar set of $\mathbf{f}$ as K in stead of $K^{\mathbf{f}}$. and write the contraction ratio of f_i as r_i in stead of $r_i^{\mathbf{f}}$ for each $1 \leq i \leq N$. Similarly, denote $r_* = \min_{1\leq i\leq N} r_i$, denote the corresponding self-similar measure as λ and denote the Hausdorff dimension and packing dimension as s. Our main result in this section, i.e., Theorem 1.2 says that if K satisfies the SOSC, then

$$\mathcal{P}^s(K \cap B(x,r)) \geq (2r)^s$$

for all $B(x,r) \subset \mathcal{O}$ centered in K. This result has several applications on densities and can also be applied to compute the exact value of the packing measure $\mathcal{P}^s(K)$ of K. Recall that in [13], Olsen also proved a density theorem for packing measure of self-similar sets which requires that the IFSs satisfy the SSC. In that setting, there exists $r_0 > 0$ such that the above formula holds for all $x \in K$ and all $r \in (0, r_0]$. However, in the SOSC case, we could not find such r_0. It is easy to check that our result is a natural generalization of the SSC case.

In order to prove Theorem 1.2, we shall need the following lemma.

Lemma 2.1. *Let $B(x,r) \subset \mathcal{O}$ be a ball centered in K and k a positive integer, then*

$$\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) = \mathcal{P}^s(K \cap \overline{B(x,r)}) > 0.$$

Proof. First, we prove that

$$K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x,r)) = \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(K \cap B(x,r)). \tag{2.1}$$

Fix $y \in K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x,r))$. Since $B(x,r) \subset \mathcal{O}$, there exists a $\mathbf{u} \in W_k$ such that $y \in f_{\mathbf{u}}(B(x,r)) \subset f_{\mathbf{u}}(\mathcal{O})$. We also have $y \in K = \bigcup_{\mathbf{i}\in W_k} K_{\mathbf{i}}$ and we therefore find $\mathbf{v} \in W_k$ such that $y \in f_{\mathbf{v}}(K) \subset f_{\mathbf{v}}(\overline{\mathcal{O}})$. Thus $y \in f_{\mathbf{u}}(\mathcal{O}) \cap f_{\mathbf{v}}(\overline{\mathcal{O}})$, and therefore $\mathbf{u} = \mathbf{v}$.

Hence $y \in f_{\mathbf{u}}(B(x,r)) \cap f_{\mathbf{u}}(K) = f_{\mathbf{u}}(K \cap B(x,r)) \subset \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(K \cap B(x,r))$. The other direction is obvious. Hence the formula (2.1) holds.

It follows from (2.1) that

$$\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x,r))) = \mathcal{P}^s(\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(K \cap B(x,r))).$$

However, since the SOSC is satisfied and $B(x,r) \subset \mathcal{O}$ the sets $f_{\mathbf{i}}(K \cap B(x,r))_{\mathbf{i}\in W_k}$ are pairwise disjoint. It therefore follows

$$\begin{aligned}\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x,r))) &= \sum_{\mathbf{i}\in W_k} \mathcal{P}^s(f_{\mathbf{i}}(K \cap B(x,r))) \\ &= \sum_{\mathbf{i}\in W_k} r_{\mathbf{i}}^s \mathcal{P}^s(K \cap B(x,r)) \\ &= \mathcal{P}^s(K \cap B(x,r)).\end{aligned}$$

Since the intersection of K with any $n-1$ dimensional C^1 manifold is an $\mathcal{P}^s$-null set. We have

$$\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x,r))) = \mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})),$$

and

$$\mathcal{P}^s(K \cap B(x,r)) = \mathcal{P}^s(K \cap \overline{B(x,r)}). \tag{2.2}$$

Using the above three equalities, we get

$$\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) = \mathcal{P}^s(K \cap \overline{B(x,r)}).$$

Moreover, since $x \in K$, we deduce that $\mathcal{P}^s(K \cap \overline{B(x,r)}) > 0$. This completes the proof of Lemma 1. □

Proof of Theorem 1.2.

In order to reach a contradiction, we assume that (1.7) is not satisfied, i.e., there exists a ball $B(x,r) \subset \mathcal{O}$ centered in K, such that

$$\mathcal{P}^s(K \cap B(x,r)) < (2r)^s.$$

From (2.2), we get

$$\mathcal{P}^s(K \cap \overline{B(x,r)}) < (2r)^s.$$

Thus we can find a number $0 < \kappa < 1$ with

$$(1+\kappa)\mathcal{P}^s(K \cap \overline{B(x,r)}) < (2r)^s. \tag{2.3}$$

Next, fix $\delta > 0$ and choose a positive integer k such that

$$2r_{\mathbf{i}} r \le \delta$$

for all $\mathbf{i} \in W_k$. Let $\eta = \frac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x,r)})$. It follows from Lemma 2.1, $\eta > 0$.

For a positive integer m, write $F_m = K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m})$, and observe that

$$F_1 \subset F_2 \subset F_3 \subset \cdots$$

and

$$\bigcup_m F_m = K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)}).$$

From this we see that there is a positive integer m_0 with $\frac{1}{m_0} < \delta$ such that

$$\mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})) = \mathcal{P}^s(F_{m_0}) \ge \mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) - \frac{\eta}{2}. \tag{2.4}$$

We can also choose a $\frac{1}{m_0}$-packing $\{B(x_i,\rho_i)\}_i$ of $K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})$ such that

$$\begin{aligned}\sum_i (2\rho_i)^s &\ge P^s_{\frac{1}{m_0}}(K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})) - \frac{\eta}{2} \\ &\ge P^s(K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})) - \frac{\eta}{2} \\ &\ge \mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})) - \frac{\eta}{2}. \end{aligned} \tag{2.5}$$

Since $x \in K$ and $B(x,r) \subset \mathcal{O}$, $f_{\mathbf{i}}(B(x,r)) \cap f_{\mathbf{j}}(B(x,r)) = \emptyset$ for all $\mathbf{i} \ne \mathbf{j}$ in W_k, and for each $\mathbf{i} \in W_k$, we have $f_{\mathbf{i}}(x) \in K_{\mathbf{i}} \subset K$ and $2r_{\mathbf{i}}r \le \delta$. Thus the family $\{f_{\mathbf{i}}(B(x,r))\}_{\mathbf{i}\in W_k}$ is a δ-packing of $K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})$.

Since $\{B(x_i,\rho_i)\}_i$ is also a $\frac{1}{m_0}$-packing of $K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})$, we conclude that $\{f_{\mathbf{i}}(B(x,r))\}_{\mathbf{i}\in W_k} \bigcup \{B(x_i,\rho_i)\}_i$ is a δ-packing of K. Using this we therefore conclude from (2.3), (2.4), (2.5), and Lemma 2.1 that

$$
\begin{aligned}
P_\delta^s(K) &\geq \sum_{\mathbf{i}\in W_k} (2r_{\mathbf{i}}r)^s + \sum_i (2\rho_i)^s \\
&\geq \sum_{\mathbf{i}\in W_k} r_{\mathbf{i}}^s (2r)^s + \mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m_0})) - \frac{\eta}{2} \\
&\geq (2r)^s + \mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) - \eta \\
&\geq (1+\kappa)\mathcal{P}^s(K \cap \overline{B(x,r)}) + \mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) - \eta \\
&= (1+\kappa)\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) + \mathcal{P}^s(K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) - \eta \\
&= \mathcal{P}^s(K) + \kappa\mathcal{P}^s(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})) - \eta \\
&= \mathcal{P}^s(K) + 2\eta - \eta \\
&= \mathcal{P}^s(K) + \frac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x,r)}).
\end{aligned}
$$

Finally, let $\delta \to 0$, we get

$$P^s(K) \geq \mathcal{P}^s(K) + \frac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x,r)}). \tag{2.6}$$

In [4] it is proved that the packing premeasure P^s coincides with the packing measure $\mathcal{P}^s$ for compact subsets with finite P^s-measure. Thus they coincide for K, and it follows from (2.6) that

$$\mathcal{P}^s(K) \geq \mathcal{P}^s(K) + \frac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x,r)}).$$

Since $\mathcal{P}^s(K)$ is positive and finite, and $\frac{1}{2}\kappa\mathcal{P}^s(K\cap\overline{B(x,r)}) > 0$, we get the contradiction. This completes the proof of Theorem 1.2. □

This result has applications on densities. For a given measure μ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, the lower α-density of μ at x is defined by

$$\Theta_*^\alpha(\mu, x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^\alpha}.$$

The upper α-density $\Theta^{*\alpha}(\mu, x)$ is defined similarly by taking the upper limit. We have the following result. If $E \subset \mathbb{R}^d$ and $\alpha > 0$ with $0 < \mathcal{P}^\alpha(E) < \infty$, then

$$\Theta_*^\alpha(\mathcal{P}^\alpha|_E, x) = 1 \text{ for } \mathcal{P}^\alpha - a.e. \quad x \in E. \tag{2.7}$$

See the proof in [9]. We then could get the following corollary on the basis of (2.7) and Theorem 1.2.

Corollary 2.2. *Let $K \subset \mathbb{R}^d$ be the self-similar set satisfying the SOSC for an open set $\mathcal{O}$, with packing and Hausdorff dimension s. Then*

$$\mathcal{P}^s(K) = \sup\{\frac{(2r)^s}{\lambda(B(x,r))}|x \in K, B(x,r) \subset \mathcal{O}\}. \tag{2.8}$$

Proof. Since $K \cap \mathcal{O} \neq \emptyset$, we can take a point $y \in K \cap \mathcal{O}$. Choose $\rho > 0$ such that the ball $B(y,\rho)$ contained in $\mathcal{O}$ and $\mathcal{P}^s(K \cap B(y,\rho)) > 0$. Hence from (2.7), there exists a point $z \in K \cap B(y,\rho)$ with $\Theta^s_*(\mathcal{P}^s|_K, z) = 1$. By the definition of $\Theta^s_*(\mathcal{P}^s|_K, z)$, there exists a sequence $\{r_n\}$ with each $r_n \leq \rho - d(z,y)$ and $r_n \to 0$ as $n \to \infty$, such that $\lim_{n\to\infty} \frac{\mathcal{P}^s(K\cap B(z,r_n))}{(2r_n)^s} = 1$. Notice that here all balls $B(z,r_n)$ are contained in $B(y,\rho) \subset \mathcal{O}$. Moreover, by Theorem 1.2, for each ball $B(x,r) \subset \mathcal{O}$ centered in K, we have $\frac{\mathcal{P}^s(K\cap B(x,r))}{(2r)^s} \geq 1$. Hence we get

$$\inf\{\frac{\mathcal{P}^s(K \cap B(x,r))}{(2r)^s}|x \in K, B(x,r) \subset \mathcal{O}\} = 1.$$

Since $\lambda = \frac{\mathcal{P}^s|_K}{\mathcal{P}^s(K)}$, (2.8) follows immediately from the above equation. □

After this work was completed, we learned that Morán [12] had proved, independently, the same result as Corollary 2.2. However, his proof is quite different of ours. In fact, in [12], the so-called self-similar tiling principle plays a central role in the proof. This principle says that any open subset U of K can be tiled by a countable set of similar copies of an arbitrarily given closed set with positive Hausdorff or packing measure while the tiling is exact in the sense that the part of U which cannot be covered by the tiles is of null measure. The continuity theorem is not studied in his paper.

The following lemma will be used in the following corollaries.

Lemma 2.3. (blow-up principle) *Let $B(x,r) \subset \mathcal{O}$ centered in K. Then for any $1 \leq j \leq N$, $f_j(B(x,r))$ has the same reciprocal density as $B(x,r)$, i.e., $\frac{(2r)^s}{\lambda(B(x,r))} = \frac{(2rr_j)^s}{\lambda(f_j(B(x,r)))}$. In other words, if $B(x,r) \subset f_j(\mathcal{O})$ centered in K_j for some $1 \leq j \leq N$, then $f_j^{-1}(B(x,r))$ has the same reciprocal density as $B(x,r)$.*

Proof. We only need to check that

$$\lambda(f_j(B(x,r))) = r_j^s \lambda(B(x,r)).$$

Actually, $\lambda(f_j(B(x,r))) = \lambda(f_j(B(x,r)) \cap K_j) + \sum_{i\neq j} \lambda(f_j(B(x,r)) \cap K_i)$. Notice that if $i \neq j$, then $f_j(B(x,r)) \cap K_i \subset f_j(\mathcal{O}) \cap f_i(K) \subset f_j(\mathcal{O}) \cap f_i(\overline{\mathcal{O}}) = \emptyset$. Hence $\lambda(f_j(B(x,r))) = \lambda(f_j(B(x,r) \cap K)) = r_j^s \lambda(B(x,r))$. □

Combining the above lemma and Corollary 2.2, we immediately get the following corollaries.

Corollary 2.4. *Let $K \subset \mathbb{R}^d$ be the self-similar set satisfying the SOSC for an open set $\mathcal{O}$, with packing and Hausdorff dimension s. Then*

$$\mathcal{P}^s(K) = \sup\{\frac{(2r)^s}{\lambda(B(x,r))} | x \in K, B(x,r) \subset \mathcal{O}, B(x,r) \nsubseteq f_j(\mathcal{O}), 1 \leq j \leq N\}.$$

Corollary 2.5. *If $\Delta > 0$ and $d(K_i, K_j) > \Delta$ for all $i \neq j$. Then*

$$\mathcal{P}^s(K) = \sup_{x\in K, \frac{1}{2}r_*\Delta \leq r \leq \frac{1}{2}\Delta} \frac{(2r)^s}{\lambda(B(x,r))},$$

where $r_* = \min_{1\leq i\leq N} r_i$.

Proof. Let $\mathcal{O} = \bigcup_{x\in K} B(x, \frac{\Delta}{2})$. Then it is obvious that this open set $\mathcal{O}$ satisfies the SOSC, and therefore the results in previous can be fully applied in the SSC case. Hence from Corollary 2.2, we get

$$\mathcal{P}^s(K) = \sup_{x\in K, 0<r\leq \frac{1}{2}\Delta} \frac{(2r)^s}{\lambda(B(x,r))}. \tag{2.9}$$

By Lemma 2.3, we can limit $B(x,r)$ not contained in each $f_j(\mathcal{O})$. Fix $x \in K$, $0 < r \leq \frac{1}{2}\Delta$. Then $B(x,r) \subset \mathcal{O}$ is a ball centered in K. Hence there exists j, such that $x \in K_j$. Obviously $B(x,r) \cap K_j \neq \emptyset$, which yields that $B(x,r) \cap f_j(\overline{\mathcal{O}}) \neq \emptyset$. Hence $B(x,r) \cap f_j(\mathcal{O}) \neq \emptyset$. But $B(x,r)$ can not contained in $f_j(\mathcal{O})$, i.e., $B(x,r) \nsubseteq \bigcup_{z\in K_j} B(z, \frac{r_j}{2}\Delta)$. So $r \geq \frac{r_j}{2}\Delta \geq \frac{r_*}{2}\Delta$. Hence

$$\mathcal{P}^s(K) = \sup_{x\in K, \frac{1}{2}r_*\Delta \leq r \leq \frac{1}{2}\Delta} \frac{(2r)^s}{\lambda(B(x,r))}. \square$$

3 Proof of Theorem 1.1.

In this section, we prove Theorem 1.1. In order to prove this theorem, we need some lemmas. Below $D_H(\cdot,\cdot)$ denotes the Hausdorff metric on the family of all compact subsets of X.

Lemma 3.1. *Let $\Delta > 0$, $\boldsymbol{f} \in M_\Delta$. Then there exists $\beta > 0$ such that if $\boldsymbol{g} \in M_\Delta$ with $D(\boldsymbol{f}, \boldsymbol{g}) \leq \beta$, then $r_*^{\boldsymbol{g}} \geq \frac{1}{2} r_*^{\boldsymbol{f}}$.*

Proof. The proof is easy since the map $\mathbf{g} \to r_*^{\mathbf{g}}$ from M_Δ into $\mathbb{R}$ is continuous. □

Lemma 3.2. *Let $\Delta > 0$, $\boldsymbol{f} \in M_\Delta$, $0 < \rho < \frac{1}{4} r_*^{\boldsymbol{f}} \Delta$, $\gamma > 0$ and let β be the same as that in Lemma 3.1. Then there exists $\delta_1 > 0$ with $\delta_1 \leq \beta$ such that if $\boldsymbol{g} \in M_\Delta$ and $\boldsymbol{h} \in M_\Delta$ with $D(\boldsymbol{f}, \boldsymbol{g}) \leq \delta_1$ and $D(\boldsymbol{f}, \boldsymbol{h}) \leq \delta_1$, then for each ball $B(x, r)$ centered in $K^{\boldsymbol{g}}$ with radius $r \in [\frac{1}{2} r_*^{\boldsymbol{g}} \Delta, \frac{1}{2}\Delta]$, there exists a ball $B(y, r - \rho)$ centered in $K^{\boldsymbol{h}}$ such that*

$$\lambda^{\boldsymbol{h}}(B(y, r-\rho)) - \gamma \leq \lambda^{\boldsymbol{g}}(B(x, r)).$$

Proof. Choose a positive integer k such that

$$\operatorname{diam}(f_{\mathbf{i}}(K^{\mathbf{f}})) \leq \frac{\rho}{8}.$$

for all $\mathbf{i} \in W_k$. By the continuity of the map $\mathbf{g} \to K^{\mathbf{g}}$ from M_Δ into $\mathcal{K}$ and the map $\mathbf{g} \to r_i^{\mathbf{g}}$ from M_Δ into $\mathbb{R}$ for each $1 \leq i \leq N$, we can choose $\delta_1 > 0$ with $\delta_1 \leq \beta$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \delta_1$, then

$$\operatorname{diam}(g_{\mathbf{i}}(K^{\mathbf{g}})) \leq \frac{\rho}{4}, \tag{3.1}$$

$$D_H(f_{\mathbf{i}}(K^{\mathbf{f}}), g_{\mathbf{i}}(K^{\mathbf{g}})) \leq \frac{\rho}{8}, \tag{3.2}$$

$$|(r_{\mathbf{i}}^{\mathbf{f}})^{s^{\mathbf{f}}} - (r_{\mathbf{i}}^{\mathbf{g}})^{s^{\mathbf{g}}}| \leq \frac{\gamma}{2N^k} \tag{3.3}$$

for all $\mathbf{i} \in W_k$.

Now fix $\mathbf{g}, \mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \delta_1$, $D(\mathbf{f}, \mathbf{h}) \leq \delta_1$, also fix a ball $B(x, r)$ centered in $K^{\mathbf{g}}$ with radius $r \in [\frac{1}{2} r_*^{\mathbf{g}} \Delta, \frac{1}{2}\Delta]$. By Lemma 3.1, $r_*^{\mathbf{g}} \geq \frac{1}{2} r_*^{\mathbf{f}}$, so $r - \rho > 0$.

Since $D_H(K^{\mathbf{f}}, K^{\mathbf{g}}) \leq \frac{\rho}{8}$, $D_H(K^{\mathbf{f}}, K^{\mathbf{h}}) \leq \frac{\rho}{8}$, we get $D_H(K^{\mathbf{g}}, K^{\mathbf{h}}) < \frac{\rho}{2}$. Hence there exists a point $y \in B(x, \frac{\rho}{2})$ with $y \in K^{\mathbf{h}}$, which yields that

$$B(y, r-\rho) \subset B(x, r - \frac{\rho}{2}). \tag{3.4}$$

If we denote $V_k = \{\mathbf{i} | \mathbf{i} \in W_k, h_{\mathbf{i}}(K^{\mathbf{h}}) \cap B(x, r - \frac{\rho}{2}) \neq \emptyset\}$, then from (3.3) and (3.4) we get that

$$
\begin{aligned}
\lambda^{\mathbf{h}}(B(y,r-\rho))-\gamma &\leq \lambda^{\mathbf{h}}(B(x,r-\frac{\rho}{2}))-\gamma \\
&\leq \lambda^{\mathbf{h}}(\bigcup_{\mathbf{i}\in V_k} h_{\mathbf{i}}(K^{\mathbf{h}}))-\gamma \\
&= \sum_{\mathbf{i}\in V_k} \lambda^{\mathbf{h}}(h_{\mathbf{i}}(K^{\mathbf{h}}))-\gamma \\
&= \sum_{\mathbf{i}\in V_k} (r_{\mathbf{i}}^{\mathbf{h}})^{s^{\mathbf{h}}}-\gamma \\
&\leq \sum_{\mathbf{i}\in V_k} ((r_{\mathbf{i}}^{\mathbf{f}})^{s^{\mathbf{f}}}+\frac{\gamma}{2N^k})-\gamma \\
&\leq \sum_{\mathbf{i}\in V_k} ((r_{\mathbf{i}}^{\mathbf{g}})^{s^{\mathbf{g}}}+\frac{\gamma}{N^k})-\gamma \\
&\leq \sum_{\mathbf{i}\in V_k} (r_{\mathbf{i}}^{\mathbf{g}})^{s^{\mathbf{g}}} \\
&= \sum_{\mathbf{i}\in V_k} \lambda^{\mathbf{g}}(g_{\mathbf{i}}(K^{\mathbf{g}})).
\end{aligned}
$$

Next, if we denote by $U_k = \{\mathbf{i}|\mathbf{i} \in W_k, g_{\mathbf{i}}(K^{\mathbf{g}}) \cap B(x, r - \frac{\rho}{4}) \neq \emptyset\}$, then we must have $V_k \subset U_k$. In fact, if $\mathbf{i} \in V_k$, then $h_{\mathbf{i}}(K^{\mathbf{h}}) \cap B(x, r - \frac{\rho}{2}) \neq \emptyset$. Combining this with (3.2) we get $f_{\mathbf{i}}(K^{\mathbf{f}}) \cap B(x, r - \frac{3\rho}{8}) \neq \emptyset$, and using (3.2) once more we could get that $g_{\mathbf{i}}(K^{\mathbf{g}}) \cap B(x, r - \frac{\rho}{4}) \neq \emptyset$ which proves $V_k \subset U_k$. Hence,

$$\lambda^{\mathbf{h}}(B(y,r-\rho))-\gamma \leq \sum_{\mathbf{i}\in U_k} \lambda^{\mathbf{g}}(g_{\mathbf{i}}(K^{\mathbf{g}})) = \lambda^{\mathbf{g}}(\bigcup_{\mathbf{i}\in U_k} g_{\mathbf{i}}(K^{\mathbf{g}})).$$

Finally, notice that if $\mathbf{i} \in W_k$ and $g_{\mathbf{i}}(K^{\mathbf{g}}) \cap B(x, r - \frac{\rho}{4}) \neq \emptyset$, then it follows from (3.1) that $g_{\mathbf{i}}(K^{\mathbf{g}}) \subset B(x,r)$, whence $\bigcup_{\mathbf{i}\in U_k} g_{\mathbf{i}}(K^{\mathbf{g}}) \subset B(x,r)$. Hence we have

$$\lambda^{\mathbf{h}}(B(y,r-\rho))-\gamma \leq \lambda^{\mathbf{g}}(B(x,r)).\square$$

Lemma 3.3. *Let $\Delta > 0$, $\boldsymbol{f} \in M_\Delta$, $\kappa > 0$, and let β be the same as that in Lemma 3.1. Then there exists $0 < \rho < \min\{\beta, \frac{1}{4} r_*^{\boldsymbol{f}} \Delta\}$ such that if $\boldsymbol{g}, \boldsymbol{h} \in M_\Delta$ with $D(\boldsymbol{f}, \boldsymbol{g}) \leq \rho$ and $D(\boldsymbol{f}, \boldsymbol{h}) \leq \rho$, and $r \in [\frac{1}{2} r_*^{\boldsymbol{g}} \Delta, \frac{1}{2}\Delta]$, then*

$$(2(r-\rho))^{s^{\boldsymbol{h}}} \geq (2r)^{s^{\boldsymbol{g}}} - \kappa.$$

Proof. Notice that the map $(r, \mathbf{g}) \to (2r)^{s^{\mathbf{g}}}$ from $(0, \frac{1}{2}\Delta) \times M_\Delta$ into $\mathbb{R}$ is continuous. Moreover, for each fixed $\mathbf{g}$, the map is uniformly continuous on $r \in (0, \frac{1}{2}\Delta)$. Hence, for fixed $\mathbf{f} \in M_\Delta$, $\kappa > 0$, there exists $0 < \rho < \min\{\beta, \frac{1}{4} r_*^{\mathbf{f}} \Delta\}$ with the following property. If $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \rho$, and $r', r'' \in (0, \frac{1}{2}\Delta)$ with $|r' - r''| \le \rho$, then

$$|(2r')^{s^{\mathbf{g}}} - (2r'')^{s^{\mathbf{f}}}| \le \frac{\kappa}{2}. \tag{3.5}$$

Hence if we take $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \rho$, and $r' = r, r'' = r - \rho$ with $r \in [\frac{1}{2} r_*^{\mathbf{g}} \Delta, \frac{1}{2}\Delta]$, then by (3.5), we get

$$|(2r)^{s^{\mathbf{g}}} - (2(r-\rho))^{s^{\mathbf{f}}}| \le \frac{\kappa}{2}.$$

And if we take $\mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{h}) \le \rho$, and $r' = r'' = r - \rho$ with $r \in [\frac{1}{4} r_*^{\mathbf{f}} \Delta, \frac{1}{2}\Delta]$, then also by (3.5), we get

$$|(2(r-\rho))^{s^{\mathbf{h}}} - (2(r-\rho))^{s^{\mathbf{f}}}| \le \frac{\kappa}{2}.$$

Combining the above two inequalities and Lemma 3.1 gives the desired result. □

Lemma 3.4. *Let c, C, $\kappa > 0$, $0 < \varepsilon < 1$ with $\kappa < c\varepsilon$ and $c < C$. Then there exists $\gamma > 0$ such that*

$$\frac{x - \kappa}{y + \gamma} \ge \frac{x}{y} - \varepsilon$$

for all $x, y \in [c, C]$.

Proof. Take $\gamma = \frac{c(c\varepsilon - \kappa)}{C}$. Without losing generality, we may assume that $\frac{x}{y} - \varepsilon > 0$, i.e., $x - y\varepsilon > 0$, then $\gamma \le \frac{y(\varepsilon y - \kappa)}{x - y\varepsilon}$. Hence $(x - y\varepsilon)\gamma \le y(\varepsilon y - \kappa)$, which yields $\kappa y + x\gamma \le \varepsilon y(y + \gamma)$. Thus, dividing the above inequality by $y(y+\gamma) > 0$ gives $\frac{\kappa y + x\gamma}{y(y+\gamma)} \le \varepsilon$, i.e., $\frac{x}{y} - \frac{x - \kappa}{y + \gamma} \le \varepsilon$. □

Proof of Theorem 1.1.

Since $M_{SSC} = \bigcup_{\Delta > 0} M_\Delta$ and that M_Δ are open subsets of M_{SSC} for all $\Delta > 0$, we only need to prove for each $\Delta > 0$, the map

$$\mathbf{f} \to \mathcal{P}^{s^{\mathbf{f}}}(K^{\mathbf{f}})$$

from M_Δ into $\mathbb{R}$ is continuous.

Fix $\Delta > 0$, $\mathbf{f} \in M_\Delta$ and let $0 < \varepsilon < 1$. We now find $\delta > 0$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta$, then

$$|\mathcal{P}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) - \mathcal{P}^{s^{\mathbf{g}}}(K^{\mathbf{g}})| \le \varepsilon.$$

By the continuity of the map $\mathbf{g} \to s^{\mathbf{g}}$ from M_Δ into $\mathbb{R}$, there exists $\delta_2 > 0$ with $\delta_2 \le \beta$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_2$, then

$$\frac{1}{2} s^{\mathbf{f}} \le s^{\mathbf{g}} \le \frac{3}{2} s^{\mathbf{f}},$$

where β is the same as that in Lemma 3.1. Hence there exists $C_1, C_2 > 0$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \delta_2$, and $r \in [\frac{1}{2} r_*^{\mathbf{g}} \Delta, \frac{1}{2}\Delta]$, then

$$C_1 \leq (2r)^{s^{\mathbf{g}}} \leq C_2.$$

In Lemma 3.1 of [14], the map $\mathbf{g} \to \lambda^{\mathbf{g}}$ from M_Δ into $\mathcal{M}$ is continuous, where $\mathcal{M}$ denotes the space consist of all Borel regular probability measures equipped with the weak topology. Hence there exists $\delta_3 > 0$, $C_3 > 0$ with $\delta_3 \leq \beta$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \delta_3$, then

$$\lambda^{\mathbf{g}}(B(x, r)) \geq C_3$$

for all ball $B(x, r)$ with radius $r \in [\frac{1}{2} r_*^{\mathbf{g}} \Delta, \frac{1}{2}\Delta]$ centered in $K^{\mathbf{g}}$.

Put $c = \min\{C_1, C_3\}$, $C = C_2 + 1 + c$, $\kappa = \frac{c\varepsilon}{4}$, $\gamma > 0$ as in Lemma 3.4. Take $\delta = \min\{\rho, \delta_1, \delta_2, \delta_3\}$. We claim that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \delta$, then

$$|\mathcal{P}^{s^{\mathbf{f}}}(K^{\mathbf{f}}) - \mathcal{P}^{s^{\mathbf{g}}}(K^{\mathbf{g}})| \leq \varepsilon.$$

To prove this we show that if $\mathbf{g}, \mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \delta$ and $D(\mathbf{f}, \mathbf{h}) \leq \delta$, then

$$\mathcal{P}^{s^{\mathbf{h}}}(K^{\mathbf{h}}) \geq \mathcal{P}^{s^{\mathbf{g}}}(K^{\mathbf{g}}) - \varepsilon. \tag{3.6}$$

We therefore fix $\mathbf{g}, \mathbf{h} \in M_\Delta$ satisfying $D(\mathbf{f}, \mathbf{g}) \leq \delta$ and $D(\mathbf{f}, \mathbf{h}) \leq \delta$. It follows from the Corollary 2.5 that there exists $B(x, r)$ centered in $K^{\mathbf{g}}$ with radius $r \in [\frac{1}{2} r_*^{\mathbf{g}} \Delta, \frac{1}{2}\Delta]$ such that

$$\frac{(2r)^{s^{\mathbf{g}}}}{\lambda^{\mathbf{g}}(B(x, r))} \geq \mathcal{P}^{s^{\mathbf{g}}}(K^{\mathbf{g}}) - \frac{\varepsilon}{2}. \tag{3.7}$$

Since $c \leq (2r)^{s^{\mathbf{g}}} \leq C$, $c \leq \lambda^{\mathbf{g}}(B(x, r)) \leq C$ and $\kappa = \frac{c\varepsilon}{4} < \frac{c\varepsilon}{2}$, then from Lemma 3.4,

$$\frac{(2r)^{s^{\mathbf{g}}} - \kappa}{\lambda^{\mathbf{g}}(B(x, r)) + \gamma} \geq \frac{(2r)^{s^{\mathbf{g}}}}{\lambda^{\mathbf{g}}(B(x, r))} - \frac{\varepsilon}{2}. \tag{3.8}$$

And from Lemma 3.3, we have

$$(2(r - \rho))^{s^{\mathbf{h}}} \geq (2r)^{s^{\mathbf{g}}} - \kappa. \tag{3.9}$$

Then from Lemma 3.2, there exists a ball $B(y, r - \rho)$ centered in $K^{\mathbf{h}}$ such that

$$\lambda^{\mathbf{h}}(B(y, r - \rho)) - \gamma \leq \lambda^{\mathbf{g}}(B(x, r)). \tag{3.10}$$

Combining (2.9) and (3.7) to (3.10), we get

$$\begin{aligned}
\mathcal{P}^{s^{\mathbf{h}}}(K^{\mathbf{h}}) &\geq \frac{2(r-\rho)^{s^{\mathbf{h}}}}{\lambda^{\mathbf{h}}(B(y,r-\rho))} \\
&\geq \frac{(2r)^{s^{\mathbf{g}}}-\kappa}{\lambda^{\mathbf{g}}(B(x,r))+\gamma} \\
&\geq \frac{(2r)^{s^{\mathbf{g}}}}{\lambda^{\mathbf{g}}(B(x,r))}-\frac{\varepsilon}{2} \\
&\geq \mathcal{P}^{s^{\mathbf{g}}}(K^{\mathbf{g}})-\varepsilon.
\end{aligned}$$

This proves (3.6) and hence the proof of Theorem 1.1 is completed. □

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