Exact spectrum of the Laplacian on a domain in the Sierpinski gasket

HUA QIU

Abstract. For a certain domain Ω in the Sierpinski gasket $\mathcal{SG}$ whose boundary is a line segment, a complete description of the eigenvalues of the Laplacian, with an exact count of dimensions of eigenspaces, under the Dirichlet and Neumann boundary conditions is presented. The method developed in this paper is a weak version of the spectral decimation method due to Fukushima and Shima, since for a lot of "bad" eigenvalues the spectral decimation method can not be used directly. Let $\rho^0(x)$, $\rho^\Omega(x)$ be the eigenvalue counting functions of the Laplacian associated to $\mathcal{SG}$ and Ω respectively. We prove a comparison between $\rho^0(x)$ and $\rho^\Omega(x)$ says that $0 \leq \rho^0(x) - \rho^\Omega(x) \leq Cx^{\log 2/\log 5}\log x$ for sufficiently large x for some positive constant C. As a consequence, $\rho^\Omega(x) = g(\log x)x^{\log 3/\log 5} + O(x^{\log 2/\log 5}\log x)$ as $x \to \infty$, for some (right-continuous discontinuous) $\log 5$-periodic function $g : \mathbb{R} \to \mathbb{R}$ with $0 < \inf_{\mathbb{R}} g < \sup_{\mathbb{R}} g < \infty$. Moreover, we explain that the asymptotic expansion of $\rho^\Omega(x)$ should admit a second term of the order $\log 2/\log 5$, that becomes apparent from the experimental data. This is very analogous to the conjectures of Weyl and Berry.

Keywords. Sierpinski gasket, Laplacian, eigenvalues, spectral decimation.

Mathematics Subject Classification (2000). 28A80, 31C99

1 Introduction

The study of the Laplacian on fractals was originated by S. Kusuoka [22] and S. Goldstein [11]. They independently constructed the Laplacian as the generator of a diffusion process on the Sierpinski gasket $\mathcal{SG}$. Later an analytic approach was developed by J. Kigami [16], who constructed the Laplacian both as a renormalized limit of difference operators and a weak formulation using the theory of Dirichlet forms.

Let V_0 be the boundary of $\mathcal{SG}$, which consists of the three vertices of the equilateral triangle containing $\mathcal{SG}$. Consider the following Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \mathcal{SG} \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

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where Δ is the standard Laplacian (with respect to the standard self-similar measure μ) on $\mathcal{SG}$. Physicists R. Rammal and G. Toulouse [28] found that an appropriate choice of a series of eigenvalues of successive difference operators produces an orbit of a dynamical system related to a quadratic polynomial, and all the eigenvalues of $-\Delta$ on $\mathcal{SG} \setminus V_0$ should be obtained by tracking back the orbits. This is the phenomenon which M. Fukushima and T. Shima [10, 31] described from the mathematical point of view, by saying that $\mathcal{SG}$ admits *spectral decimation* with respect to a quadratic polynomial. Using the spectral decimation, all the eigenvalues and eigenfunctions of $-\Delta$ on $\mathcal{SG} \setminus V_0$ have been determined exactly. This method also works for the eigenvalue problem of $-\Delta$ with Neumann boundary condition.

Later the theory of the Laplacian was developed for nested fractals and p.c.f. self-similar sets by T. Lindstrøm [24] and Kigami [17] by introducing the notion of *harmonic structure*. Every p.c.f. self-similar set is approximated by an increasing sequence of finite graphs and the harmonic structure determines a sequence of difference operators on the successive graphs, which converges to the Laplacian. Then some generalizations of the spectral decimation to a class of p.c.f. self-similar sets were developed by Shima [32], L. Malozemov and A. Teplyaev [25], in which some strong symmetry conditions are supposed to be satisfied to ensure the spectral decimation applies to the corresponding graph sequences. Under such strong symmetry conditions, the spectrum of the Laplacian can also be determined in terms of the iteration of a rational function. Recently, the spectrum of the Laplacian on some other fractals has been analyzed either numerically [1] or using the spectral decimation method [7, 8, 38, 40] by R. S. Strichartz (with co-authors), D. Zhou and Teplyaev. In all the references mentioned above, spectral decimation plays a key role in the theoretical study of the spectrum of the Laplacian.

The Weyl asymptotic behavior of the eigenvalue counting function of $-\Delta$ on $\mathcal{SG} \setminus V_0$ has also been studied by Fukushima and Shima [10]. Afterwards, a general spectral distribution theory on p.c.f. self-similar sets was obtained by Kigami and M. L. Lapidus [18, 19]. Denote by $\rho^0(x)$ the number of eigenvalues of $-\Delta$(taking the multiplicities into account) on $\mathcal{SG} \setminus V_0$ not exceeding x, with Dirichlet boundary condition at the three vertices. As proved in [10, 18], there exist positive constants c, C such that

$$cx^{d_S/2} \leq \rho^0(x) \leq Cx^{d_S/2} \tag{1.1}$$

for sufficiently large x, where $d_S = \log 9/\log 5$ is the *spectral dimension* of $\mathcal{SG}$. In particular, $\rho^0(x)$ varies highly irregularly at ∞ due to the high multiplicities of localized eigenfunctions,

$$0 < \liminf_{x\to\infty} \rho^0(x)x^{-d_S/2} < \limsup_{x\to\infty} \rho^0(x)x^{-d_S/2} < \infty. \tag{1.2}$$

Furthermore, using a refinement of the Renewal Theorem, Kigami [19] showed that the remainder of $\rho^0(x)$ is bounded,

$$\rho^0(x) = g(\log x)x^{ds/2} + O(1) \quad as \quad x \to \infty, \tag{1.3}$$

for some (right-continuous discontinuous) $\log 5$-periodic function $g : \mathbb{R} \to \mathbb{R}$ with $0 < \inf_{\mathbb{R}} g < \sup_{\mathbb{R}} g < \infty$. Exactly the same results hold for the eigenvalue counting function for the Neumann Laplacian.

In this paper, we are mainly concerned with eigenvalue problems for a domain in $\mathcal{SG}$. Although analysis on fractals has been made possible by the definition of the Laplacian, there has been little research into differential equations on bounded subsets of fractals. Recall that $\mathcal{SG}$ is the attractor of the *iterated function system* $\{F_0, F_1, F_2\}$ with $F_i x = \frac{1}{2}(x + q_i)$ where q_0, q_1, q_2 are the vertices of an equilateral triangle in the plane,

$$\mathcal{SG} = \bigcup_{i=0}^{2} F_i(\mathcal{SG}).$$

In Kigami's theory the boundary of $\mathcal{SG}$ consists of the three points q_0, q_1, q_2, and the space of *harmonic functions* (solutions of $\Delta u = 0$) is three dimensional, with u determined explicitly by its boundary values $u(q_i)$. (Note that this boundary is not a topological boundary.) Thus the harmonic function theory on $\mathcal{SG}$ is more closely related to the theory of linear functions on the unit interval than to harmonic functions on the disk. To get a richer theory we should take an open set Ω in $\mathcal{SG}$ and restrict the Laplacian on $\mathcal{SG} \setminus V_0$ to functions defined on Ω. Hence we believe it is appropriate to begin the study of differential equations related to a bounded domain Ω in $\mathcal{SG}$.

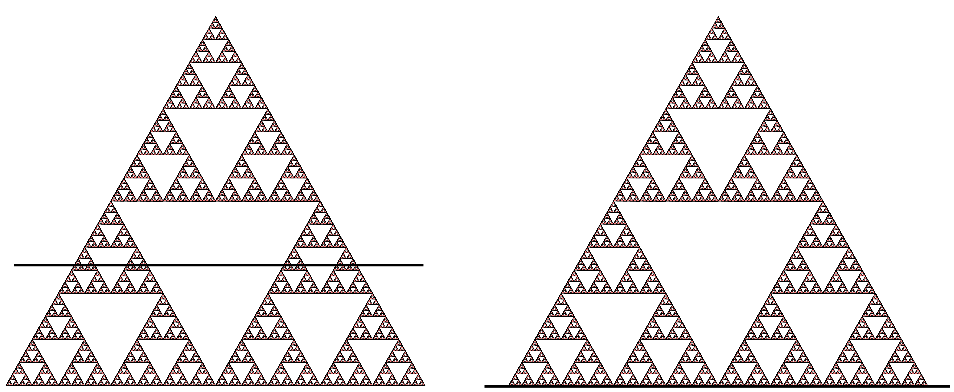


Fig. 1.1. Ω_x and Ω_1.

For simplicity, here we particularly focus on the certain domain Ω_x which is a triangle obtained by cutting $\mathcal{SG}$ with a horizontal line at any vertical height x ($0 < x \leq 1$ if we suppose that the height of $\mathcal{SG}$ is equal to 1.) below the top vertex q_0. See Fig. 1.1. An important motivation for studying this kind of domains is that they are the simplest examples which could serve as a testing ground for questions and conjectures on analysis of more general fractal domains with fractal boundaries. These domains were first introduced by Strichartz [33] and later studied by J. Owen and Strichartz [26], where they gave an explicit analog of the *Poisson integral formula* to recover a harmonic function u on Ω_x from its boundary values. It is also natural to calculate an explicit *Green's function* for the Laplacian on Ω_x. This was studied by Z. Guo, R. Kogan, H. Qiu and Strichartz in [12] which is completely similar to the construction of the Green's function on $\mathcal{SG} \setminus V_0$ given by Kigami in [16, 17, 20]. For some other analytic topics related to this kind of domains, see [14, 15, 21].

In the present paper, we study the spectral properties of the Laplacian on Ω_x, which is an open problem posed in [26]. For the simplicity of description, we mainly concentrate our attention to a particular domain Ω_1 (We drop the subscript 1 on Ω in all that follows without causing any confusion.) which is the complement of $\{q_0\} \cup L$, where L is the line segment joining q_1 and q_2 (in this case $\partial\Omega = \{q_0\} \cup L$). We give a complete description of the Dirichlet and Neumann spectra of the Laplaician on Ω.

In our context, for a number of "bad" eigenvalues (whose associated eigenfunctions have supports touching the bottom boundary line L) the spectral decimation method can not be used directly, which makes things much more complicated. By choosing a sequence of appropriate graph approximations, we describe a phenomenon on those eigenvalues called *weak spectral decimation* which approximates to spectral decimation when the levels of the successive graphs go to infinity. And we use this weak spectral decimation to replace the role of spectral decimation in the original Fukushima and Shima's work [10]. Actually, similarly to the standard case, weak spectral decimation can also produce a "weak" orbit related to the same quadratic polynomial by an appropriate series of eigenvalues of successive difference operators on graph approximations. We can then trace back those "weak" orbits to capture all the "bad" eigenvalues. More precisely, we classify the eigenvalues of $-\Delta$ on Ω into three types, the *localized eigenvalues*, *primitive eigenvalues* and *miniaturized eigenvalues*. The localized eigenfunctions associated to localized eigenvalues on Ω are just a subspace of the localized eigenfunctions on $\mathcal{SG} \setminus V_0$, whose supports are disjoint from L. This type of eigenvalues can be dealt with in a same way as the $\mathcal{SG} \setminus V_0$ case, for which the spectral decimation can apply. The primitive and miniaturized eigenvalues are the so-called "bad" eigenvalues. They are the eigenvalues that need to be paid particular attention to. We will give a precise description of the structure of the Dirichlet and Neumann spectra of $-\Delta$ on Ω in Section 3, before giving the technical proofs.

Now what happens to the asymptotic behavior of the eigenvalue counting function $\rho^{\Omega}(x)$(with Dirichlet boundary condition on $\partial\Omega$) on Ω? A natural analogue of (1.1) holds. Namely, there exists some positive constants c, C such that for sufficiently large x,

$$cx^{d_S/2} \leq \rho^{\Omega}(x) \leq Cx^{d_S/2}, \tag{1.4}$$

which can be proved by first considering the asymptotic behavior of the eigenvalue counting function for each type of eigenvalues separately, then adding them up together. In fact, (1.4) can be even easily proved without involving the structure of the Dirichlet spectrum on Ω, as follows: the Dirichlet eigenvalue counting function on the top cell $F_0(\mathcal{SG})$ is given by $\rho^0(x/5)$ by the self-similarity of both the Dirichlet form and the measure μ. Therefore it follows from the minmax principle that $\rho^0(x/5) \leq \rho^{\Omega}(x) \leq \rho^0(x)$, which together with (1.1), also yields (1.4). Moreover, the high multiplicities of localized eigenfunctions immediately imply that $\rho^{\Omega}(x)$ does not vary regularly at ∞, similarly to (1.2). Thus,

$$0 < \liminf_{x\to\infty} \rho^{\Omega}(x)x^{-d_S/2} < \limsup_{x\to\infty} \rho^{\Omega}(x)x^{-d_S/2} < \infty.$$

Since most eigenvalues are localized, $\rho^0(x)$ and $\rho^{\Omega}(x)$ are very close. We are interested in the difference $\rho^0(x) - \rho^{\Omega}(x)$. More precisely, is there some power β such that $\rho^0(x) - \rho^{\Omega}(x) \approx x^{\beta}$? For this question, we have the following partial result:

Theorem 3.10. *There exists some constant $C > 0$ such that for sufficiently large x,*

$$0 \leq \rho^0(x) - \rho^{\Omega}(x) \leq Cx^{\log 2/\log 5}\log x.$$

As a consequence, it then follows from (1.3) that

$$\rho^{\Omega}(x) = g(\log x)x^{\log 3/\log 5} + O(x^{\log 2/\log 5}\log x) \quad \text{as } x \to \infty. \tag{1.5}$$

The same argument also works for the Neumann Laplacian.

Nevertheless, this should not be the entire story for the Weyl asymptotic behavior of $\rho^{\Omega}(x)$. Recall the classical case. Suppose D is an arbitrary nonempty bounded open set in $\mathbb{R}^n$ with smooth boundary ∂D, then Weyl's classical asymptotic formula can be stated as follows:

$$\rho(x) = (2\pi)^{-n}c_n|D|_n x^{n/2} + O(x^{(n-1)/2})$$

as $x \to \infty$, where c_n depends only on n. See details in [27, 29, 30]. The above remainder estimate constitutes an important step on the way to H. Weyl's conjecture [39] which states that if ∂D is sufficiently "smooth", then the asymptotic expansion of $\rho(x)$ admits a second term, proportional to $x^{(n-1)/2}$. Extending Weyl's conjecture to the fractal case, M. V. Berry [3, 4] conjectured that if D has a fractal boundary ∂D with Hausdorff dimension (which later was revised into Minkowski dimension in [6, 23]) $d_{\partial D} \in (n-1, n]$, then the order of the second term should be replaced by $d_{\partial D}/2$. See further discussion and a partial resolution of the conjectures of Weyl and Berry in Lapidus's work [23]. Hence it is natural to ask that whether there is an analogue result in $\mathcal{SG} \setminus V_0$ or Ω setting. For $\mathcal{SG} \setminus V_0$ case, Kigami [19] showed that the remainder is bounded, see (1.3). Note that this is consistent with the fact that the boundary of $\mathcal{SG}$ consists of three points, hence has dimension zero. This was refined by Strichartz in [37], where an exact formula was presented with no remainder term at all, provided we restrict attention to almost every x. As for the Ω case, (1.5) can be viewed as a weak analog of the Weyl-Berry's conjecture. Moreover, we will see that although we are unable to provide a proof, it becomes apparent there is a second term of order $\log 2/\log 5$ in the expansion of the eigenvalue counting function on Ω from observing the experimental data.

We note that our work deals with the vibrations of "drums with fractal membrane" since the domain itself is a fractal. The order of the second term should have a close connection with the dimension of the boundary $\partial\Omega$ due to Weyl-Berry's conjectures. Moreover, when we consider a more general domain Ω_x, we will meet "drums with fractal membrane" with also fractal boundary.

The paper is organized as follows. In Section 2 we will briefly introduce some key notions from analysis on fractals and give a concise description of the Dirichlet and Neumann spectra of the Laplacian for the standard $\mathcal{SG} \setminus V_0$ case, which will be used in the rest of the paper.

In Section 3, we will present the exact structure of the Dirichlet spectrum of $-\Delta$ on Ω in a self-contained and precise way before going into the technical details. We will find an appropriate sequence of graph approximations for the fractal domain Ω, and describe the exact structures of the discrete Dirichlet spectra of the corresponding successive difference operators on them. Accordingly, for each graph all the graph eigenvalues are also divided into three types, localized, primitive and miniaturized. By using an eigenspace dimensional counting argument, we will show that they should make up the whole discrete Dirichlet spectrum. We will also briefly describe how to relate the spectra of consecutive levels and how to pass the graph approximations to the limit by using spectral decimation for localized eigenvalues and weak spectral decimation for other types of eigenvalues. We will also present analogous results for Laplacians with Neumann boundary conditions.

In Section 4, we will describe the discrete graph primitive Dirichlet eigenvalues on the graph approximations for each level. We will divide our discussion into symmetric case and skew-symmetric case. In each case, we will prove that for each level the primitive graph eigenvalues are exactly the total roots of a high degree polynomial. And we will describe the weak spectral decimation phenomenon by studying the relation between roots of consecutive polynomials. Moreover, we will prove that for each level, the complete discrete spectrum is made up of the three types of eigenvalues as expected.

In Section 5, we will discuss the primitive Dirichlet eigenvalues of $-\Delta$ on Ω by passing the results of Section 4 on graph approximations to the limit. Since we can only use weak spectral decimation which is essentially based on estimates, comparing to the $\mathcal{SG} \setminus V_0$ case, some trivial results become nontrivial and need to be proved in this section.

In Section 6, first we will prove that the whole Dirichlet spectrum on Ω is made up of the three types of eigenvalues as expected, following the basic idea of Fukushima and Shima's work. Then we will give a comparison concerning the eigenvalue asymptotics of the eigenvalue counting functions between $\mathcal{SG} \setminus V_0$ case and Ω case.

In Section 7, we will give a brief discussion on how to deal with the Neumann spectrum. We will find a similar weak spectral decimation for primitive eigenvalues by establishing a relation between symmetric (or skew-symmetric) primitive graph eigenvalues with some high degree polynomials, but the proof is quite different from that in the Dirichlet case.

Then in Section 8, we will list some conjectures concerning eigenvalue asymptotics (especially the existence of the second term of the expansion of the eigenvalue counting function), gaps in the ratios of consecutive eigenvalues and eigenvalue clusters, which become apparent from observing the experimental data.

We will also give a brief discussion on how to extend our method from Ω to Ω_x with $0 < x < 1$ in Section 9.

The purpose of this paper is to work out the details for one specific example. We hope this example will provide insights which will inspire future work on a more general theory.

2 Spectral decimation on $\mathcal{SG} \setminus V_0$

First we collect some key facts from analysis on $\mathcal{SG}$ that we need to state and prove our results. These come from Kigami's theory of analysis on fractals, and can be found in [16, 17, 20]. An elementary exposition can be found in [34, 36]. The fractal $\mathcal{SG}$ will be realized as the limit of a sequence of graphs $\Gamma_0, \Gamma_1, \cdots$ with vertices $V_0 \subseteq V_1 \subseteq \cdots$. The initial graph Γ_0 is just the complete graph on $V_0 = \{q_0, q_1, q_2\}$, the vertices of an equilateral triangle in the plane, which is considered the boundary of $\mathcal{SG}$. See Fig. 2.1. The entire fractal is the only 0-cell, which has V_0 as its boundary. At stage m of the construction, all the cells of level $m-1$ lie in triangles whose vertices make up V_{m-1}. Each cell of level $m-1$ splits into three cells of level m, adding three new vertices to V_m. The graph Γ_m is the graph with vertices V_m by defining the edge relation $x \sim_m y$ if there is a cell of level m containing both x and y.

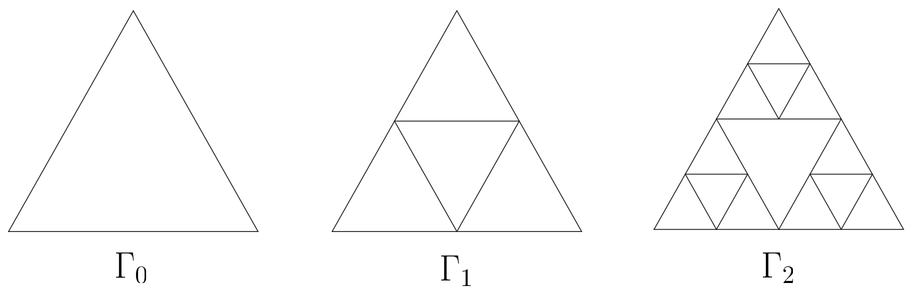


Fig. 2.1. The first 3 graphs, $\Gamma_0, \Gamma_1, \Gamma_2$ in the approximations to the Sierpinski gasket.

We define the unrenormalized energy of a function u on Γ_m by

$$E_m(u) := \sum_{x \sim_m y} (u(x) - u(y))^2.$$

The energy renormalization factor is $r = \frac{3}{5}$, so the renormalized graph energy on Γ_m is

$$\mathcal{E}_m(u) := r^{-m} E_m(u),$$

and we can define the *fractal energy* $\mathcal{E}(u) := \lim_{m\to\infty} \mathcal{E}_m(u)$. We define $\mathcal{F}$ as the space of continuous functions with finite energy modulus constants. Then $\mathcal{E}$ extends by polarization to a bilinear form $\mathcal{E}(u, v)$ which serves as an inner product in this space. The energy $\mathcal{E}$ gives rise to a natural distance on $\mathcal{SG}$ called the *effective resistance metric* on $\mathcal{SG}$, which is defined by

$$d(x, y) := (\min\{\mathcal{E}(u) : u(x) = 0 \text{ and } u(y) = 1\})^{-1} \tag{2.1}$$

for $x, y \in \mathcal{SG}$. It is known that $d(x, y)$ is bounded above and below by constant multiples of $|x-y|^{\log(5/3)/\log 2}$, where $|x-y|$ is the Euclidean distance. Furthermore, the definition (2.1) implies that functions on $\mathcal{F}$ are Hölder continuous of order $\frac{1}{2}$ in the effective resistance metric.

We let μ denote the standard probability measure on $\mathcal{SG}$ that assigns the measure 3^{-m} to each cell of m level. The standard Laplacian may then be defined using the weak formulation: $u \in dom\Delta$ with $-\Delta u = f$ if f is continuous, $u \in \mathcal{F}$, and

$$\mathcal{E}(u, v) = \int_{\mathcal{SG}} f v d\mu \tag{2.2}$$

for all $v \in \mathcal{F}_0$, where $\mathcal{F}_0 = \{v \in \mathcal{F} : v|_{V_0} = 0\}$. There is also a pointwise formula (which is proven to be equivalent in [36]) which, for nonboundary points in $V_* = \bigcup_m V_m$ (not in V_0) computes

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x),$$

where Δ_m is a discrete Laplacian associated to the graph Γ_m, defined by

$$\Delta_m u(x) := \sum_{y \sim_m x} (u(y) - u(x))$$

for x not on the boundary.

The Laplacian satisfies the scaling property

$$\Delta(u \circ F_i) = \frac{1}{5}(\Delta u) \circ F_i$$

and by iteration

$$\Delta(u \circ F_w) = \frac{1}{5^m}(\Delta u) \circ F_w$$

for $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$.

Although there is no satisfactory analogue of gradient, there is a *normal derivative* $\partial_n u(q_i)$ defined at boundary points by

$$\partial_n u(q_i) := \lim_{m \to \infty} \sum_{y \sim_m q_i} r^{-m}(u(q_i) - u(y)),$$

the limit existing for all $u \in dom\Delta$. The definition may be localized to boundary points of cells. For each point $x \in V_m \setminus V_0$, there are two cells containing x as a boundary point, hence two normal derivatives at x. For $u \in dom\Delta$, the normal derivatives at x satisfy the *matching condition* that their sum is zero. The matching condition allows us to glue together local solutions to $-\Delta u = f$.

The above matching condition property follows easily from a local version of the following *Gauss-Green formula*, which is an extension of (2.2) to the case when v doesn't vanish on the boundary:

$$\mathcal{E}(u, v) = \int_{\mathcal{SG}} (-\Delta u) v d\mu + \sum_{V_0} v \partial_n u.$$

The local version of the Gauss-Green formula is

$$\mathcal{E}_A(u, v) = \int_A (-\Delta u) v d\mu + \sum_{\partial A} v \partial_n u,$$

where A is any finite union of cells and $\mathcal{E}_A(u, v)$ is the restriction of the energy bilinear form $\mathcal{E}(u, v)$ to A, which can also be defined directly by

$$\mathcal{E}_A(u, v) := \lim_{m \to \infty} \sum_{\substack{x \sim_m y \\ inA}} (u(x) - u(y))(v(x) - v(y)).$$

Now we come to a brief recap of the spectral decimation on $\mathcal{SG}$. Our goal is to find all solutions of the eigenvalue equation

$$-\Delta u = \lambda u \quad \text{on } \mathcal{SG} \setminus V_0$$

as limits of solutions of the discrete version

$$-\Delta_m u_m = \lambda_m u_m \quad \text{ on } V_m \setminus V_0.$$

In the $\mathcal{SG} \setminus V_0$ case, we are lucky that we may always take $u_m = u|_{V_m}$, which turns out to be very convenient for spectral decimation. We should emphasize that this is not true for Ω case.

The method of spectral decimation on $\mathcal{SG}$ was invented by Fukushima and Shima [10] to relate eigenfunctions and eigenvalues of the discrete Laplacian $-\Delta_m$'s on the graph approximation Γ_m's for different values of m to each other and the eigenfunctions and eigenvalues of the fractal Laplacian $-\Delta$ on $\mathcal{SG} \setminus V_0$. In essence, an eigenfunction on Γ_m with eigenvalue λ_m can be extended to an eigenfunction on Γ_{m+1} with eigenvalue λ_{m+1}, where $\lambda_m = f(\lambda_{m+1})$ for an explicit function f defined by

$$f(x) := x(5-x), \tag{2.3}$$

except for certain specified *forbidden eigenvalues*, and all eigenfunctions on $\mathcal{SG} \setminus V_0$ arise as limits of this process starting at some level m_0 which is called the *generation of birth*. This is true regardless of the boundary conditions, but if we specify Dirichlet or Neumann boundary condition we can describe explicitly all eigenspaces and their multiplicities.

Denote the real valued inverse functions of $f(x)$ by $\phi_\pm(x)$. That is

$$\phi_\pm(x) := \frac{5 \pm \sqrt{25-4x}}{2}. \tag{2.4}$$

We describe the procedure briefly here. First, there is a *local extension algorithm* that shows how to uniquely extend an eigenfunction u_m defined on V_m to a function defined on V_{m+1} such that the λ-eigenvalue equations hold on all points of $V_{m+1} \setminus V_m$. For $\mathcal{SG}$, the extension algorithm is: Suppose u_m is an eigenfunction on Γ_m with eigenvalue λ_m. Let $\lambda_{m+1} = \phi_\pm(\lambda_m)$. Consider an m-cell with boundary points x_0, x_1, x_2 and let y_0, y_1, y_2 denote the points in $V_{m+1} \setminus V_m$ in that cell, with y_i opposite x_i. Extend u_m to a function u_{m+1} on V_{m+1} by defining (for simplicity of notation, we drop the subscripts on u)

$$u(y_i) = \frac{(4-\lambda_{m+1})((u(x_{i+1}) + u(x_{i-1}))) + 2u(x_i)}{(2-\lambda_{m+1})(5-\lambda_{m+1})}, \quad i = 0, 1, 2. \tag{2.5}$$

Then we have the following proposition taken from [36].

Proposition 2.1. *Suppose $\lambda_{m+1} \neq 2, 5$ or 6, and $\lambda_m = f(\lambda_{m+1})$. If u_m is a λ_m-eigenfunction of $-\Delta_m$ and is extended to a function u_{m+1} on V_{m+1} by (2.5), then u_{m+1} is a λ_{m+1}-eigenfunction of $-\Delta_{m+1}$, Conversely, if u_{m+1} is a λ_{m+1}-eigenfunction of $-\Delta_{m+1}$ and is restricted to a function u_m on V_m, then u_m is a λ_m-eigenfunction of $-\Delta_m$.*

The forbidden eigenvalues $\{2, 5, 6\}$ are singularities of the spectral decimation function f. It is "forbidden" to decimate to a forbidden eigenvalue. Because forbidden eigenvalues have no predecessor, we speak of forbidden eigenvalues being "born" at a level of approximation m.

Next we want to take the limit as $m \to \infty$. We assume that we have an infinite sequence $\{\lambda_m\}_{m \geq m_0}$ related by $\lambda_{m+1} = \phi_\pm(\lambda_m)$ with all but a finite number of ϕ_-'s. Then we may define

$$\lambda := \frac{3}{2} \lim_{m \to \infty} 5^m \lambda_m.$$

It is easy to see that the limit exists since

$$\phi_-(x) = \frac{1}{5}x + O(x^2) \tag{2.6}$$

as $x \to 0$. We start with a λ_{m_0}-eigenfunction u of $-\Delta_{m_0}$ on V_{m_0}, and extend u to V_* successively using (2.5), assuming that none of λ_m is a forbidden eigenvalue. Since (2.6) implies $\lambda_m = O(\frac{1}{5^m})$ as $m \to \infty$, it is easy to see that u is uniformly continuous on V_* and so extends to a continuous function on $\mathcal{SG}$. Moreover, it satisfies the λ-eigenvalue equation for $-\Delta$.

A proof in [10] guarantees that this spectral decimation produces all possible eigenvalues and eigenfunctions (up to linear combination).

To describe the explicit Dirichlet and Neumann spectra, we have to describe all possible generations of birth and values for λ_{m_0}, and describe the multiplicity of the eigenvalue by giving an explicit basis for the λ_{m_0}-eigenspace of $-\Delta_{m_0}$. For each m, we have to add up the dimensions of eigenspaces with generation of birth $m_0 \leq m$, extended to Γ_m in all allowable ways. This total must be $\sharp V_m$ (Neumann) or $\sharp V_m - 3$ (Dirichlet), the dimension of the space on which the symmetric operator $-\Delta_m$ acts. Now we give a brief description of the structure of the Dirichlet and Neumann spectra of $-\Delta$ on $\mathcal{SG} \setminus V_0$ respectively.

Dirichlet spectrum.

We denote by $\mathcal{D}$ the Dirichlet spectrum of $-\Delta$ on $\mathcal{SG} \setminus V_0$ and by $\mathcal{D}_m$ the discrete Dirichlet spectrum of $-\Delta_m$ on Γ_m for $m \geq 1$. Due to the above discussion, we only need to make clear the spectrum $\mathcal{D}_m$ for each level m. There are two kinds of eigenvalues, *initial* and *continued*. The continued eigenvalues will be those that arise from eigenvalues of $\mathcal{D}_{m-1}$ by spectral decimation. Those that remain, the initial eigenvalues, must be some of the forbidden eigenvalues by Proposition 2.1.

In [31], it is proved that $\mathcal{D}_1$ consists of two eigenvalues 2 and 5 with multiplicities 1 and 2 respectively, and for $m \geq 2$, the only possible initial eigenvalues in $\mathcal{D}_m$ are the two forbidden eigenvalues 5 and 6 with multiplicities $\frac{3^{m-1}+3}{2}$ and $\frac{3^m-3}{2}$ respectively. Hence we may classify eigenvalues into three series, which we call the 2-series, 5-series, and 6-series, depending on the value of λ_{m_0}. The eigenvalues in the 2-series all have multiplicity 1, while the eigenvalues in the other series all exhibit higher multiplicity. Also, if λ is an eigenvalue in the 5-series or 6-series, then $5^m\lambda$ is also an eigenvalue, corresponding to a generation of birth $m_0 + m$, with the same choice of $\phi_\pm$ relations (suitably reindexed).

Neumann spectrum.

We impose a Neumann condition on the graph Γ_m by imagining that it is embedded in a larger graph by reflecting in each boundary vertex and imposing the λ_m-eigenvalue equation on the even extension of u. This just means that we impose the equation

$$(4 - \lambda_m)u(q_i) = 2u(F_i^m q_{i+1}) + 2u(F_i^m q_{i-1})$$

at q_i for $i = 0, 1, 2$. Then the Neumann λ_m-eigenvalue equations consist of exactly $\sharp V_m$ equations in $\sharp V_m$ unknowns. Similarly to the Dirichlet case, we also only need to make clear all the discrete spectra. The result is very similar to the Dirichlet spectrum, with only a few changes. We omit it.

It should be emphasized here that those eigenfunctions which are simultaneously Dirichlet and Neumann play an important role in the spectral analysis of $\mathcal{SG}$. Here we call them *localized eigenfunctions* since all of them have small supports. (Here this definition of localized eigenfunctions is slightly different from that of [2, 19, 36] for the convenience of further discussion for Ω case.) Similarly to $\mathcal{D}$, to describe the structure of localized eigenfunctions, we only need to make clear the structure of all initial localized eigenvalues, which consists of 5-series and 6-series eigenvalues. In fact, the multiplicity of a 5-series eigenvalue with generation of birth m, is $\rho_m(5) := \frac{3^{m-1}-1}{2}$ with an eigenfunction associated to each m-level loop (a m-level circuit around an empty upside-down triangle in the graph Γ_m). The eigenfunction u associated to each loop takes value 0 on all m-level points not lying in that loop. Moreover, the support of u is exactly the union of all m-cells intersecting that loop. The multiplicity of a 6-series eigenvalue with generation of birth m, is $\rho_m(6) := \frac{3^m-3}{2}$ with an eigenfunction associated to each point x in $V_{m-1} \setminus V_0$. Each such eigenfunction u takes value 0 on all points in V_{m-1} except x. Moreover, u is supported in the union of two $(m-1)$-level cells containing x. The existence of localized eigenfunctions is unprecedented in all of smooth mathematics. However, for a class of p.c.f. self-similar sets, including $\mathcal{SG}$, localized eigenfunctions dominate global eigenfunctions. See more details in [2, 19]. See also Section 4 of Kigami's book [20], where most results are explained in detail.

3 The structures of Dirichlet spectrum and Neumann spectrum on Ω

To give the readers an intuitive understanding of the structure of the spectrum of $-\Delta$ on Ω in advance, in this section we describe all Dirichlet and Neumann eigenvalues and eigenfunctions on Ω avoiding involving technical proofs. We will go to the details in the remaining sections.

3.1 Dirichlet spectrum

We begin with the Dirichlet case. First we formulate the eigenvalue problem of $-\Delta$ on Ω with Dirichlet boundary condition(for short, the *Dirichlet Laplacian*).

Definition 3.1. *Let $\mathcal{F}_\Omega := \{u \in \mathcal{F} : u|_{\partial\Omega} = 0\}$. The Dirichlet Laplacian Δ_D on Ω with domain $\mathcal{D}[\Delta_D]$ is formulated as follows: for $u \in \mathcal{F}_\Omega$ and $f \in L^2(\Omega, \mu|_\Omega)$,*

$$u \in \mathcal{D}[\Delta_D] \text{ and } -\Delta_D u = f \qquad \text{if and only if} \qquad \mathcal{E}(u,v) = \int_\Omega f v d\mu \text{ for any } v \in \mathcal{F}_\Omega.$$

If we replace Ω by $\mathcal{SG} \setminus V_0$ in the above definition, then we get the standard Dirichlet Laplacian which is introduced in [20].

Definition 3.2. *For $\lambda \in \mathbb{R}$ and $u \in \mathcal{D}[\Delta_D]$, suppose*

$$-\Delta_D u = \lambda u.$$

Then λ is called an eigenvalue of $-\Delta_D$ on Ω (or, a Dirichlet eigenvalue of $-\Delta$ on Ω), and u is called an associated (Dirichlet) eigenfunction.

Let $\mathcal{S}$ denote the spectrum of $-\Delta_D$ on Ω ($\mathcal{S}$ is also called the Dirichlet spectrum of $-\Delta$ on Ω). We will consider three kinds of Dirichlet eigenfunctions, *localized*, *primitive*, and *miniaturized* eigenfunctions.

In the following, we will always use u to denote an eigenfunction of $-\Delta_D$ on Ω and λ to denote the associated eigenvalue of u.

Definition 3.3. *u is called a localized eigenfunction if it is a localized eigenfunction on $\mathcal{SG} \setminus V_0$ whose support is disjoint from L (the line segment joining q_1 and q_2).*

The associated eigenvalue λ is called a *localized eigenvalue*. Denote by $\mathcal{L}$ the set consisting of all such eigenvalues. Obviously, all the eigenvalues in $\mathcal{L}$ have generation of birth $m_0 \geq 3$ (the ones with $m_0 = 2$ all have supports intersecting L) and $\lambda_{m_0} = 5$ or 6.

Compared to the $\mathcal{SG} \setminus V_0$ case, instead of the eigenfunctions associated to the 2-series eigenvalues, there is also a type of global eigenfunctions in the Ω case, and these will be sorted into symmetric and skew-symmetric parts according to the reflection symmetry fixing q_0.

Definition 3.4. *u is called a symmetric primitive eigenfunction if it is symmetric under the reflection symmetry fixing q_0 and also locally symmetric in each cell $F_w(\mathcal{SG})$ under the reflection symmetry fixing $F_w q_0$ with word w taking symbols only from* $\{1,2\}$.

Fig. 3.1. gives a symbolic picture of the above mentioned symmetries, indicated by dotted lines. The associated eigenvalue λ is called a *symmetric primitive eigenvalue*. Denote by $\mathcal{P}^+$ the set consisting of all such eigenvalues.

Similarly,

Definition 3.5. *If u is skew-symmetric under the reflection symmetry fixing q_0, but still local symmetric in small cells, then it is called a skew-symmetric eigenfunction.*

The associated eigenvalue λ is called a *skew-symmetric eigenvalue*. Denote by $\mathcal{P}^-$ the set consisting of all such eigenvalues.

Both the symmetric and skew-symmetric primitive eigenfunctions are called *primitive eigenfunctions*. All the associated eigenvalues are called *primitive eigenvalues*. Let $\mathcal{P}$ denote the set consisting of all of them. Namely,

$$\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^-.$$

The primitive eigenfunction u (either the symmetric or skew-symmetric case) is uniquely determined by the values denoted by $(b_0, b_1, b_2, \cdots)$ of u on vertices $(q_0, F_1 q_0, F_1^2 q_0, \cdots)$ by using the eigenfunction extension algorithm described in (2.5). Due to the Dirichlet boundary condition, $\forall \lambda \in \mathcal{P}$, for the associated eigenfunction u of λ, we always have $b_0 = 0$ and $\lim_{m\to\infty} b_m = 0$. We call $(q_0, F_1 q_0, F_1^2 q_0, \cdots)$ a *skeleton* of Ω since it plays a critical role in the study of primitive eigenfunctions.

Theorem 3.1. *All the primitive eigenvalues are of multiplicity* 1.

This theorem will be proved in Section 6.

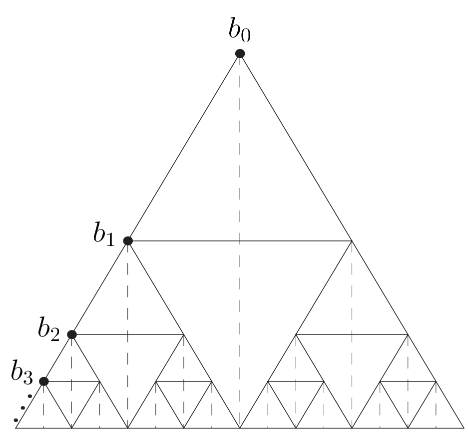


Fig. 3.1. The first 4 level symmetries and the skeleton of Ω.

The following argument will show that there is another type of eigenfunction. For each skew-symmetric eigenvalue $\lambda \in \mathcal{P}^-$, there is a family of eigenfunctions with eigenvalue $5^k\lambda$ and multiplicity 2^k for $k = 1, 2, 3, \cdots$. To get such an eigenfunction, just take the λ-eigenfunction u, contract it k times, place it in any one of the 2^k bottom cells of level k, and take value 0 elsewhere. See Fig 3.2. The reason we can do this is that on the boundary point q_0, $u(q_0) = 0$ and $\partial_n u(q_0) = 0$ which makes the matching condition hold automatically.

Definition 3.6. *We call all the above obtained eigenfunctions miniaturized eigenfunctions.*

If u is a miniaturized eigenfunction obtained by contracting a skew-symmetric primitive eigenfunction k times, then we call u a *k-contracted miniaturized eigenfunction.* Let $\mathcal{M}$ denote all the eigenvalues associated to them. Obviously, $\mathcal{M}$ is determined by $\mathcal{P}^-$.

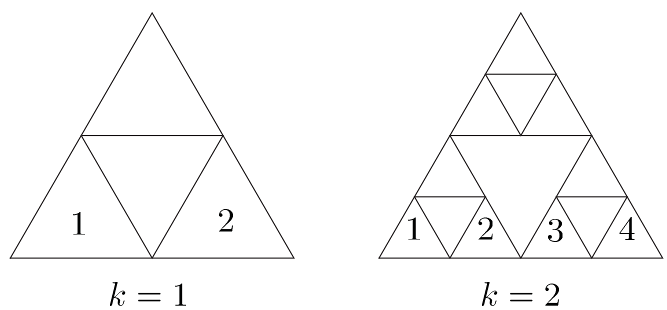


Fig. 3.2. The first 2 level miniaturized eigenfunctions.

We will prove in Section 6 that all the eigenfunctions of $-\Delta_D$ on Ω fall into one of these three types, and there are no coincidences of eigenvalues among different types.

For the purpose of studying the exact structure of the spectrum $\mathcal{S}$, the first thing we should consider is to describe the Dirichlet spectra of $-\Delta_m$'s on the associated graph approximations. Similar to the $\mathcal{SG} \setminus V_0$ case, the fractal domain Ω can be realized as the limit of a sequence of graphs Ω_m. More precisely, $\forall m \geq 1$, let V_m^Ω be a subset of V_m with all vertices lying along L removed. Let Ω_m be the subgraph of Γ_m restricted to V_m^Ω. Denote by $\partial\Omega_m$ the boundary of the finite graph Ω_m. It is easy to see that $V_m^\Omega \setminus \partial\Omega_m$ and $\partial\Omega_m$ approximate to Ω and $\partial\Omega$ as m goes to infinity, respectively. See Fig. 3.3.

A routing argument shows that the Dirichlet Laplaician Δ_D could be viewed as the limit of suitably scaled graph Laplaicians Δ_m on Ω_m, as is done in [16, 17, 20, 34, 36] for the standard $\mathcal{SG} \setminus V_0$ case. Hence, there is also a pointwise formula which, for nonboundary points in $V_* \cup \Omega$, computes

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x),$$

where Δ_m is a discrete Laplacian associated to the graph Ω_m, defined by

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x))$$

for x in $V_m^\Omega \setminus \partial\Omega_m$.

We denote by $\mathcal{S}_m$ the discrete Dirichlet spectrum of $-\Delta_m$ on Ω_m for $m \in \mathbb{N}$. On Ω_m the Dirichlet λ_m-eigenvalue equations consist of exactly $\sharp(V_m^\Omega \setminus \partial\Omega_m)$ equations in $\sharp(V_m^\Omega \setminus \partial\Omega_m)$ unknowns. We start from m=2 since there is no Dirichlet λ_1-eigenvalue equation. For simplicity, let $a_m = \sharp(V_m^\Omega \setminus \partial\Omega_m)$. It is easy to check that $a_2 = 5$, $a_3 = 24$, and more generally,

Proposition 3.1. $a_m = \frac{3^{m+1}-1}{2} - 2^{m+1}$, $\forall m \in \mathbb{N}$.

Proof. Notice that $a_m = a_{m-1} + 3^m + 2^{m-1} - 3 \cdot 2^{m-1}$, where $3^m = \sharp(V_m \setminus V_{m-1})$, 2^{m-1} is the number of points lying on the bottom boundary of Ω_{m-1}, and $3 \cdot 2^{m-1}$ is the number of points in $V_m \setminus V_{m-1}$ lying on L or $\partial\Omega_m$. $\square$

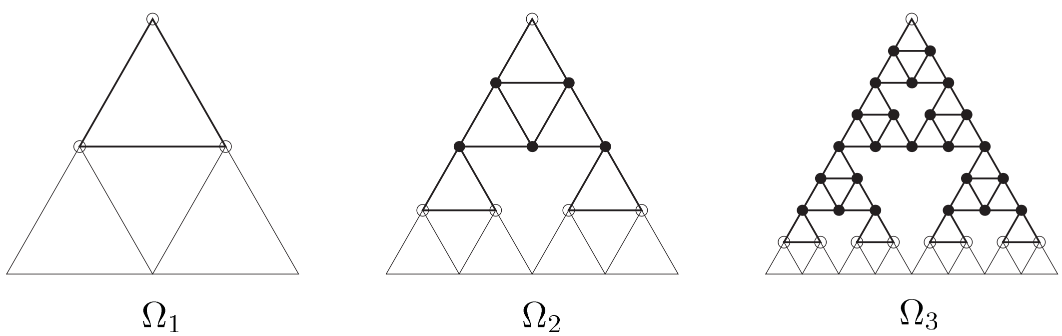


Fig. 3.3. The first 3 graphs, $\Omega_1, \Omega_2, \Omega_3$ in the approximations to Ω with interior points and boundary points represented by dots and circles respectively.

Due to different types of eigenvalues of $-\Delta_D$ on Ω, we should consider the associated different types of graph Dirichlet eigenvalues of $-\Delta_m$. We now describe how to define $\mathcal{L}_m$, $\mathcal{P}_m$ and $\mathcal{M}_m$ respectively. For simplicity, in the rest of this subsection, without causing any confusion, we omit the term "Dirichlet" for graph eigenfunctions and eigenvalues.

In the following, we will always use u_m to denote an eigenfunction of $-\Delta_m$ on Ω_m satisfying the Dirichlet boundary condition, and λ_m to denote the associated eigenvalue of u_m.

In fact, by the spectral decimation recipe, each localized eigenfunction u of $-\Delta_D$ on Ω with generation of birth $m_0 \leq m$, can be restricted to Ω_m to get a graph eigenfunction u_m of $-\Delta_m$, with the Dirichlet boundary condition of u_m on $\partial\Omega_m$ holding automatically.

Definition 3.7. *Let u be a localized eigenfunction of $-\Delta_D$ on Ω with generation of birth $m_0 \leq m$. Then its restricted graph function u_m on Ω_m is called a m-level localized* *graph eigenfunctions of* $-\Delta_m$ *on* Ω_m.

All the associated eigenvalues are called m-level *localized graph eigenvalues.* We use $\mathcal{L}_m$ to denote the set consisting of all these type eigenvalues.

We can not imitate the above process to get the m-level primitive graph eigenfunctions since the Dirichlet boundary condition would be destroyed if we do the similar restriction. But we can define m-level primitive graph eigenfunctions on Ω_m directly in the following way.

Definition 3.8. *A Dirichlet eigenfunction* u_m *on* Ω_m *is called a* m*-level symmetric primitive graph eigenfunction if it is symmetric under the reflection symmetry fixing* q_0 *and also locally symmetric in* $F_w(\mathcal{SG}) \cap V_m^\Omega$ *under the reflection symmetry fixing* $F_w q_0$ *for each word* w *taking symbols only from* $\{1,2\}$.

The associated eigenvalue λ_m is called a m-level *symmetric primitive graph eigenvalue.* Denote by $\mathcal{P}_m^+$ the set of all eigenvalues of this type.

Similarly,

Definition 3.9. *If* u_m *is skew-symmetric under the reflection symmetry fixing* q_0*, but still local symmetric in small cells, then it is called a* m*-level skew-symmetric primitive graph eigenfunction.*

$\mathcal{P}_m^-$ can be defined in a similar way. Let $\mathcal{P}_m$ denote all the m-level primitive graph eigenvalues. Namely,

$$\mathcal{P}_m = \mathcal{P}_m^+ \cup \mathcal{P}_m^-.$$

Similarly to the limit case, the primitive graph eigenfunction u_m (either the symmetric or skew-symmetric case) is uniquely determined by the values denoted by $(b_0, b_1, b_2, \cdots, b_m)$ of u_m on vertex points $(q_0, F_1q_0, F_1^2q_0, \cdots, F_1^m q_0)$ by using the eigenfunction extension algorithm (2.5). Due to the Dirichlet boundary condition, we always have $b_0 = b_m = 0$ for an m-level primitive graph eigenfunction u_m. We call $(q_0, F_1q_0, F_1^2q_0, \cdots, F_1^m q_0)$ a *skeleton* of Ω_m. It also plays a critical role in the study of m-level primitive graph eigenfunctions.

Miniaturized graph eigenfunctions on Ω_m can be defined in a similar way by using miniaturization of skew-symmetric primitive graph eigenfunctions whose level is strictly less than m.

Definition 3.10. *For a Dirichlet eigenfunction* u_m *on* Ω_m*, if there exists an integer* $m' < m$ *and a* m'*-level skew-symmetric primitive graph eigenfunction* $u_{m'}$ *such that after contracting* $u_{m'}$ $m - m'$ *times, placing it in one of the* $2^{m-m'}$ *bottom copies of* $\Omega_{m'}$ *in* Ω_m*, and taking value* 0 *elsewhere, one can obtain* u_m*, then* u_m *is called a* m*-level miniaturized graph eigenfunction.*

The associated eigenvalue λ_m is called a m-level *miniaturized graph eigenvalue.* m' is called the *type* of λ_m. Denote by $\mathcal{M}_m$ the set of all such eigenvalues. Obviously, $\mathcal{M}_m$ is determined by all $\mathcal{P}_k^-$'s with $k < m$.

It is not difficult to make clear all the localized graph eigenvalues in $\mathcal{L}_m$, since they are almost the same as the $\mathcal{SG} \setminus V_0$ case. There are two kinds of eigenvalues in $\mathcal{L}_m$, *initial* and *continued.*

Theorem 3.2. *Let $m \geq 2$, then $\sharp\mathcal{L}_m = \frac{3^{m+1}-1}{2} - (m-2)\cdot 2^m - 26\cdot 2^{m-3}$. Moreover, the initial eigenvalues in $\mathcal{L}_m$ are 5 and 6 with multiplicity $\rho_m^\Omega(5) := \frac{3^{m-1}+1}{2} - 2^{m-1}$ and $\rho_m^\Omega(6) := \frac{3^m-1}{2} - 2^m$ respectively.*

Proof. Similarly to the $\mathcal{SG} \setminus V_0$ case, the initial eigenvalues are 5 and 6. For the 6-eigenfunctions of $-\Delta_m$ on Ω_m, comparing to the 6-eigenfunctions of $-\Delta_m$ on Γ_m, the only difference is that those eigenfunctions whose support intersects the boundary $\partial\Omega_m$ should be removed. A similar analysis shows that they are indexed by points in $V_{m-1}^\Omega \setminus \partial\Omega_{m-1}$. Hence the multiplicity of 6 is

$$\rho_m^\Omega(6) = a_{m-1} = \frac{3^m-1}{2} - 2^m.$$

Similarly, the 5-eigenfunctions of $-\Delta_m$ on Ω_m are indexed by m-level loops except those loops touching $\partial\Omega_m$. Hence the multiplicity of 5 is

$$\rho_m^\Omega(5) = \rho_m(5) - (1+2+2^2+\cdots+2^{m-2}) = \frac{3^{m-1}+1}{2} - 2^{m-1}.$$

The continued eigenvalues will be those that arise from eigenvalues of $\mathcal{L}_{m-1}$ by spectral decimation. Note that every eigenvalue λ_{m-1} of $-\Delta_{m-1}$ bifurcates into two choices of λ_m of $-\Delta_m$ by (2.4), except $\lambda_{m-1} = 6$, which just yields the single choice $\lambda_m = 3$ since the other is a forbidden eigenvalue 2. We know that $\rho_{m-1}^\Omega(6)$ of $\mathcal{L}_{m-1}$ correspond to eigenvalue 6 of $-\Delta_{m-1}$, while the remaining $\sharp\mathcal{L}_{m-1} - \rho_{m-1}^\Omega(6)$ of them correspond to other eigenvalues, leading to a space of continued eigenfunctions of dimension $2\cdot(\sharp\mathcal{L}_{m-1} - \rho_{m-1}^\Omega(6)) + \rho_{m-1}^\Omega(6) = 2\cdot\sharp\mathcal{L}_{m-1} - \frac{3^{m-1}-1}{2} + 2^{m-1}$. If we add to this $\rho_m^\Omega(6) = \frac{3^m-1}{2} - 2^m$ and $\rho_m^\Omega(5) = \frac{3^{m-1}+1}{2} - 2^{m-1}$, we should obtain $\sharp\mathcal{L}_m$. Hence we have

$$\begin{aligned}
\sharp\mathcal{L}_m &= 2\cdot\sharp\mathcal{L}_{m-1} - \frac{3^{m-1}-1}{2} + 2^{m-1} + \frac{3^m-1}{2} - 2^m + \frac{3^{m-1}+1}{2} - 2^{m-1} \\
&= 2\cdot\sharp\mathcal{L}_{m-1} + \frac{3^m+1}{2} - 2^m.
\end{aligned}$$

Combining this with $\sharp\mathcal{L}_2 = 0$, we can easily get

$$\sharp\mathcal{L}_m = \frac{3^{m+1}-1}{2} - (m-2)\cdot 2^m - 26\cdot 2^{m-3} \quad \text{for } m \geq 2. \square$$

As for primitive graph eigenvalues $\mathcal{P}_m$, things become more complicated. We consider $\mathcal{P}_m^+$ and $\mathcal{P}_m^-$ respectively. We will show in the next section the spectral decimation recipe for this type of eigenvalues can not be used directly. In fact there is even not an analytic relation between elements in $\mathcal{P}_m^+$ (or $\mathcal{P}_m^-$) and elements in $\mathcal{P}_{m+1}^+$ (or $\mathcal{P}_{m+1}^-$). A rough but intuitive explanation of why this "bad" thing happens is that the Dirichlet boundary condition will be destroyed when we use the eigenfunction extension algorithm (2.5) to extend a λ_m-eigenfunction u_m from Ω_m to Ω_{m+1} or restrict a λ_{m+1}-eigenfunction u_{m+1} from Ω_{m+1} to Ω_m. However, a weak but useful relation between $\mathcal{P}_m^+$ (or $\mathcal{P}_m^-$) and $\mathcal{P}_{m+1}^+$ (or $\mathcal{P}_{m+1}^-$) will be found in the next section, which will take the place of spectral decimation in the further discussion. Let $\phi_\pm(x)$ be the same functions as defined in (2.4). We will prove that:

Theorem 3.3. *For each $m \geq 2$, $\mathcal{P}_m^+$ consists of $r_m := 2^m + 2^{m-2} - 2$ distinct eigenvalues with multiplicity 1, between 0 and 6 strictly, denoted by $\lambda_{m,1}, \lambda_{m,2}, \cdots, \lambda_{m,r_m}$ in increasing order, satisfying*

$$0 < \lambda_{m,1} < \lambda_{m,2} < \cdots < \lambda_{m,r_m-1} < 5 < \lambda_{m,r_m} < 6.$$

Moreover, $r_{m+1} = 2r_m + 2$ and

$$\begin{aligned}
0 &< \lambda_{m+1,1} < \phi_-(\lambda_{m,1}), \\
\phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \ \textit{if } 2 \leq k \leq r_m, \\
\phi_-(\lambda_{m,r_m}) &< \lambda_{m+1,r_m+1} < \phi_+(\lambda_{m,r_m}), \\
\phi_+(\lambda_{m,2r_m+2-k}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2r_m+1-k}), \ \textit{if } r_m + 2 \leq k \leq 2r_m, \\
\phi_+(\lambda_{m,1}) &< \lambda_{m+1,2r_m+1} < 5, \\
5 &< \lambda_{m+1,2r_m+2} < 6.
\end{aligned}$$

Similar properties hold for $\mathcal{P}_m^-$ with r_m replaced by $s_m := 2^m - 2$.

In order to study the relation between $\mathcal{P}_m^+$ (or $\mathcal{P}_m^-$) and $\mathcal{P}_{m+1}^+$ (or $\mathcal{P}_{m+1}^-$), we introduce the following notations. In the symmetric case, let $\widetilde{\phi}_-(\lambda_{m,1})$ denote the $(m+1)$-level eigenvalue between 0 and $\phi_-(\lambda_{m,1})$. Let $\widetilde{\phi}_-(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_-(\lambda_{m,k-1})$ and $\phi_-(\lambda_{m,k})$ for $2 \leq k \leq r_m$. Let $\widetilde{\phi}_+(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_+(\lambda_{m,k})$ and $\phi_+(\lambda_{m,k-1})$ for $2 \leq k \leq r_m$. Let $\widetilde{\phi}_+(\lambda_{m,1})$ denote the $(m+1)$-level eigenvalue between $\phi_+(\lambda_{m,1})$ and 5. Call this kind of $(m+1)$-level eigenvalues *continued eigenvalues*. There remain two other $(m+1)$-level eigenvalues: one is between $\phi_-(\lambda_{m,r_m})$ and $\phi_+(\lambda_{m,r_m})$, the other is between 5 and 6. Call these two $(m+1)$-level eigenvalues *initial eigenvalues* with generation of birth $m+1$. For the 2 level, all $r_2 = 3$ symmetric primitive eigenvalues $\lambda_{2,1}, \lambda_{2,2}$ and $\lambda_{2,3}$ are called initial eigenvalues with generation of birth 2. We define similar notations for the skew-symmetric case in an obvious way with r_m replaced by s_m. From this point of view, the continued primitive eigenvalues in $\mathcal{P}_{m+1}^+$ (or $\mathcal{P}_{m+1}^-$) will be those arise from eigenvalues in $\mathcal{P}_m^+$ (or $\mathcal{P}_m^-$) by a $\widetilde{\phi}_\pm$ bifurcation similar (but never equal) to the $\phi_\pm$ bifurcation. We call this phenomenon *weak spectral decimation*, which will be proved to play a critical role in the study of the exact structure of primitive eigenvalues on Ω instead of spectral decimation.

We should emphasize here that $\widetilde{\phi}_{\pm}$ is not a real function relation. (It is just a notation for simplicity.) See the following diagram for the relation between $\mathcal{P}_m^+$ and $\mathcal{P}_{m+1}^+$. The skew-symmetric case is similar.

$$
\begin{array}{cccccccccc}
 & \lambda_{m,1} & & & \lambda_{m,2} & & \cdots & & \lambda_{m,r_m} & & & \\
\swarrow & & \searrow & \swarrow & & \searrow & \cdots & \swarrow & & \searrow & & \\
\widetilde{\phi}_-(\lambda_{m,1}) & & \widetilde{\phi}_+(\lambda_{m,1}) & \widetilde{\phi}_-(\lambda_{m,2}) & & \widetilde{\phi}_+(\lambda_{m,2}) & \cdots & \widetilde{\phi}_-(\lambda_{m,2}) & & \widetilde{\phi}_+(\lambda_{m,2}) & \lambda_{m+1,r_m+1} & \lambda_{m+1,r_{m+1}}
\end{array}
$$

The structure of $\mathcal{M}_m$ depends on the structure of all $\mathcal{P}_k^-$'s with $k < m$ by the definition of $\mathcal{M}_m$. In fact, it is easy to check that

Theorem 3.4. *Let $m \geq 2$, then for each eigenvalue λ_m in $\mathcal{M}_m$, it has multiplicity $2^{m-m'}$, where m' is the type of λ_m. Moreover, $\sharp\mathcal{M}_m = (m-3)\cdot 2^m + 4$.*

Proof. Let $\lambda_{m'}$ be the graph eigenvalue associated to $u_{m'}$ as used in Definition 3.10. Then obviously, $\lambda_m = 5^{m-m'}\lambda_{m'}$ and λ_m has multiplicity $2^{m-m'}$. Hence by using Theorem 3.3,

$$\sharp\mathcal{M}_m = \sum_{m'=2}^{m-1} 2^{m-m'}\sharp\mathcal{P}_{m'}^- = \sum_{m'=2}^{m-1} 2^{m-m'}(2^{m'}-2) = (m-3)\cdot 2^m + 4. \quad \square$$

From Theorem 3.2, Theorem 3.3 and Theorem 3.4, it is easy to check that $\sharp\mathcal{L}_m$, $\sharp\mathcal{P}_m$ and $\sharp\mathcal{M}_m$ add up to $\sharp(V_m^\Omega \setminus \partial\Omega_m)$ since

$$\sharp\mathcal{L}_m+\sharp\mathcal{P}_m+\sharp\mathcal{M}_m = \frac{3^{m+1}-1}{2}-(m-2)\cdot 2^m-26\cdot 2^{m-3}+r_m+s_m+(m-3)\cdot 2^m+4 = a_m. \quad (3.1)$$

This means that there are no more Dirichlet eigenvalues, except in the case of localized, primitive and miniaturized types. Or, more precisely, we will prove in the next section that

Theorem 3.5. *Let $m \geq 2$, then all the above mentioned three types of Dirichlet eigenfunctions of $-\Delta_m$ on Ω_m are linearly independent. Moreover, the Dirichlet spectrum $\mathcal{S}_m$ of $-\Delta_m$ on Ω_m satisfies*

$$\mathcal{S}_m = \mathcal{L}_m \cup \mathcal{P}_m^+ \cup \mathcal{P}_m^- \cup \mathcal{M}_m,$$

where the union is disjoint.

Hence we have the complete Dirichlet spectrum $\mathcal{S}_m$ of $-\Delta_m$ on Ω_m. In Table 3.1, we list the eigenspace dimensions of all different types of eigenvalues in $\mathcal{S}_m$ for level $m = 2, 3, 4, 5$.

level	$\sharp\mathcal{L}_m$	$\sharp\mathcal{P}_m^+$	$\sharp\mathcal{P}_m^-$	$\sharp\mathcal{M}_m$	$\sharp\mathcal{S}_m$
m	$\frac{3^{m+1}-1}{2}-(m-2)\cdot 2^m-26\cdot 2^{m-3}$	$2^m+2^{m-2}-2$	2^m-2	$(m-3)\cdot 2^m+4$	$\frac{3^{m+1}-1}{2}-2^{m+1}$
2	0	3	2	0	5
3	6	8	6	4	24
4	37	18	14	20	89
5	164	38	30	68	300

Table 3.1. Eigenspace dimensions of different types of eigenvalues in $\mathcal{S}_m$.

Next we want to pass from the approximations to the limit.

For the $\mathcal{L}$ case, assume that $\{\lambda_m\}_{m\geq m_0}$ is an infinite sequence of localized graph eigenvalues related by $\phi_\pm$ relations, with all but a finite number of ϕ_-'s. Then we define

$$\lambda = \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m.$$

By successively using the eigenfunction extension algorithm (2.5) from a λ_{m_0}-eigenfunction u_{m_0} of $-\Delta_{m_0}$ on Ω_{m_0}, one can extend u_{m_0} to a localized eigenfunction u of $-\Delta_D$ on Ω associated to λ. This method generates all the localized eigenvalues $\mathcal{L}$ as in the $\mathcal{SG}\setminus V_0$ case.

For the $\mathcal{P}^+$ case, for an infinite sequence of $\mathcal{P}^+$ type graph eigenvalues $\{\lambda_m\}_{m\geq m_0}$ related by $\widetilde{\phi}_\pm$ relations, with all but a finite number of $\widetilde{\phi}_-$'s, we define

$$\lambda = \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m.$$

We will also show the existence of the limit λ. As pointed out before, now we could not use the eigenfunction extension algorithm directly. However, we will prove that λ is still a $\mathcal{P}^+$ type eigenvalue of $-\Delta_D$ on Ω by a nonconstructive method. That is, in Section 5, we will prove

Theorem 3.6. *For each sequence of symmetric primitive eigenvalues $\{\lambda_m\}_{m\geq m_0}$ related by $\widetilde{\phi}_\pm$ relations, with all but a finite number of $\widetilde{\phi}_-$'s, the limit $\lambda = \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m$ exists. Moreover, $\lambda\in\mathcal{P}^+$.*

Furthermore, all $\mathcal{P}^+$ type eigenvalues arise in this way. This will be done in Section 6.

Theorem 3.7. *For each element $\lambda\in\mathcal{P}^+$, there is uniquely a sequence of symmetric primitive eigenvalues $\{\lambda_m\}_{m\geq m_0}$ as described in Theorem 3.6 such that $\lambda = \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m$.*

The $\mathcal{P}^-$ and $\mathcal{M}$ cases are completely similar to the $\mathcal{P}^+$ case.

Now we could consider the complete spectrum $\mathcal{S}$ of $-\Delta_D$ on Ω. That is, we will prove

Theorem 3.8. *$\mathcal{S} = \mathcal{L}\cup\mathcal{P}\cup\mathcal{M}$ where the union is disjoint.*

The following is a description of the multiplicity of each of the above mentioned three types of eigenvalues.

Theorem 3.9. *If $\lambda\in\mathcal{L}$ with generation of birth of m_0, then λ is either a 5-series eigenvalue with multiplicity $\frac{3^{m_0-1}+1}{2} - 2^{m_0-1}$ or a 6-series eigenvalue with multiplicity $\frac{3^{m_0}-1}{2} - 2^{m_0}$; if $\lambda\in\mathcal{P}$, then the multiplicity of λ is 1; if $\lambda\in\mathcal{M}$ is a k-contracted miniaturized eigenvalue, then the multiplicity of λ is 2^k.*

Proof. This follows directly from Theorem 3.1, Theorem 3.2 and Definition 3.6. □

Now we describe the Weyl's eigenvalue asymptotics on Ω. As introduced in the introduction section, let $\rho^0(x)$ and $\rho^\Omega(x)$ be the Dirichlet eigenvalue counting functions for the

$\mathcal{SG} \setminus V_0$ case and the Ω case, respectively. Then we will prove in Section 6 the following comparison between $\rho^0(x)$ and $\rho^\Omega(x)$.

Theorem 3.10. *There exists a positive constant C such that for sufficiently large x, we have*

$$0 \leq \rho^0(x) - \rho^\Omega(x) \leq C x^{\log 2/\log 5} \log x. \tag{3.2}$$

In Section 8, we present the eigenvalues and their multiplicities in $\mathcal{S}_m$ for level $m = 2, 3, 4, 5$(see Table 8.1, 8.2, 8.3 and 8.4).

We will prove Theorem 3.3 and Theorem 3.5 in Section 4, Theorem 3.6 in Section 5, Theorem 3.1, Theorem 3.7, Theorem 3.8 and Theorem 3.10 in Section 6.

3.2 Neumann spectrum

Before going to the Neumann spectrum of $-\Delta$ on Ω, we should give a precise description of the non-positive self-adjoint operator Δ_N, the Neumann Laplacian, on Ω under consideration.

For $m \in \mathbb{N}$, define the renormalization energy on Ω_m by

$$\mathcal{E}_m^\Omega(u,v) := r^{-m} \sum_{x,y \in \Omega_m, x \sim_m y} (u(x) - u(y))(v(x) - v(y)),$$

for $u, v \in \mathbb{R}^{\Omega_m}$, where $r = \frac{3}{5}$ is the energy renormalization factor. Then it is easy to see that $\{\mathcal{E}_m^\Omega\}_{m \in \mathbb{N}}$ forms a compatible sequence of discrete Dirichlet forms in the sense of Kigami [20](Definition 2.2.1), and hence we can define a resistance form $(\mathcal{E}^\Omega, \mathcal{F}^\Omega)$ on $\Omega_* := \bigcup_{m \in \mathbb{N}} \Omega_m$ by

$$\mathcal{F}^\Omega := \{u \in \mathbb{R}^{\Omega_*} | \lim_{m \to \infty} \mathcal{E}_m^\Omega(u|_{\Omega_m}, v|_{\Omega_m}) < \infty\},$$

$$\mathcal{E}^\Omega(u,v) := \lim_{m \to \infty} \mathcal{E}_m^\Omega(u|_{\Omega_m}, v|_{\Omega_m}), \quad u, v \in \mathcal{F}^\Omega.$$

Let R^Ω be the resistance metric on Ω_* associated with the resistance form $(\mathcal{E}^\Omega, \mathcal{F}^\Omega)$, and let $\widetilde{\Omega}$ be the R^Ω-completion of Ω_*. Then each $u \in \mathcal{F}^\Omega$ is naturally identified with its unique extension to a continuous function on $\widetilde{\Omega}$ by virtue of its 1/2-Hölder continuity with respect to R^Ω. Kigami[20](Definition 2.3.10) assures that the resulting bilinear form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$, defined for functions on $\widetilde{\Omega}$, is a resistance form on $\widetilde{\Omega}$ with resistance metric (the completion of) R^Ω. On the other hand, it is not difficult to show that $\widetilde{\Omega}$ can be identified as $(\mathcal{SG} \setminus L) \cup \widetilde{L}$, where L denotes the bottom line segment of $\mathcal{SG}$ and $\widetilde{L}$ is the Cantor set naturally appearing as the "boundary" at the bottom of the graphs Ω_m. Roughly speaking, $(\mathcal{SG} \setminus L) \cup \widetilde{L}$ is obtained from $\mathcal{SG}$ by distinguishing the left and right sides of each dyadic rational on the bottom line L to regard L as a Cantor set $\widetilde{L}$.

From here on we will identify $\widetilde{\Omega}$ and $(\mathcal{SG} \setminus L) \cup \widetilde{L} = \Omega \cup \{q_0\} \cup \widetilde{L}$. Call $\{q_0\} \cup \widetilde{L}$ the boundary of $\widetilde{\Omega}$, denoted by $\widetilde{\Omega} \setminus \Omega$. Define a Borel measure $\widetilde{\mu}$ on $\widetilde{\Omega}$ by $\widetilde{\mu}(A) = \mu(A \cap \Omega)$ for each Borel subset A of $\widetilde{\Omega}$. Now we define the Neumann Laplacian as the non-positive self-adjoint operator associated with the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(\widetilde{\Omega}, \widetilde{\mu})$ in the following way.

Definition 3.11. *The Neumann Laplacian Δ_N on Ω with domain $\mathcal{D}[\Delta_N]$ is formulated as follows: for $u \in \widetilde{\mathcal{F}}$ and $f \in L^2(\widetilde{\Omega}, \widetilde{\mu})$,*

$$u \in \mathcal{D}[\Delta_N] \text{ and } -\Delta_N u = f \qquad \text{if and only if} \qquad \widetilde{\mathcal{E}}(u,v) = \int_\Omega f v d\mu \text{ for any } v \in \widetilde{\mathcal{F}}.$$

Definition 3.12. *For $\lambda \in \mathbb{R}$ and $u \in \mathcal{D}[\Delta_N]$, if*

$$-\Delta_N u = \lambda u,$$

then λ is called an eigenvalue of $-\Delta_N$ on Ω (or, a Neumann eigenvalue of $-\Delta$ on Ω), and u is called an associated (Neumann) eigenfunction.

Let $\mathcal{S}^N$ denote the spectrum of $-\Delta_N$ on Ω ($\mathcal{S}^N$ is also called the Neumann spectrum of $-\Delta$ on Ω).

Similarly to the Dirichlet case, the Neumann Laplacian Δ_N on Ω could also be realized by the limit of graph Laplacians Δ_m on Ω_m. Hence it is natural to believe that the discrete Neumann spectrum of $-\Delta_m$ on Ω_m should converge to the spectrum of the Neumann Laplacian on Ω. Thus we need to analyze the discrete Neumann spectra first. We denote $\mathcal{S}_m^N$ the Neumann spectrum of $-\Delta_m$ on Ω_m for $m \in \mathbb{N}$.

To study the Neumann spectrum we impose a Neumann condition on the graph Ω_m by extending functions from Ω_m by even reflection, and imposing the pointwise eigenvalue equation at the boundary points in $\partial\Omega_m$, which now have 4 neighbors. Then the Neumann λ_m-eigenvalue equations consist of exactly $\sharp V_m^\Omega$ equations in $\sharp V_m^\Omega$ unknowns. It is even convenient to allow $m = 1$, in which case there are three equations associated to the boundary $\partial\Omega_1$ and no others. In particular, on Ω_1 we find eigenvalues $\lambda_1 = 0$ corresponding to the constant function, and $\lambda_1 = 6$ corresponding to the two dimensional space of functions satisfying $u(q_0) + u(F_1 q_0) + u(F_2 q_0) = 0$ which can be split into an one dimensional symmetric space and an one dimensional skew-symmetric space under the reflection symmetry fixing q_0. For simplicity, let $b_m = \sharp V_m^\Omega$. It is easy to check that $b_1 = 3$, $b_2 = 10$, and more generally,

Proposition. 3.2. $b_m = \frac{3^{m+1}+1}{2} - 2^m, \forall m \in \mathbb{N}$.

Proof. $b_m = a_m + \sharp\partial\Omega_m = \frac{3^{m+1}-1}{2} - 2^{m+1} + 2^m + 1 = \frac{3^{m+1}+1}{2} - 2^m$. □

Similarly to the Dirichlet case, $\mathcal{S}^N$ will also consist of three types of eigenvalues, localized, primitive and miniaturized, with obvious modifications, denoted by $\mathcal{L}^N$, $\mathcal{P}^N$

and $\mathcal{M}^N$ respectively. And correspondingly, $\mathcal{S}_m^N$ will consist of three types of graph Neumann eigenvalues, denoted by $\mathcal{L}_m^N$, $\mathcal{P}_m^N$ and $\mathcal{M}_m^N$ respectively. Moreover, $\mathcal{P}^N(\mathcal{P}_m^N)$ can also be split into a symmetric part $\mathcal{P}^{+,N}(\mathcal{P}_m^{+,N})$ and a skew-symmetric part $\mathcal{P}^{-,N}(\mathcal{P}_m^{-,N})$.

The structure of localized (graph) Neumann eigenvalues is very similar to the Dirichlet case, with only a few changes:

Theorem 3.11. *Let $m \geq 1$, then $\sharp\mathcal{L}_m^N = \frac{3^{m+1}-1}{2} - 2^{m+1} - (m-1)\cdot 2^m$. Moreover, the initial eigenvalues in $\mathcal{L}_m^N$ are 5 and 6 with multiplicity $\rho_m^{\Omega,N}(5) := \frac{3^{m-1}+1}{2} - 2^{m-1}$ and $\rho_m^{\Omega,N}(6) := \frac{3^m+1}{2} - 2^m$ respectively.*

Proof. Comparing to the Dirichlet case(see Theorem 3.2), the 6-series has multiplicity increased by 1, namely the eigenfunction associated to q_0, while the 5-series is unchanged. Hence $\rho_m^{\Omega,N}(6) = \rho_m^{\Omega}(6) + 1 = \frac{3^m+1}{2} - 2^m$ and $\rho_m^{\Omega,N}(5) = \rho_m^{\Omega}(5) = \frac{3^{m-1}+1}{2} - 2^{m-1}$, $\forall m \geq 1$. A similar discussion shows that

$$\sharp\mathcal{L}_m^N = 2\cdot\sharp\mathcal{L}_{m-1}^N - \rho_{m-1}^{\Omega,N}(6) + \rho_m^{\Omega,N}(6) + \rho_m^{\Omega,N}(5).$$

Hence we have

$$\sharp\mathcal{L}_m^N = 2\cdot\sharp\mathcal{L}_{m-1}^N + \frac{3^m+1}{2} - 2^m,$$

which yields

$$\sharp\mathcal{L}_m^N = \frac{3^{m+1}-1}{2} - 2^{m+1} - (m-1)\cdot 2^m \quad \text{for } m \geq 1,$$

since $\sharp\mathcal{L}_1^N = 0$. $\square$

The structure of primitive (graph) Neumann eigenvalues $\mathcal{P}^N(\mathcal{P}_m^N)$ is also similar to the Dirichlet case. We consider the symmetric and skew-symmetric case respectively. We will prove that:

Theorem 3.12. *For each $m \geq 1$, $\mathcal{P}_m^{+,N}$ consists of 2^m distinct eigenvalues with multiplicity 1, between 0 and 6 with $0, 6$ included, denoted by $\lambda_{m,1}, \lambda_{m,2}, \cdots, \lambda_{m,r_m}$ in increasing order, satisfying*

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < 5 < \lambda_{m,2^m} = 6.$$

Moreover,

$$\begin{aligned} \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \ \textit{if } 2 \leq k \leq 2^m, \\ \phi_+(\lambda_{m,2^{m+1}-k+1}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2^{m+1}-k}), \ \textit{if } 2^m+1 \leq k \leq 2^{m+1}-1. \end{aligned}$$

Similar properties hold for $\mathcal{P}_m^{-,N}$ with 2^m replaced by $2^m - 1$, and $\lambda_{m,1} > 0$ in that case.

For the symmetric case, there is a weak spectral decimation which relates $\mathcal{P}_m^{+,N}$ and $\mathcal{P}_{m+1}^{+,N}$ by introducing the following notations. Let $\widetilde{\phi}_-(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_-(\lambda_{m,k-1})$ and $\phi_-(\lambda_{m,k})$ for $2 \leq k \leq 2^m$. Let $\widetilde{\phi}_+(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_+(\lambda_{m,k})$ and $\phi_+(\lambda_{m,k-1})$ for $2 \leq k \leq 2^m$. Call this kind of

$(m+1)$-level eigenvalues *continued eigenvalues*. Hence the continued primitive eigenvalues in $\mathcal{P}_{m+1}^{+,N}$ will be those that arise from eigenvalues of $\mathcal{P}_m^{+,N} \setminus \{0\}$ by a $\widetilde{\phi}_\pm$ bifurcation similar (but never equal) to $\phi_\pm$ bifurcation. There remain two other $(m+1)$-level eigenvalues: one is 0, called *zero eigenvalue*, the other is 6, called *initial eigenvalue* with generation of birth $m+1$. See the following diagram of eigenvalues in $\mathcal{P}_m^{+,N}$.

$$
\begin{array}{lcccccccccccc}
\mathcal{P}_1^{+,N}: & 0 & & & & & 6 & & & & & & \\
 & \downarrow & & & & \swarrow & & \searrow & & & & & \\
\mathcal{P}_2^{+,N}: & 0 & & \widetilde{\phi}_-(6) & & & & & \widetilde{\phi}_+(6) & & & 6 & \\
 & \downarrow & \swarrow & & \searrow & & & \swarrow & & \searrow & \swarrow & & \searrow \\
\mathcal{P}_3^{+,N}: & 0 & \widetilde{\phi}_-\widetilde{\phi}_-(6) & & \widetilde{\phi}_+\widetilde{\phi}_-(6) & & \widetilde{\phi}_-\widetilde{\phi}_+(6) & & & \widetilde{\phi}_+\widetilde{\phi}_+(6) & \widetilde{\phi}_-(6) & & \widetilde{\phi}_+(6) \quad 6 \\
\vdots & \vdots & & & & & \vdots & & & & & &
\end{array}
$$

For the skew-symmetric case, the only difference is that there is no zero eigenvalue in $\mathcal{P}_m^{-,N}$. $\mathcal{P}_m^{-,N}$ consists of $2^m - 1$ distinct eigenvalues between 0 and 6, including 6, where 6 is an initial eigenvalue with generation of birth m and the others are continued eigenvalues arise from previous level eigenvalues by a similar weak bifurcation. We have now the following decimation diagram of eigenvalues in $\mathcal{P}_m^{-,N}$.

$$
\begin{array}{lcccccccccccc}
\mathcal{P}_1^{-,N}: & & & & & & 6 & & & & & & \\
 & & & & & \swarrow & & \searrow & & & & & \\
\mathcal{P}_2^{-,N}: & & & \widetilde{\phi}_-(6) & & & & & \widetilde{\phi}_+(6) & & & 6 & \\
 & & \swarrow & & \searrow & & & \swarrow & & \searrow & \swarrow & & \searrow \\
\mathcal{P}_3^{-,N}: & & \widetilde{\phi}_-\widetilde{\phi}_-(6) & & \widetilde{\phi}_+\widetilde{\phi}_-(6) & & \widetilde{\phi}_-\widetilde{\phi}_+(6) & & & \widetilde{\phi}_+\widetilde{\phi}_+(6) & \widetilde{\phi}_-(6) & & \widetilde{\phi}_+(6) \quad 6 \\
\vdots & & \vdots & & & & & & & \vdots & & &
\end{array}
$$

As for the miniaturized Neumann eigenvalues, the structure of $\mathcal{M}_m^N$ depends on the structure of all $\mathcal{P}_k^{-,N}$'s with $k < m$ in a completely similar way to the Dirichlet case. In fact, it is easy to check that

Theorem 3.13. *Let $m \geq 1$. Then each eigenvalue λ_m in $\mathcal{M}_m^N$ has multiplicity $2^{m-m'}$, where m' is the type of λ_m. Moreover, $\sharp\mathcal{M}_m^N = (m-2)\cdot 2^m + 2$.*

Proof. Let $\lambda_{m'}$ be the graph eigenvalue associated to $u_{m'}$, as defined in the Neumann version of Definition 3.10. Then obviously, $\lambda_m = 5^{m-m'}\lambda_{m'}$ and λ_m has multiplicity $2^{m-m'}$. Hence by using the fact that $\sharp\mathcal{P}_k^{-,N} = 2^k - 1$ for all $k \geq 1$,

$$\sharp\mathcal{M}_m^N = \sum_{k=1}^{m-1} 2^{m-k}\sharp\mathcal{P}_k^{-,N} = \sum_{k=1}^{m-1} 2^{m-k}(2^k - 1) = (m-2)\cdot 2^m + 2 \quad \text{for } m \geq 1. \quad \square$$

It is easy to check $\sharp\mathcal{L}_m^N$, $\sharp\mathcal{P}_m^N$ and $\sharp\mathcal{M}_m^N$ add up to $\sharp V_m^\Omega$, since

$$\sharp\mathcal{L}_m^N + \sharp\mathcal{P}_m^N + \sharp\mathcal{M}_m^N = \frac{3^{m+1}-1}{2} - 2^{m+1} - (m-1)\cdot 2^m + 2^m + 2^m - 1 + (m-2)\cdot 2^m + 2 = b_m. \quad (3.3)$$

A similar argument to the Dirichlet case will yield:

Theorem 3.14. *Let $m \geq 1$, then all the above mentioned three types of Neumann eigenfunctions of $-\Delta_m$ on Ω_m are linearly independent. Moreover, the spectrum $\mathcal{S}_m^N$ of $-\Delta_m$ on Ω_m satisfies*

$$\mathcal{S}_m^N = \mathcal{L}_m^N \cup \mathcal{P}_m^{+,N} \cup \mathcal{P}_m^{-,N} \cup \mathcal{M}_m^N,$$

where the union is disjoint.

Hence we have the complete Neumann spectrum of $-\Delta_m$.

In Table 3.2, we list the eigenspace dimensions of all different types of eigenvalues in $\mathcal{S}_m^N$ for level $m = 1, 2, 3, 4, 5$.

level	$\sharp\mathcal{L}_m^N$	$\sharp\mathcal{P}_m^{+,N}$	$\sharp\mathcal{P}_m^{-,N}$	$\sharp\mathcal{M}_m^N$	$\sharp\mathcal{S}_m^N$
m	$\frac{3^{m+1}-1}{2} - 2^{m+1} - (m-1)\cdot 2^m$	2^m	$2^m - 1$	$(m-2)\cdot 2^m + 2$	$\frac{3^{m+1}+1}{2} - 2^m$
1	0	2	1	0	3
2	1	4	3	2	10
3	8	8	7	10	33
4	41	16	15	34	106
5	172	32	31	98	333

Table 3.2. Eigenspace dimensions of different types of eigenvalues in $\mathcal{S}_m^N$.

Then a similar discussion on how to pass to the limit leads to the spectrum $\mathcal{S}^N$ of $-\Delta_N$ on Ω.

The counterpart of Theorem 3.6 becomes:

Theorem 3.15. *For each sequence of symmetric primitive Neumann eigenvalues $\{\lambda_m\}_{m\geq m_0}$ related by $\widetilde{\phi}_\pm$ relations, with all but a finite number of $\widetilde{\phi}_-$'s, the limit $\lambda := \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m$ exists. Moreover, $\lambda \in \mathcal{P}^{+,N}$.*

Similarly, we will have

Theorem 3.16. $\mathcal{S}^N = \mathcal{L}^N \cup \mathcal{P}^N \cup \mathcal{M}^N$ *where the union is disjoint.*

The following is a description of the multiplicity of each of the above mentioned three types of eigenvalues.

Theorem 3.17. *If $\lambda \in \mathcal{L}^N$ with generation of birth of m_0, then λ is either a 5-series eigenvalue with multiplicity $\frac{3^{m_0-1}+1}{2} - 2^{m_0-1}$ or a 6-series eigenvalue with multiplicity $\frac{3^{m_0}+1}{2} - 2^{m_0}$; If $\lambda \in \mathcal{P}^N$, then the multiplicity of λ is 1; If $\lambda \in \mathcal{M}^N$ is a k-contracted miniaturized eigenvalue, then the multiplicity of λ is 2^k.*

Proof. It follows directly from Theorem 3.11, Theorem 3.12 and the Neumann version of Definition 3.6. □

As for the Weyl's eigenvalue asymptotics on Ω, a similar argument as in Theorem 3.10 works also for the Neumann Laplacian.

We will prove Theorem 3.12 and Theorem 3.14 in Section 7. We will also make some comments in Section 7 on the proof of Theorem 3.15, since compared to its Dirichlet counterpart Theorem 3.6, there is no direct analogue of Green's function for the Neumann

Laplacian, which is used in the proof of Theorem 3.6. Other proofs are omitted since they can be easily modified suitably from the Dirichlet case.

4 Primitive graph Dirichlet eigenvalues of $-\Delta_m$

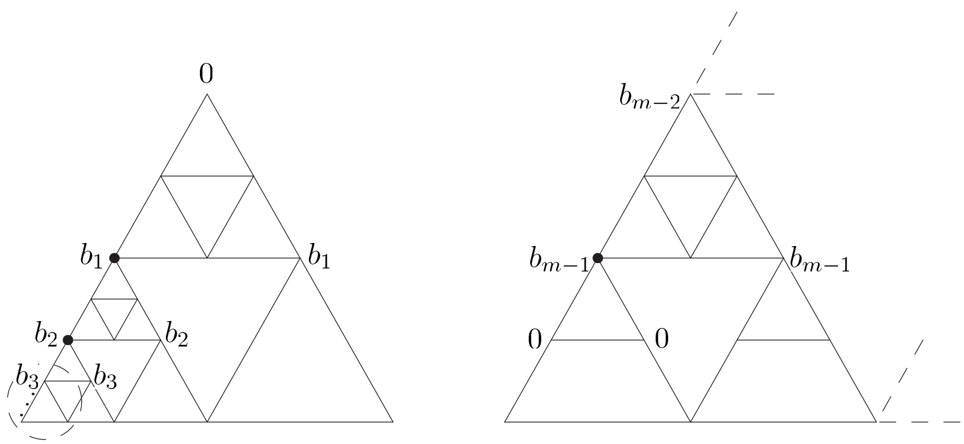


Fig. 4.1. The values of the λ_m-eigenfunction u_m on the skeleton of Ω_m with $\lambda_m \in \mathcal{P}_m^+$.

In this section, we work with m-level graph approximation Ω_m, $m = 2, 3, 4 \cdots$. As introduced in Section 3, we use $\mathcal{P}_m$ to denote the totality of the primitive graph Dirichlet eigenvalues of the discrete Laplacian $-\Delta_m$ on Ω_m. Throughout this section, for simplicity, we omit the terms "graph" and "Dirichlet" without causing any confusion. The main object in this section is to prove Theorem 3.3 and Theorem 3.5.

Let f and $\phi_\pm(x)$ be the same functions as defined in (2.3) and (2.4) respectively. In the following we use $f^{(n)}$ to denote the n'th iteration of f, $n \geq 1$. In particular, $f^{(0)}$ is the identity map of $\mathbb{R}$. If $w = f^{(n)}(x)$, w is called a *successor* of x of order n with respect to f, and x is called a *predecessor* of w of order n with respect to f.

We begin with $\mathcal{P}_m^+$, the symmetric eigenvalues in $\mathcal{P}_m$. Let u_m be a λ_m-eigenfunction of $-\Delta_m$ with $\lambda_m \in \mathcal{P}_m^+$. Denote by $(b_0, b_1, b_2, \cdots, b_m)$ the values of u_m on the skeleton $(q_0, F_1q_0, F_1^2q_0, \cdots, F_1^mq_0)$ of Ω_m where $b_0 = b_m = 0$ by the Dirichlet boundary condition. See Fig. 4.1. Write $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $2 \leq i \leq m$ for simplicity. Assume that none of $\lambda_i^{(m)}$'s is equal to 2 or 5. (Later we will show this assumption automatically holds for any $\lambda_m \in \mathcal{P}_m^+$.) The eigenfunction extension algorithm (2.5) gives the value of u_m on the four $(i+1)$-level neighbors of $F_1^iq_0$ for each $1 \leq i \leq m-1$, shown in Fig. 4.2. Hence the $\lambda_{i+1}^{(m)}$-eigenvalue equation at the vertex $F_1^iq_0$ gives

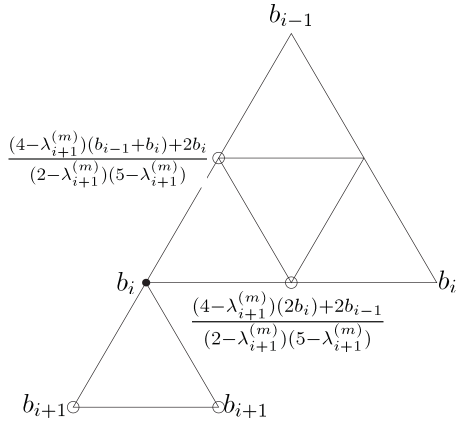


Fig. 4.2. Values of u_m on neighbors of $F_1^i q_0$.

$$(4-\lambda_{i+1}^{(m)})b_i = 2b_{i+1} + \frac{(14-3\lambda_{i+1}^{(m)})b_i + (6-\lambda_{i+1}^{(m)})b_{i-1}}{(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})}, \quad \forall 1 \le i \le m-1, \tag{4.1}$$

which can be modified into

$$l(\lambda_{i+1}^{(m)})b_{i-1} + s(\lambda_{i+1}^{(m)})b_i + r(\lambda_{i+1}^{(m)})b_{i+1} = 0, \quad \forall 1 \le i \le m-1, \tag{4.2}$$

with $l(x) := x-6$, $s(x) := (2-x)(4-x)(5-x)-(14-3x)$ and $r(x) := -2(2-x)(5-x)$. Still from the eigenfunction extension algorithm, u_m is uniquely determined by $(b_1, b_2, \cdots, b_{m-1})$. Here $(b_1, b_2, \cdots, b_{m-1})$ can be viewed as a non-zero vector solution of either of the above two systems of equations consisting of $m-1$ equations in $m-1$ unknowns. Hence the determinants of them should both be equal to 0. For simplicity, we are interested in the second determinant, although comparing to the first one, it brings the possibility that $\lambda_i^{(m)}$ $(2 \le i \le m)$ could be 2 or 5, which should be removed.

The determinant associated to system (4.2) is a tridiagonal determinant,

$$\begin{vmatrix} s(\lambda_2^{(m)}) & r(\lambda_2^{(m)}) & & & \\ l(\lambda_3^{(m)}) & s(\lambda_3^{(m)}) & r(\lambda_3^{(m)}) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_{m-1}^{(m)}) & s(\lambda_{m-1}^{(m)}) & r(\lambda_{m-1}^{(m)}) \\ & & & l(\lambda_m^{(m)}) & s(\lambda_m^{(m)}) \end{vmatrix}.$$

Hence λ_m should be a solution of the following equation

$$q_m(x) := \begin{vmatrix} s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & & \\ l(f^{(m-3)}(x)) & s(f^{(m-3)}(x)) & r(f^{(m-3)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(f(x)) & s(f(x)) & r(f(x)) \\ & & & l(x) & s(x) \end{vmatrix} = 0. \tag{4.3}$$

Conversely, if λ_m is a root of the polynomial $q_m(x)$ and none of $f^{(i)}(\lambda_m)$'s with $0 \leq i \leq m-2$ is equal to 2 or 5, then $\lambda_m \in \mathcal{P}_m^+$. Hence we are particular interested in all the root x's of the polynomial $q_m(x)$ excluding those satisfying $f^{(i)}(x) = 2$ or 5 for some $0 \leq i \leq m-2$.

We list some useful facts about the polynomial $q_m(x)$.

Proposition 4.1. *Let $m \geq 2$, then*

(1) $q_m(0) > 0$;

(2) $q_m(5) > 0$;

(3) $q_m(6) < 0$;

(4) $q_m(\phi_-^{(m-1)}(5)) < 0$;

(5) $q_m(\phi_-^{(m-1)}(2)) > 0$;

(6) $q_{m+2}(\phi_-^{(m-1)}(3)) < 0$ and $q_3(3) > 0$.

Proof. (1) We will prove a stronger result,

$$q_{m+1}(0) > 20q_m(0) > 0 \tag{4.4}$$

for $m \geq 2$. This can be proved by induction. It is easy to check that $q_2(0) = 26 > 0$ and $q_3(0) = 556 > 20q_2(0)$ by a direct computation. If we assume $q_m(0) > 20q_{m-1}(0) > 0$, then the expansion along the first row of $q_{m+1}(0)$ yields that

$$q_{m+1}(0) = 26q_m(0) - 6 \cdot 20q_{m-1}(0) > 26q_m(0) - 6q_m(0) = 20q_m(0) > 0.$$

(2) It is easy to compute that $q_2(5) = 1 > 0$ and $q_3(5) = 6 > 0$. For $m \geq 4$, $q_m(5) = q_{m-1}(0) - 20q_{m-2}(0) > 0$ by using (4.4).

(3) It is easy to compute that $q_2(6) = -4 < 0$, $q_3(6) = -3392 \leq q_2(6) < 0$ and

$$q_m(6) = s(f^{(m-2)}(6)) \cdot q_{m-1}(6) - r(f^{(m-2)}(6)) \cdot l(f^{(m-3)}(6)) \cdot q_{m-2}(6)$$

for $m \geq 4$ by the expansion along the first row of $q_m(6)$.

Consider a polynomial defined by $g_1(x) := s(f(x)) - r(f(x))l(x)$; it is easy to check that $g_1(x) \geq 1$ whenever $x \leq -6$. In fact, we can write $g_1(x) = (2-f(x))(5-f(x))(4-f(x)+2(x-6)) - (14-3f(x))$ by substituting the expressions for $s(f(x))$, $r(f(x))$ and $l(x)$. Noticing that $4-f(x)+2(x-6) = x^2-3x-8 \geq 46$ and $f(x) < 0$ whenever $x \leq -6$, we have $g_1(x) \geq 46(2-f(x))(5-f(x))-(14-3f(x)) = 46(f(x))^2-319f(x)+446$. Moreover, since $f(x) \leq -66$ whenever $x \leq -6$, we finally have $g_1(x) \geq 46(-66)^2 - 319 \cdot (-66) + 446 \geq 1$.

Then we can prove $q_m(6) \leq q_{m-1}(6) < 0$ by induction. Suppose $q_{m-1}(6) \leq q_{m-2}(6) < 0$. (This is true for $m = 4$.) Write $q_m(6) = aq_{m-1}(6) + bq_{m-2}(6)$ with $a = s(f^{(m-2)}(6))$ and $b = -r(f^{(m-2)}(6)) \cdot l(f^{(m-3)}(6))$. Noticing that $m \geq 4$, we have $f^{(m-3)}(6) \leq -6$ and $f^{(m-2)}(6) < 0$. Hence $a + b = g_1(f^{(m-3)}(6)) \geq 1$ and $b < 0$. So by the induction assumption, we have

$$q_m(6) \leq aq_{m-1}(6) + bq_{m-1}(6) = (a+b)q_{m-1}(6) \leq q_{m-1}(6) < 0.$$

Hence we always have $q_m(6) < 0$ for $m \geq 2$.

(4) For simplicity, denote $\alpha_m = q_m(\phi_-^{(m-1)}(5))$. By direct computation, we have $\alpha_2 = -4 < 0$ and $\alpha_3 \approx -92.10 < 0$. We will prove a stronger result, $\alpha_{m+1} \leq 10\alpha_m < 0$, $\forall m \geq 2$. It holds for $m = 2$. In order to use the induction, we assume $\alpha_{m+1} \leq 10\alpha_m < 0$. An expansion of α_{m+2} along the last row yields that

$$\alpha_{m+2} = s(\phi_-^{(m+1)}(5))\alpha_{m+1} - r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5))\alpha_m.$$

Since $2 - \phi_-^{(m)}(5) > 0$, $5 - \phi_-^{(m)}(5) > 0$ and $\phi_-^{(m+1)}(5) - 6 < 0$, we have

$$\begin{aligned} \alpha_{m+2} &= s(\phi_-^{(m+1)}(5))\alpha_{m+1} - \frac{1}{10}r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5)) \cdot (10\alpha_m) \\ &\leq s(\phi_-^{(m+1)}(5))\alpha_{m+1} - \frac{1}{10}r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5))\alpha_{m+1} \\ &= [s(\phi_-^{(m+1)}(5)) - \frac{1}{10}r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5))]\alpha_{m+1}. \end{aligned}$$

Consider a polynomial

$$g_2(x) := s(x) - \frac{1}{10}r(f(x))l(x) = 14 + 9x - \frac{172}{5}x^2 + \frac{87}{5}x^3 - \frac{16}{5}x^4 + \frac{1}{5}x^5.$$

It is easy to compute that

$$g_2'(x) = 9 - \frac{344}{5}x + \frac{261}{5}x^2 - \frac{64}{5}x^3 + x^4 \geq 9 - \frac{344}{5}(\phi_-^{(3)}(5)) - \frac{64}{5}(\phi_-^{(3)}(5))^3 \approx 4.91 > 0$$

whenever $0 \leq x \leq \phi_-^{(3)}(5)$. Hence $g_2(x)$ is monotone increasing in the interval $[0, \phi_-^{(3)}(5)]$.

Since $0 < \phi_-^{(m+1)}(5) \leq \phi_-^{(3)}(5)$, we have $g_2(\phi_-^{(m+1)}(5)) \geq g_2(0) \geq 10$. Hence $\alpha_{m+2} \leq g_2(\phi_-^{(m+1)}(5))\alpha_{m+1} \leq 10\alpha_{m+1} < 0$.

The proofs of (5) and (6) are similar to that of (4). □

Now we discuss the possibility of the roots of $q_m(x)$ satisfying $f^{(i)}(x) = 2$ or 5 for some $0 \leq i \leq m-2$. The following well-known basic algebra lemma will be useful.

Lemma 4.1. *Let g, h be two polynomials whose coefficients all belong to $\mathbb{Q}$ (the field of rational numbers), i.e., $g, h \in \mathbb{Q}[x]$. If g is irreducible in $\mathbb{Q}[x]$ and g, h have a common root in $\mathbb{R}$, then g divides h in $\mathbb{Q}[x]$, i.e., all real roots of g belong to those of h.*

Lemma 4.2. *Let x be a predecessor of 2 of order i with $0 \leq i \leq m-3$. Then $q_m(x) = 0$. Let x be a predecessor of 2 of order $m-2$. Then $q_m(x) \neq 0$.*

Proof. Firstly, let $m \geq 3$ and x be a predecessor of 2 of order i with $0 \leq i \leq m-3$. Then $f^{(i)}(x) = 2$ and $f^{(i+1)}(x) = 6$. Substituting them into (4.3), noticing $s(2) = r(6) = -8$,

$s(6) = l(2) = -4$ and $r(2) = l(6) = 0$, we get

$$
\begin{aligned}
q_m(x) &= \begin{vmatrix}
\ddots & \ddots & \ddots & & & \\
& l(f^{(i+1)}(x)) & s(f^{(i+1)}(x)) & r(f^{(i+1)}(x)) & & \\
& & l(f^{(i)}(x)) & s(f^{(i)}(x)) & r(f^{(i)}(x)) & \\
& & & \ddots & \ddots & \ddots
\end{vmatrix} \\
&= \begin{vmatrix}
\ddots & \ddots & \ddots & & & \\
& 0 & -4 & -8 & & \\
& & -4 & -8 & 0 & \\
& & & \ddots & \ddots & \ddots
\end{vmatrix} = 0.
\end{aligned}
$$

Secondly, let x be a predecessor of 2 of order $m-2$. Then $f^{(m-2)}(x) = 2$. If $m = 2$, then $x = 2$. It is easy to check that $x = 2$ is not a root of $q_2(x)$. If $m \geq 3$, suppose x is a root of $q_m(x)$. Then using Lemma 4.1, all roots of $f^{(m-2)}(x) - 2$ are roots of $q_m(x)$. Noticing that $\phi_-^{(m-2)}(2)$ is also a root of $f^{(m-2)}(x) - 2$, we have $q_m(\phi_-^{(m-2)}(2)) = 0$. But

$$
\begin{aligned}
q_m(\phi_-^{(m-2)}(2)) &= \begin{vmatrix}
s(2) & r(2) & & & \\
l(\phi_-(2)) & s(\phi_-(2)) & r(\phi_-(2)) & & \\
& \ddots & \ddots & \ddots & \\
& & & l(\phi_-^{(m-2)}(2)) & s(\phi_-^{(m-2)}(2))
\end{vmatrix} \\
&= s(2) \cdot \begin{vmatrix}
s(\phi_-(2)) & r(\phi_-(2)) & \\
\ddots & \ddots & \ddots \\
& l(\phi_-^{(m-2)}(2)) & s(\phi_-^{(m-2)}(2))
\end{vmatrix} \\
&= (-8) \cdot q_{m-1}(\phi_-^{(m-2)}(2)),
\end{aligned}
$$

since $r(2) = 0$. By using Proposition 4.1(5), we get $q_m(\phi_-^{(m-2)}(2)) < 0$ which contradicts $q_m(\phi_-^{(m-2)}(2)) = 0$. Hence $q_m(x) \neq 0$. $\square$

Lemma 4.3. *Let x be a predecessor of 5 of order i with $0 \leq i \leq m-2$. Then $q_m(x) \neq 0$.*

Proof. Let x be a predecessor of 5 of order i with $0 \leq i \leq m-2$. Then $f^{(i)}(x) = 5$. Hence if x is a root of $q_m(x)$, then using Lemma 4.1, all roots of $f^{(i)}(x) - 5$ are roots of $q_m(x)$. Noticing that $\phi_-^{(i)}(5)$ is also a root of $f^{(i)}(x) - 5$, we have $q_m(\phi_-^{(i)}(5)) = 0$. But

$q_m(\phi_-^{(0)}(5)) = q_m(5) > 0$ by Proposition 4.1(2). More generally, for $0 < i \leq m-2$,

$$\begin{aligned} q_m(\phi_-^{(i)}(5)) &= \begin{vmatrix} s(f^{(m-2-i)}(5)) & r(f^{(m-2-i)}(5)) & & & & \\ \ddots & \ddots & \ddots & & & \\ & l(5) & s(5) & r(5) & & \\ & & l(\phi_-(5)) & s(\phi_-(5)) & r(\phi_-(5)) & \\ & & & \ddots & \ddots & \ddots \\ & & & & l(\phi_-^{(i)}(5)) & s(\phi_-^{(i)}(5)) \end{vmatrix} \\ &= \begin{vmatrix} s(f^{(m-2-i)}(5)) & r(f^{(m-2-i)}(5)) & & & & \\ \ddots & \ddots & \ddots & & & \\ & l(5) & s(5) & 0 & & \\ & & l(\phi_-(5)) & s(\phi_-(5)) & r(\phi_-(5)) & \\ & & & \ddots & \ddots & \ddots \\ & & & & l(\phi_-^{(i)}(5)) & s(\phi_-^{(i)}(5)) \end{vmatrix}, \end{aligned}$$

since $r(5) = 0$. Thus

$$\begin{aligned} q_m(\phi_-^{(i)}(5)) &= \begin{vmatrix} s(f^{(m-2-i)}(5)) & r(f^{(m-2-i)}(5)) & \\ \ddots & \ddots & \ddots \\ & l(5) & s(5) \end{vmatrix} \\ &\cdot \begin{vmatrix} s(\phi_-(5)) & r(\phi_-(5)) & \\ \ddots & \ddots & \ddots \\ & l(\phi_-^{(i)}(5)) & s(\phi_-^{(i)}(5)) \end{vmatrix} \\ &= q_{m-i}(5) \cdot q_{i+1}(\phi_-^{(i)}(5)) < 0 \end{aligned}$$

by the 2'nd and 4'th statements in Proposition 4.1. Hence $\forall 0 \leq i \leq m-2$, we have proved $q_m(\phi_-^{(i)}(5)) \neq 0$ which yields a contradiction to $q_m(\phi_-^{(i)}(5)) = 0$. So $q_m(x) \neq 0$. □

From Lemma 4.2 and Lemma 4.3, for $m \geq 3$, all unwanted roots of $q_m(x)$ are those predecessors of 2 of order i with $0 \leq i \leq m-3$. $q_2(x)$ does not have any unwanted root. Hence to eliminate them, we define

$$p_m(x) := \frac{q_m(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)} \qquad \text{for } m \geq 3,$$

and

$$p_2(x) := q_2(x) = s(x).$$

$p_m(x)$ is still a polynomial from Lemma 4.2, although it looks like a rational function.

Now we can say if λ_m is a root of the polynomial $p_m(x)$, then $\lambda_m \in \mathcal{P}_m^+$. Note that the degree of the polynomial $q_m(x)$ is $3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{m-2} = 3(2^{m-1} - 1)$ and the number of all the unwanted roots of $q_m(x)$ is $1+2+\cdots+2^{m-3}=2^{m-2}-1$ for $m\geq 3$ and 0 for $m=2$. Hence it is easy to check that the degree of $p_m(x)$ is $r_m := 2^m+2^{m-2}-2$.

The following is a list of some useful facts about the polynomial $p_m(x)$.

Proposition 4.2. *(1)* $(-1)^m p_m(0)>0$, $\forall m\geq 2$;

(2) $p_2(5)>0$ *and* $(-1)^{m-1}p_m(5)>0$, $\forall m\geq 3$;

(3) $p_2(6)<0$ *and* $(-1)^m p_m(6)>0$, $\forall m\geq 3$.

Proof. It can be checked by a direct computation when $m=2$. When $m\geq 3$, noticing that by the definition of $p_m(x)$,

$$p_m(0)=\frac{q_m(0)}{(-2)^{m-2}},\quad p_m(5)=\frac{q_m(5)}{3\cdot(-2)^{m-3}}$$

and

$$p_m(6)=\frac{q_m(6)}{(6-2)(f(6)-2)\cdots(f^{(m-3)}(6)-2)}.$$

Using Proposition 4.1(1)-(3), we get the desired result. □

We now present a more precise result about the distribution of the roots of $p_m(x)$ and show a useful relation between roots of two consecutive polynomials.

Lemma 4.4. *Let* $m\geq 2$*. Then* $p_m(x)$ *has* r_m *distinct real roots, denoted by* $\lambda_{m,1}$, $\lambda_{m,2},\cdots,\lambda_{m,r_m}$ *in increasing order, satisfying*

$$0<\lambda_{m,1}<\lambda_{m,2}<\cdots<\lambda_{m,r_m-1}<5<\lambda_{m,r_m}<6.$$

Moreover, $(-1)^{m+k-1}p_{m+1}(\phi_-(\lambda_{m,k}))>0$ *and* $(-1)^{m+k}p_{m+1}(\phi_+(\lambda_{m,k}))>0$, *for* $1\leq k\leq r_m$.

Proof. We prove it by using induction on m.

When $m=2$, $p_2(x)=s(x)$ has 3 distinct roots: $\lambda_{2,1}\approx 1.0646$, $\lambda_{2,2}\approx 4.4626$ and $\lambda_{2,3}\approx 5.4728$ by a direct computation.

Let λ be one of the $\lambda_{2,k}$'s. Then $p_2(\lambda)=0$, i.e., $s(\lambda)=0$, and $p_3(\phi_-(\lambda))=\frac{q_3(\phi_-(\lambda))}{\phi_-(\lambda)-2}=\frac{2(\phi_-(\lambda)-6)(2-\lambda)(5-\lambda)}{\phi_-(\lambda)-2}$ by using $s(\lambda)=0$. Since $0<\lambda<6$, we have $\phi_-(\lambda)-2<0$ and $\phi_-(\lambda)-6<0$. Hence $p_3(\phi_-(\lambda))\sim(2-\lambda)(5-\lambda)$ where "$\sim$" means both sides of "$\sim$" have the same signs. Similarly, $p_3(\phi_+(\lambda))=\frac{2(\phi_+(\lambda)-6)(2-\lambda)(5-\lambda)}{\phi_+(\lambda)-2}$ and $p_3(\phi_+(\lambda))\sim-(2-\lambda)(5-\lambda)$.

Hence $0<\lambda_{2,1}<2$ yields that $p_3(\phi_-(\lambda_{2,1}))>0$ and $p_3(\phi_+(\lambda_{2,1}))<0$; $2<\lambda_{2,2}<5$ yields that $p_3(\phi_-(\lambda_{2,2}))<0$ and $p_3(\phi_+(\lambda_{2,2}))>0$; $\lambda_{2,1}>5$ yields that $p_3(\phi_-(\lambda_{2,3}))>0$ and $p_3(\phi_+(\lambda_{2,3}))<0$. So our lemma holds for $m=2$.

We now assume our lemma holds for m, and prove it for $m+1$.

Notice that from Proposition 4.2, we have $p_{m+1}(0)\sim(-1)^{m-1}$, $p_{m+1}(5)\sim(-1)^m$ and $p_{m+1}(6)\sim(-1)^{m-1}$. Hence if we write

$$0,\phi_-(\lambda_{m,1}),\phi_-(\lambda_{m,2}),\cdots,\phi_-(\lambda_{m,r_m}),\phi_+(\lambda_{m,r_m}),\cdots,\phi_+(\lambda_{m,2}),\phi_+(\lambda_{m,1}),5,6 \tag{4.5}$$

in increasing order, then the values of p_{m+1} on them have alternating signs by the induction assumption. Hence there exist at least $2r_m + 2 = r_{m+1}$ distinct roots of $p_{m+1}(x)$, with each located strictly between each two consecutive points in (4.5). Moreover, these are the totality of the roots of $p_{m+1}(x)$ since the degree of $p_{m+1}(x)$ is also r_{m+1}. Hence we can write them in increasing order:

$$0 < \lambda_{m+1,1} < \lambda_{m+1,2} < \cdots < \lambda_{m+1,r_{m+1}-1} < 5 < \lambda_{m+1,r_{m+1}} < 6.$$

Now we study the signs of $p_{m+2}(\phi_\pm(\lambda_{m+1,k}))$'s. Let λ be one of the $\lambda_{m+1,k}$'s. Then $p_{m+1}(\lambda) = 0$. Moreover,

$$\begin{aligned} p_{m+2}(\phi_-(\lambda)) &= \frac{q_{m+2}(\phi_-(\lambda))}{(\phi_-(\lambda)-2)(\lambda-2)\cdots(f^{(m-2)}(\lambda)-2)} \\ &= \frac{s(\phi_-(\lambda))q_{m+1}(\lambda) + 2(\phi_-(\lambda)-6)(2-\lambda)(5-\lambda)q_m(f(\lambda))}{(\phi_-(\lambda)-2)(\lambda-2)\cdots(f^{(m-2)}(\lambda)-2)} \end{aligned}$$

by using the expansion of $q_{m+2}(\phi_-(\lambda))$ along the last row. Since $p_{m+1}(\lambda) = 0$, we have $q_{m+1}(\lambda) = 0$. Hence

$$p_{m+2}(\phi_-(\lambda)) = \frac{2(\phi_-(\lambda)-6)(2-\lambda)(5-\lambda)q_m(f(\lambda))}{(\phi_-(\lambda)-2)(\lambda-2)\cdots(f^{(m-2)}(\lambda)-2)} = \frac{-2(\phi_-(\lambda)-6)(5-\lambda)p_m(f(\lambda))}{\phi_-(\lambda)-2}.$$

Since $0 < \lambda < 6$, we have $\phi_-(\lambda) - 2 < 0$ and $\phi_-(\lambda) - 6 < 0$, hence

$$p_{m+2}(\phi_-(\lambda)) \sim (\lambda - 5)p_m(f(\lambda)).$$

Similarly,

$$p_{m+2}(\phi_+(\lambda)) = \frac{-2(\phi_+(\lambda)-6)(5-\lambda)p_m(f(\lambda))}{\phi_+(\lambda)-2}$$

and

$$p_{m+2}(\phi_+(\lambda)) \sim (5-\lambda)p_m(f(\lambda)).$$

When $\lambda = \lambda_{m+1,1}$, we have $0 < \lambda < \phi_-(\lambda_{m,1})$, hence $0 < f(\lambda) < \lambda_{m,1}$. Noticing that $\lambda_{m,1}$ is the least root of $p_m(x)$ and $\lambda_{m,1} > 0$ by the induction assumption, we have $p_m(f(\lambda)) \sim p_m(0) \sim (-1)^m$. Hence $p_{m+2}(\phi_-(\lambda_{m+1,1})) \sim (-1)^{m+1}$ and $p_{m+2}(\phi_+(\lambda_{m+1,1})) \sim (-1)^m$ since $\lambda_{m+1,1} < 5$.

When $\lambda = \lambda_{m+1,k}$ with $2 \le k \le r_m$, we have $\phi_-(\lambda_{m,k-1}) < \lambda < \phi_-(\lambda_{m,k})$, hence $\lambda_{m,k-1} < f(\lambda) < \lambda_{m,k}$. Noticing that $p_m(\lambda_{m,k-1}) = 0$ and $p_m(0) \sim (-1)^m$, we have $p_m(f(\lambda)) \sim (-1)^{m+k+1}$ by using the induction assumption. Hence $p_{m+2}(\phi_-(\lambda_{m+1,k})) \sim (-1)^{m+k}$ and $p_{m+2}(\phi_+(\lambda_{m+1,k})) \sim (-1)^{m+k+1}$ since $\lambda_{m+1,k} < 5$.

When $\lambda = \lambda_{m+1,r_m+1}$, we have $\phi_-(\lambda_{m,r_m}) < \lambda < \phi_+(\lambda_{m,r_m})$, hence $f(\lambda) > \lambda_{m,r_m}$. Noticing that λ_{m,r_m} is the last root of $p_m(x)$ and $p_m(0) \sim (-1)^m$, we have $p_m(f(\lambda)) \sim$

$(-1)^{m+r_m}$ by using the induction assumption. Hence $p_{m+2}(\phi_-(\lambda_{m+1,r_m+1})) \sim (-1)^{m+1+r_m}$ and $p_{m+2}(\phi_+(\lambda_{m+1,r_m+1})) \sim (-1)^{m+r_m}$ since $\lambda_{m+1,r_m+1} < 5$.

When $\lambda = \lambda_{m+1,k}$ with $r_m+2 \leq k \leq 2r_m$, we have $\phi_+(\lambda_{m,r_{m+1}-k}) < \lambda < \phi_+(\lambda_{m,r_{m+1}-k-1})$, hence $\lambda_{m,r_{m+1}-k-1} < f(\lambda) < \lambda_{m,r_{m+1}-k}$. Noticing that $p_m(\lambda_{m,r_{m+1}-k-1}) = 0$ and $p_m(0) \sim (-1)^m$, we have $p_m(f(\lambda)) \sim (-1)^{m+r_{m+1}-k-1} \sim (-1)^{m+k-1}$ by using the induction assumption. Hence $p_{m+2}(\phi_-(\lambda_{m+1,k})) \sim (-1)^{m+k}$ and $p_{m+2}(\phi_+(\lambda_{m+1,k})) \sim (-1)^{m+k-1}$ since $\lambda_{m+1,k} < 5$.

When $\lambda = \lambda_{m+1,2r_m+1}$, we have $\phi_+(\lambda_{m,1}) < \lambda < 5$, hence $f(\lambda) < \lambda_{m,1}$. So we have $p_m(f(\lambda)) \sim p_m(0) \sim (-1)^m$. Hence $p_{m+2}(\phi_-(\lambda_{m+1,2r_m+1})) \sim (-1)^{m+1}$ and $p_{m+2}(\phi_+(\lambda_{m+1,2r_m+1})) \sim (-1)^m$ since $\lambda_{m+1,2r_m+1} < 5$.

When $\lambda = \lambda_{m+1,2r_m+2}$, we have $5 < \lambda < 6$, hence $f(\lambda) < 0$. So we have $p_m(f(\lambda)) \sim p_m(0) \sim (-1)^m$. But now $\lambda > 5$, hence $p_{m+2}(\phi_-(\lambda_{m+1,2r_m+2})) \sim (-1)^m$ and $p_{m+2}(\phi_+(\lambda_{m+1,2r_m+2})) \sim (-1)^{m-1}$.

Hence we have proved $(-1)^{m+1+k-1}p_{m+2}(\phi_-(\lambda_{m+1,k})) > 0$ and $(-1)^{m+1+k}p_{m+2}(\phi_+(\lambda_{m+1,k})) > 0$, $\forall 1 \leq k \leq r_{m+1}$. So our lemma holds for $m+1$. □

Thus by Lemma 4.4, in particular the proof of Lemma 4.4 and the fact that each root of $p_m(x)$ belongs to $\mathcal{P}_m^+$, we have the following result:

Lemma 4.5. *For each $m \geq 2$, $\mathcal{P}_m^+$ consists of at least r_m distinct eigenvalues satisfying*

$$0 < \lambda_{m,1} < \lambda_{m,2} < \cdots < \lambda_{m,r_m-1} < 5 < \lambda_{m,r_m} < 6. \tag{4.6}$$

Moreover,

$$\begin{aligned}
0 &< \lambda_{m+1,1} < \phi_-(\lambda_{m,1}), \\
\phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \ \textit{if } 2 \leq k \leq r_m, \\
\phi_-(\lambda_{m,r_m}) &< \lambda_{m+1,r_m+1} < \phi_+(\lambda_{m,r_m}), \\
\phi_+(\lambda_{m,2r_m+2-k}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2r_m+1-k}), \ \textit{if } r_m+2 \leq k \leq 2r_m, \\
\phi_+(\lambda_{m,1}) &< \lambda_{m+1,2r_m+1} < 5, \\
5 &< \lambda_{m+1,2r_m+2} < 6.
\end{aligned} \tag{4.7}$$

Remark. *The third inequality in* (4.7) *can be refined into* $2 < \lambda_{m+1,r_m+1} < \phi_+(\lambda_{m,r_m})$. *See details in Theorem A in the Appendix.*

Moreover, we have

Lemma 4.6. *Let λ_m be a root of $p_m(x)$, u_m a primitive λ_m-eigenfunction on Ω_m, and $(b_0, b_1, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m. Then $b_1 \neq 0$ and $b_{m-1} \neq 0$.*

Proof. Without loss of generality, assume $m \geq 3$. We still use $\lambda_i^{(m)}$ to denote the successor of λ_m of order $(m-i)$ with $2 \leq i \leq m$. From the definition of $p_m(x)$, $\lambda_i^{(m)} \neq 2$

or 5, for each $2 \le i \le m$. From the discussion in the beginning of this section, the vector $(b_1, b_2, \cdots, b_{m-1})$ can be viewed as a non-zero vector solution of system (4.2) of equations.

Suppose $b_{m-1} = 0$. Then $(b_1, b_2, \cdots, b_{m-2})$ can be viewed as a non-zero vector solution of the system of equations consisting of the first $(m-2)$ equations of (4.2) in $(m-2)$ unknowns. Hence the determinant of this system $q_{m-1}(\lambda_{m-1}^{(m)})$ should be equal to 0. Thus $\lambda_{m-1}^{(m)}$ is a root of $p_{m-1}(x)$ since its all successors $\lambda_2^{(m)}, \cdots, \lambda_{m-1}^{(m)}$ obviously do not take value 2 or 5. Then Lemma 4.4 says that neither of $\phi_\pm(\lambda_{m-1}^{(m)})$ should be a root of $p_m(x)$. This contradicts to $p_m(\lambda_m) = 0$ since λ_m is equal to either of $\phi_\pm(\lambda_{m-1}^{(m)})$. Hence $b_{m-1} \neq 0$.

On the other hand, if $b_1 = 0$, then by substituting it into (4.2), and noticing that none of the $\lambda_i^{(m)}$'s is equal to 2 or 5, we get $b_2 = 0, \cdots, b_{m-1} = 0$ successively, which contradicts $b_{m-1} \neq 0$. Hence $b_1 \neq 0$. $\square$

Next we give a brief discussion of the skew-symmetric case. It is very similar to the symmetric case. Let u_m be a λ_m-eigenfunction of $-\Delta_m$ with $\lambda_m \in \mathcal{P}_m^-$. Denote by $(b_0, b_1, b_2, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m where $b_0 = b_m = 0$ by the Dirichlet boundary condition. Write $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $2 \le i \le m$. Comparing to the symmetric case, the eigenvalue equations at the vertex $F_1^i q_0$'s are unchanged except the one at $F_1 q_0$, since now the values of u_m on the four 2-level neighbors of $F_1 q_0$ are modified as shown in Fig. 4.3. Hence we still have the same

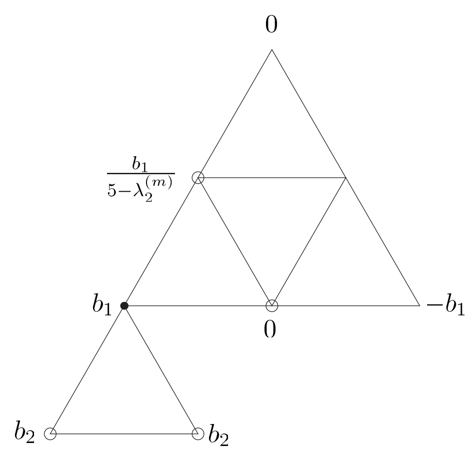


Fig. 4.3. Values of u_m on neighbors of $F_1 q_0$.

modified eigenvalue equation

$$l(\lambda_{i+1}^{(m)}) b_{i-1} + s(\lambda_{i+1}^{(m)}) b_i + r(\lambda_{i+1}^{(m)}) b_{i+1} = 0, \quad \forall 2 \le i \le m-1,$$

while the first equation in (4.2) is replaced by

$$\widetilde{s}(\lambda_2^{(m)}) b_1 + \widetilde{r}(\lambda_2^{(m)}) b_2 = 0,$$

with $\widetilde{s}(x) := (4-x)(5-x)-1$ and $\widetilde{r}(x) := -2(5-x)$. Now we assume $\lambda_2^{(m)} \neq 5$ and none of $\lambda_i^{(m)}$'s is equal to 2 or 5 for $3 \leq i \leq m$. (Later we will show this assumption automatically holds for any $\lambda_m \in \mathcal{P}_m^-$.) Then by the eigenfunction extension algorithm, u_m is unique and determined by its values on the skeleton of Ω_m. Using a similar discussion, λ_m should be a solution of the following equation

$$\widetilde{q}_m(x) := \begin{vmatrix} \widetilde{s}(f^{(m-2)}(x)) & \widetilde{r}(f^{(m-2)}(x)) & & & \\ l(f^{(m-3)}(x)) & s(f^{(m-3)}(x)) & r(f^{(m-3)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(f(x)) & s(f(x)) & r(f(x)) \\ & & & l(x) & s(x) \end{vmatrix} = 0, \qquad (4.8)$$

instead of $q_m(x) = 0$ in the symmetric case. Hence if λ_m is a root of $\widetilde{q}_m(x)$, $f^{(m-2)}(\lambda_m) \neq 5$ and none of $f^{(i)}(\lambda_m)$'s is equal to 2 or 5 for $0 \leq i \leq m-3$, then $\lambda_m \in \mathcal{P}_m^-$. Similarly to Proposition 4.1, we have

Proposition 4.3. *(1)* $\widetilde{q}_m(0) > 0$, $\forall m \geq 2$;

(2) $\widetilde{q}_2(5) < 0$ *and* $\widetilde{q}_m(5) > 0$, $\forall m \geq 3$;

(3) $\widetilde{q}_2(6) > 0$ *and* $\widetilde{q}_m(6) < 0$, $\forall m \geq 3$.

Proof. The first two statements follow from a very similar argument in the proof of Proposition 4.1. We only need to prove the third one.

It is easy to check that $\widetilde{q}_2(6) = 1 > 0$ and $\widetilde{q}_3(6) = -436 < 0$ by a direct computation. For $m \geq 4$, an expansion along the first row yields that

$$\widetilde{q}_m(6) = \widetilde{s}(f^{(m-2)}(6))q_{m-1}(6) + 2(5 - f^{(m-2)}(6))(f^{(m-3)}(6) - 6)q_{m-2}(6).$$

Recall that in the proof of Proposition 4.1(3), we have proved that $q_{m-1}(6) \leq q_{m-2}(6) < 0$. Hence

$$\widetilde{q}_m(6) \leq (\widetilde{s}(f^{(m-2)}(6)) + 2(5 - f^{(m-2)}(6))(f^{(m-3)}(6) - 6))q_{m-1}(6),$$

noticing that $f^{(m-2)}(6) < f^{(m-3)}(6) \leq -6$. An easy calculation shows that $\widetilde{s}(f^{(m-2)}(6)) + 2(5 - f^{(m-2)}(6))(f^{(m-3)}(6) - 6) \geq 1$, hence $\widetilde{q}_m(6) \leq q_{m-1}(6) < 0$. □

Similarly to the symmetric case, the following two lemmas focus on the possibility of the roots of $\widetilde{q}_m(x)$ satisfying $f^{(m-2)}(x) = 5$, or $f^{(i)}(x) = 2$ or 5 for some $0 \leq i \leq m-3$.

Lemma 4.7. *Let x be a predecessor of 2 of order i with $0 \leq i \leq m-3$. Then* $\widetilde{q}_m(x) = 0$.

Proof. If $0 \leq i < m-3$, the proof is the same as that of Lemma 4.2. So we only need to check the $i = m-3$ case. In this case, $f^{(m-3)}(x) = 2$ and $f^{(m-2)}(x) = 6$. Substituting them into (4.8), noticing $s(2) = -8$, $l(2) = -4$, $r(2) = 0$, $\widetilde{s}(6) = 1$ and $\widetilde{r}(6) = 2$, we get

$$\widetilde{q}_m(x) = \begin{vmatrix} \widetilde{s}(f^{(m-2)}(x)) & \widetilde{r}(f^{(m-2)}(x)) & & \\ l(f^{(m-3)}(x)) & s(f^{(m-3)}(x)) & r(f^{(m-3)}(x)) & \\ & \ddots & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} 1 & 2 & & \\ -4 & -8 & 0 & \\ & \ddots & \ddots & \ddots \end{vmatrix} = 0. \quad \square$$

Lemma 4.8. *Let x be a predecessor of 5 of order i with $0 \leq i \leq m-2$. Then $\widetilde{q}_m(x) \neq 0$.*

Proof. Similarly to the proof of Lemma 4.3, we only need to prove $\widetilde{q}_m(\phi_-^{(i)}(5)) \neq 0$. A similar argument yields that $\widetilde{q}_m(\phi_-^{(0)}(5)) = \widetilde{q}_m(5)$ and $\widetilde{q}_m(\phi_-^{(i)}(5)) = \widetilde{q}_{m-i}(5) \cdot q_{i+1}(\phi_-^{(i)}(5))$ for $0 < i \leq m-2$. Combined with Proposition 4.1(4) and Proposition 4.3(2), the desired result follows. $\square$

Hence the total unwanted roots of $\widetilde{q}_m(x)$ consist of those predecessors of 2 of order i with $0 \leq i \leq m-3$ for $m \geq 3$. Note that $\widetilde{q}_2(x)$ does not have any unwanted root. This is exactly the same as in the symmetric case. To eliminate them, we define

$$\widetilde{p}_m(x) := \frac{\widetilde{q}_m(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)}, \qquad \text{for } m \geq 3,$$

and

$$\widetilde{p}_2(x) := \widetilde{q}_2(x) = \widetilde{s}(x).$$

These polynomials play a very similar role to the $p_m(x)$'s in the symmetric case. It is easy to check that the degree of $\widetilde{p}_m(x)$ is $s_m := 2^m - 2$, since the degree of the polynomial $\widetilde{q}_m(x)$ is $3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{m-3} + 2 \cdot 2^{m-2} = 3(2^{m-2} - 1) + 2^{m-1}$, and the number of all the unwanted roots of $\widetilde{q}_m(x)$ is $1 + 2 + \cdots + 2^{m-3} = 2^{m-2} - 1$ for $m \geq 3$ and 0 for $m = 2$. The following is a list of some facts about $\widetilde{p}_m(x)$ similar to Proposition 4.2, which can be easily obtained from Proposition 4.3.

Proposition 4.4. *Let $m \geq 2$. Then*

(1) $(-1)^m \widetilde{p}_m(0) > 0$;

(2) $(-1)^{m-1} \widetilde{p}_m(5) > 0$;

(3) $(-1)^m \widetilde{p}_m(6) > 0$.

Then following a similar argument, the results in Lemma 4.5 and Lemma 4.6 still hold with $\mathcal{P}_m^+$, $p_m(x)$ and r_m replaced by $\mathcal{P}_m^-$, $\widetilde{p}_m(x)$ and s_m respectively.

Hence we have found r_m distinct eigenvalues in $\mathcal{P}_m^+$ and s_m distinct eigenvalues in $\mathcal{P}_m^-$. We will show these eigenvalues are the totality of $\mathcal{P}_m$. To prove this, the following lemma is needed.

Lemma 4.9. *Let $\mathcal{P}_m^{+,*}$ and $\mathcal{P}_m^{-,*}$ be the sets of total roots of $p_m(x)$ and $\widetilde{p}_m(x)$ respectively. Let $\mathcal{M}_m^*$ be the set of miniaturized eigenvalues generated by $\mathcal{P}_k^{-,*}$ with $2 \leq k < m$. Let $\mathcal{L}_m$ denote the set of m-level localized eigenvalues. Then all eigenfunctions associated to these eigenvalues are linearly independent.*

Proof. Without loss of generality, assume $m \geq 3$. It is easy to check that for each m-level localized eigenfunction $u_m^{\mathcal{L}}$, it must be 0 on $\partial\Omega_{m-1}$. Lemma 4.6 says that each m-level symmetric primitive λ_m-eigenfunction $u_m^{\mathcal{P},+}$ with $\lambda_m \in \mathcal{P}_m^{+,*}$ must be a non-zero constant on $\partial\Omega_{m-1} \setminus \{q_0\}$ and be a non-zero constant on $\partial\Omega_1 \setminus \{q_0\}$. The skew-symmetric analog of Lemma 4.6 says that each m-level skew-symmetric primitive λ_m-eigenfunction $u_m^{\mathcal{P},-}$ with

$\lambda_m \in \mathcal{P}_m^{-,*}$ must be a non-zero constant on each symmetric part of $\partial\Omega_{m-1} \setminus \{q_0\}$ under the symmetry fixing q_0, and take a non-zero value on $F_1 q_0$ and $F_2 q_0$ only different in signs. From the construction of the miniaturized eigenfunctions, for each m-level miniaturized eigenfunction $u_m^{\mathcal{M}}$ with eigenvalue in $\mathcal{M}_m^*$, $u_m^{\mathcal{M}}$ must take a non-zero value on a subset of $\partial\Omega_{m-1} \setminus \{q_0\}$ and be 0 on $\partial\Omega_1$. These observations implies the linearly independence of eigenfunctions among different types. □

Hence we have

Lemma 4.10. *For each $m \geq 2$, $\mathcal{P}_m^{+,*} = \mathcal{P}_m^+$ and $\mathcal{P}_m^{-,*} = \mathcal{P}_m^-$.*

Proof. Lemma 4.5 and its skew-symmetric analog say that $\sharp\mathcal{P}_m^{+,*} = r_m$ and $\sharp\mathcal{P}_m^{-,*} = s_m$. Accordingly, by a similar argument to the proof of Theorem 3.4, it follows that $\sharp\mathcal{M}_m^* = (m-3)\cdot 2^m + 4$.

Then it is easy to check that $\sharp\mathcal{L}_m$, $\sharp\mathcal{P}_m^{+,*}$, $\sharp\mathcal{P}_m^{-,*}$ and $\sharp\mathcal{M}_m^*$ add up to $\sharp(V_m^\Omega \setminus \partial\Omega_m)$ with a suitable modification of the eigenspace dimensional counting formula (3.1). Using Lemma 4.9, we then get the desired result. □

By this lemma, it is easy to see that the assumptions we made before on symmetric and skew-symmetric m-level primitive eigenvalues hold automatically.

Next we will prove each primitive eigenvalue $\lambda \in \mathcal{P}_m$ has multiplicity 1. For this purpose, we need the following lemma.

Lemma 4.11. *For each $m \geq 2$, $\mathcal{P}_m^+ \cap \mathcal{P}_{m+1}^+ = \emptyset$.*

Proof. For $m = 2$, it can be checked by a direct computation. In order to use induction, we assume that $\mathcal{P}_m^+ \cap \mathcal{P}_{m+1}^+ = \emptyset$ and will prove $\mathcal{P}_{m+1}^+ \cap \mathcal{P}_{m+2}^+ = \emptyset$.

Suppose there is a $\lambda \in \mathcal{P}_{m+1}^+ \cap \mathcal{P}_{m+2}^+$. Then $p_{m+1}(\lambda) = p_{m+2}(\lambda) = 0$ (hence $q_{m+1}(\lambda) = q_{m+2}(\lambda) = 0$). Moreover, none of $f^{(i)}(\lambda)$ $(0 \leq i \leq m)$ is equal to 2 or 5.

The expansion along the first row of $q_{m+2}(\lambda)$ gives

$$q_{m+2}(\lambda) = s(f^{(m)}(\lambda))q_{m+1}(\lambda) - r(f^{(m)}(\lambda))l(f^{(m-1)}(\lambda))q_m(\lambda).$$

Noticing that $q_{m+1}(\lambda) = q_{m+2}(\lambda) = 0$, we have

$$r(f^{(m)}(\lambda))l(f^{(m-1)}(\lambda))q_m(\lambda) = -2(2 - f^{(m)}(\lambda))(5 - f^{(m)}(\lambda))(f^{(m-1)}(\lambda) - 6)q_m(\lambda) = 0.$$

Hence $q_m(\lambda) = 0$ or $f^{(m-1)}(\lambda) = 6$, since $f^{(m)}(\lambda) \neq 2$ or 5.

If $q_m(\lambda) = 0$, then $\lambda \in \mathcal{P}_m^+$, hence $\mathcal{P}_m^+ \cap \mathcal{P}_{m+1}^+ \neq \emptyset$. This contradicts our induction assumption.

Hence we have $f^{(m-1)}(\lambda) = 6$, i.e., $f^{(m-2)}(\lambda) = 3$. Noticing that λ is also a root of $p_{m+1}(x)$, Lemma 4.1 says that $\phi_-^{(m-2)}(3)$ is a root of $p_{m+1}(x)$. Hence $p_{m+1}(\phi_-^{(m-2)}(3)) = 0$, which contradicts Proposition 4.1(6). Hence such λ can not exist. So we get the desired result. □

Then we can prove:

Lemma 4.12. *For each $m \geq 2$, $\mathcal{P}_m^+ \cap \mathcal{P}_m^- = \emptyset$.*

Proof. For $m = 2$ or 3, it can be checked by a direct computation. Let $m \geq 4$. Suppose there is an eigenvalue $\lambda_m \in \mathcal{P}_m^+ \cap \mathcal{P}_m^-$. Then by Lemma 4.10, $p_m(\lambda_m) = \widetilde{p}_m(\lambda_m) = 0$. For each $2 \leq i \leq m$, denote by $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$. Obviously we have $q_m(\lambda_m) = \widetilde{q}_m(\lambda_m) = 0$ and $\lambda_i^{(m)} \neq 2$ or 5 for $2 \leq i \leq m$. Furthermore, by Lemma 4.11, we have $p_{m-1}(\lambda_m) \neq 0$, hence $q_{m-1}(\lambda_m) \neq 0$.

Using the expansions of $q_m(\lambda_m)$ and $\widetilde{q}_m(\lambda_m)$ along their first rows respectively, we have

$$s(\lambda_2^{(m)})q_{m-1}(\lambda_m) - r(\lambda_2^{(m)})l(\lambda_3^{(m)})q_{m-2}(\lambda_m) = 0$$

and

$$\widetilde{s}(\lambda_2^{(m)})q_{m-1}(\lambda_m) - \widetilde{r}(\lambda_2^{(m)})l(\lambda_3^{(m)})q_{m-2}(\lambda_m) = 0.$$

Hence, the vector $(q_{m-1}(\lambda_m), q_{m-2}(\lambda_m))$ can be viewed as a non-zero solution of the system of linear equations,

$$\begin{cases} s(\lambda_2^{(m)})x - r(\lambda_2^{(m)})l(\lambda_3^{(m)})y = 0 \\ \widetilde{s}(\lambda_2^{(m)})x - \widetilde{r}(\lambda_2^{(m)})l(\lambda_3^{(m)})y = 0. \end{cases}$$

Thus

$$\begin{vmatrix} s(\lambda_2^{(m)}) & 2(2-\lambda_2^{(m)})(5-\lambda_2^{(m)})(\lambda_3^{(m)}-6) \\ \widetilde{s}(\lambda_2^{(m)}) & 2(5-\lambda_2^{(m)})(\lambda_3^{(m)}-6) \end{vmatrix} = 0.$$

Since $\lambda_2^{(m)} \neq 5$, we have $\lambda_3^{(m)} = 6$ or $s(\lambda_2^{(m)}) = (2-\lambda_2^{(m)})\widetilde{s}(\lambda_2^{(m)})$. By substituting the expressions for $s(x)$ and $\widetilde{s}(x)$, we get $\lambda_2^{(m)} = 6$ or $\lambda_3^{(m)} = 6$. Hence we have $\lambda_3^{(m)} = 3$ or $\lambda_4^{(m)} = 3$, i.e., $f^{(m-3)}(\lambda_m) = 3$, or $f^{(m-4)}(\lambda_m) = 3$.

Noticing that λ_m is a root of $q_m(x)$, by using Lemma 4.1, we can see that either $\phi_-^{(m-3)}(3)$ or $\phi_-^{(m-4)}(3)$ is a root of $q_m(x)$, i.e., $q_m(\phi_-^{(m-3)}(3)) = 0$ or $q_m(\phi_-^{(m-4)}(3)) = 0$. An expansion of $q_m(\phi_-^{(m-4)}(3))$ along the first row yields that

$$q_m(\phi_-^{(m-4)}(3)) = s(f^{(2)}(3))q_{m-1}(\phi_-^{(m-4)}(3)) = 848q_{m-1}(\phi_-^{(m-4)}(3))$$

since $l(f(3)) = 0$. Hence we have either $q_m(\phi_-^{(m-3)}(3)) = 0$ or $q_{m-1}(\phi_-^{(m-4)}(3)) = 0$. By Proposition 4.1(6), this is impossible. Hence such λ_m can not exist. So $\mathcal{P}_m^+ \cap \mathcal{P}_m^- = \emptyset$. □

Proof of Theorem 3.3 and Theorem 3.5.

It is an immediate consequence, by using Lemma 4.5 and its skew-symmetric analog, Lemma 4.9, Lemma 4.10, Lemma 4.12 and the eigenspace dimension counting formula (3.1). □

5 Primitive Dirichlet eigenvalues of $-\Delta$

Having found the primitive Dirichlet eigenvalues and eigenfunctions for $-\Delta_m$, it is natural to believe that the primitive Dirichlet eigenvalues of $-\Delta$ could be obtained in the limit as m goes to infinity. This is true for the spectrum in the $\mathcal{SG} \setminus V_0$ case, benefiting from the spectral decimation method and the eigenfunction extension algorithm (2.5). Our goal in this section is to extend this recipe to Ω case by instead using the weak spectral decimation introduced in Section 3. Comparing to the $\mathcal{SG} \setminus V_0$ case, our method is more based on estimates. We focus on the symmetric case, since the skew-symmetric case can be obtained by using a similar discussion. We will prove Theorem 3.6 in this section.

We use the $\widetilde{\phi}_\pm$ notations introduced in Section 3. Recall that if α_m, β_m are two consecutive eigenvalues in $\mathcal{P}_m^+$ with $\alpha_m < \beta_m$, then we always have

$$\phi_-(\alpha_m) < \widetilde{\phi}_-(\beta_m) < \phi_-(\beta_m) \text{ and } \phi_+(\beta_m) < \widetilde{\phi}_+(\beta_m) < \phi_+(\alpha_m), \tag{5.1}$$

and if β_m is the least eigenvalue in $\mathcal{P}_m^+$, then instead we have

$$0 < \widetilde{\phi}_-(\beta_m) < \phi_-(\beta_m) \text{ and } \phi_+(\beta_m) < \widetilde{\phi}_+(\beta_m) < 5.$$

Let $m_0 \geq 2$, λ_{m_0} be an m_0-level symmetric primitive eigenvalue, $\{\lambda_m\}_{m \geq m_0}$ be an infinite sequence related by $\lambda_{m+1} = \widetilde{\phi}_-(\lambda_m)$ or $\widetilde{\phi}_+(\lambda_m)$, $\forall m \geq m_0$, assuming that there are only a finite number of $\widetilde{\phi}_+$ relations. Call the minimum value m_1, such that $\forall m \geq m_1$, $\lambda_{m+1} = \widetilde{\phi}_-(\lambda_m)$, the *generation of fixation* of the sequence $\{\lambda_m\}_{m \geq m_0}$. In all that follows in this section, we always use $\{\lambda_m\}_{m \geq m_0}$ as such a sequence without specifical declaration.

The first fact about this sequence is:

Lemma 5.1. $\lim_{m\to\infty} 5^m \lambda_m$ *exists.*

Proof. Without loss of generality, assume $\lambda_{m_1} < 5$, otherwise, we could choose $\widetilde{m}_1 = m_1 + 1$ and use $\widetilde{m}_1$ to replace m_1 in the following proof.

Let $m \geq m_1$, then $\frac{\lambda_{m+1}}{\lambda_m} = \frac{\widetilde{\phi}_-(\lambda_m)}{\lambda_m} \leq \frac{\phi_-(\lambda_m)}{\lambda_m} = \frac{\phi_-(\lambda_m)}{\phi_-(\lambda_m)(5-\phi_-(\lambda_m))} = \frac{1}{5-\phi_-(\lambda_m)}$. Since $0 < \lambda_m < 5$, we have $0 < \phi_-(\lambda_m) < 2$, hence $\frac{1}{5-\phi_-(\lambda_m)} < \frac{1}{3}$. Thus $\sum_{m \geq m_1} \lambda_m < \infty$.

Furthermore, $\frac{5^{m+1}\lambda_{m+1}}{5^m \lambda_m} = 5\frac{\lambda_{m+1}}{\lambda_m} \leq \frac{5}{5-\phi_-(\lambda_m)} = 1 + \frac{\phi_-(\lambda_m)}{5-\phi_-(\lambda_m)}$. Noticing that $\sum_{m \geq m_1} \frac{\phi_-(\lambda_m)}{5-\phi_-(\lambda_m)} \leq \frac{1}{3}\sum_{m \geq m_1} \phi_-(\lambda_m) \leq \frac{1}{3}\sum_{m \geq m_1} \lambda_m < \infty$ since $\phi_-'(x) < 1$ whenever $0 < x < 5$, we get that $\Pi_{m \geq m_1} \frac{5^{m+1}\lambda_{m+1}}{5^m \lambda_m}$ converges. Hence $\lim_{m\to\infty} 5^m \lambda_m$ exists. $\square$

The following is an estimate of the difference between $\widetilde{\phi}_-(\lambda_m)$ and $\phi_-(\lambda_m)$ for λ_m in the sequence $\{\lambda_m\}_{m \geq m_0}$.

Proposition 5.1.

$$\sum_{m \geq m_1} 5^m (\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)) < \infty.$$

In particular, $\lim_{m\to\infty} 5^m (\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)) = 0$.

Proof. Without loss of generality, assume $\lambda_{m_1} < 5$. From Lemma 5.1, we have

$\sum_{m\geq m_1}(5^{m+1}\lambda_{m+1}-5^m\lambda_m)<\infty$. Hence

$$\begin{aligned}
& \sum_{m\geq m_1} 5^m(5\lambda_{m+1}-\lambda_m) \\
= & \sum_{m\geq m_1} 5^m(5\widetilde{\phi}_-(\lambda_m)-\phi_-(\lambda_m)(5-\phi_-(\lambda_m))) \\
= & \sum_{m\geq m_1} (5^{m+1}(\widetilde{\phi}_-(\lambda_m)-\phi_-(\lambda_m))+5^m(\phi_-(\lambda_m))^2)<\infty. \qquad (5.2)
\end{aligned}$$

Since $0<\phi_-'(x)<1$ whenever $0<x<5$, we have $5^m(\phi_-(\lambda_m))^2\leq 5^m\lambda_m^2$. Still from Lemma 5.1, we have $\lambda_m=O(\frac{1}{5^m})$, hence $5^m(\phi_-(\lambda_m))^2\leq\frac{c}{5^m}$ for some constant c. Thus $\sum_{m\geq m_1}5^m(\phi_-(\lambda_m))^2<\infty$. Combining this with (5.2), we get $\sum_{m\geq m_1}5^m(\widetilde{\phi}_-(\lambda_m)-\phi_-(\lambda_m))<\infty$. $\square$

To reveal some further properties of the limit $\lim_{m\to\infty}5^m\lambda_m$, the following lemma is required, which is a generalization of formula (5.1).

Lemma 5.2. *Let $m\geq 2$. α_m,β_m be two consecutive eigenvalues in $\mathcal{P}_m^+$ with $\alpha_m<\beta_m$. Then $\forall l\in\mathbb{N}$,*

$$\phi_-^{(l)}(\alpha_m)<\widetilde{\phi}_-^{(l)}(\beta_m). \qquad (5.3)$$

Proof. First we need to prove the following relation.

$$p_{m+l}(\phi_-^{(l)}(\alpha_m))\sim(-1)^{l-1}p_{m+1}(\phi_-(\alpha_m)),\quad \forall l\in\mathbb{N}. \qquad (5.4)$$

In fact, when $l\geq 3$, using the Laplace theorem to expand the determinant $q_{m+l}(\phi_-^{(l)}(\alpha_m))$ according to the last $(l-1)$ rows, we have

$$q_{m+l}(\phi_-^{(l)}(\alpha_m))=q_l(\phi_-^{(l)}(\alpha_m))q_{m+1}(\phi_-(\alpha_m))-l(\phi_-^{(2)}(\alpha_m))q_{l-1}(\phi_-^{(l)}(\alpha_m))r(\phi_-(\alpha_m))q_m(\alpha_m).$$

Since $q_m(\alpha_m)=0$, we have

$$q_{m+l}(\phi_-^{(l)}(\alpha_m))=q_l(\phi_-^{(l)}(\alpha_m))q_{m+1}(\phi_-(\alpha_m)).$$

This equality also holds for $l=2$ by instead using an expansion along the last row of $q_{m+2}(\phi_-^{(2)}(\alpha_m))$. Hence for each $l\geq 2$, we always have $q_{m+l}(\phi_-^{(l)}(\alpha_m))=q_l(\phi_-^{(l)}(\alpha_m))q_{m+1}(\phi_-(\alpha_m))$. Then from Lemma B in the Appendix, we have $q_l(\phi_-^{(l)}(\alpha_m))>0$, hence $q_{m+l}(\phi_-^{(l)}(\alpha_m))\sim q_{m+1}(\phi_-(\alpha_m))$. By the relation between $p_{m+l}(x)$ and $q_{m+l}(x)$, we easily get (5.4).

Now we prove (5.3). When $l=1$, (5.3) follows from (5.1) directly. In order to use the induction, assuming (5.3) holds for l, we turn to prove

$$\phi_-^{(l+1)}(\alpha_m)<\widetilde{\phi}_-^{(l+1)}(\beta_m).$$

Suppose α_m and β_m are the k'th and $(k+1)$'th eigenvalues in $\mathcal{P}_m^+$ respectively. Recall that in Lemma 4.4, we have proved that $p_{m+1}(\phi_-(\alpha_m)) \sim (-1)^{m+k-1}$. Combining this with (5.4), we have

$$p_{m+l+1}(\phi_-^{(l+1)}(\alpha_m)) \sim (-1)^{m+k+l-1}. \tag{5.5}$$

On the other hand, if we denote $\alpha_{m+l} = \widetilde{\phi}_-^{(l)}(\alpha_m)$ and $\beta_{m+l} = \widetilde{\phi}_-^{(l)}(\beta_m)$, then it is easy to see that α_{m+l} and β_{m+l} are the k'th and $(k+1)$'th eigenvalues in $\mathcal{P}_{m+l}^+$ respectively. Lemma 4.4 says that

$$p_{m+l+1}(\phi_-(\alpha_{m+l})) \sim (-1)^{m+l+k-1} \tag{5.6}$$

and

$$p_{m+l+1}(\phi_-(\beta_{m+l})) \sim (-1)^{m+l+k}. \tag{5.7}$$

Furthermore, if we denote $\beta_{m+l+1} = \widetilde{\phi}_-^{(l+1)}(\beta_m)$, then β_{m+l+1} is the only root of $p_{m+l+1}(x)$ located between $\phi_-(\alpha_{m+l})$ and $\phi_-(\beta_{m+l})$, i.e.,

$$\phi_-(\alpha_{m+l}) < \beta_{m+l+1} < \phi_-(\beta_{m+l}). \tag{5.8}$$

Noticing that from the induction assumption, we have $\phi_-^{(l+1)}(\alpha_m) < \phi_-(\beta_{m+l})$ since $\beta_{m+l} = \widetilde{\phi}_-^{(l)}(\beta_m)$. Moreover, (5.5) and (5.7) say that there exists at least one root of $p_{m+l+1}(x)$, denoted by β^*_{m+l+1}, between $\phi_-^{(l+1)}(\alpha_m)$ and $\phi_-(\beta_{m+l})$, i.e.,

$$\phi_-^{(l+1)}(\alpha_m) < \beta^*_{m+l+1} < \phi_-(\beta_{m+l}). \tag{5.9}$$

Since $\phi_-(\alpha_{m+l}) = \phi_-(\widetilde{\phi}_-^{(l)}(\alpha_m)) < \phi_-^{(l+1)}(\alpha_m)$, we have

$$\phi_-(\alpha_{m+l}) < \phi_-^{(l+1)}(\alpha_m) < \beta^*_{m+l+1} < \phi_-(\beta_{m+l}).$$

Combing this with (5.8), from the uniqueness of β_{m+l+1}, we have $\beta_{m+l+1} = \beta^*_{m+l+1}$. Hence substituting it into (5.9), we finally get $\phi_-^{(l+1)}(\alpha_m) < \beta_{m+l+1}$, i.e., $\phi_-^{(l+1)}(\alpha_m) < \widetilde{\phi}_-^{(l+1)}(\beta_m)$, which is the desired result. □

The following is an application of Lemma 5.2.

Lemma 5.3. *Let $m_1 \geq 2$, $\alpha_{m_1}, \beta_{m_1}$ be two consecutive eigenvalues in $\mathcal{P}_{m_1}^+$ with $\alpha_{m_1} < \beta_{m_1}$. $\{\alpha_m\}_{m \geq m_1}$ is an infinite sequence related by $\alpha_{m+1} = \widetilde{\phi}_-(\alpha_m), \forall m \geq m_1$; $\{\beta_m\}_{m\geq m_1}$ is an infinite sequence related by $\beta_{m+1} = \widetilde{\phi}_-(\beta_m), \forall m \geq m_1$. Then $\forall m \geq m_1$, $\alpha_m < \beta_m$. Moreover,*

$$\lim_{m\to\infty} 5^m \alpha_m < \lim_{m\to\infty} 5^m \beta_m.$$

Remark. *In the $\mathcal{SG} \setminus V_0$ case, this is a direct result since $\phi_-(x)$ is a definite strictly increasing continuous function.*

Proof of Lemma 5.3. Let $m > m_1$. Since $\alpha_m = \widetilde{\phi}_-^{(m-m_1)}(\alpha_{m_1})$ and $\beta_m = \widetilde{\phi}_-^{(m-m_1)}(\beta_{m_1})$, we have

$$\alpha_m < \phi_-^{(m-m_1)}(\alpha_{m_1}) < \widetilde{\phi}_-^{(m-m_1)}(\beta_{m_1}) = \beta_m \tag{5.10}$$

by Lemma 5.2. Hence $\forall m > m_1$, $\alpha_m < \beta_m$.

Now we prove $\lim_{m\to\infty} 5^m\alpha_m < \lim_{m\to\infty} 5^m\beta_m$.

Let $m > m_1$. Then from (5.10), we have

$$\alpha_m < \phi_-^{(m-m_1-1)}(\widetilde{\phi}_-(\alpha_{m_1})) < \phi_-^{(m-m_1)}(\alpha_{m_1}) < \beta_m.$$

Hence $\beta_m - \alpha_m > \phi_-^{(m-m_1-1)}(\phi_-(\alpha_{m_1})) - \phi_-^{(m-m_1-1)}(\widetilde{\phi}_-(\alpha_{m_1}))$. Since $\phi_-'(x) \geq \frac{1}{5}$ whenever $0 < x < 5$, and $0 < \widetilde{\phi}_-(\alpha_{m_1}) < \phi_-(\alpha_{m_1}) < 5$, we have

$$\beta_m - \alpha_m > \frac{1}{5^{m-m_1-1}}(\phi_-(\alpha_{m_1}) - \widetilde{\phi}_-(\alpha_{m_1})).$$

Hence $5^m(\beta_m - \alpha_m) > 5^{m_1+1}(\phi_-(\alpha_{m_1}) - \widetilde{\phi}_-(\alpha_{m_1}))$ which yields that

$$\lim_{m\to\infty} 5^m(\beta_m - \alpha_m) \geq 5^{m_1+1}(\phi_-(\alpha_{m_1}) - \widetilde{\phi}_-(\alpha_{m_1})) > 0.$$

Thus $\lim_{m\to\infty} 5^m\alpha_m < \lim_{m\to\infty} 5^m\beta_m$. □

Lemma 5.4. $\lim_{m\to\infty} 5^m\lambda_m > 0$.

Remark. *In the $\mathcal{SG} \setminus V_0$ case, this is also a direct result, since $\{5^m\lambda_m\}_{m\geq m_1}$ is then a monotone increasing sequence.*

Proof of Lemma 5.4. Without loss of generality, we assume that λ_{m_1} is the least eigenvalue in $\mathcal{P}_{m_1}^+$, since Lemma 5.3 says that it suffices to prove it for this special case. Then $\forall m \geq m_1$, λ_m is also the least eigenvalue in $\mathcal{P}_m^+$. Note that Lemma B in the Appendix says that $\forall m \geq m_1$, we have $\lambda_m \geq \phi_-^{(m)}(6)$. Hence

$$\lim_{m\to\infty} 5^m\lambda_m \geq \lim_{m\to\infty} 5^m\phi_-^{(m)}(6) > 0,$$

where the existence and positivity of the second limit are already shown in the $\mathcal{SG} \setminus V_0$ case. See [10]. □

Now we define

$$\lambda = \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m.$$

We will prove λ is an primitive Dirichlet eigenvalue of $-\Delta$ on the fractal domain Ω.

Note that $\forall m \geq m_0$, $\lambda_m \in \mathcal{P}_m^+$, i.e., λ_m is a root of both $p_m(x)$ and $q_m(x)$ by Lemma 4.5 and Theorem 3.3. As in Section 4, denote by $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $2 \leq i \leq m$. Lemma 4.6 and Theorem 3.3 say that the system (4.2) of equations has 1-dimensional solutions $(b_1, b_2, \cdots, b_{m-1})$ with $b_1 \neq 0$ and $b_{m-1} \neq 0$. We normalize the solution by requiring $b_1 = 1$, and write it as $(b_1^{(m)}, b_2^{(m)}, \cdots, b_{m-1}^{(m)})$ with $b_1^{(m)} = 1$ to specify its relation to λ_m. We always denote $b_0^{(m)}=0$ for convenience. As described in Section 4, from $(b_1^{(m)},b_2^{(m)},\cdots,b_{m-1}^{(m)})$ one can recover the unique (up to a constant) λ_m-eigenfunction u_m on Ω_m (noticing that $\lambda_i^{(m)}\neq 2$ or 5, $\forall 2\leq i\leq m$). Hence

$$\begin{cases} -\Delta_m u_m=\lambda_m u_m \text{ on } \Omega_m, \\ u_m|_{\partial\Omega_m}=0. \end{cases}$$

For each $m\geq m_0$, we start with the λ_m-eigenfunction u_m on Ω_m, and extend u_m to Ω by successively using the eigenfunction extension algorithm (2.5) corresponding to the revised eigenvalue sequence $\{\lambda_m,\phi_-(\lambda_m),\phi_-^{(2)}(\lambda_m),\cdots\}$ (starting from λ_m, but continued with the standard spectral decimation eigenvalues) to get a primitive eigenfunction (possessing the symmetry in each cell $F_w(\mathcal{SG})$ under the reflection symmetry fixing F_wq_0 with word w taking symbols only from $\{1,2\}$) on Ω. We still denote u_m for this function. Of course, u_m may not satisfy the Dirichlet boundary condition on L. $\forall i>m$, we use $\lambda_i^{(m)}=\phi_-^{(i-m)}(\lambda_m)$ to denote the i-level revised eigenvalue. Hence for each $m\geq m_0$, u_m is an eigenfunction of $-\Delta$ on Ω(not satisfying the Dirichlet boundary condition), associated to the eigenvalue sequence $\{\lambda_i^{(m)}\}_{i\geq 2}$, where $\lambda_i^{(m)}=f^{(m-i)}(\lambda_m)$, $\forall 2\leq i\leq m$, and $\lambda_i^{(m)}=\phi_-^{(i-m)}(\lambda_m)$, $\forall i>m$. We use $b_i^{(m)}$ ($\forall i\geq m$) to denote the value of u_m at vertex $F_1^iq_0$. Hence $\{b_i^{(m)}\}_{i\geq 0}$ are the values of u_m on the skeleton of Ω which conversely determine u_m on Ω. We have the following relationship between $\{\lambda_i^{(m)}\}_{i\geq 2}$ and $\{b_i^{(m)}\}_{i\geq 0}$.

$$(4-\lambda_{i+1}^{(m)})b_i^{(m)}=2b_{i+1}^{(m)}+\frac{(14-3\lambda_{i+1}^{(m)})b_i^{(m)}+(6-\lambda_{i+1}^{(m)})b_{i-1}^{(m)}}{(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})},\quad \forall i\geq 1, \tag{5.11}$$

which follows from the eigenvalue equation at the vertex $F_1^iq_0$. Note that when $1\leq i\leq m-1$, these are exactly the equations in (4.1). Moreover, u_m on Ω satisfies

$$\begin{cases} -\Delta u_m=5^m\Phi(\lambda_m)u_m \text{ on } \Omega, \\ u_m(q_0)=0, \\ u_m|_L=\lim_{i\to\infty}b_i^{(m)}<\infty, \end{cases}$$

where $\Phi(z)$ is a function defined by $\Phi(z):=\frac{3}{2}\lim_{k\to\infty}5^k\phi_-^{(k)}(z)$. The existence of the limit $\lim_{i\to\infty}b_i^{(m)}$ will be given later.

It is easy to find that $5^m\Phi(\lambda_m)\to\lambda$ as m goes to infinity. Moreover, we have the following lemmas.

Lemma 5.5. *There exists a constant $C_1>0$ depending only on m_1, such that $\forall i\in\mathbb{N}$, $\forall p\in\mathbb{N}$, we have $|b_{i+p}^{(m)}-b_i^{(m)}|\leq C_1(\frac{3}{10})^i\|u_m\|_\infty$ uniformly on $m\geq m_1$.*

Proof. Without loss of generality, assume $i>m_1$ and λ_{m_1} is not the largest eigenvalue in $\mathcal{P}_{m_1}^+$. Denote by γ_{m_1} the next eigenvalue of λ_{m_1} in $\mathcal{P}_{m_1}^+$. Let $\{\gamma_m\}_{m\geq m_1}$ be the infinite sequence staring from γ_{m_1} related by $\gamma_{m+1}=\widetilde{\phi}_-(\gamma_m)$, $\forall m\geq m_1$. We now show

$$\lambda_{i+1}^{(m)}<\gamma_{i+1}<\phi_-(2),\quad \forall m\geq m_1. \tag{5.12}$$

In fact if $m \geq i+1$, then

$$\lambda_{i+1}^{(m)} = f^{(m-i-1)}(\lambda_m) = f^{(m-i-1)}(\widetilde{\phi}_{-}^{(m-i-1)}(\lambda_{i+1})) \leq f^{(m-i-1)}(\phi_{-}^{(m-i-1)}(\lambda_{i+1})) = \lambda_{i+1} < \gamma_{i+1}.$$

If $m < i+1$, then $\lambda_{i+1}^{(m)} = \phi_{-}^{(i+1-m)}(\lambda_m) < \widetilde{\phi}_{-}^{(i+1-m)}(\gamma_m) = \gamma_{i+1}$ by using Lemma 5.2. The right inequality of (5.12) is obvious. Hence (5.12) always holds.

On the other hand, notice that from (5.11),

$$\begin{aligned} b_{i+1}^{(m)} - b_i^{(m)} &= \frac{s(\lambda_{i+1}^{(m)})b_i^{(m)} - (6-\lambda_{i+1}^{(m)})b_{i-1}^{(m)}}{2(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})} - b_i^{(m)} \\ &= \frac{(6-\lambda_{i+1}^{(m)})(b_i^{(m)} - b_{i-1}^{(m)}) - (20\lambda_{i+1}^{(m)} - 9(\lambda_{i+1}^{(m)})^2 + (\lambda_{i+1}^{(m)})^3)b_i^{(m)}}{2(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})}. \end{aligned}$$

Hence

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq \frac{|6-\lambda_{i+1}^{(m)}|}{2|2-\lambda_{i+1}^{(m)}| \cdot |5-\lambda_{i+1}^{(m)}|}|b_i^{(m)} - b_{i-1}^{(m)}| + \frac{|20 - 9\lambda_{i+1}^{(m)} + (\lambda_{i+1}^{(m)})^2|}{2|2-\lambda_{i+1}^{(m)}| \cdot |5-\lambda_{i+1}^{(m)}|}|\lambda_{i+1}^{(m)}| \cdot |b_i^{(m)}|.$$

In the remaining proof, we use c to denote different constants.

By (5.12), we have

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq \frac{3}{(2-\gamma_{i+1})(5-\gamma_{i+1})}|b_i^{(m)} - b_{i-1}^{(m)}| + c\gamma_{i+1}|b_i^{(m)}|.$$

Noticing that $\gamma_i = O(\frac{1}{5^i})$ and $|b_i^{(m)}| \leq \|u_m\|_\infty$, we get

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq (\frac{3}{10} + \frac{c}{5^i})|b_i^{(m)} - b_{i-1}^{(m)}| + \frac{c}{5^i}\|u_m\|_\infty.$$

Hence

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq \frac{3}{10}|b_i^{(m)} - b_{i-1}^{(m)}| + \frac{c}{5^i}\|u_m\|_\infty.$$

Similarly we have the estimates

$$|b_i^{(m)} - b_{i-1}^{(m)}| \leq \frac{3}{10}|b_{i-1}^{(m)} - b_{i-2}^{(m)}| + \frac{c}{5^{i-1}}\|u_m\|_\infty$$

till

$$|b_{m_1+2}^{(m)} - b_{m_1+1}^{(m)}| \leq \frac{3}{10}|b_{m_1+1}^{(m)} - b_{m_1}^{(m)}| + \frac{c}{5^{m_1+1}}\|u_m\|_\infty.$$

A routine argument shows that

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq (\frac{3}{10})^{i-m_1}|b_{m_1+1}^{(m)} - b_{m_1}^{(m)}| + (\frac{3}{10})^{i-m_1-1}\frac{c}{5^{m_1+1}}\|u_m\|_\infty.$$

Hence we have proved that

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq c(\frac{3}{10})^i\|u_m\|_\infty$$

where c depends only on m_1.

Similarly, we have

$$|b_{i+2}^{(m)} - b_{i+1}^{(m)}| \leq c(\frac{3}{10})^{i+1} \|u_m\|_\infty,$$

till

$$|b_{i+p}^{(m)} - b_{i+p-1}^{(m)}| \leq c(\frac{3}{10})^{i+p-1} \|u_m\|_\infty.$$

By adding up the above estimates, we finally get $|b_{i+p}^{(m)} - b_i^{(m)}| \leq C_1(\frac{3}{10})^i \|u_m\|_\infty$.□

Lemma 5.6. *For each $m \geq m_1$, $\lim_{i\to\infty} b_i^{(m)}$ exists. Moreover, there exists a constant $C_2 > 0$ depending only on m_1, such that $|\lim_{i\to\infty} b_i^{(m)}| \leq C_2(\frac{3}{10})^m \|u_m\|_\infty$ uniformly on $m \geq m_1$.*

Proof. For each $m \geq m_1$, Lemma 5.5 says that each sequence $\{b_i^{(m)}\}_{i\geq 1}$ is a Cauchy sequence, hence $\lim_{i\to\infty} b_i^{(m)}$ exists.

Taking $i = m$, $p = 1$ in Lemma 5.5, noticing that $b_m^{(m)} = 0$, we get that $|b_{m+1}^{(m)}| \leq C_1(\frac{3}{10})^m \|u_m\|_\infty$.

On the other hand, $\forall i > m+1$, notice that $|b_i^{(m)}| \leq |b_i^{(m)} - b_{m+1}^{(m)}| + |b_{m+1}^{(m)}|$. By using Lemma 5.5 again, we have

$$|b_i^{(m)}| \leq C_1(\frac{3}{10})^{m+1} \|u_m\|_\infty + C_1(\frac{3}{10})^m \|u_m\|_\infty = C_2(\frac{3}{10})^m \|u_m\|_\infty.$$

Letting $i \to \infty$, we get the desired result. □

In the following context, for each $m \geq m_1$, let θ_m denote the limit $\lim_{i\to\infty} b_i^{(m)}/\|u_m\|_\infty$. Lemma 5.6 guarantees the existence of this limit, and furthermore, $|\theta_m| \leq C_2(\frac{3}{10})^m$. Let $v_m := \frac{u_m}{\|u_m\|_\infty}$. Then v_m on Ω satisfies

$$\begin{cases} -\Delta v_m = 5^m \Phi(\lambda_m) v_m \text{ on } \Omega, \\ v_m(q_0) = 0, \\ v_m|_L = \theta_m. \end{cases}$$

We will prove that $\{v_m\}_{m\geq m_1}$ contains a subsequence converging uniformly to a continuous function on Ω, which is a Dirichlet eigenfunction associated to λ.

Lemma 5.7. *$\{\partial_n v_m(q_0)\}_{m\geq m_1}$ is uniformly bounded, i.e., there exist a constant $C_3 > 0$ depending only on m_1, such that $|\partial_n v_m(q_0)| \leq C_3$.*

Proof. Let $m \geq m_1$. Choosing a harmonic function h such that $h(q_0) = 1$, $h(F_1 q_0) = h(F_2 q_0) = 0$, the local Gauss-Green formula on $F_0(\mathcal{SG})$ says that

$$\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h) = \int_{F_0(\mathcal{SG})} (-\Delta v_m) h d\mu + \sum_{\partial F_0(\mathcal{SG})} h \partial_n v_m.$$

Hence $|\partial_n v_m(q_0)| \leq |\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h)| + |\int_{F_0(\mathcal{SG})} (-\Delta v_m) h d\mu|$.

Since h is harmonic on $F_0(\mathcal{SG})$, we have $\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h) = \frac{5}{3}\mathcal{E}(v_m \circ F_0, h \circ F_0) = \frac{5}{3}\mathcal{E}_0(v_m \circ F_0, h \circ F_0)$. Noticing that $h(q_0) = 1$, $h(F_1 q_0) = h(F_2 q_0) = 0$, we get $|\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h)| \leq c_1$, since $\|v_m\|_\infty = 1$.

On the other hand, since $-\Delta v_m = 5^m \Phi(\lambda_m) v_m$, we have $|\int_{F_0(\mathcal{SG})}(-\Delta v_m) h d\mu| \leq 5^m \Phi(\lambda_m)\|v_m\|_\infty \cdot \|h\|_\infty \mu(F_0(\mathcal{SG})) \leq c_2$, since $5^m\Phi(\lambda_m) \to \lambda$.

Hence $|\partial_n v_m(q_0)| \leq c_1 + c_2 \triangleq C_3$. □

Lemma 5.8. *$\{\mathcal{E}(v_m)\}_{m \geq m_1}$ is uniformly bounded, i.e., there exists a constant $C_4 > 0$ depending only on m_1, such that $\mathcal{E}(v_m) \leq C_4$.*

Proof. $\forall n \geq m_1$, let K_n be the part of Ω above $\partial\Omega_n \setminus \{q_0\}$. We first prove $\{\mathcal{E}_{K_n}(v_m)\}_{m \geq m_1}$ is uniformly bounded and the upper bound is independent of n.

Fix $n \geq m_1$, $m \geq m_1$. The Gauss-Green formula says that $\int_{K_n} \Delta v_m d\mu = \sum_{\partial K_n} \partial_n v_m$. From the symmetry property of v_m, $\partial_n v_m$ takes same value along $\partial K_n \setminus \{q_0\}$. Hence we get

$$-5^m \Phi(\lambda_m) \int_{K_n} v_m d\mu = \partial_n v_m(q_0) + 2^n \partial_n v_m(F_1^n(q_0)). \tag{5.13}$$

On the other hand, the Gauss-Green formula also says that

$$\begin{aligned} \mathcal{E}_{K_n}(v_m) &= \int_{K_n} (-\Delta v_m) v_m d\mu + \sum_{\partial K_n} v_m \partial_n v_m \\ &= 5^m \Phi(\lambda_m) \int_{K_n} v_m^2 d\mu + 2^n v_m(F_1^n q_0) \partial_n v_m(F_1^n q_0), \end{aligned}$$

since $v_m(q_0) = 0$. Combined with (5.13), it follows that

$$\mathcal{E}_{K_n}(v_m) = 5^m \Phi(\lambda_m) \int_{K_n} v_m^2 d\mu + v_m(F_1^n q_0)(-5^m \Phi(\lambda_m) \int_{K_n} v_m d\mu - \partial_n v_m(q_0)).$$

Since $5^m \Phi(\lambda_m) \to \lambda$, there exists a constant $c > 0$, such that $5^m \Phi(\lambda_m) \leq c$. Hence

$$\mathcal{E}_{K_n}(v_m) \leq c\|v_m\|_\infty^2 + \|v_m\|_\infty (c\|v_m\|_\infty + |\partial_n v_m(q_0)|).$$

Using Lemma 5.7, we get $\mathcal{E}_{K_n}(v_m) \leq 2c + C_3 \triangleq C_4$. Since the above inequality is independent on n, we then get the desired result by taking the limit as $n \to \infty$. □

Now we come to the main purpose of this section.

Proof of Theorem 3.6.

For each $m \geq m_1$, since $v_m \in \mathcal{F}$, we have

$$|v_m(x) - v_m(y)| \leq \mathcal{E}(v_m)^{1/2} d(x,y)^{1/2}, \quad \forall x, y \in \Omega,$$

where $d(\cdot,\cdot)$ is the effective resistance metric on Ω. Hence by Lemma 5.8,

$$|v_m(x) - v_m(y)| \leq C_4^{1/2} d(x,y)^{1/2}, \quad \forall x, y \in \Omega$$

holds uniformly on $m \geq m_1$. Thus $\{v_m\}_{m \geq m_1}$ is equicontinuous. Moreover, notice that $\{v_m\}_{m \geq m_1}$ is also uniformly bounded. Then using the Arzelà-Ascoli theorem, there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ which converges uniformly to a continuous function v on Ω.

Let $G_\Omega(x, y)$ denote the Green's function associated to Ω. See the explicit expression for $G_\Omega(x, y)$ in [12]. Then $\forall k$, we have

$$v_{m_k}(x) = \int_\Omega G_\Omega(x, y) 5^{m_k} \Phi(\lambda_{m_k}) v_{m_k}(y) d\mu(y) + h_{m_k}(x), \tag{5.14}$$

where h_{m_k} is a harmonic function on Ω taking the same boundary values as v_{m_k}. Namely, $h_{m_k}(q_0) = 0$ and $h_{m_k}|_L = \theta_{m_k}$. If $k \to \infty$, then $\theta_{m_k} \to 0$ and hence h_{m_k} goes to 0 uniformly on Ω by the maximum principle. Hence by letting $k \to \infty$ on both side of (5.14), we get

$$v(x) = \int_\Omega G_\Omega(x, y)(\lambda v(y)) d\mu(y).$$

Thus we finally get

$$\begin{cases} -\Delta v = \lambda v \text{ in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Hence v is a Dirichlet eigenfunction associated to λ. $\square$

Thus for each sequence $\{\lambda_m\}_{m \geq m_0}$, we have proved that $\lambda = \frac{3}{2} \lim_{m \to} 5^m \lambda_m$ is a symmetric primitive Dirichlet eigenvalue of $-\Delta$ on Ω. We denote by $\mathcal{P}_*^+$ the totality of all these kinds of eigenvalues. Of course, $\mathcal{P}_*^+ \subset \mathcal{P}^+$. Lemma 5.3 and Lemma 5.4 guarantee that all eigenvalues in $\mathcal{P}_*^+$ are distinct and they are all greater than 0. In the next section, we will prove that all eigenvalues in $\mathcal{P}^+$ arise in this way. Namely, $\mathcal{P}_*^+ = \mathcal{P}^+$.

The skew-symmetric case is similar. We denote by $\mathcal{P}_*^-$ the set of skew-symmetric eigenvalues generated in this way. Let $\mathcal{P}_* = \mathcal{P}_*^+ \cup \mathcal{P}_*^-$ denote all the associated primitive eigenvalues. Let $\mathcal{M}_*$ be the set of miniaturized eigenvalues generated by $\mathcal{P}_*^-$. Accordingly, $\mathcal{P}_*^- \subset \mathcal{P}^-$, $\mathcal{P}_* \subset \mathcal{P}$ and $\mathcal{M}_* \subset \mathcal{M}$.

6 Complete Dirichlet spectrum of $-\Delta$

It is clear that the weak spectral decimation recipe constructs many primitive eigenvalues (hence also many miniaturized eigenvalues) of $-\Delta$ on Ω. Recall that the standard spectral decimation recipe also constructs many localized eigenvalues of $-\Delta$ on Ω. It is natural to ask: do these recipes construct the whole Dirichlet spectrum $\mathcal{S}$? In this section, we will give an affirmative answer to this question.

Till now, for each $m \geq 2$, we have proved that the Dirichlet spectrum $\mathcal{S}_m$ of the discrete Laplacian $-\Delta_m$ on Ω_m consists of $\mathcal{L}_m$, $\mathcal{P}_m$ and $\mathcal{M}_m$ the three types of eigenvalues. After passing to the limit, we have proved that there are at least three types of eigenvalues $\mathcal{L}$, $\mathcal{P}_*$ and $\mathcal{M}_*$ in the Dirichlet spectrum $\mathcal{S}$ of $-\Delta$, which can be generated by the (weak) spectral decimation recipe. Namely, $\mathcal{S} \supset \mathcal{L} \cup \mathcal{P}_* \cup \mathcal{M}_*$. We call all of the above three types of eigenvalues *raw eigenvalues.* By the *raw multiplicity* of the raw eigenvalue λ, we mean the multiplicity of the associated eigenvalue λ_{m_0} of $-\Delta_{m_0}$, where m_0 is the generation of birth. Since linearly independent eigenfunctions of $-\Delta_{m_0}$ belonging to λ_{m_0} give rise to linearly independent eigenfunctions of $-\Delta$, and the fact that all primitive graph eigenvalues have only raw multiplicity 1, the raw multiplicity of λ is not greater than the true multiplicity of λ.

Denote by $\mathcal{S}_*$ the collection of raw eigenvalues of $-\Delta$, then $\mathcal{S}_* = \mathcal{L} \cup \mathcal{P}_* \cup \mathcal{M}_*$ and $\mathcal{S}_* \subset \mathcal{S}$. Hence we need to prove $\mathcal{S}_* = \mathcal{S}$, $\mathcal{P}_* = \mathcal{P}$ and $\mathcal{M}_* = \mathcal{M}$ and the raw multiplicity of each element of $\mathcal{S}_*$ coincides with its true multiplicity.

Comparing with the proof of the analogous problem for the standard $\mathcal{SG} \setminus V_0$ case (see details in [10]), we see that to prove the above results, the following proposition will play a vital role. Recall that $a_m = \sharp(V_m^\Omega \setminus \partial\Omega_m) = \frac{3^{m+1}-1}{2} - 2^{m+1}$.

Proposition 6.1. *Let* $0 < \kappa_1 \leq \kappa_2 \leq \cdots$ *be the rearrangement of elements of* $\mathcal{S}_*$ *each repeated according to its raw multiplicity. Let* $\{\kappa_{m,i}\}_{1\leq i\leq a_m}$ *be the* m*-level graph eigenvalues of* $-\Delta_m$ *on* Ω_m *including multiplicities. Then*

$$\lim_{m\to\infty} \sum_{1\leq i\leq a_m} \frac{1}{\frac{3}{2}5^m \kappa_{m,i}} = \sum_{i=1}^{\infty} \frac{1}{\kappa_i} < \infty.$$

In order to prove this proposition, we first list some notations and lemmas. It is more convenient to consider the following slightly different classification of all the raw eigenvalues of $-\Delta$,

$$\mathcal{S}_* = \mathcal{L} \cup \mathcal{P}_*^+ \cup \widetilde{\mathcal{P}}_*^-$$

where $\widetilde{\mathcal{P}}_*^- = \mathcal{P}_*^- \cup \mathcal{M}_*$, since miniaturized eigenvalues have the same generation mechanism as the skew-symmetric primitive eigenvalues. In the following, we always use α, β, γ to denote $\mathcal{L}$, $\mathcal{P}_*^+$, $\widetilde{\mathcal{P}}_*^-$ type eigenvalues respectively. Accordingly, $\forall m \geq 2$, all the m-level graph eigenvalues are classified into the three types $\mathcal{L}_m$, $\mathcal{P}_m^+$ and $\widetilde{\mathcal{P}}_m^-$, where $\widetilde{\mathcal{P}}_m^- = \mathcal{P}_m^- \cup \mathcal{M}_m$, and we always use $\alpha_m, \beta_m, \gamma_m$ to denote eigenvalues in them respectively. For simplicity, we denote $A_m = \sharp\mathcal{L}_m$, $B_m = \sharp\mathcal{P}_m^+$ and $C_m = \sharp\widetilde{\mathcal{P}}_m^-$. Of course, $a_m = A_m + B_m + C_m$. Moreover, recall that $\rho_m^\Omega(5)$ and $\rho_m^\Omega(6)$ are the multiplicities of m-level initial eigenvalues 5 and 6 respectively. See the exact values of them in Section 3.

Lemma 6.1. $\mathcal{L} = \bigcup_{k=3}^\infty \mathcal{L}^k$ *(disjoint union) where* $\mathcal{L}^k \subset [5^k\Phi(3), 5^k\Phi(5)]$.

Proof. $\forall \alpha \in \mathcal{L}$, let $\{\alpha_m\}_{m\geq m_0}$ be the corresponding sequence of eigenvalues with a generation of fixation m_1. Then $\alpha = \frac{3}{2}\lim_{m\to\infty} 5^m\alpha_m = 5^{m_1}\Phi(\alpha_{m_1})$.

If α_{m_1} is an initial eigenvalue, then α_{m_1} can only be equal to 5. If α_{m_1} is a continued eigenvalue, then $\alpha_{m_1} = \phi_+(\alpha_{m_1-1})$, which yields that $3 \leq \alpha_{m_1} \leq 5$. Hence we always have $3 \leq \alpha_{m_1} \leq 5$.

Noticing that each localized eigenvalue has generation of birth at least 3, denote by $\mathcal{L}^k$ the set of eigenvalues with $m_1 = k$, $k = 3, 4, \cdots$. Then $\mathcal{L} = \bigcup_{k=3}^{\infty} \mathcal{L}^k$ and $\mathcal{L}^k \subset [5^k\Phi(3), 5^k\Phi(5)]$. Since $\phi_-(5) < 3$, we have $\Phi(5) < 5\Phi(3)$. Hence $\mathcal{L} = \bigcup_{k=3}^{\infty} \mathcal{L}^k$ is a disjoint union. $\square$

Lemma 6.2. *$\mathcal{P}_*^+ = \bigcup_{k=2}^{\infty} \mathcal{P}_*^{+,k}$ (disjoint union) where $\mathcal{P}_*^{+,2} \subset (0, 5^2\Phi(6)]$ and $\mathcal{P}_*^{+,k} \subset [5^k\Phi(\phi_-(3)), 5^k\Phi(6)]$ for $k \geq 3$.*

Proof. $\forall \beta \in \mathcal{P}_*^+$, let $\{\beta_m\}_{m \geq m_0}$ be the corresponding sequence of eigenvalues with a generation of fixation m_1. Then $\beta = \frac{3}{2}\lim_{m\to\infty} 5^m \beta_m = 5^{m_1} \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\beta_{m_1})$.

If β_{m_1} is a continued eigenvalue (hence $m_1 \geq 3$), then we must have $\beta_{m_1} = \widetilde{\phi}_+(\beta_{m_1-1})$, which obviously yields that $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1})$ where $\beta^*_{m_1-1}$ denotes the largest eigenvalue in $\mathcal{P}^+_{m_1-1}$. If β_{m_1} is an initial eigenvalue with $m_1 \geq 3$, then obviously $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1})$. Hence we always have $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1})$ if $m_1 \geq 3$.

Moreover, when $m_1 > 3$, if we denote by $\beta^{**}_{m_1-1}$ the largest eigenvalue in $\mathcal{P}^+_{m_1-1}$ except for $\beta^*_{m_1-1}$, then we have $\widetilde{\phi}_-(\beta^*_{m_1-1}) > \phi_-(\beta^{**}_{m_1-1})$. It is easy to check that $\beta^{**}_{m_1-1} > \phi_+(\beta^*_{m_1-2}) > 3$ since $m_1 > 3$. Thus $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1}) > \phi_-(3)$. When $m_1 = 3$, it can be checked directly that $\beta_3 > \widetilde{\phi}_-(\beta^*_2) \approx 1.33 > \phi_-(3)$. Hence we always have $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1}) > \phi_-(3)$ if $m_1 \geq 3$. By Lemma 5.2, we have $\widetilde{\phi}_-^{(n)}(\beta_{m_1}) > \phi_-^{(n)}(\widetilde{\phi}_-(\beta^*_{m_1-1})), \forall n \in \mathbb{N}$. Hence if $m_1 \geq 3$, we have

$$\begin{aligned}
\beta &= 5^{m_1} \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\beta_{m_1}) \\
&\geq 5^{m_1} \lim_{n\to\infty} \frac{3}{2} 5^n \phi_-^{(n)}(\widetilde{\phi}_-(\beta^*_{m_1-1})) \\
&\geq 5^{m_1} \lim_{n\to\infty} \frac{3}{2} 5^n \phi_-^{(n)}(\phi_-(3)) \\
&= 5^{m_1}\Phi(\phi_-(3)).
\end{aligned}$$

On the other hand, when $m_1 \geq 2$, we always have

$$\beta = 5^{m_1} \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\beta_{m_1}) \leq 5^{m_1} \lim_{n\to\infty} \frac{3}{2} 5^n \phi_-^{(n)}(6) = 5^{m_1}\Phi(6).$$

Denote by $\mathcal{P}_*^{+,k}$ the set of eigenvalues with $m_1 = k$, $k = 2, 3, \cdots$. Then $\mathcal{P}_*^+ = \bigcup_{k=2}^{\infty} \mathcal{P}_*^{+,k}$ where $\mathcal{P}_*^{+,2} \subset (0, 5^2\Phi(6)]$ and $\mathcal{P}_*^{+,k} \subset [5^k\Phi(\phi_-(3)), 5^k\Phi(6)]$ for $k \geq 3$.

Next we need to prove $\mathcal{P}_*^+ = \bigcup_{k=2}^{\infty} \mathcal{P}_*^{+,k}$ is a disjoint union. $\forall 2 \leq k < k'$, take an element β in $\mathcal{P}_*^{+,k}$, β' in $\mathcal{P}_*^{+,k'}$ respectively. Then $\beta = 5^k \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\beta_k)$ for some eigenvalue β_k in $\mathcal{P}_k^+$, and $\beta' = 5^{k'} \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\beta'_{k'})$ for some eigenvalue $\beta'_{k'}$ in $\mathcal{P}^+_{k'}$.

Note that $\widetilde{\phi}_-^{(k'-k)}(\beta_k)$ and $\beta'_{k'}$ both belong to $\mathcal{P}^+_{k'}$. Since k' is the generation of fixation of β', we can easily get $\widetilde{\phi}_-^{(k'-k)}(\beta_k) < \beta'_{k'}$. Then by using Lemma 5.3, we have $\beta < \beta'$.

From the arbitrariness of β, β' and k, k', we finally get that $\mathcal{P}_*^+ = \bigcup_{k=2}^\infty \mathcal{P}_*^{+,k}$ is a disjoint union. $\square$

Lemma 6.3. *Let $0 < \alpha_1 \le \alpha_2 \le \cdots$ be the rearrangement of elements of $\mathcal{L}$ each repeated according to its raw multiplicity. Let $\{\alpha_{m,i}\}_{1\le i\le A_m}$ be the m-level localized eigenvalues of $-\Delta_m$ on Ω_m including multiplicities. Then*

$$\lim_{m\to\infty}\sum_{1\le i\le A_m}\frac{1}{\frac{3}{2}5^m\alpha_{m,i}}=\sum_{i=1}^\infty\frac{1}{\alpha_i},$$

providing $\sum_{i=1}^\infty \frac{1}{\alpha_i} < \infty$.

Proof. Since $\lim_{m\to\infty}\frac{\rho_m^\Omega(6)}{5^m}=0$, it suffices to show that

$$\sum_{\substack{1\le i\le A_m\\ \alpha_{m,i}\ne 6}}\frac{1}{\frac{3}{2}5^m\alpha_{m,i}}-\sum_{i=1}^{A_m-\rho_m^\Omega(6)}\frac{1}{\alpha_i}\to 0 \text{ as } m\to\infty. \tag{6.1}$$

For $m\ge 2$, denote $D_m = A_m - \rho_m^\Omega(6)$. By Lemma 6.1, $\{\alpha_1,\alpha_2,\cdots,\alpha_{D_m}\}$ is an arrangement of elements of $\bigcup_{k=3}^m \mathcal{L}^k$ each being repeated according to its raw multiplicity. The first sum of (6.1) has also D_m terms, which can be rearranged so that

$$\lim_{n\to\infty}\frac{3}{2}5^{m+n}\phi_-^{(n)}(\alpha_{m,i})=\alpha_i,\quad 1\le i\le D_m.$$

Hence by using Lemma 6.1, (6.1) is equal to $\sum_{k=3}^m\sum_{\alpha_i\in\mathcal{L}^k}(\frac{1}{\frac{3}{2}5^m\alpha_{m,i}}-\frac{1}{\alpha_i})$. If $\alpha_i\in\mathcal{L}^k$ $(k=3,\cdots,m)$, then $\alpha_i = 5^k\Phi(\theta)$ for some $\theta\in[3,5]$ and accordingly the corresponding $\alpha_{m,i}$ is of the form $\alpha_{m,i}=\phi_-^{(m-k)}(\theta)$. Hence

$$0<\frac{1}{\frac{3}{2}5^m\alpha_{m,i}}-\frac{1}{\alpha_i}=\frac{1}{5^k}\Big(\frac{1}{\frac{3}{2}5^{m-k}\phi_-^{(m-k)}(\theta)}-\frac{1}{\Phi(\theta)}\Big).$$

Since $\frac{1}{\frac{3}{2}5^n\phi_-^{(n)}(x)}$ converges to $\frac{1}{\Phi(x)}$ uniformly on $[3,5]$ as n goes to infinity, $\forall\varepsilon>0$, the last expression is dominated by $\frac{\varepsilon}{5^k}$ whenever $m-k$ is greater than some number N. When $m-k\le N$, the same expression is dominated by $\frac{1}{5^mR}$ for $R=\frac{3}{2}\inf_{3\le x\le 5}\phi_-^{(N)}(x)$. The number of α_i's in $\mathcal{L}^k$ is less than $A_{k-1}+\rho_k^\Omega(5)$, so (6.1) is dominated by

$$\sum_{k=3}^{m-N-1}\frac{A_{k-1}+\rho_k^\Omega(5)}{5^k}\varepsilon+\sum_{k=m-N}^m\frac{A_{k-1}+\rho_k^\Omega(5)}{5^mR}\le c_1\varepsilon+c_2\Big(\frac{3}{5}\Big)^m\frac{1}{R}$$

for some constants $c_1,c_2>0$. If m is large enough, (6.1) can be dominated by $(c_1+c_2)\varepsilon$. Hence we have proved $\sum_{\substack{1\le i\le A_m\\ \alpha_{m,i}\ne 6}}\frac{1}{\frac{3}{2}5^m\alpha_{m,i}}-\sum_{i=1}^{D_m}\frac{1}{\alpha_i}$ converges to 0 as m goes to infinity. $\square$

Lemma 6.4. *Let $0<\beta_1<\beta_2<\cdots$ be the elements of $\mathcal{P}_*^+$ in increasing order. Let $\{\beta_{m,i}\}_{1\le i\le B_m}$ be the m-level symmetric primitive eigenvalues of $-\Delta_m$ on Ω_m. Then*

$$\lim_{m\to\infty}\sum_{1\le i\le B_m}\frac{1}{\frac{3}{2}5^m\beta_{m,i}}=\sum_{i=1}^\infty\frac{1}{\beta_i},$$

providing $\sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty$.

Proof. It suffices to prove that

$$\sum_{1 \le i \le B_m} \frac{1}{\frac{3}{2} 5^m \beta_{m,i}} - \sum_{i=1}^{B_m} \frac{1}{\beta_i} \to 0 \ as\ m \to \infty. \tag{6.2}$$

By Lemma 6.2, $\{\beta_1, \beta_2, \cdots, \beta_{B_m}\}$ is an arrangement of elements of $\bigcup_{k=2}^m \mathcal{P}_*^{+,k}$. The first sum of (6.2) can be rearranged so that

$$\lim_{n\to\infty} \frac{3}{2} 5^{m+n} \widetilde{\phi}_-^{(n)}(\beta_{m,i}) = \beta_i, \quad 1 \le i \le B_m.$$

Hence by using Lemma 6.2, (6.2) is equal to $\sum_{k=2}^m \sum_{\beta_i \in \mathcal{P}_*^{+,k}} (\frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \frac{1}{\beta_i})$. The $k = 2$ term converges to 0 as m goes to infinity since $\sharp \mathcal{P}_*^{+,2} = B_2 = 3$.

Hence we only need to prove

$$\sum_{k=3}^{m} \sum_{\beta_i \in \mathcal{P}_*^{+,k}} |\frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \frac{1}{\beta_i}| \to 0 \ as\ m \to \infty. \tag{6.3}$$

If $\beta_i \in \mathcal{P}_*^{+,k}$ $(k = 3, \cdots, m)$, then $\beta_i = 5^k \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\theta)$ for some $\theta \in \mathcal{P}_k^+$ and accordingly the corresponding $\beta_{m,i}$ is of the form $\beta_{m,i} = \widetilde{\phi}_-^{(m-k)}(\theta)$. Hence

$$|\frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \frac{1}{\beta_i}| = \frac{1}{5^k} |\frac{1}{\frac{3}{2}5^{m-k} \widetilde{\phi}_-^{(m-k)}(\theta)} - \frac{1}{\lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\theta)}|. \tag{6.4}$$

From the proof of Lemma 6.2, we have

$$\frac{3}{2} 5^n \phi_-^{(n)}(\phi_-(3)) < \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\theta) < \frac{3}{2} 5^n \phi_-^{(n)}(6).$$

Then by the proof of Lemma 5.1, $\forall \varepsilon > 0$, the right side of formula (6.4) is dominated by $\frac{1}{5^k}\varepsilon$ whenever $m - k$ is greater than some number N. When $m - k \le N$, $\frac{1}{\frac{3}{2}5^m \widetilde{\phi}_-^{(m-k)}(\theta)}$ is dominated by $\frac{1}{5^m R}$ for $R = \frac{3}{2}\phi_-^{(N+1)}(3)$. The number of β_i's in $\mathcal{P}_*^{+,k}$ is controlled by B_k, so the sum (6.3) is dominated by

$$\begin{aligned} & \sum_{k=3}^{m-N-1} \frac{B_k}{5^k}\varepsilon + \sum_{k=m-N}^{m} \frac{B_k}{5^m R} + \sum_{k=m-N}^{m} \sum_{\beta_i \in \mathcal{P}_*^{+,k}} \frac{1}{\beta_i} \\ \le \quad & c_1 \varepsilon + c_2 (\frac{2}{5})^m \frac{1}{R} + \sum_{k=m-N}^{m} \sum_{\beta_i \in \mathcal{P}_*^{+,k}} \frac{1}{\beta_i}. \end{aligned} \tag{6.5}$$

Noticing that $\sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty$, the last term goes to 0 as m goes to infinity. Hence for large m, (6.5) is less than $(c_1 + c_2 + 1)\varepsilon$. Thus we have proved $\sum_{1 \le i \le B_m} \frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \sum_{i=1}^{B_m} \frac{1}{\beta_i}$ converges to 0 as m goes to infinity. □

Lemma 6.5. *Let $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ be the elements of $\widetilde{\mathcal{P}}_*^-$ in increasing order repeated according to their raw multiplicities. Let $\{\gamma_{m,i}\}_{1\leq i\leq C_m}$ be the m-level $\widetilde{\mathcal{P}}_m^-$ type eigenvalues of $-\Delta_m$ on Ω_m including multiplicities. Then*

$$\lim_{m\to\infty} \sum_{1\leq i\leq C_m} \frac{1}{\frac{3}{2}5^m \gamma_{m,i}} = \sum_{i=1}^{\infty} \frac{1}{\gamma_i},$$

providing $\sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty$.

The proof is similar to those of Lemma 6.3 and Lemma 6.4.

Proof of Proposition 6.1.

Let $0 < \widetilde{\kappa}_1 \leq \widetilde{\kappa}_2 \leq \cdots$ be the rearrangement of elements of $\mathcal{S}$ each repeated according to its true multiplicity. Let $\widetilde{v}_1, \widetilde{v}_2, \cdots$ be the associated eigenfunctions. Let $G_\Omega(x,y)$ be the Green's function for Ω. Then $G_\Omega(x,y)$ can be expanded as a uniformly convergence series

$$G_\Omega(x,y) = \sum_{i=1}^{\infty} \frac{\widetilde{v}_i(x)\widetilde{v}_i(y)}{\widetilde{\kappa}_i}, \quad \forall x,y \in \Omega.$$

Since $\mathcal{S}_* \subset \mathcal{S}$ and the raw multiplicity is not greater than the true one, we get that $\sum_{i=1}^{\infty} \frac{1}{\kappa_i} < \infty$. Hence $\sum_{i=1}^{\infty} \frac{1}{\alpha_i} < \infty$, $\sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty$, and $\sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty$. The by adding up the results in Lemma 6.3, Lemma 6.4 and Lemma 6.5, we have

$$\lim_{m\to\infty} \sum_{1\leq i\leq a_m} \frac{1}{\frac{3}{2}5^m \kappa_{m,i}} = \sum_{i=1}^{\infty} \frac{1}{\kappa_i} < \infty.\square$$

Proof of Theorem 3.8.

Based on Proposition 6.1, following a similar argument in [10], we finally get $\mathcal{S}_* = \mathcal{S}$, hence $\mathcal{S} = \mathcal{L} \cup \mathcal{P}_*^+ \cup \widetilde{\mathcal{P}}_*^-$. As a immediate consequence, we have $\mathcal{P}_*^+ = \mathcal{P}^+$, $\mathcal{P}_*^- = \mathcal{P}^-$ and $\mathcal{M}_* = \mathcal{M}$. Thus we have $\mathcal{S} = \mathcal{L} \cup \mathcal{P} \cup \mathcal{M}$ where the union is disjoint. $\square$

Proof of Theorem 3.7.

The proof follows immediately from Lemma 5.3, Theorem 3.6 and Theorem 3.8. $\square$

The skew-symmetric analog of Theorem 3.7 is obvious. We omit it.

Proof of Theorem 3.1.

We only need to prove $\mathcal{P}^+ \cap \mathcal{P}^- = \emptyset$.

Let λ belong to both $\mathcal{P}^+$ and $\mathcal{P}^-$. Then there exist a symmetric primitive eigenfunction u_1 and a skew-symmetric primitive eigenfunction u_2 on Ω, both associated to λ. Let $m \geq 1$. Consider the subdomain $F_1^m(\Omega)$ of Ω, which is a m-times contraction of Ω, with the boundary $F_1^m q_0$ and $F_1^m(L)$. From Lemma 4.12, the restriction of u_1, u_2 to $F_1^m(\Omega)$ should be linear independent. Notice that on the bottom line segment $F_1^m(L)$, both u_1 and u_2 satisfy the Dirichlet condition. Let u be a linear composition of u_1 and u_2 such that u satisfy the Dirichlet boundary condition on the vertex $F_1^m q_0$. Then obviously we have that $-\Delta u = \lambda u$ on $F_1^m(\Omega)$ satisfying the Dirichlet boundary condition on $\partial F_1^m(\Omega)$.

Hence $u \circ F_1^m$ becomes a symmetric primitive Dirichlet eigenfunction on Ω associated to an eigenvalue $\frac{1}{5^m}\lambda$ by using the scaling property of Δ. Thus $\frac{1}{5^m}\lambda \in \mathcal{P}^+$. However, from Lemma 5.3 and Lemma 5.4, we have a smallest positive element in $\mathcal{P}^+$. Hence we get a contradiction by choosing m sufficiently large. $\Box$

Thus we have constructed the complete Dirichlet spectrum of $-\Delta$ on Ω, and the raw multiplicity of each element of $\mathcal{S}$ coincides with its true multiplicity.

Finally we turn to the Weyl's eigenvalue asymptotics on Ω. As before, we use $\rho^0(x)$ and $\rho^\Omega(x)$ to denote the Dirichlet eigenvalue counting function with respect to $\mathcal{SG} \setminus V_0$ and Ω, respectively.

Proof of Theorem 3.10.

We divide $\rho^\Omega(x)$ into four parts $\rho^{\mathcal{L}}(x)$, $\rho^{\mathcal{P}^+}(x)$, $\rho^{\mathcal{P}^-}(x)$ and $\rho^{\mathcal{M}}(x)$ corresponding to different types of eigenvalues. The exact definitions are: $\rho^{\mathcal{L}}(x) = \sharp\{\lambda \in \mathcal{L} : \lambda \leq x\}$, $\rho^{\mathcal{P}^+}(x) = \sharp\{\lambda \in \mathcal{P}^+ : \lambda \leq x\}$, $\rho^{\mathcal{P}^-}(x) = \sharp\{\lambda \in \mathcal{P}^- : \lambda \leq x\}$ and $\rho^{\mathcal{M}}(x) = \sharp\{\lambda \in \mathcal{M} : \lambda \leq x\}$. Obviously,

$$\rho^\Omega(x) - \rho^{\mathcal{L}}(x) = \rho^{\mathcal{P}^+}(x) + \rho^{\mathcal{P}^-}(x) + \rho^{\mathcal{M}}(x).$$

For $\rho^{\mathcal{P}^+}(x)$, denote β_m^* the largest eigenvalue in $\mathcal{P}_m^+$, and $\beta^{(m)}$ the eigenvalue in $\mathcal{P}^+$ corresponding to the sequence $\{\widetilde{\phi}_-^{(n)}(\beta_m^*)\}_{n\geq 0}$, i.e., $\beta^{(m)} = \lim_{n\to\infty} \frac{3}{2} 5^{n+m} \widetilde{\phi}_-^{(n)}(\beta_m^*)$. By using Lemma 5.2, it is easy to check that

$$c_1 5^m = \lim_{n\to\infty} \frac{3}{2} 5^{n+m} \phi_-^{(n)}(2) \leq \lim_{n\to\infty} \frac{3}{2} 5^{n+m} \phi_-^{(n)}(\beta_m^{**}) \leq \beta^{(m)} \leq \lim_{n\to\infty} \frac{3}{2} 5^{n+m} \phi_-^{(n)}(6) = c_2 5^m, \tag{6.6}$$

for appropriate constants $c_1, c_2 > 0$, where β_m^{**} denote the largest eigenvalue in $\mathcal{P}_m^+$ except β_m^*.

Notice that the bottom r_m eigenvalues in $\mathcal{P}^+$ are generated from eigenvalues in $\mathcal{P}_m^+$ by extending these eigenvalues by choosing the $\widetilde{\phi}_-$ relation for all $m' > m$. Hence we get

$$\rho^{\mathcal{P}^+}(\beta^{(m)}) = r_m, \quad \forall m \geq 2.$$

Using (6.6), we get $\rho^{\mathcal{P}^+}(c_1 5^m) \leq r_m$, and $\rho^{\mathcal{P}^+}(c_2 5^m) \geq r_m$.

Denote by k_0 the least number such that $5^{k_0} c_1 \geq c_2$. $\forall x \geq 25 c_2$, choose a number m such that $c_2 5^m \leq x < c_2 5^{m+1}$. Then $c_2 5^m \leq x < c_1 5^{m+k_0+1}$. Hence

$$c_3 x^{\log 2/\log 5} \leq r_m \leq \rho^{\mathcal{P}^+}(c_2 5^m) \leq \rho^{\mathcal{P}^+}(x) \leq \rho^{\mathcal{P}^+}(c_1 5^{m+k_0+1}) \leq r_{m+k_0+1} \leq c_4 x^{\log 2/\log 5},$$

for appropriate constants $c_3, c_4 > 0$. Thus we have proved that for x large enough,

$$c_3 x^{\log 2/\log 5} \leq \rho^{\mathcal{P}^+}(x) \leq c_4 x^{\log 2/\log 5}.$$

A similar argument yields that for x large enough,

$$c_5 x^{\log 2/\log 5} \leq \rho^{\mathcal{P}^-}(x) \leq c_6 x^{\log 2/\log 5},$$

for appropriate constants $c_5, c_6 > 0$.

Now we consider $\rho^{\mathcal{M}}(x)$. Notice that for each $\lambda' \in \{\lambda \in \mathcal{M} : \lambda \leq x\}$, there exists a $k \geq 1$, such that λ' has multiplicity 2^k in $\mathcal{M}$, and $\frac{1}{5^k}\lambda' \in \{\lambda \in \mathcal{P}^- : \lambda \leq \frac{x}{5^k}\}$. Hence

$$\rho^{\mathcal{M}}(x) \leq \sum_k 2^k \rho^{\mathcal{P}^-}(\frac{x}{5^k}).$$

Denote λ_* the least eigenvalue in $\mathcal{P}^-$. Then

$$\rho^{\mathcal{M}}(x) \leq \sum_{k=1}^{[\log(x/\lambda_*)/\log 5]} 2^k \rho^{\mathcal{P}^-}(\frac{x}{5^k}) \leq c_6 \sum_{k=1}^{[\log(x/\lambda_*)/\log 5]} 2^k (\frac{x}{5^k})^{\log 2/\log 5} \leq c_7 (\log x) x^{\log 2/\log 5},$$

for an appropriate constant $c_7 > 0$.

Taking the above estimates into account, we get

$$\rho^{\Omega}(x) - \rho^{\mathcal{L}}(x) = O(x^{\log 2/\log 5} \log x) \quad \text{as } x \to \infty. \tag{6.7}$$

On the other hand, by using the consequences of the usual spectral decimation for the Dirichlet Laplacian on $\mathcal{SG} \setminus V_0$, we will also prove that

$$\rho^{0}(x) - \rho^{\mathcal{L}}(x) = O(x^{\log 2/\log 5} \log x) \quad \text{as } x \to \infty. \tag{6.8}$$

(a similar result can be found in Kigami [19].)

In fact, as showed before, $\mathcal{L}$ is also a subset of the Dirichlet spectrum $\mathcal{D}$ of $-\Delta$ on $\mathcal{SG} \setminus V_0$. Besides $\mathcal{L}$, there remain some 2-series, 5-series and 6-series eigenvalues in $\mathcal{D}$, whose associated eigenfunction having support touching the line segment L. We denote them by $\mathcal{R}^2$, $\mathcal{R}^5$ and $\mathcal{R}^6$ respectively. Thus we have

$$\rho^{0}(x) - \rho^{\mathcal{L}}(x) = \rho^{\mathcal{R}^2}(x) + \rho^{\mathcal{R}^5}(x) + \rho^{\mathcal{R}^6}(x),$$

where $\rho^{\mathcal{R}^2}(x), \rho^{\mathcal{R}^5}(x), \rho^{\mathcal{R}^6}(x)$ are the eigenvalue counting functions of the associated type eigenvalues.

For $\mathcal{R}^5$, we use $\mathcal{R}^5_m$ to denote the associated total m-level graph eigenvalues. Then it is easy to verify that $\mathcal{R}^5_m$ consists of $1 + 2^{m-1}$ initial eigenvalues and $(m+1)2^{m-1} - 2$ continued eigenvalues. Notice that the bottom $\sharp\mathcal{R}^5_m$ eigenvalues in $\mathcal{R}^5$ are generated from eigenvalues in $\mathcal{R}^5_m$ by extending these eigenvalues by choosing ϕ_- relations for all $m' > m$. Since 5 is the largest eigenvalue in $\mathcal{R}^5_m$, we get

$$\rho^{\mathcal{R}^5}(c_8 5^m) = \sharp\mathcal{R}^5_m = (1 + 2^{m-1}) + ((m+1)2^{m-1} - 2) = (m+2)2^{m-1} - 1,$$

for appropriate constant $c_8 > 0$. Similar to the analysis on $\rho^{\mathcal{P}^+}(x)$, we then get

$$\rho^{\mathcal{R}^5}(x) = O(x^{\log 2/\log 5} \log x) \quad \text{as } x \to \infty.$$

Following a similar argument, we also have

$$\rho^{\mathcal{R}^2}(x) = O(x^{\log 2/\log 5}) \quad \text{as } x \to \infty,$$

and

$$\rho^{\mathcal{R}^6}(x) = O(x^{\log 2/\log 5}\log x) \quad \text{as } x \to \infty.$$

Taking these estimates into account, we then get (6.8).

Thus Theorem 3.10 follows from (6.7) and (6.8). $\square$

7 The Neumann case

In this section, we give a brief discussion on the Neumann spectrum of $-\Delta$ on Ω. Throughout this section, for simplicity, we omit the terms "graph" and "Neumann" without causing any confusion. The main object in this section is to prove Theorem 3.12 and Theorem 3.14. We will also give a comment on how to modify the proof of Theorem 3.6 suitably to prove its Neumann counterpart, Theorem 3.15, at the end of this section.

As indicated in Section 3, we want to impose a Neumann condition on the graph Ω_m by imagining that it is embedded in a larger graph by reflecting in each boundary vertex and imposing the λ_m-eigenvalue equation on the even extension of u_m. It is convenient to allow $m = 1$, in which case there are only three boundary points in Ω_1 and no others. As introduced in Section 3, $\mathcal{P}_m^N$ denotes the totality of primitive Neumann eigenvalues of the discrete Laplacian $-\Delta_m$ on Ω_m. Due to the eigenspace dimensional counting argument in Section 3, this time we need to find 2^m symmetric primitive eigenvalues and $2^m - 1$ skew-symmetric primitive eigenvalues.

We focus our discussion on $\mathcal{P}_m^{+,N}$, the symmetric case, and describe a similar weak spectral decimation which relates $\mathcal{P}_m^{+,N}$ with $\mathcal{P}_{m+1}^{+,N}$. Let u_m be a λ_m-eigenfunction of $-\Delta_m$ on Ω_m with $\lambda_m \in \mathcal{P}_m^{+,N}$. Still denote by $(b_0, b_1, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m. Write $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $1 \le i \le m$. (This time we begin with $\lambda_1^{(m)}$.) Assume that none of $\lambda_i^{(m)}$'s is equal to 2 or 5 for $2 \le i \le m$. Then u_m is uniquely determined by $(b_0, b_1, \cdots, b_m)$. In addition to the eigenvalue equations at the vertex $F_1q_0, F_1^2q_0, \cdots, F_1^{m-1}q_0$ as described in Section 4, we impose the equations

$$(4 - \lambda_1^{(m)})b_0 = 4b_1 \tag{7.1}$$

at q_0 and

$$(4 - \lambda_m)b_m = 2b_{m-1} + 2b_m \tag{7.2}$$

at $F_1^m q_0$ according to the Neumann boundary condition. Hence $(b_0, b_1, \cdots, b_m)$ can be viewed as a non-zero vector solution of a system of equations consisting of $m+1$ equations in $m+1$ unknowns, whose determinant is

$$\begin{vmatrix} 4-\lambda_1^{(m)} & -4 & & & \\ l(\lambda_2^{(m)}) & s(\lambda_2^{(m)}) & r(\lambda_2^{(m)}) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_m^{(m)}) & s(\lambda_m^{(m)}) & r(\lambda_m^{(m)}) \\ & & & -2 & 2-\lambda_m^{(m)} \end{vmatrix}.$$

Hence λ_m should be a solution of the following equation

$$q_m^N(x) := \begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(x) & s(x) & r(x) \\ & & & -2 & 2-x \end{vmatrix} = 0. \tag{7.3}$$

Thus if λ_m is a root of $q_m^N(x)$ and none of $f^{(i)}(\lambda_m)$'s with $0 \le i \le m-2$ is equal to 2 or 5, then $\lambda_m \in \mathcal{P}_m^{+,N}$. We should mention here that when $m \ge 2$, comparing to $q_m(x)$ in the Dirichlet case, $q_m^N(x)$ is a $(m+1)\times(m+1)$ tridiagonal determinant, containing $q_m(x)$ in the center as a $(m-1)\times(m-1)$ minor. Namely, we can write

$$q_m^N(x) = \begin{vmatrix} 4-f^{(m-1)}(x) & -4 & 0 & \cdots & 0 & 0 \\ l(f^{(m-2)}(x)) & & & & & 0 \\ \vdots & & q_m(x) & & & \vdots \\ 0 & & & & & r(x) \\ 0 & 0 & \cdots & 0 & -2 & 2-x \end{vmatrix}.$$

The degree of $q_m^N(x)$ is $3(2^{m-1}-1)+2^{m-1}+1 = 2^{m+1}-2$, since the degree of $q_m(x)$ is $3(2^{m-1}-1)$. The analysis on $q_m^N(x)$ is more complicated than that on $q_m(x)$ since for $q_m(x)$ we can always use the expansion of $q_m(x)$ along the first or last row to get a relation between two polynomials in same type but with smaller degree.

The following lemma is a slight modification of the form of $q_m^N(x)$ from an $(m+1)\times(m+1)$ determinant to an $m\times m$ determinant.

Lemma 7.1. *Let $m \ge 2$. Then*

$$q_m^N(x) = (2-x)(x-6)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(f(x)) & s(f(x)) & r(f(x)) \\ & & & 1 & f(x)-1 \end{vmatrix}.$$

Proof. Substituting the expression for $r(x)$ into (7.3), we get

$$
\begin{aligned}
q_m^N(x) &= (2-x)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(x) & s(x) & -2(5-x) \\ & & & -2 & 1 \end{vmatrix} \\
&= (2-x)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(x) & s(x)-4(5-x) & -2(5-x) \\ & & & 0 & 1 \end{vmatrix} \\
&= (2-x)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & \\ & \ddots & \ddots & \ddots \\ & & l(x) & s(x)-4(5-x) \end{vmatrix}.
\end{aligned}
$$

Noticing that $s(x)-4(5-x)=(x-6)(f(x)-1)$ and $l(x)=x-6$, we get the desired result. □

The following lemma focuses on the possibility of the roots of $q_m^N(x)$ satisfying $f^{(i)}(x)=2$ or 5 for some $0\le i\le m-2$.

Lemma 7.2. *Let $m\ge 2$, and x be a predecessor of 2 or 5 of order i with $0\le i\le m-2$. Then $q_m^N(x)=0$.*

Proof. First, let x be a predecessor of 2 of order i with $0\le i\le m-2$. Then $f^{(i)}(x)=2$ and $f^{(i+1)}(x)=6$. If $0\le i<m-2$, the proof is the same as that of Lemma 4.2. So we only need to check the $i=m-2$ case. In this case we have

$$
q_m^N(x)=\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & \\ & \ddots & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} -2 & -4 & \\ -4 & -8 & 0 \\ & \ddots & \ddots & \ddots \end{vmatrix} = 0.
$$

Second, let x be a predecessor of 5 of order i with $0\le i\le m-2$. Then $f^{(i)}(x)=5$

and $f^{(i+1)}(x) = \cdots f^{(m-1)}(x) = 0$. Hence we have

$$q_m^N(x) = \begin{vmatrix} 4 & -4 & & & & & \\ l(0) & s(0) & r(0) & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & l(0) & s(0) & r(0) & & \\ & & & l(5) & s(5) & r(5) & \\ & & & & \ddots & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} 4 & -4 & & & & & \\ -6 & 26 & -20 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -6 & 26 & -20 & & \\ & & & -1 & 1 & 0 & \\ & & & & \ddots & \ddots & \ddots \end{vmatrix} = 0.$$

Thus we always have $q_m^N(x) = 0$. $\square$

This lemma means that for $m \geq 2$, all the predecessors of 2 or 5 of order i with $0 \leq i \leq m-2$ are unwanted roots of $q_m^N(x)$. To eliminate them, we define

$$p_m^N(x) := \frac{q_m^N(x)}{(x-2)(x-5)\cdots(f^{(m-2)}(x)-2)(f^{(m-2)}(x)-5)} \quad \text{for } m \geq 2,$$

and

$$p_1^N(x) := q_1^N(x).$$

Now we can say if λ_m is a root of the polynomial $p_m^N(x)$, then $\lambda_m \in \mathcal{P}_m^{+,N}$. It is easy to check that the degree of $p_m^N(x)$ is 2^m, since the degree of $q_m^N(x)$ is $2^{m+1}-2$ and the number of all the unwanted roots of $q_m^N(x)$ is $2(1+2+\cdots 2^{m-2}) = 2^m - 2$ for $m \geq 2$ and 0 for $m = 1$. The following is an easy observation on $p_m^N(x)$.

Lemma 7.3. *For $m \geq 1$, $p_m^N(x)$ always has roots 0 and 6.*

Proof. We only need to check $q_m^N(0) = q_m^N(6) = 0$. It is easy to see that

$$q_m^N(0) = \begin{vmatrix} 4 & -4 & & & \\ l(0) & s(0) & r(0) & & \\ & \ddots & \ddots & \ddots & \\ & & l(0) & s(0) & r(0) \\ & & & -2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -4 & & & \\ -6 & 26 & -20 & & \\ & \ddots & \ddots & \ddots & \\ & & -6 & 26 & -20 \\ & & & -2 & 2 \end{vmatrix} = 0.$$

$q_m^N(6) = 0$ follows from Lemma 7.1 for $m \geq 2$, and from direct computation for $m = 1$.$\square$

In order to study the distribution of roots of $p_m^N(x)$, we now introduce auxiliary polynomials $l_m(x)$ associated to $p_m^N(x)$. First, $\forall m \geq 1$, let $\widetilde{l}_m(x)$ denote the $m \times m$ minor located in the upper left corner of $q_m^N(x)$. Namely, $\widetilde{l}_1(x) := 4 - x$ and for $m \geq 2$,

$$\widetilde{l}_m(x) := \begin{vmatrix} 4 - f^{(m-1)}(x) & -4 & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & \\ & \ddots & \ddots & \ddots \\ & & l(x) & s(x) \end{vmatrix}.$$

Note that the $(m-1)\times(m-1)$ minor located in the bottom right corner of $\widetilde{l}_m(x)$ is $q_m(x)$. The degree of $\widetilde{l}_m(x)$ is $2^{m+1}-3$ since it is reduced by 1 compared to the degree of $q_m^N(x)$. With a similar argument as in the proof of Lemma 7.2, all the predecessors of 2 or 5 of order i with $0\le i\le m-2$ are roots of $\widetilde{l}_m(x)$. To eliminate them, we define

$$l_m(x) := \frac{\widetilde{l}_m(x)}{(x-2)(x-5)\cdots(f^{(m-2)}(x)-2)(f^{(m-2)}(x)-5)} \qquad \text{for } m\ge 2,$$

and

$$l_1(x) := \widetilde{l}_1(x).$$

It is easy to check that the degree of $l_m(x)$ is 2^m-1, since the degree of $\widetilde{l}_m(x)$ is $2^{m+1}-3$ and the number of all the unwanted roots of $\widetilde{l}_m(x)$ is $2(1+2+\cdots 2^{m-2}) = 2^m-2$ for $m\ge 2$ and 0 for $m=1$.

Based on the property

$$\widetilde{l}_m(x) = s(x)\widetilde{l}_{m-1}(f(x)) - r(f(x))l(x)\widetilde{l}_{m-2}(f^{(2)}(x)),$$

$l_m(x)$ can be analyzed in a similar way to $p_m(x)$ or $\widetilde{p}_m(x)$ in the Dirichlet case. We then have:

Lemma 7.4. *$l_m(0)>0$ and $l_m(6)<0$, $\forall m\ge 1$.*

Proof. $l_m(0)>0$ follows from a similar argument as in the proof of Proposition 4.1 and Proposition 4.2.

To prove $l_m(6)<0$, we only need to prove $\widetilde{l}_m(6)<0$ by the definition of $l_m(x)$. It can be checked that $\widetilde{l}_1(6)=-2<0$ and $\widetilde{l}_2(6)=-40<0$ by a direct computation. For $m\ge 3$, an expansion of $\widetilde{l}_m(6)$ along the first row yields

$$\widetilde{l}_m(6) = (4-f^{(m-1)}(6))q_m(6) + 4(f^{(m-2)}(6)-6)q_{m-1}(6).$$

Recall that in the proof of Proposition 4.1(3), we have proved that $q_m(6)\le q_{m-1}(6)<0$, $\forall m\ge 3$. Hence

$$\widetilde{l}_m(6) \le (4-f^{(m-1)}(6)+4f^{(m-2)}(6)-24)q_m(6) = (f^{(m-2)}(6)-5)(f^{(m-2)}(6)+4)q_m(6) < 0,$$

because $f^{(m-2)}(6)\le -6$ whenever $m\ge 3$. □

Lemma 7.5. *For each $m\ge 1$, $l_m(x)$ has 2^m-1 distinct real roots between 0 and 6 satisfying*

$$0<\beta_{m,1}<\beta_{m,2}<\cdots<\beta_{m,2^m-1}<6.$$

Moreover,

$$\begin{aligned}
0 &< \beta_{m+1,1} < \phi_-(\beta_{m,1}), \\
\phi_-(\beta_{m,k-1}) &< \beta_{m+1,k} < \phi_-(\beta_{m,k}), \;\; \textit{if } 2 \le k \le 2^m - 1, \\
\phi_-(\beta_{m,2^m-1}) &< \beta_{m+1,2^m} < \phi_+(\beta_{m,2^m-1}), \\
\phi_+(\beta_{m,2^{m+1}-k}) &< \beta_{m+1,k} < \phi_+(\beta_{m,2^{m+1}-k-1}), \;\; \textit{if } 2^m + 1 \le k \le 2^{m+1} - 2, \\
\phi_+(\beta_{m,1}) &< \beta_{m+1,2^{m+1}-1} < 6.
\end{aligned}$$

Proof. It follows from a similar argument as in the proof of Lemma 4.4. □

The following lemma shows a relation between $p_m^N(x)$'s and $l_m(x)$'s.

Lemma 7.6. *Let $m \ge 2$. Then $p_m^N(x) = (2-x)l_m(x) - 4l_{m-1}(f(x))$.*

Proof. This is easy to get since we have

$$q_m^N(x) = (2-x)\widetilde{l}_m(x) - 4(2-x)(5-x)\widetilde{l}_{m-1}(f(x)), \;\; \textit{for } m \ge 2,$$

using the expansion along the last row of $q_m^N(x)$. □

Now we consider the distribution of roots of $p_m^N(x)$.

Lemma 7.7. *For each $m \ge 1$, $p_m^N(x)$ has 2^m distinct roots between 0 and 6 (including 0 and 6). Moreover, $p_m^N(0+) < 0$ and $p_m^N(6-) < 0$.*

Proof. When $m = 1$, it is obvious.

Let $m \ge 2$. From Lemma 7.5, $l_m(x)$ has $2^m - 1$ distinct real roots between 0 and 6 satisfying

$$0 < \beta_{m,1} < \beta_{m,2} < \cdots < \beta_{m,2^m-1} < 6.$$

For each $1 \le k \le 2^m - 1$, using Lemma 7.6, we have

$$p_m^N(\beta_{m,k}) = (2 - \beta_{m,k})l_m(\beta_{m,k}) - 4l_{m-1}(f(\beta_{m,k})) = -4l_{m-1}(f(\beta_{m,k})).$$

As in the proof of Lemma 4.4, we use the notation "$A \sim B$" to indicate that both the numbers A, B have the same signs.

When $k = 1$, by Lemma 7.5, $0 < \beta_{m,1} < \phi_-(\beta_{m-1,1})$, hence $0 < f(\beta_{m,1}) < \beta_{m-1,1}$. Combined with $l_{m-1}(0) > 0$ from Lemma 7.4, it follows that $l_{m-1}(f(\beta_{m,1})) > 0$, hence $p_m^N(\beta_{m,1}) < 0$.

When $2 \le k \le 2^{m-1} - 1$, it follows from Lemma 7.5 that we have $\phi_-(\beta_{m-1,k-1}) < \beta_{m,k} < \phi_-(\beta_{m-1,k})$, hence $\beta_{m-1,k-1} < f(\beta_{m,k}) < \beta_{m-1,k}$. Combined with $l_{m-1}(0) > 0$, it follows that $l_{m-1}(f(\beta_{m,k})) \sim (-1)^{k-1}$, hence $p_m^N(\beta_{m,k}) \sim (-1)^k$.

When $k = 2^{m-1}$, it follows from Lemma 7.5 that we have $\phi_-(\beta_{m-1,2^{m-1}-1}) < \beta_{m,2^{m-1}} < \phi_+(\beta_{m-1,2^{m-1}-1})$, hence $f(\beta_{m,2^{m-1}}) > \beta_{m-1,2^{m-1}-1}$. Combined with $l_{m-1}(0) > 0$, it follows that $l_{m-1}(f(\beta_{m,2^{m-1}})) < 0$, hence $p_m^N(\beta_{m,2^{m-1}}) > 0$.

When $2^{m-1}+1 \leq k \leq 2^m-2$, it follows from Lemma 7.5 that we have $\phi_+(\beta_{m-1,2^m-k}) < \beta_{m,k} < \phi_+(\beta_{m-1,2^m-k-1})$, hence $\beta_{m-1,2^m-k-1} < f(\beta_{m,k}) < \beta_{m-1,2^m-k}$. Combined with $l_{m-1}(0) > 0$, it follows that $l_{m-1}(f(\beta_{m,k})) \sim (-1)^{k-1}$, hence $p_m^N(\beta_{m,k}) \sim (-1)^k$.

When $k = 2^m - 1$, it follows from Lemma 7.5 that we have $\phi_+(\beta_{m-1,1}) < \beta_{m,2^m-1} < 6$, hence $f(\beta_{m,2^m-1}) < \beta_{m-1,1}$. Combined with $l_{m-1}(0) > 0$, it follows that $l_{m-1}(f(\beta_{m,2^m-1})) > 0$, hence $p_m^N(\beta_{m,2^m-1}) < 0$.

Hence we have proved $p_m^N(\beta_{m,k}) \sim (-1)^k$, for $1 \leq k \leq 2^m - 1$. So there exist at least 2^m-2 roots of $p_m^N(x)$, each located strictly between each two consecutive $\beta_{m,k}$'s. Moreover, Lemma 7.3 says that 0 and 6 are also roots of $p_m^N(x)$. Thus we have found 2^m distinct roots of $p_m^N(x)$. Since the order of $p_m^N(x)$ is also 2^m, these are the total roots of $p_m^N(x)$.

Furthermore, from the fact that $p_m^N(\beta_{m,1}) < 0$ and $p_m^N(\beta_{m,2^m-1}) < 0$, we have $p_m^N(0+) < 0$ and $p_m^N(6-) < 0$. □

In all that follows, we denote

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < \lambda_{m,2^m} = 6$$

the 2^m distinct roots of $p_m^N(x)$ in increasing order, $\forall m \geq 1$. In order to study the relation of roots of two consecutive $p_m^N(x)$'s, we prove the following two lemmas:

Lemma 7.8. *Let $m \geq 1$ and $2 \leq k \leq 2^m$, then*

$$p_{m+1}^N(\phi_-(\lambda_{m,k})) = -\frac{\lambda_{m,k}}{2}\frac{\phi_-(\lambda_{m,k})-6}{\phi_-(\lambda_{m,k})-5} \cdot l_m(\lambda_{m,k}),$$

and

$$p_{m+1}^N(\phi_+(\lambda_{m,k})) = -\frac{\lambda_{m,k}}{2}\frac{\phi_+(\lambda_{m,k})-6}{\phi_+(\lambda_{m,k})-5} \cdot l_m(\lambda_{m,k}).$$

Proof. For simplicity we only prove the first equality. The second will follow from a similar argument. It is easy to see that $\lambda_{m,k}$ is also a root of $q_m^N(x)$ and none of $f^{(i)}(\lambda_{m,k})$'s $(0 \leq i \leq m-2)$ is equal to 2 or 5.

By Lemma 7.1, $q_{m+1}^N(\phi_-(\lambda_{m,k})) = (2 - \phi_-(\lambda_{m,k}))(\phi_-(\lambda_{m,k}) - 6) \cdot A$ where

$$A = \begin{vmatrix} 4 - f^{(m-1)}(\lambda_{m,k}) & -4 & & & \\ l(f^{(m-2)}(\lambda_{m,k})) & s(f^{(m-2)}(\lambda_{m,k})) & r(f^{(m-2)}(\lambda_{m,k})) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_{m,k}) & s(\lambda_{m,k}) & r(\lambda_{m,k}) \\ & & & 1 & \lambda_{m,k}-1 \end{vmatrix}.$$

Noticing that from $q_m^N(\lambda_{m,k})=0$, we have

$$\begin{vmatrix} 4-f^{(m-1)}(\lambda_{m,k}) & -4 & & & & \\ l(f^{(m-2)}(\lambda_{m,k})) & s(f^{(m-2)}(\lambda_{m,k})) & r(f^{(m-2)}(\lambda_{m,k})) & & & \\ & \ddots & \ddots & \ddots & & \\ & & & l(\lambda_{m,k}) & s(\lambda_{m,k}) & r(\lambda_{m,k}) \\ & & & & -1 & 1-\lambda_{m,k}/2 \end{vmatrix} = 0.$$

The sum of the above two determinants yields that $A=\frac{\lambda_{m,k}}{2}\widetilde{l}_m(\lambda_{m,k})$. Hence

$$q_{m+1}^N(\phi_-(\lambda_{m,k}))=\frac{\lambda_{m,k}}{2}(2-\phi_-(\lambda_{m,k}))(\phi_-(\lambda_{m,k})-6)\cdot\widetilde{l}_m(\lambda_{m,k}),$$

which yields the desired result. □

Lemma 7.9. *Let $m\geq 1$. Then $(-1)^{k-1}l_m(\lambda_{m,k})>0$, for $1\leq k\leq 2^m$.*

Proof. Let $\beta_{m,1},\beta_{m,2},\cdots\beta_{m,2^m-1}$ denote the 2^m-1 distinct roots of $l_m(x)$ in increasing order as described in Lemma 7.5. Then by the proof of Lemma 7.7, we have

$$\lambda_{m,1}=0<\beta_{m,1}<\lambda_{m,2}<\beta_{m,2}<\cdots<\lambda_{m,2^m-1}<\beta_{m,2^m-1}<\lambda_{m,2^m}=6.$$

Combined with the fact that $l_m(\lambda_{m,1})=l_m(0)>0$ by Lemma 7.4, the desired result follows. □

Now we can prove the following lemma.

Lemma 7.10. *For each $m\geq 1$, $\mathcal{P}_m^{+,N}$ consists of at least 2^m distinct eigenvalues satisfying*

$$\lambda_{m,1}=0<\lambda_{m,2}<\cdots<\lambda_{m,2^m-1}<\lambda_{m,2^m}=6.$$

Moreover,

$$\begin{aligned} \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k}<\phi_-(\lambda_{m,k}),\ \textit{for } 2\leq k\leq 2^m, \\ \phi_+(\lambda_{m,2^{m+1}-k+1}) &< \lambda_{m+1,k}<\phi_+(\lambda_{m,2^{m+1}-k}),\ \textit{for } 2^m+1\leq k\leq 2^{m+1}-2, \\ \phi_+(\lambda_{m,2}) &< \lambda_{m+1,2^{m+1}-1}<6. \end{aligned} \tag{7.4}$$

Proof. Since each root of $p_m^N(x)$ belongs to $\mathcal{P}_m^{+,N}$, we only need to prove the results for the roots of $p_m^N(x)$. The first statement follows from Lemma 7.7. We now prove the second statement. From Lemma 7.8 and Lemma 7.9, we have

$$p_{m+1}^N(\phi_-(\lambda_{m,k}))=-\frac{\lambda_{m,k}}{2}\frac{\phi_-(\lambda_{m,k})-5}{\phi_-(\lambda_{m,k})-6}\cdot l_m(\lambda_{m,k})\sim -l_m(\lambda_{m,k})\sim(-1)^k,\quad \forall 2\leq k\leq 2^m,$$

and similarly,

$$p_{m+1}^N(\phi_+(\lambda_{m,k}))=-\frac{\lambda_{m,k}}{2}\frac{\phi_+(\lambda_{m,k})-5}{\phi_+(\lambda_{m,k})-6}\cdot l_m(\lambda_{m,k})\sim -l_m(\lambda_{m,k})\sim(-1)^k,\quad \forall 2\leq k\leq 2^m.$$

Following the above facts and Lemma 7.7, we can list the signs of the values of $p^N_{m+1}(x)$ at different point x in the following table.

$x:$	0	$0+$	$\phi_-(\lambda_{m,2})$	$\phi_-(\lambda_{m,3})$	$\cdots$	$\phi_-(\lambda_{m,2^m})$	$\phi_+(\lambda_{m,2^m})$	$\cdots$	$\phi_+(\lambda_{m,3})$	$\phi_+(\lambda_{m,2})$	$6-$	6
$p^N_{m+1}(x):$	0	$-$	$+$	$-$	$\cdots$	$+$	$+$	$\cdots$	$-$	$+$	$-$	0

Hence there exist at least 2^{m+1} distinct roots of $p^N_{m+1}(x)$ satisfying (7.4). Moreover, these are the totality of the roots of $p^N_{m+1}(x)$ since the degree of $p^N_{m+1}(x)$ is also 2^{m+1}. Hence we get the desired distribution of roots of $p^N_{m+1}(x)$. $\square$

The estimate $\phi_+(\lambda_{m,2}) < \lambda_{m+1,2^{m+1}-1} < 6$ in Lemma 7.10 can be refined into

$$\phi_+(\lambda_{m,2}) < \lambda_{m+1,2^{m+1}-1} < 5 \tag{7.5}$$

by using the following lemma.

Lemma 7.11. *For $m \geq 2$, let $\lambda_{m,1} = 0, \lambda_{m,2}, \cdots, \lambda_{m,2^m-1}, \lambda_{m,2^m} = 6$ be the 2^m distinct roots of $p^N_m(x)$ in increasing order. Then*

$$\lambda_{m,k} + \lambda_{m,2^m-k+1} = 5, \text{ for } 2 \leq k \leq 2^m - 1.$$

Proof. From Lemma 7.1, it is easy to see that if $q^N_m(x) = 0$ and $x \neq 2$ or 6, then $q^N_m(5-x) = 0$. Obviously, each $\lambda_{m,k}$ $(2 \leq k \leq 2^m - 1)$ satisfies this property. $\square$

Hence we have the following Neumann analog of Lemma 4.5.

Lemma 7.12. *For each $m \geq 1$, $\mathcal{P}^{+,N}_m$ consists of at least 2^m distinct eigenvalues satisfying*

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < 5 < \lambda_{m,2^m} = 6.$$

Moreover,

$$\begin{aligned} \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \text{ for } 2 \leq k \leq 2^m, \\ \phi_+(\lambda_{m,2^{m+1}-k+1}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2^{m+1}-k}), \text{ for } 2^m + 1 \leq k \leq 2^{m+1} - 1. \end{aligned} \tag{7.6}$$

Proof. This follows from Lemma 7.10 and Lemma 7.11. $\square$

The following is a Neumann analog of Lemma 4.6.

Lemma 7.13. *Let λ_m be a root of $p^N_m(x)$, u_m a primitive λ_m-eigenfunction on Ω_m, and $(b_0, b_1, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m. Then $b_0 \neq 0$ and $b_m \neq 0$.*

Proof. Without loss of generality, assume $m \geq 3$. We still use $\lambda^{(m)}_i$ to denote the successor of λ_m of order $(m-i)$ with $1 \leq i \leq m$. From the definition of $p^N_m(x)$, $\lambda^{(m)}_i \neq 2$ or 5, for each $2 \leq i \leq m$. The vector $(b_0, b_1, \cdots, b_m)$ can be viewed as a non-zero vector solution of system (4.2) of equations and in addition the two Neumann boundary eigenvalue equations (7.1) and (7.2).

Suppose $b_m = 0$. Then from (7.2), $b_{m-1} = 0$. It is easy to check that the determinant of the remaining equations in $m-1$ unknowns $(b_0, b_1, \cdots, b_{m-2})$ is $\widetilde{l}_{m-1}(\lambda_{m-1}^{(m)})$. Since $(b_0, b_1, \cdots, b_{m-2})$ should be a non-zero vector, we have $\widetilde{l}_{m-1}(\lambda_{m-1}^{(m)}) = 0$, hence $l_{m-1}(\lambda_{m-1}^{(m)}) = 0$. Note that Lemma 7.6 implies $p_m^N(\lambda_m) = (2-\lambda_m)l_m(\lambda_m) - 4l_{m-1}(\lambda_{m-1}^{(m)})$. Hence we get that $l_m(\lambda_m) = 0$ since both $l_{m-1}(\lambda_{m-1}^{(m)})$ and $p_m^N(\lambda_m)$ are equal to 0. But this is impossible, since Lemma 7.5 says that if $l_{m-1}(\lambda_{m-1}^{(m)}) = 0$ then $l_m(\lambda_m)$ could not equal to 0. Hence $b_m \neq 0$.

On the other hand, if $b_0 = 0$, then by substituting it into the system, noticing that none of $\lambda_i^{(m)}$'s is equal to 2 or 5, we can get $b_1 = 0, \cdots, b_m = 0$ successively, which contradicts to $b_m \neq 0$. Hence $b_0 \neq 0$. $\Box$

This is the whole story of the symmetric case. The skew-symmetric case is slightly different but very similar. The result is shown in Section 3, but the proof is omitted.

Proof of Theorem 3.12 and Theorem 3.14.

The results follows by using Lemma 7.12, Lemma 7.13 and their skew-symmetric analogs, and the eigenspace dimension counting formula (3.3), following a similar argument as in the Dirichlet case. $\Box$

We should remark that Lemma 7.13 and its skew-symmetric analog show that there is no primitive eigenfunction (or miniaturized eigenfunction) that is simultaneously Dirichlet and Neumann $(D-N)$. Hence the only possible $D-N$ eigenfunctions are localized eigenfunctions. This is same as the $\mathcal{SG} \setminus V_0$ case.

Before closing this section, we will make a comment on how to prove Theorem 3.15, by a suitable modification of the proof of Theorem 3.6.

Proof of Theorem 3.15.

Let $\{\lambda_m\}_{m \geq m_0}$ be a sequence of symmetric primitive graph Neumann eigenvalues related by $\widetilde{\phi}_\pm$ relations, with all but a finite number of $\widetilde{\phi}_-$'s. An argument similar to Lemma 5.1 says that the limit $\lambda := \frac{3}{2}\lim_{m\to\infty} 5^m \lambda_m$ exists. We will prove $\lambda \in \mathcal{P}^{+,N}$.

Let m_1 be the generation of fixation of $\{\lambda_m\}_{m \geq m_0}$. For $m \geq m_1$, we still use u_m to denote the associated λ_m-eigenfunction on Ω_m, and extend it to the whole domain Ω using a similar recipe as for the Dirichlet case. Then u_m is also defined on Ω, satisfying

$$\begin{cases} -\Delta u_m = 5^m \Phi(\lambda_m) u_m \text{ on } \Omega, \\ \partial_n u_m|_{\partial \Omega_m} = 0, \end{cases}$$

with $5^m \Phi(\lambda_m) \to \lambda$ as m goes to infinity.

As in the Dirichlet case, defined $v_m := \frac{u_m}{\|u_m\|_\infty}$. With suitable modification of Lemma 5.7 and Lemma 5.8, we still have the equicontinuity of the sequence $\{v_m\}_{m \geq m_1}$. Hence a similar argument still yields that there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ which converges uniformly to a continuous function v on Ω. Hence we only need to prove that v is the associated Neumann eigenfunction of λ. We will use the notations defined in Definition 3.11.

Fix an integer $n \geq m_1$. We still use K_n to denote the part of Ω above $\partial\Omega_n \setminus \{q_0\}$. Let $G_{K_n}(x,y)$ denote the Green's function associate to the simple domain K_n.

Then $\forall k$ we have

$$v_{m_k}(x) = \int_{K_n} G_{K_n}(x,y) 5^{m_k}\Phi(\lambda_{m_k}) v_{m_k}(y) d\mu(y) + h^{(n)}_{m_k}(x) \text{ on } K_n,$$

where $h^{(n)}_{m_k}$ is a harmonic function on K_n taking the same boundary values as v_{m_k} on ∂K_n. Noticing that the sequence $\{h^{(n)}_{m_k}\}$ converges uniformly on K_n to $h^{(n)}$ as k goes to infinity. We then get

$$v(x) = \lambda \int_{K_n} G_{K_n}(x,y) v(y) d\mu(y) + h^{(n)} \text{ on } K_n.$$

Thus $-\Delta v = \lambda v$ on K_n.

Let φ be a test function in $\widetilde{\mathcal{F}}$. Now we calculate $\mathcal{E}_{K_n}(v,\varphi)$.

The Gauss-Green formula says that

$$\mathcal{E}_{K_n}(v,\varphi) = \lambda \int_{K_n} v\varphi d\mu + \sum_{\partial K_n} \varphi \partial_n v.$$

It is easy to check that $\partial_n v(q_0) = 0$ since $\partial_n v_{m_k}(q_0) = 0$ for each m_k. Hence we have

$$\mathcal{E}_{K_n}(v,\varphi) = \lambda \int_{K_n} v\varphi d\mu + \partial_n v(F_1^n q_0) \cdot \sum_{\partial K_n \setminus \{q_0\}} \varphi, \tag{7.7}$$

since v takes same value along $\partial K_n \setminus \{q_0\}$.

On the other hand, the Gauss-Green formula also says that $\int_{K_n} \Delta v d\mu = \sum_{\partial K_n} \partial_n v$, hence $\partial_n v(F_1^n q_0) = -\frac{1}{2^n}\lambda \int_{K_n} v d\mu$. Substituting this into (7.7), we then have

$$\begin{aligned} \mathcal{E}_{K_n}(v,\varphi) &= \lambda \int_{K_n} v\varphi d\mu - \lambda \int_{K_n} v d\mu \cdot \left(\frac{1}{2^n} \sum_{K_n \setminus \{q_0\}} \varphi\right) \\ &= \lambda \int_{K_n} v\varphi d\mu - \lambda \int_{K_n} (v - v_n) d\mu \cdot \left(\frac{1}{2^n} \sum_{K_n \setminus \{q_0\}} \varphi\right). \end{aligned}$$

The last equality follows from the fact that $\int_{K_n} \Delta v_n d\mu = \sum_{\partial K_n} \partial_n v_n = 0$ and $-\Delta v_n = 5^n \Phi(\lambda_n) v_n$ on K_n.

Taking $n = m_k$, we then get $\forall k$,

$$\mathcal{E}_{K_{m_k}}(v,\varphi) = \lambda \int_{K_{m_k}} v\varphi d\mu - \lambda \int_{K_{m_k}} (v - v_{m_k}) d\mu \cdot \left(\frac{1}{2^{m_k}} \sum_{K_{m_k} \setminus \{q_0\}} \varphi\right).$$

Letting k goes to infinity, we get

$$\widetilde{\mathcal{E}}(v,\varphi)=\lambda\int_{\widetilde{\Omega}}v\varphi d\mu,$$

since v_{m_k} converges uniformly to v and $|\frac{1}{2^{m_k}}\sum_{K_{m_k}\setminus\{q_0\}}\varphi|$ is controlled by $\|\varphi\|_\infty$.

Hence v is an eigenfunction of the Neumann Laplacian $-\Delta_N$ associated to λ from the arbitrariness of the test function φ. $\Box$

8 Spectral asymptotics, ratio gaps and clusters

In this section, we list some unproved conjectures related to the structure of the spectrum of $-\Delta$ on Ω. For simplicity, we only discuss the Dirichlet spectrum $\mathcal{S}$ on Ω.

In Tables 8.1, 8.2, 8.3 and 8.4 we present the eigenvalues and their multiplicities in $\mathcal{S}_m$ for level $m=2,3,4,5$, where we use $\lambda^+_{m,k}$, $\lambda^-_{m,k}$, $\lambda_{m,k}$ to denote the k'th $\mathcal{P}^+_m$, $\mathcal{P}^-_m$, $\mathcal{L}_m$ type eigenvalues respectively, and use $\mathcal{M}_m(\lambda^-_{m',k})$ to denote the miniaturized eigenvalue generated from $\lambda^-_{m',k}$.

The following conjectures list some interesting phenomena we observed from the data.

Conjecture 8.1. *Let $\rho^\Omega_m(x)$ denote the eigenvalue counting function of $\mathcal{S}_m$, i.e., $\rho^\Omega_m(x)=\sharp\{\lambda_m\in\mathcal{S}_m:\lambda_m\leq x\}$. Then $\rho^\Omega_m(\phi_-^{(m-k)}(5))=3^k-2^k$ for $k<m$.*

Remark. *Here 3^k-2^k is the difference between a_k and a_{k-1}.*

This conjecture suggests that the bottom 3^k-2^k eigenvalues of the Dirichlet spectrum of Ω should be generated from the bottom 3^k-2^k eigenvalues in $\mathcal{S}_m$ and the largest of these eigenvalues should be $\lim_{n\to\infty}\frac{3}{2}5^n\phi_-^{(n-k)}(5)=c5^k$ for the appropriate choice of c. For the eigenvalue counting function $\rho^\Omega(x)$ on Ω, we then have $\rho^\Omega(c5^k)=3^k-2^k$. Roughly this suggests an asymptotic growth rate $\rho^\Omega(x)\sim x^{\log 3/\log 5}$ as $x\to\infty$, which is similar to the $\mathcal{SG}\setminus V_0$ case. But more precisely, this also implies that

$$\rho^\Omega(x)=c_1x^{\log 3/\log 5}-c_2x^{\log 2/\log 5}$$

along the sequence $x=c5^k$ for some appropriate constants c_1 and c_2. Hence, in analogy with the $\mathcal{SG}\setminus V_0$ case, we believe the following more precise conjecture.

Conjecture 8.2. *There exist two periodic functions $g_1(t)$ and $g_2(t)$ of period* $\log 5$, *which are bounded above, bounded away from zero, and necessarily discontinuous at the value* $\log c$, *such that*

$$\rho^\Omega(x)=g_1(\log x)x^{\log 3/\log 5}+g_2(\log x)x^{\log 2/\log 5}+o(x^{\log 2/\log 5}).\tag{8.1}$$

Here, comparing to formula (1.5), besides the leading term $g_1(\log x)x^{\log 3/\log 5}$, the asymptotic second term of the eigenvalue counting function appears. This is very analogous to the conjectures of Weyl and Berry [3,4].

Conjecture 8.3. *There exist gaps in the ratios of eigenvalues from the Dirichlet spectrum $\mathcal{S}$ of $-\Delta$. That is, we can find infinitely many pairs of consecutive eigenvalues λ, λ' with $\frac{\lambda'}{\lambda} \geq c$ for some constant $c > 1$.*

Remark. *In fact, in the discrete spectrum $\mathcal{S}_m$, one can observe that gap appears above each $\phi_-^{(m-k)}(5)$ for $k < m$. Moreover, there are also smaller gaps below miniaturized eigenvalues.*

In [5] it was shown that on $\mathcal{SG} \setminus V_0$ there exist gaps in the ratios of eigenvalues. The existence of gaps is an interesting phenomenon in itself, but it also has important applications to analysis on fractals. See details in [5], [19], [37]. Thus it is of great interest to know whether similar phenomenon exists for fractals other than $\mathcal{SG}$. In fact [40] shows that this is the case for Vicsek set. Also [8, 13, 41] investigates this question for a variant of the $\mathcal{SG}$ type fractal.

Conjecture 8.4. *In the spectrum $\mathcal{S}_m$, between consecutive 5 and 6 type localized eigenvalues, there is exactly one $\mathcal{P}^+$ and one $\mathcal{P}^-$ type eigenvalue (except the case that the two consecutive eigenvalues are $\phi_-(5)$ and $\phi_+(6) = 3$, where there is nothing in between).*

Conjecture 8.5. *In the spectrum $\mathcal{S}_m$, the number of distinct eigenvalues between $5-\varepsilon$ and 5 goes to ∞ as $m \to \infty$ for any $\varepsilon > 0$.*

Remark. *This means in $\mathcal{S}$ there exist eigenvalue clusters, that is, arbitrarily many distinct eigenvalues in an arbitrarily small interval.*

We say the spectrum $\mathcal{S}$ exhibits spectral clustering. Clustering does not occur on the $\mathcal{SG} \setminus V_0$ case. Experimental evidence suggests that it does occur on the pentagasket [1] and on the Julia sets [9]. It is proved that in [7] it also does occur on the Vicsek set.

$m=2$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=2$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{2,1}^+$=1.064568	39.92	1	$\mathcal{P}^+$	4	$\lambda_{2,3}^+$=5.472834	205.23	1	$\mathcal{P}^+$
2	$\lambda_{2,1}^-$=3.381966	126.82	1	$\mathcal{P}^-$	5	$\lambda_{2,2}^-$=5.618034	210.68	1	$\mathcal{P}^-$
3	$\lambda_{2,2}^+$=4.462598	167.35	1	$\mathcal{P}^+$					

Table 8.1. The 2-level eigenvalues in $\mathcal{S}_2$ in increasing order.

$m=3$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=3$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{3,1}^{+}$=0.187518	35.16	1	$\mathcal{P}^+$	11	$\lambda_{3,4}^{-}$=3.902230	731.67	1	$\mathcal{P}^-$
2	$\lambda_{3,1}^{-}$=0.558733	104.76	1	$\mathcal{P}^-$	12	$\lambda_{3,6}^{+}$=4.517231	846.98	1	$\mathcal{P}^+$
3	$\lambda_{3,2}^{+}$=0.805532	151.04	1	$\mathcal{P}^+$	13	$\lambda_{3,5}^{-}$=4.803115	900.58	1	$\mathcal{P}^-$
4	$\lambda_{3,2}^{-}$=1.247636	233.93	1	$\mathcal{P}^-$	14	$\lambda_{3,7}^{+}$=4.946726	927.51	1	$\mathcal{P}^+$
5	$\lambda_{3,3}^{+}$=1.329287	249.24	1	$\mathcal{P}^+$	15	$\lambda_{3,1}$=5	937.50	1	$\mathcal{L}$
6	$\lambda_{3,3}^{-}$=3.059152	573.59	1	$\mathcal{P}^-$	16	$\lambda_{3,8}^{+}$=5.424059	1017.01	1	$\mathcal{P}^+$
7	$\lambda_{3,4}^{+}$=3.075910	576.73	1	$\mathcal{P}^+$	17	$\lambda_{3,6}^{-}$=5.429135	1017.96	1	$\mathcal{P}^-$
8,9	$\mathcal{M}_3(\lambda_{2,1}^{-})$=3.381966	634.12	2	$\mathcal{M}$	18,19	$\mathcal{M}_3(\lambda_{2,2}^{-})$=5.618034	1053.38	2	$\mathcal{M}$
10	$\lambda_{3,5}^{+}$=3.713736	696.33	1	$\mathcal{P}^+$	20–24	$\lambda_{3,2}$=6	1125.00	5	$\mathcal{L}$

Table 8.2. The 3-level eigenvalues in $\mathcal{S}_3$ in increasing order.

$m=4$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=4$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{4,1}^{+}$=0.035755	33.52	1	$\mathcal{P}^+$	34	$\lambda_{4,10}^{+}$=3.631877	3404.88	1	$\mathcal{P}^+$
2	$\lambda_{4,1}^{-}$=0.100554	94.27	1	$\mathcal{P}^-$	35	$\lambda_{4,8}^{-}$=3.656967	3428.41	1	$\mathcal{P}^-$
3	$\lambda_{4,2}^{+}$=0.146945	137.76	1	$\mathcal{P}^+$	36	$\lambda_{4,11}^{+}$=3.760496	3525.46	1	$\mathcal{P}^+$
4	$\lambda_{4,2}^{-}$=0.249495	233.90	1	$\mathcal{P}^-$	37,38	$\mathcal{M}_4(\lambda_{3,4}^{-})$=3.902230	3658.34	2	$\mathcal{M}$
5	$\lambda_{4,3}^{+}$=0.277423	260.08	1	$\mathcal{P}^+$	39	$\lambda_{4,9}^{-}$=3.982762	3733.84	1	$\mathcal{P}^-$
6,7	$\mathcal{M}_4(\lambda_{3,1}^{-})$=0.558733	523.81	2	$\mathcal{M}$	40	$\lambda_{4,12}^{+}$=4.074531	3819.87	1	$\mathcal{P}^+$
8	$\lambda_{4,4}^{+}$=0.645454	605.11	1	$\mathcal{P}^+$	41	$\lambda_{4,13}^{+}$=4.223191	3959.24	1	$\mathcal{P}^+$
9	$\lambda_{4,3}^{-}$=0.652593	611.81	1	$\mathcal{P}^-$	42	$\lambda_{4,10}^{-}$=4.241362	3976.28	1	$\mathcal{P}^-$
10	$\lambda_{4,4}^{-}$=0.843591	790.87	1	$\mathcal{P}^-$	43	$\lambda_{4,11}^{-}$=4.573615	4287.76	1	$\mathcal{P}^-$
11	$\lambda_{4,5}^{+}$=0.857718	804.11	1	$\mathcal{P}^+$	44	$\lambda_{4,14}^{+}$=4.586787	4300.11	1	$\mathcal{P}^+$
12	$\lambda_{4,6}^{+}$=0.965805	905.44	1	$\mathcal{P}^+$	45	$\lambda_{4,15}^{+}$=4.735683	4439.70	1	$\mathcal{P}^+$
13	$\lambda_{4,5}^{-}$=1.065699	999.09	1	$\mathcal{P}^-$	46	$\lambda_{4,12}^{-}$=4.793032	4493.47	1	$\mathcal{P}^-$
14,15	$\mathcal{M}_4(\lambda_{3,2}^{-})$=1.247636	1169.66	2	$\mathcal{M}$	47,48	$\mathcal{M}_4(\lambda_{3,5}^{-})$=4.803115	4502.92	2	$\mathcal{M}$
16	$\lambda_{4,7}^{+}$=1.263652	1184.67	1	$\mathcal{P}^+$	49	$\lambda_{4,16}^{+}$=4.926848	4618.92	1	$\mathcal{P}^+$
17	$\lambda_{4,6}^{-}$=1.358256	1273.37	1	$\mathcal{P}^-$	50	$\lambda_{4,13}^{-}$=4.979948	4668.70	1	$\mathcal{P}^-$
18	$\lambda_{4,8}^{+}$=1.372367	1286.59	1	$\mathcal{P}^+$	51	$\lambda_{4,17}^{+}$=4.993259	4681.18	1	$\mathcal{P}^+$
19	$\lambda_{4,1}$=1.381966	1295.59	1	$\mathcal{L}$	52–57	$\lambda_{4,4}$=5	4687.50	6	$\mathcal{L}$
20–24	$\lambda_{4,2}$=3	2812.50	5	$\mathcal{L}$	58	$\lambda_{4,18}^{+}$=5.423778	5084.79	1	$\mathcal{P}^+$
25,26	$\mathcal{M}_4(\lambda_{3,3}^{-})$=3.059152	2867.96	2	$\mathcal{M}$	59	$\lambda_{4,14}^{-}$=5.423779	5084.79	1	$\mathcal{P}^-$
27	$\lambda_{4,7}^{-}$=3.078348	2885.95	1	$\mathcal{P}^-$	60,61	$\mathcal{M}_4(\lambda_{3,6}^{-})$=5.429135	5089.81	2	$\mathcal{M}$
28	$\lambda_{4,9}^{+}$=3.078431	2886.03	1	$\mathcal{P}^+$	62–65	$\mathcal{M}_4(\lambda_{2,2}^{-})$=5.618034	5266.91	4	$\mathcal{M}$
29–32	$\mathcal{M}_4(\lambda_{2,1}^{-})$=3.381966	3170.59	4	$\mathcal{M}$	66–89	$\lambda_{4,5}$=6	5625.00	24	$\mathcal{L}$
33	$\lambda_{4,3}$=3.618034	3391.91	1	$\mathcal{L}$					

Table 8.3. The 4-level eigenvalues in $\mathcal{S}_4$ in increasing order.

$m=5$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=5$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{5,1}^{+}$=0.007039	33.00	1	$\mathcal{P}^+$	112	$\lambda_{5,20}^{+}$=3.620288	16970.1	1	$\mathcal{P}^+$
2	$\lambda_{5,1}^{-}$=0.019385	90.87	1	$\mathcal{P}^-$	113	$\lambda_{5,16}^{-}$=3.623927	16987.2	1	$\mathcal{P}^-$
3	$\lambda_{5,2}^{+}$=0.028430	133.27	1	$\mathcal{P}^+$	114	$\lambda_{5,21}^{+}$=3.644882	17085.4	1	$\mathcal{P}^+$
4	$\lambda_{5,2}^{-}$=0.049571	232.36	1	$\mathcal{P}^-$	115,116	$\mathcal{M}_5(\lambda_{4,8}^{-})$=3.656967	17142.0	2	$\mathcal{M}$
5	$\lambda_{5,3}^{+}$=0.055860	261.84	1	$\mathcal{P}^+$	117	$\lambda_{5,17}^{-}$=3.694772	17319.2	1	$\mathcal{P}^-$
6,7	$\mathcal{M}_5(\lambda_{4,1}^{-})$=0.100554	471.35	2	$\mathcal{M}$	118	$\lambda_{5,22}^{+}$=3.720985	17442.1	1	$\mathcal{P}^+$
8	$\lambda_{5,4}^{+}$=0.123515	578.98	1	$\mathcal{P}^+$	119	$\lambda_{5,23}^{+}$=3.749413	17575.4	1	$\mathcal{P}^+$
9	$\lambda_{5,3}^{-}$=0.125398	587.80	1	$\mathcal{P}^-$	120	$\lambda_{5,18}^{-}$=3.753145	17592.9	1	$\mathcal{P}^-$
10	$\lambda_{5,4}^{-}$=0.166319	779.62	1	$\mathcal{P}^-$	121–124	$\mathcal{M}_5(\lambda_{3,4}^{-})$=3.902230	18291.7	4	$\mathcal{M}$
11	$\lambda_{5,5}^{+}$=0.170850	800.86	1	$\mathcal{P}^+$	125	$\lambda_{5,19}^{-}$=3.908588	18321.5	1	$\mathcal{P}^-$
12	$\lambda_{5,6}^{+}$=0.196017	918.83	1	$\mathcal{P}^+$	126	$\lambda_{5,24}^{+}$=3.912510	18339.9	1	$\mathcal{P}^+$
13	$\lambda_{5,5}^{-}$=0.217665	1020.30	1	$\mathcal{P}^-$	127	$\lambda_{5,25}^{+}$=3.971467	18616.3	1	$\mathcal{P}^+$
14,15	$\mathcal{M}_5(\lambda_{4,2}^{-})$=0.249495	1169.51	2	$\mathcal{M}$	128,129	$\mathcal{M}_5(\lambda_{4,9}^{-})$=3.982762	18669.2	2	$\mathcal{M}$
16	$\lambda_{5,7}^{+}$=0.264441	1239.57	1	$\mathcal{P}^+$	130	$\lambda_{5,20}^{-}$=3.997137	18736.6	1	$\mathcal{P}^-$
17	$\lambda_{5,6}^{-}$=0.286684	1343.83	1	$\mathcal{P}^-$	131	$\lambda_{5,26}^{+}$=4.069518	19075.9	1	$\mathcal{P}^+$
18	$\lambda_{5,8}^{+}$=0.290993	1364.03	1	$\mathcal{P}^+$	132	$\lambda_{5,21}^{-}$=4.103862	19236.9	1	$\mathcal{P}^-$
19	$\lambda_{5,1}$=0.293638	1376.43	1	$\mathcal{L}$	133	$\lambda_{5,27}^{+}$=4.116582	19296.5	1	$\mathcal{P}^+$
20–23	$\mathcal{M}_5(\lambda_{3,1}^{-})$=0.558733	2619.06	4	$\mathcal{M}$	134	$\lambda_{5,7}$=4.122334	19323.4	1	$\mathcal{L}$
24	$\lambda_{5,9}^{+}$=0.644676	3021.92	1	$\mathcal{P}^+$	135	$\lambda_{5,28}^{+}$=4.219041	19776.8	1	$\mathcal{P}^+$
25	$\lambda_{5,7}^{-}$=0.644693	3022.00	1	$\mathcal{P}^-$	136	$\lambda_{5,22}^{-}$=4.219295	19777.9	1	$\mathcal{P}^-$
26,27	$\mathcal{M}_5(\lambda_{4,3}^{-})$=0.652593	3059.03	2	$\mathcal{M}$	137,138	$\mathcal{M}_5(\lambda_{4,10}^{-})$=4.241362	19881.4	2	$\mathcal{M}$
28–32	$\lambda_{5,2}$=0.697224	3268.24	5	$\mathcal{L}$	139–143	$\lambda_{5,8}$=4.302776	20169.3	5	$\mathcal{L}$
33,34	$\mathcal{M}_5(\lambda_{4,4}^{-})$=0.843591	3954.33	2	$\mathcal{M}$	144,145	$\mathcal{M}_5(\lambda_{4,11}^{-})$=4.573615	21438.8	2	$\mathcal{M}$
35	$\lambda_{5,8}^{-}$=0.864034	4050.16	1	$\mathcal{P}^-$	146	$\lambda_{5,23}^{-}$=4.588806	21510.0	1	$\mathcal{P}^-$
36	$\lambda_{5,10}^{+}$=0.866936	4063.76	1	$\mathcal{P}^+$	147	$\lambda_{5,29}^{+}$=4.588882	21510.4	1	$\mathcal{P}^+$
37	$\lambda_{5,3}$=0.877666	4114.06	1	$\mathcal{L}$	148	$\lambda_{5,9}$=4.706362	22061.1	1	$\mathcal{L}$
38	$\lambda_{5,11}^{+}$=0.890579	4174.59	1	$\mathcal{P}^+$	149	$\lambda_{5,30}^{+}$=4.710126	22078.7	1	$\mathcal{P}^+$
39	$\lambda_{5,9}^{-}$=0.921042	4317.38	1	$\mathcal{P}^-$	150	$\lambda_{5,24}^{-}$=4.717827	22114.8	1	$\mathcal{P}^-$
40	$\lambda_{5,12}^{+}$=0.951360	4459.50	1	$\mathcal{P}^+$	151	$\lambda_{5,31}^{+}$=4.742035	22228.3	1	$\mathcal{P}^+$
41	$\lambda_{5,10}^{-}$=1.013289	4749.79	1	$\mathcal{P}^-$	152	$\lambda_{5,25}^{-}$=4.791572	22460.5	1	$\mathcal{P}^-$
42	$\lambda_{5,13}^{+}$=1.031636	4835.79	1	$\mathcal{P}^+$	153,154	$\mathcal{M}_5(\lambda_{4,12}^{-})$=4.793032	22467.3	2	$\mathcal{M}$
43,44	$\mathcal{M}_5(\lambda_{4,5}^{-})$=1.065699	4995.46	2	$\mathcal{M}$	155–158	$\mathcal{M}_5(\lambda_{3,5}^{-})$=4.803115	22514.6	4	$\mathcal{M}$
45	$\lambda_{5,14}^{+}$=1.095777	5136.45	1	$\mathcal{P}^+$	159	$\lambda_{5,32}^{+}$=4.809185	22543.1	1	$\mathcal{P}^+$
46	$\lambda_{5,11}^{-}$=1.097686	5145.40	1	$\mathcal{P}^-$	160	$\lambda_{5,33}^{+}$=4.844770	22709.9	1	$\mathcal{P}^+$
47–50	$\mathcal{M}_5(\lambda_{3,2}^{-})$=1.247636	5848.29	4	$\mathcal{M}$	161	$\lambda_{5,26}^{-}$=4.847489	22722.6	1	$\mathcal{P}^-$
51	$\lambda_{5,15}^{+}$=1.259109	5902.07	1	$\mathcal{P}^+$	162	$\lambda_{5,27}^{-}$=4.932207	23119.7	1	$\mathcal{P}^-$
52	$\lambda_{5,12}^{-}$=1.260744	5909.74	1	$\mathcal{P}^-$	163	$\lambda_{5,34}^{+}$=4.934639	23131.1	1	$\mathcal{P}^+$
53	$\lambda_{5,16}^{+}$=1.291565	6054.21	1	$\mathcal{P}^+$	164	$\lambda_{5,35}^{+}$=4.950036	23203.3	1	$\mathcal{P}^+$
54	$\lambda_{5,13}^{-}$=1.314754	6162.91	1	$\mathcal{P}^-$	165	$\lambda_{5,28}^{-}$=4.963126	23264.7	1	$\mathcal{P}^-$
55	$\lambda_{5,17}^{+}$=1.358055	6365.88	1	$\mathcal{P}^+$	166,167	$\mathcal{M}_5(\lambda_{4,13}^{-})$=4.979948	23343.5	2	$\mathcal{M}$
56,57	$\mathcal{M}_5(\lambda_{4,6}^{-})$=1.358256	6366.83	2	$\mathcal{M}$	168	$\lambda_{5,36}^{+}$=4.987488	23378.9	1	$\mathcal{P}^+$

$m=5$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=5$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
58	$\lambda^-_{5,14}$=1.377582	6457.42	1	$\mathcal{P}^-$	169	$\lambda^-_{5,29}$=4.997193	23424.3	1	$\mathcal{P}^-$
59	$\lambda^+_{5,18}$=1.380161	6469.50	1	$\mathcal{P}^+$	170	$\lambda^+_{5,37}$=4.998947	23432.6	1	$\mathcal{P}^+$
60–65	$\lambda_{5,4}$=1.381966	6477.97	6	$\mathcal{L}$	171–195	$\lambda_{5,10}$=5	23437.5	25	$\mathcal{L}$
66–89	$\lambda_{5,5}$=3	14063.0	24	$\mathcal{L}$	196	$\lambda^+_{5,38}$=5.423778	25424.0	1	$\mathcal{P}^+$
90–93	$\mathcal{M}_5(\lambda^-_{3,3})$=3.059152	14339.8	4	$\mathcal{M}$	197	$\lambda^-_{5,30}$=5.423778	25424.0	1	$\mathcal{P}^-$
94,95	$\mathcal{M}_5(\lambda^-_{4,7})$=3.078348	14429.8	2	$\mathcal{M}$	198,199	$\mathcal{M}_5(\lambda^-_{4,14})$=5.423779	25424.0	2	$\mathcal{M}$
96	$\lambda^+_{5,19}$=3.078432	14430.2	1	$\mathcal{P}^+$	200–203	$\mathcal{M}_5(\lambda^-_{3,6})$=5.429135	25449.1	4	$\mathcal{M}$
97	$\lambda^-_{5,15}$=3.078432	14430.2	1	$\mathcal{P}^-$	204–211	$\mathcal{M}_5(\lambda^-_{2,2})$=5.618034	26334.5	8	$\mathcal{M}$
98–105	$\mathcal{M}_5(\lambda^-_{2,1})$=3.381966	15853.0	8	$\mathcal{M}$	212–300	$\lambda_{5,11}$=6	28125.0	89	$\mathcal{L}$
106–111	$\lambda_{5,6}$=3.618034	16959.5	6	$\mathcal{L}$					

Table 8.4. The 5-level eigenvalues in $\mathcal{S}_5$ in increasing order.

9 Further discussion

In this section, we discuss to what extent our method can be extended to other domains in $\mathcal{SG}$. In particular, we will focus on Ω_x $(0<x<1)$. It seems that we can analyze the spectrum of $-\Delta$ on Ω_x case by case following the similar recipe for the Ω_1 case. However, it is hard to develop a general method which is suitable for all cases, although we believe that we could make clear the structures of the spectra. We let L_x denote the bottom boundary of Ω_x. Then L_x will be a Cantor set for generic x, and an union of intervals if x is a dyadic rational. We may assume without loss of generality that $\frac{1}{2}<x<1$, for if not we may first solve the problem for Ω_{2x}, and then simply dilate the solution to Ω_x.

For simplicity, we only discuss the Dirichlet spectrum of $-\Delta$. Obviously, it will suffice to describe the discrete Dirichlet spectra of $-\Delta_m$'s for all m. Hence the first problem is how to define the graph approximations. Similar to Ω_1, the fractal domain Ω_x can be realized as the limit of a sequence of graphs $\Omega_{x,m}$. More precisely, $\forall m\geq 1$, let $V_m^{\Omega_x}$ be a subset of V_m with all vertices lying along or under L_x removed. Let $\Omega_{x,m}$ be the subgraph of Γ_m restricted to $V_m^{\Omega_x}$. Denote by $\partial\Omega_{x,m}$ the boundary of the finite graph $\Omega_{x,m}$. It is easy to find that $V_m^{\Omega_x}\setminus\partial\Omega_{x,m}$, $\partial\Omega_{x,m}$ approximate to Ω_x and $\partial\Omega_x$ as m goes to infinity, respectively. See Fig. 9.1 and Fig. 9.2 for Ω_x and $\Omega_{x,m}$ where $x=3/4$.

On $\Omega_{x,m}$ the Dirichlet λ_m-eigenvalue equations consists of exactly $\sharp(V_m^{\Omega_x}\setminus\partial\Omega_{x,m})$ equations in $\sharp(V_m^{\Omega_x}\setminus\partial\Omega_{x,m})$ unknowns. We denote by $\mathcal{S}_m(x)$ the spectrum of $-\Delta_m$ on $\Omega_{x,m}$ for each $m\geq 1$. Accordingly, $\mathcal{S}_m(x)$ should consists of (at most) three types of eigenvalues, denoted by $\mathcal{L}_m(x)$, $\mathcal{P}_m(x)$ and $\mathcal{M}_m(x)$ respectively. $\mathcal{P}_m(x)$ can also be split into symmetric part $\mathcal{P}_m^+(x)$ and skew-symmetric part $\mathcal{P}_m^-(x)$. The precise definitions should be obvious. To ensure that there is no other eigenvalue in $\mathcal{S}_m(x)$, the following eigenspace dimensional counting formula should hold,

$$\sharp(V_m^{\Omega_x}\setminus\partial\Omega_{x,m})=\sharp\mathcal{L}_m(x)+\sharp\mathcal{P}_m(x)+\sharp\mathcal{M}_m(x).$$

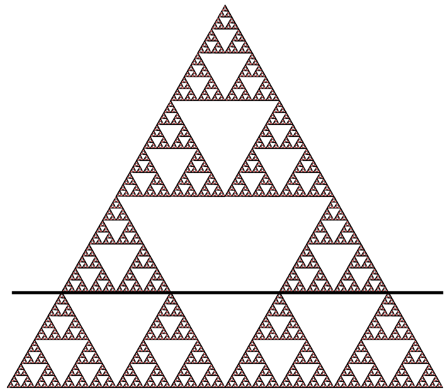
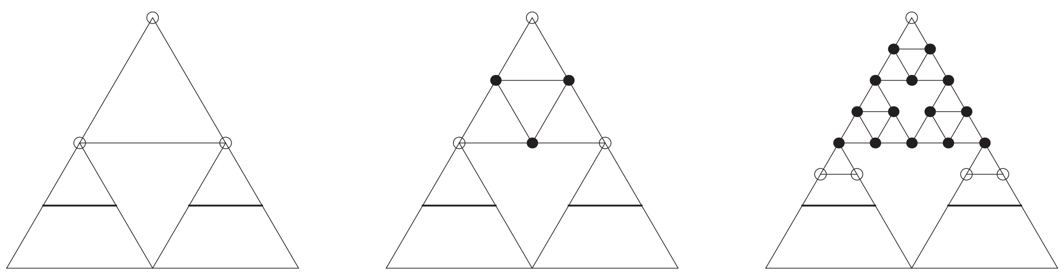


Fig. 9.1. $\Omega_{3/4}$.



$\Omega_{3/4,1}$ $\Omega_{3/4,2}$ $\Omega_{3/4,3}$

Fig. 9.2. The first 3 graphs, $\Omega_{3/4,1}, \Omega_{3/4,2}, \Omega_{3/4,3}$ in the approximations to $\Omega_{3/4}$ with interior points and boundary points represented by dots and circles respectively.

We now focus on a particular example $\Omega_{3/4}$ to illustrate how to extend the recipe for Ω_1 to the general case. We are particular interested in the primitive eigenvalues. We begin with $\mathcal{P}_m^+(3/4)$, the symmetric case. It is convenient to define the skeleton of $\Omega_m(3/4)$ by $(q_0, F_1q_0, F_{10}F_1q_0, \cdots, F_{10}F_1^{m-2}q_0)$ for $m \geq 3$ and (q_0, F_1q_0) for $m = 1$ or 2. Let u_m be a λ_m-eigenfunction of $-\Delta_m$ with $\lambda_m \in \mathcal{P}_m^+(3/4)$. Denote by $(b_0, b_1, b_2, \cdots, b_m)$ the values of u_m on the skeleton of $\Omega_{3/4,m}$ where $b_0 = b_m = 0$ by the Dirichlet boundary condition. It is easy to observe that when $i \geq 2$, the eigenvalue equation at the vertex $F_{10}F_1^{i-1}q_0$ is exactly the same as that of Ω_1 case with suitable reindexing. Hence the generation mechanism of symmetric primitive eigenvalues is quite similar to the Ω_1 case. Based on this observation, one can easily find that $\sharp\mathcal{P}_m^+(3/4) = 2^m - 2$ for $m \geq 2$ by still using the weak spectral decimation method. A similar argument yields that $\sharp\mathcal{P}_m^-(3/4) = 2^m - 2^{m-2} - 2$ for $m \geq 2$.

To verify the eigenspace dimensional counting formula, we only look at the first 4 levels of approximations since the continued process is similar.

When $m = 1$, the result is trivial since there is no inside point in $\Omega_{3/4,1}$. Hence $\sharp\mathcal{S}_1(3/4) = 0 = \sharp V_1^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,1}$.

When $m = 2$, it is easy to check that there are only primitive eigenvalues. Hence $\sharp\mathcal{S}_2(3/4) = \sharp\mathcal{P}_2^+(3/4) + \sharp\mathcal{P}_2^-(3/4) = 2 + 1 = \sharp V_2^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,2}$.

When $m = 3$, it is easy to check that there are 4 initial localized eigenvalues, i.e., 5 with multiplicity 1 and 6 with multiplicity 3; there are 6 symmetric primitive eigenvalues and 4 skew-symmetric primitive eigenvalues; there are no miniaturized eigenvalues. Hence $\sharp\mathcal{S}_3(3/4) = \sharp\mathcal{L}_3(3/4) + \sharp\mathcal{P}_3^+(3/4) + \sharp\mathcal{P}_3^-(3/4) = 4 + 6 + 4 = \sharp V_3^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,3}$.

When $m = 4$, it is easy to check that besides $1 \cdot 2 + 3 \cdot 1 = 5$ continued localized eigenvalues, there are 18 initial localized eigenvalues, i.e., 5 with multiplicity 4 and 6 with multiplicity 14. Hence $\sharp\mathcal{L}_4(3/4) = 5 + 18 = 23$. There are 14 symmetric primitive eigenvalues and 10 skew-symmetric primitive eigenvalues. Hence $\sharp\mathcal{P}_4(3/4) = 14+10 = 24$. Moreover, there are some miniaturized eigenvalues which come from the miniaturizations of eigenvalues in $\mathcal{P}_2^-(1)$. Hence $\sharp\mathcal{M}_4(3/4) = 2 \cdot \mathcal{P}_2^-(1) = 2 \cdot 2 = 4$. Thus $\sharp\mathcal{S}_4(3/4) = 23 + 24 + 4 = \sharp V_4^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,4}$.

It is easy to verify the general formula for general m. We will not attempt to list the details here. However, a more important fact should be pointed out is that for $\Omega_{3/4}$ case, the miniaturized eigenvalues in $\mathcal{M}_m(3/4)$ are generated not from those in $\mathcal{P}_k^-(3/4)$ but from those in $\mathcal{P}_k^-(1)$ for $k \leq m-2$. This means that to study $\mathcal{S}_m(3/4)$, one should first make clear $\mathcal{S}_m(1)$. Things will be more complicated for general Ω_x.

Next we briefly present another observation. Still consider a domain Ω_x with a series of graph approximations $\{\Omega_{x,m}\}$. Notice that there are only two possible patterns when passing from the m-level graph approximation to its next level. One is that the boundary $\partial\Omega_{x,m+1}$ remains unchanged, i.e., $\partial\Omega_{x,m+1} = \partial\Omega_{x,m}$, the other is that $\partial\Omega_{x,m} \setminus \{q_0\}$ becomes a collection of interior points of $\Omega_{x,m+1}$, i.e., each point in $\partial\Omega_{x,m} \setminus \{q_0\}$ is connected with two new $(m+1)$-level points in $\partial\Omega_{x,m+1}$. In fact, for the $\mathcal{SG} \setminus V_0$ case, when passing from one level to the next level, the boundaries of graphs are always V_0, keeping unchanged. This is also the reason why spectral decimation can work for 2-series eigenvalues (which should be considered as the primitive eigenvalues in the $\mathcal{SG} \setminus V_0$ case). As for the Ω_1 case, when passing from one level to the next level, the boundaries always change. Due to this phenomenon, the spectral decimation recipe should be replaced by the weak spectral decimation recipe for primitive and miniaturized eigenvalues since their supports always touch the boundaries. For general Ω_x $(0 < x < 1)$, these two possible patterns can both exist. It is natural to expect that under the first pattern, the two levels of primitive eigenvalues are related by the spectral decimation (it is obviously true), while under the second pattern, they are related by a weak spectral decimation instead. Thus we expect:

Conjecture 9.1. *For a domain Ω_x $(0 < x < 1)$ with a series of graph approximations $\{\Omega_{x,m}\}$, if the boundaries change when passing from m-level to $(m+1)$-level, then there is a weak spectral decimation relating the two levels of symmetric (or skew-symmetric) primitive eigenvalues.*

10 Appendix

Theorem A. *For each $m \geq 2$, let $\lambda_{m,1}, \lambda_{m,2}, \cdots, \lambda_{m,r_m}$ be the r_m distinct eigenvalues in $\mathcal{P}_m^+$ in increasing order. Then $\lambda_{m+1,r_m+1} > 2$.*

To prove this theorem, we need the following lemma:

Lemma A. $p_2(2) < 0$, $p_3(2) > 0$ and $(-1)^m p_m(2) > 0$, $\forall m \geq 4$.

Proof. It is easy to check that $p_2(2) = -8 < 0$ and $p_3(2) = 68 > 0$.

Let $m \geq 4$. Then

$$\begin{aligned} p_m(x) &= \frac{q_m(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)} \\ &= \frac{s(f^{(m-2)}(x))q_{m-1}(x) - l(f^{(m-3)}(x))r(f^{(m-2)}(x))q_{m-2}(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)} \\ &= \frac{s(f^{(m-2)}(x))}{f^{(m-3)}(x)-2}p_{m-1}(x) + \frac{-l(f^{(m-3)}(x))r(f^{(m-2)}(x))}{(f^{(m-4)}(x)-2)(f^{(m-3)}(x)-2)}p_{m-2}(x). \end{aligned}$$

Note that $l(f^{(m-3)}(x)) = f^{(m-3)}(x) - 6 = (f^{(m-4)}(x)-2)(3-f^{(m-4)}(x))$. Then we have

$$p_m(2) = \frac{s(f^{(m-2)}(2))p_{m-1}(2) + 2(2-f^{(m-2)}(2))(5-f^{(m-2)}(2))(3-f^{(m-4)}(2))p_{m-2}(2)}{f^{(m-3)}(2)-2}. \tag{10.1}$$

We will prove the following stronger result than that stated in Lemma A.

$$p_m(2) \sim (-1)^m \text{ and } p_{m+1}(2) + p_m(2) \sim (-1)^{m+1}, \forall m \geq 4. \tag{10.2}$$

Using (10.1), it is easy to check that $p_4(2) = 14064 > 0$ and $p_5(2) = -593514756 < 0$ by a direct computation. Hence (10.2) holds for $m = 4$. In order to use induction, we assume (10.2) holds for m and will prove it for $m+1$.

First, it is easy to get that $p_{m+1}(2) \sim (-1)^{m+1}$, since otherwise $p_{m+1}(2) + p_m(2) \sim (-1)^m$, which contradicts the induction assumption. Hence we only need to prove $p_{m+2}(2) + p_{m+1}(2) \sim (-1)^m$.

Note that from (10.1),

$$
\begin{aligned}
& p_{m+2}(2)+p_{m+1}(2) \\
= \quad & \frac{(s(f^{(m)}(2))+f^{(m-1)}(2)-2)p_{m+1}(2)+2(2-f^{(m)}(2))(5-f^{(m)}(2))(3-f^{(m-2)}(2))p_m(2)}{f^{(m-1)}(2)-2} \\
= \quad & a_m p_{m+1}(2)+b_m(p_{m+1}(2)+p_m(2)),
\end{aligned}
$$

where

$$
a_m=\frac{s(f^{(m)}(2))+f^{(m-1)}(2)-2-2(2-f^{(m)}(2))(5-f^{(m)}(2))(3-f^{(m-2)}(2))}{f^{(m-1)}(2)-2}
$$

and

$$
b_m=\frac{2(2-f^{(m)}(2))(5-f^{(m)}(2))(3-f^{(m-2)}(2))}{f^{(m-1)}(2)-2}.
$$

It is easy to check that $b_m<0$, since $f^{(m)}(2)<f^{(m-1)}(2)<f^{(m-2)}(2)<0$ and $f^{(2)}(2)=-6$ and $m\geq 4$. We will prove that $a_m<0$ also. In fact, the numerator of a_m is $s(\gamma)+f(\beta)-2-2(2-\gamma)(5-\gamma)(3-\beta)$, where we write $\gamma:=f^{(m)}(2)$ and $\beta:=f^{(m-2)}(2)$ for simplicity. By using $\gamma<f(\beta)<\beta\leq-6$, it is easy to get

$$
\begin{aligned}
s(\gamma)+f(\beta)-2 &= (2-\gamma)(4-\gamma)(5-\gamma)-14+3\gamma+f(\beta)-2 \\
&> (2-\gamma)(4-\gamma)(5-\gamma)-16+4\gamma \\
&> (2-\gamma)(4-\gamma)(5-\gamma)-(2-\gamma)(5-\gamma) \\
&= (3-\gamma)(2-\gamma)(5-\gamma),
\end{aligned}
$$

and

$$
3-\gamma>3-f(\beta)=3-\beta(5-\beta)>3-5\beta>2(3-\beta).
$$

Hence we have $s(\gamma)+f(\beta)-2>2(2-\gamma)(5-\gamma)(3-\beta)$. Thus the numerator of a_m is positive. Since the denominator of a_m is obviously negative, we get $a_m<0$.

Now $p_{m+1}(2)\sim(-1)^{m+1}$ as we have proved before, and $p_{m+1}(2)+p_m(2)\sim(-1)^{m+1}$ by the induction assumption. Thus we finally get $p_{m+2}(2)+p_{m+1}(2)\sim(-1)^m$. □

Proof of Theorem A. Recall that in Lemma 4.4, we have proved that $p_{m+1}(\phi_-(\lambda_{m,r_m}))\sim(-1)^{m+r_m-1}$ and $p_{m+1}(\phi_+(\lambda_{m,r_m}))\sim(-1)^{m+r_m}$. Furthermore, λ_{m+1,r_m+1} is the only root of $p_{m+1}(x)$ between $\phi_-(\lambda_{m,r_m})$ and $\phi_+(\lambda_{m,r_m})$.

When $m=2$, we have $p_3(\phi_-(\lambda_{2,r_2}))>0$ and $p_3(\phi_+(\lambda_{2,r_2}))<0$ since r_2 is odd. By Lemma A, we have $p_3(2)>0$. Since λ_{3,r_2+1} is the only root between $\phi_-(\lambda_{2,r_2})$ and $\phi_+(\lambda_{2,r_2})$, we get $\lambda_{3,r_2+1}>2$.

When $m\geq 3$, we have $p_{m+1}(\phi_-(\lambda_{m,r_m}))\sim(-1)^{m-1}$ and $p_{m+1}(\phi_+(\lambda_{m,r_m}))\sim(-1)^m$ since r_m is always even. Still by Lemma A, we have $p_{m+1}(2)\sim(-1)^{m-1}$. Since λ_{m+1,r_m+1} is the only root between $\phi_-(\lambda_{m,r_m})$ and $\phi_+(\lambda_{m,r_m})$, we get $\lambda_{m+1,r_m+1}>2$. □

Remark. *This theorem says that when $m \geq 3$, the first m-level initial eigenvalue is always greater than* 2.

Lemma B. *Let $m \geq 2$. Then $q_m(x) > 0$ whenever $0 < x < \phi_-^{(m)}(6)$.*

Proof. Define $\theta_m(z) = q_m(\phi_-^{(m)}(z))$ on $0 < z < 6$, $\forall m \geq 2$.

When $m = 2$, $\theta_2(z) = q_2(\phi_-^{(2)}(z))$. Noticing that $q_2(x) = s(x)$ and $q_2'(x) = -3x^2+22x-35$, an easy calculation shows that $q_2(x)$ is monotone decreasing when $0 < x < \phi_-^{(2)}(6)$. Hence for $0 < x < \phi_-^{(2)}(6)$, we have $q_2(0) = 26 > q_2(x) > q_2(\phi_-^{(2)}(6)) \approx 12.68$. Thus

$$26 > \theta_2(z) > 12.68, \text{ for } 0 < z < 6. \tag{10.3}$$

When $m = 3$, $\theta_3(z) = q_3(\phi_-^{(3)}(z)) = s(\phi_-^{(3)}(z))\theta_2(z) - l(\phi_-^{(3)}(z))r(\phi_-^{(2)}(z))$ on $0 < z < 6$. Noticing that $s(\phi_-^{(3)}(z)) = q_2(\phi_-^{(3)}(z))$ and $q_2(x)$ is monotone decreasing when $0 < x < \phi_-^{(2)}(6)$, we have

$$s(0) = 26 > s(\phi_-^{(3)}(z)) > s(\phi_-^{(3)}(6)) \approx 22.96.$$

The monotone property of $-l(\phi_-^{(3)}(z))r(\phi_-^{(2)}(z))$ on $0 < z < 6$ implies that

$$-84.21 > -l(\phi_-^{(3)}(z))r(\phi_-^{(2)}(z)) > -120.$$

Hence by using (10.3), we get

$$26 \cdot 26 - 84.21 = 591.80 > \theta_3(z) > 22.96 \cdot 12.68 - 120 = 171.16, \text{ for } 0 < z < 6.$$

Hence $\theta_3(z) \geq 6\theta_2(z) > 0$ on $0 < z < 6$.

We now use induction to prove:

$$\theta_{m+1}(z) \geq 6\theta_m(z) > 0 \text{ on } 0 < z < 6, \text{ for } m \geq 2. \tag{10.4}$$

Of course, it holds for $m = 2$. To use induction, assuming $\theta_{m+1}(z) \geq 6\theta_m(z) > 0$ on $0 < z < 6$, we will prove $\theta_{m+2}(z) \geq 6\theta_{m+1}(z) > 0$ on $0 < z < 6$.

Consider the polynomial $g(x) = s(x) - \frac{1}{6}l(x)r(f(x)) = 6 + \frac{115}{3}x - \frac{194}{3}x^2 + \frac{89}{3}x^3 - \frac{16}{3}x^4 + \frac{1}{3}x^5$. It is easy to compute that

$$g'(x) = \frac{115}{3} - \frac{388}{3}x + 89x^2 - \frac{64}{3}x^3 + \frac{5}{3}x^4 \geq \frac{115}{3} - \frac{388}{3}\phi_-^{(4)}(6) - \frac{64}{3}(\phi_-^{(4)}(6))^3 \approx 36.02 > 0$$

on $0 < x < \phi_-^{(4)}(6)$. Hence $g(x)$ is a monotone increasing function on the interval $[0, \phi_-^{(4)}(6)]$. So $g(x) \geq g(0) = 6$ on $0 < x < \phi_-^{(4)}(6)$.

By using an expansion along the last row of $\theta_{m+2}(z) = q_{m+2}(\phi_-^{(m+2)}(z))$, we have

$$\theta_{m+2}(z) = s(\phi_-^{(m+2)}(z))\theta_{m+1}(z) - \frac{1}{6}l(\phi_-^{(m+2)}(z))r(\phi_-^{(m+1)}(z)) \cdot 6\theta_m(z).$$

By the induction assumption and the fact that $\phi_-^{(m+2)}(z) < \phi_-^{(m+1)}(z) < 2$, we have

$$\begin{aligned}\theta_{m+2}(z) &\geq s(\phi_-^{(m+2)}(z))\theta_{m+1}(z) - \frac{1}{6}l(\phi_-^{(m+2)}(z))r(\phi_-^{(m+1)}(z))\theta_{m+1}(z) \\ &= g(\phi_-^{m+2}(z))\theta_{m+1}(z).\end{aligned}$$

Since $0 < \phi_-^{(m+2)}(z) < \phi_-^{(4)}(6)$ on $0 < z < 6$ when $m \geq 2$, we have $g(\phi_-^{(m+2)}(z)) \geq 6$. Hence

$$\theta_{m+2}(z) \geq 6\theta_{m+1}(z) > 0.$$

Hence we have proved (10.4) holds for $m+1$. From (10.4), we get the desired result. □

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(Hua Qiu) DEPARTMENT OF MATHEMATICS, NANJING UNIVERISITY, NANJING, 210093, CHINA

E-mail address: huaqiu@nju.edu.cn