RESTRICTIONS OF LAPLACIAN EIGENFUNCTIONS TO EDGES IN THE SIERPINSKI GASKET

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ABSTRACT. In this paper, we study the restrictions of both the harmonic functions and the eigenfunctions of the symmetric Laplacian to edges of pre-gaskets contained in the Sierpinski gasket. For a harmonic function, its restriction to any edge is either monotone or having a single extremum. For an eigenfunction, it may have several local extrema along edges. We prove general criteria, involving the values of any given function at the endpoints and midpoint of any edge, to determine which case it should be, as well as the asymptotic behavior of the restriction near the endpoints. Moreover, for eigenfunctions, we use spectral decimation to calculate the exact numbers of the local extrema along edges. This confirm, in a more general situation, a conjecture of K. Dalrymple, R.S. Strichartz and J.P. Vinson [4] on the behavior of the restrictions to edges of the basis Dirichlet eigenfunctions, suggested by the numerical data.

1. INTRODUCTION

The study of analysis on *post critically finite (p.c.f.) self-similar sets* has been extensively developed since Kigami's analytic construction of the fractal Laplacian on the *Sierpinski gasket* [8, 9]. Such a Laplacian has also been obtained independently by Goldstein [6] and Kusuoka [13] using an indirect probabilistic approach. The harmonic functions and Laplacian eigenfunctions play important roles in the analysis on the Sierpinski gasket, as the Laplacian Δ is the fundamental differential operator on which the analysis is based. Harmonic functions are the solutions of $\Delta f = 0$, and Laplacian eigenfunctions are non-zero functions satisfying $-\Delta f = \lambda f$ for some eigenvalue λ. Fukushima and Shima [5, 17] studied the spectrum of the Dirichlet Laplacian, whose eigenfunctions vanish on the boundary, and determined explicitly all the eigenvalues and eigenfunctions by using a *spectral decimation* method introduced by physicists Rammal and Toulouse [16]. See [1, 10, 12, 21] and the references therein for related works on the eigenvalues and eigenfunctions. In this paper, we consider the local structure of these special functions on the Sierpinski gasket. More precisely, for any edge of the pre-gaskets of the Sierpinski gasket, we are interested in the local behavior of the restrictions of these special functions.

Recall the Sierpinski gasket, denoted by $\mathcal{SG}$, is the attractor of the *iterated function system (i.f.s.)* in $\mathbb{R}^2$ consisting of three contractive mappings F_0, F_1, F_2 with contraction ratio $\frac{1}{2}$ and fixed points q_0, q_1, q_2 which are the three vertices of an equilateral triangle. So

$$\mathcal{SG} = \bigcup_{i=0}^{2} F_i \mathcal{SG},$$

see Fig. 1. We denote by $V_0 = \{q_0, q_1, q_2\}$ the *boundary* of $\mathcal{SG}$ and $V_m = \bigcup_{i=0}^{2} F_i V_{m-1}$ for $m \geq 1$, inductively. Write $V_* = \bigcup_{m \geq 0} V_m$. For a *word* $w = w_1 \cdots w_m$ of *length* $|w| = m$ with $w_i \in \{0, 1, 2\}$, denote $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$, and call $F_w \mathcal{SG}$ a *m-cell* of $\mathcal{SG}$. Obviously, $V_m = \bigcup_{|w|=m} F_w V_0$. We approximate $\mathcal{SG}$ from within by a sequence of *graphs* $\Gamma_m = (V_m, \sim_m)$ with vertices V_m and edge relation $\sim_m$ defined as $x \sim_m y$ if and only if there is a word w of length m such that $x \neq y$ and $x, y \in F_w V_0$.
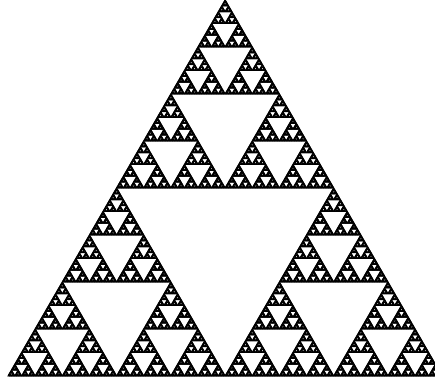

Figure 1. The Sierpinski gasket

The *(symmetric) Laplacian* Δ on $\mathcal{SG}$ is the renormalized limit of *graph Laplacians* Δ_m on Γ_m defined as:

$$\Delta_m u(x) = \sum_{y \sim_m x} \big(u(y) - u(x) \big) \text{ for } x \in V_m \setminus V_0,$$

and

(1.1) $$\Delta u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x)$$

with $u \in dom\Delta$ if and only if the right side of (1.1) converges uniformly on $V_* \setminus V_0$ to a continuous function.

A function h is called *harmonic* if it minimize the graph energy $\sum_{x \sim_m y} \big(h(x) - h(y) \big)^2$ on Γ_m from each level to its next level. The space of harmonic functions on $\mathcal{SG}$ is 3-dimensional and the values at vertices in V_0 can be freely assigned. The harmonic functions are also graph harmonic, i.e., they are the solutions of $\Delta_m h = 0$ for each $m \geq 1$. So it is elementary to calculate the values of h on V_1 from those on V_0 and recursively to obtain the values on V_m for any m. See Fig. 2 for some pictures of harmonic functions on $\mathcal{SG}$.
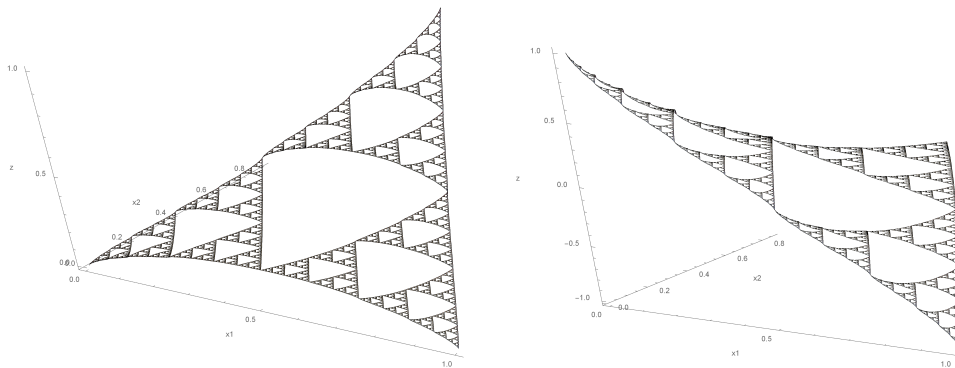
Figure 2. The harmonic functions with boundary values $(0,0,1)$ **and** $(1,-1,0)$

Similar to that of the harmonic functions, for an *eigenfunction* u satisfying $-\Delta u = \lambda u$, there is a minimal integer m_0, called the *level of birth*, such that for $m \geq m_0$, the restriction of u to V_m is also an eigenfunction of the graph Laplacian associated with an eigenvalue λ_m, i.e., $-\Delta_m u = \lambda_m u$, satisfying that

(1.2) $$\lambda_m = \lambda_{m+1}(5 - \lambda_{m+1})$$

and

(1.3) $$\lambda = \frac{3}{2} \lim_{m \to \infty} 5^m \lambda_m.$$

The identity (1.2) implies that

$$\lambda_{m+1} = \frac{5 + \varepsilon_{m+1}\sqrt{25 - 4\lambda_m}}{2} \quad \text{for } \varepsilon_{m+1} = \pm 1, m \geq m_0.$$

Call $\{\varepsilon_m\}_{m > m_0}$ the ε-*sequence* of u and λ. Note that to ensure the limit in (1.3) exists, ε_m is only permitted to be positive 1 for finitely many values of m. So there is an minimal integer $m_1 > m_0$, called the *level of fixation* such that for each $m \geq m_1$, $\varepsilon_m = -1$.

For the Dirichlet eigenfunctions, we can classify them into three families, which we called 2-*series*, 5-*series* and 6-*series* eigenfunctions, depending on the initial eigenvalue $\lambda_{m_0} = 2, 5$ or 6. The 2-series eigenfunctions all have $m_0 = 1$, the 5-series eigenfunctions have $m_0 \geq 1$, and the 6-series eigenfunctions have $m_0 \geq 2$ together with $\lambda_{m_0+1} = 3$ ($\varepsilon_{m_0+1} = 1$). The 2-series eigenvalues all have multiplicity 1, while the eigenvalues in the other series exhibit higher multiplicity. The case for the Neumann eigenfunctions is very similar. See Section 3.3 in [19] for a complete description of the Dirichlet (Neumann) spectrum for the $\mathcal{SG}$ and a basis (not orthonormal) for all Dirichlet (Neumann) eigenfunctions. Here we only mention that all such basis functions are formed from gluing of scaled and rotated copies of functions as shown in Fig. 3. See Fig. 4 for some pictures of the Dirichlet eigenfunctions.

The reader is referred to the books [11] and [19] for exact definitions and any unexplained notations.
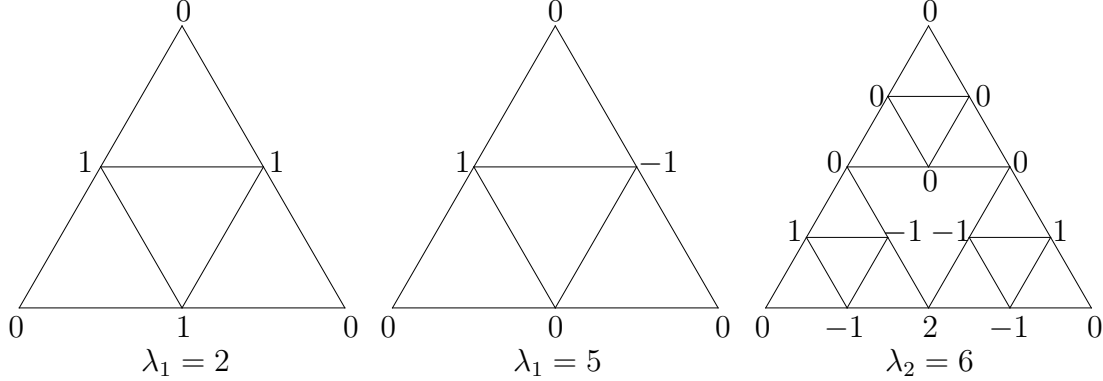
Figure 3. Three typical Dirichlet eigenfunctions with $\lambda_1 = 2$ or 5 and $\lambda_2 = 6$
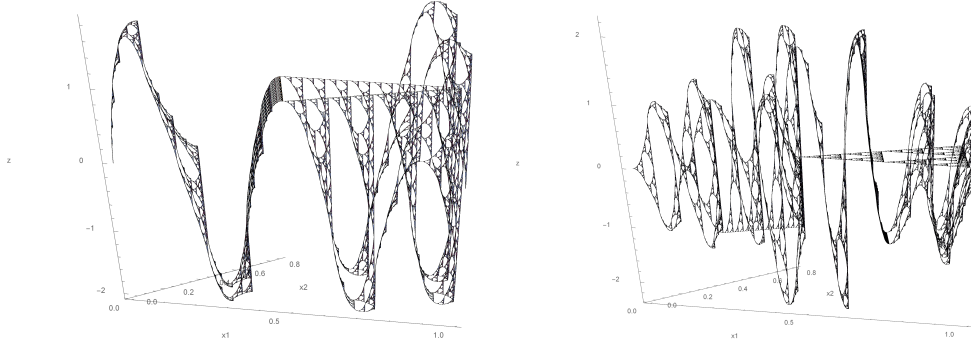
Figure 4. The eigenfunctions $\psi_2^{(2)}$ and $\psi_1^{(6)}$

In [4], it was proved that the restriction of any harmonic function along any edge is either monotone or having a single extremum. And from the numerical computations shown in Fig. 4, it appears that there is a much more complicated structure for the Laplacian eigenfunctions. The restrictions sometimes are monotone, and sometimes may attain several local extrema. Moreover, for a graph eigenfunction on Γ_m for some m, if we index all the eigenfunctions, initialed from it and continued by the spectral decimation (choosing all but a finite number of $\varepsilon_{m'} = -1$ with $m' \geq m+1$), by a natural number n so that the corresponding eigenvalue is increasing of n, then the exact number of the local extrema along edges seems to satisfy some pattern, which is a function of n. In particular, let's start from the three typical Dirichlet eigenfunctions shown in Fig. 3 to get three basic families of eigenfunctions, denoted by $\{\psi_n^{(2)}\}_{n \geq 0}$, $\{\psi_n^{(5)}\}_{n \geq 0}$ and $\{\psi_n^{(6)}\}_{n \geq 0}$, with $\psi_0^{(2)}$, $\psi_0^{(5)}$ choosing all $\varepsilon_m = -1$ for $m \geq 2$, and $\psi_0^{(6)}$ choosing $\varepsilon_3 = 1$ and all $\varepsilon_m = -1$ for $m \geq 4$. It was conjectured in [4] that the numbers of the local extrema of the restrictions of $\psi_n^{(2)}$, $\psi_n^{(5)}$ and $\psi_n^{(6)}$ to the bottom edge of $\mathcal{SG}$ are $2n + 1$, $4[\frac{n}{2}] + 2$ and $8n + 3$ respectively, and the number of local extrema of the function $\psi_n^{(5)}$ restricted to the left edge of $\mathcal{SG}$ is $2[\frac{n}{2}] + n + 1$, where we use $[x]$ to denote the largest integer no larger than x. This conjecture was observed by numerical computations, see Fig. 10, Fig. 12 and Fig. 14 for the patterns.

Let E be an edge of a m-cell in $\mathcal{SG}$, call it a m-edge. Denote the endpoint of E by p_0^E, p_1^E and the midpoint by p_{01}^E. Let f be either a harmonic function or an eigenfunction on $\mathcal{SG}$, which is non-constant on E. Denote

$$r^E(f) = \frac{f(p_1^E) - f(p_{01}^E)}{f(p_{01}^E) - f(p_0^E)}$$

$\left(\text{allow } r^E(f) \text{ to be } \infty, \text{ not distinguishing between } +\infty \text{ and } -\infty, \text{ when } f(p_{01}^E) = f(p_0^E)\right)$. We will prove criteria involving the parameter $r^E(f)$ to determine whether the restriction of f to E is monotone or not.

For $i = 0, 1$, denote $\phi_i^E : \mathbb{R}^2 \to \mathbb{R}^2$ the contractive mapping by $\phi_i^E(x) = \frac{1}{2}(x - p_i^E) + p_i^E$, and write $E_\tau = \phi_\tau^E E := \phi_{\tau_1}^E \circ \cdots \phi_{\tau_l}^E E$ for any word $\tau = \tau_1 \cdots \tau_l$ of length $|\tau| = l$, with $\tau_i = 0$ or 1. Obivously, E_τ is a $(m + l)$-edge in $\mathcal{SG}$.

For the function f, denote

$$\alpha^E(f) = \lim_{l \to \infty} \frac{|f(p_1^{E_{0^l}}) - f(p_0^E)|}{|p_1^{E_{0^l}} - p_0^E|} \text{ and } \beta^E(f) = \lim_{l \to \infty} \frac{|f(p_0^{E_{1^l}}) - f(p_1^E)|}{|p_0^{E_{1^l}} - p_1^E|}.$$

The values of $\alpha^E(f)$ and $\beta^E(f)$ reflect the asymptotic behavior of the restriction $f|_E$ near the boundary point p_0^E and p_1^E of the edge E. They are closely related to the fractal derivatives of f at the points p_0^E, p_1^E introduced in [20]. In fact, if we denote by $\partial_n f(p_0^E)$ and $\partial_T f(p_0^E)$ the normal derivative and tangential derivative of f at p_0^E (inside the cell containing E) respectively, it is not hard to check that for a function f in $dom\Delta$, $\alpha^E(f) = \infty$ whenever $\partial_n f(p_0^E) \neq 0$ and $\partial_T f(p_0^E)$ exists; and $\alpha^E(f) = 0$ whenever $\partial_n f(p_0^E) = 0$. The conclusion for $\beta^E(f)$ is similar. See [19, 20] for further information for the fractal derivatives.

Then for harmonic functions and eigenfunctions, we have:

Theorem 1.1. *Let h be a non-constant harmonic function and E be a m-edge in $\mathcal{SG}$. Then*

(1) if $\frac{1}{4} \leq r^E(h) \leq 4$, then $h|_E$ is strictly monotone;

(2) otherwise $h|_E$ has a single extremum.

Moreover, we have $\alpha^E(h) = 0$ if $r^E(h) = 4$, $\beta^E(h) = 0$ if $r^E(h) = \frac{1}{4}$, and otherwise $\alpha^E(h) = \infty$, $\beta^E(h) = \infty$.

Theorem 1.2. *Let u be a Dirichlet (or Neumann) eigenfunction and E be a m-edge in $\mathcal{SG}$ with $m \geq m_1 - 1$. Suppose u is non-constant on E, then*

(1) if $\frac{1}{4 - \lambda_{m+1}} \leq r^E(u) \leq 4 - \lambda_{m+1}$, then $u|_E$ is strictly monotone;

(2) otherwise $u|_E$ has a single extremum.

Moreover, we have $\alpha^E(u) = 0$ if $r^E(u) = 4 - \lambda_{m+1}$, $\beta^E(u) = 0$ if $r^E(u) = \frac{1}{4 - \lambda_{m+1}}$, and otherwise $\alpha^E(u) = \infty$, $\beta^E(u) = \infty$.

We remark here that part (1) in Theorem 1.1 is exact Theorem 1 in [4]. The assumption that u is non-constant along E in Theorem 1.2 is necessary since there indeed exist eigenfunctions which are constant along certain edges, see Theorem 5 in [4], see

also Fig. 4. The result in Theorem 1.2 also holds for general eigenfunctions when m is sufficiently large since $\lambda_m \to 0$ as m goes to ∞.

Note that the assumption $m \geq m_1 - 1$ in Theorem 1.2 makes all $\varepsilon_{m+i} = -1$ for $i \geq 1$. Without this assumption, things are much more complicated. In fact, when $m < m_1 - 1$, in general, the restriction of u to E will attain several local extrema. The following theorem calculate the exact number of extrema that occur in this case.

Theorem 1.3. *Let u be a Dirichlet (or Neumann) eigenfunction and E be a m-edge in $\mathcal{SG}$ with $m \geq m_0$ (require $m > m_0$ if u is a 6-series eigenfunction). Suppose u is non-constant on E, and let $\{\psi_n\}_{n \geq 0}$ be the sequence of all eigenfunctions initialed from $u|_{V_m}$ with the ordering making the corresponding eigenvalue increasing of n. Then*

$$N^E(\psi_n) = \begin{cases} 2[\frac{n+1}{2}], & if \ \frac{1}{4 - \lambda_{m+1}^{(0)}} < r^E(\psi_0) < 4 - \lambda_{m+1}^{(0)}, \\ n, & if \ r^E(\psi_0) = \frac{1}{4 - \lambda_{m+1}^{(0)}} \ or \ 4 - \lambda_{m+1}^{(0)}, \\ 2[\frac{n}{2}] + 1, & otherwise, \end{cases}$$

where $N^E(\psi_n)$ denotes the number of local extrema of ψ_n in E, and $\lambda_{m+1}^{(0)}$ is the $(m+1)$-th graph eigenvalue of ψ_0, i.e., $\lambda_{m+1}^{(0)} = \frac{5 - \sqrt{25 - 4\lambda_m}}{2}$.

Note that Theorem 1.3 is a natural extension of Theorem 1.2, since in case of Theorem 1.2, u is ψ_0. By using Theorem 1.3, with a bit of more work, we can check that the conjecture in [4] which we mentioned before is true. That is

Theorem 1.4. *Let $\{\psi_n^{(2)}\}_{n \geq 0}$, $\{\psi_n^{(5)}\}_{n \geq 0}$ and $\{\psi_n^{(6)}\}_{n \geq 0}$ be the three basic families of eigenfunctions initialed from the graph eigenfunctions in Fig. 3. The numbers of local extrema along the bottom edge of $\mathcal{SG}$ of these functions are $2n + 1$, $4[\frac{n}{2}] + 2$ and $8n + 3$ respectively, and the number of local extrema of $\psi_n^{(5)}$ along the left edge of $\mathcal{SG}$ is $2[\frac{n}{2}] + n + 1$.*

There are a number of interesting features observable in the numerical computations of harmonic functions and Laplacian eigenfunctions for $\mathcal{SG}$. In this paper we have deliberately restricted our attention to the local behavior of these special functions restricted to edges. See [3] for a related result concerning the gradients of the eigenfunctions on the Sierpinski gasket. In a series of papers [7, 14, 15] of Strichartz and his coauthors, a complete theory of another class of special functions, the polynomials on the Sierpinski gasket, has been developed. They are defined as solutions of higher order Laplace equations. See [15] for some numerical observations on the zero sets of the restrictions of orthogonal polynomials to edges of $\mathcal{SG}$. It seems to indicate that the zeros behave like the zeros of the restriction of eigenfunctions considered in our setting. All these special functions are closely related to analogs of fractal Fourier series.

The paper is organized as follows. In Section 2, we will deal with the restrictions of harmonic functions to edges and prove Theorem 1.1. In Section 3, we come to the Laplacian eigenfunctions and prove Theorem 1.2 and 1.3. As an application of Theorem 1.3, we will prove Theorem 1.4 in Section 4.

2. RESTRICTIONS OF HARMONIC FUNCTIONS TO EDGES

In this section, we study the local behavior of the restrictions of harmonic functions to edges. We will prove Theorem 1.1.

Let h be a harmonic function. There is an explicit *extension algorithm*, the "$\frac{1}{5} - \frac{2}{5}$" rule, for computing the values of harmonic functions at all vertices in V_* in terms of the boundary values. That is, for any cell in $\mathcal{SG}$ whose boundary vertices are denoted by p_0, p_1, p_2, we have $h(p_{ij}) = \frac{2}{5}h(p_i) + \frac{2}{5}h(p_j) + \frac{1}{5}h(p_k)$, where p_{ij} is the midpoint of the edge joining p_i and p_j, for distinct $i, j, k \in \{0, 1, 2\}$.

From now on, we will assume that h is a non-constant. Let E be a m-edge in $\mathcal{SG}$. It is known that h is also non-constant on E. We denote p_0^E, p_1^E, p_{01}^E, $r^E(h)$, and E_τ as we did in Section 1, where τ is any word $\tau = \tau_1 \cdots \tau_l$ with $\tau_i = 0$ or 1 and $|\tau| = l \geq 0$. Without causing any confusion, we will remove the superscript E for simplicity, and write r, r^τ, p_0^τ, p_1^τ, p_{01}^τ instead of $r^E(h)$, $r^{E_\tau}(h)$, $p_0^{E_\tau}$, $p_1^{E_\tau}$, $p_{01}^{E_\tau}$ for short. Obviously, for any word τ, $p_{01}^\tau = p_0^{\tau 1} = p_1^{\tau 0}$, and

$$r = \frac{h(p_1) - h(p_{01})}{h(p_{01}) - h(p_0)} \text{ and } r^\tau = \frac{h(p_1^\tau) - h(p_{01}^\tau)}{h(p_{01}^\tau) - h(p_0^\tau)},$$

where we allow r, r^τ to take ∞, gluing $\pm\infty$ together as a single point, when $h(p_{01}) = h(p_0)$ or $h(p_{01}^\tau) = h(p_0^\tau)$. If we denote by p_2 the other vertex in the m-cell containing E, then we also have

$$r = \frac{3h(p_1) - 2h(p_0) - h(p_2)}{2h(p_1) + h(p_2) - 3h(p_0)}$$

by using the "$\frac{1}{5} - \frac{2}{5}$" extension rule.

There is an algorithm which enables us to calculate the restriction of a harmonic function to any edge in $\mathcal{SG}$. Let p_{001}^τ be the midpoint of the edge connecting p_0^τ and p_{01}^τ. Then

$$(2.1) \qquad h(p_{001}^\tau) = \frac{4}{5}h(p_{01}^\tau) + \frac{8}{25}h(p_0^\tau) - \frac{3}{25}h(p_1^\tau).$$

See Algorithm 2.2 in [4] for a proof of (2.1) by using the "$\frac{1}{5} - \frac{2}{5}$" rule.

To prove Theorem 1.1, we need some lemmas.

Lemma 2.1. *For any word τ, we have*

$$(2.2) \qquad r^{\tau 0} = \frac{8 + 3r^\tau}{17 - 3r^\tau} \text{ and } r^{\tau 1} = \frac{17r^\tau - 3}{8r^\tau + 3},$$

where the two mappings in (2.2) are both bijections: $\mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$. In particular, $r^{\tau 0} = -1$, $r^{\tau 1} = \frac{17}{8}$ when $r^\tau = \infty$. See Fig. 5 for the graphs of $r^{\tau 0}$ and $r^{\tau 1}$ of r^τ.

Proof. The first equality in (2.2) follows from (2.1) by an elementary calculation. The second equality follows similarly by a symmetric consideration. It can be directly checked that the two mappings are bijections. $\square$
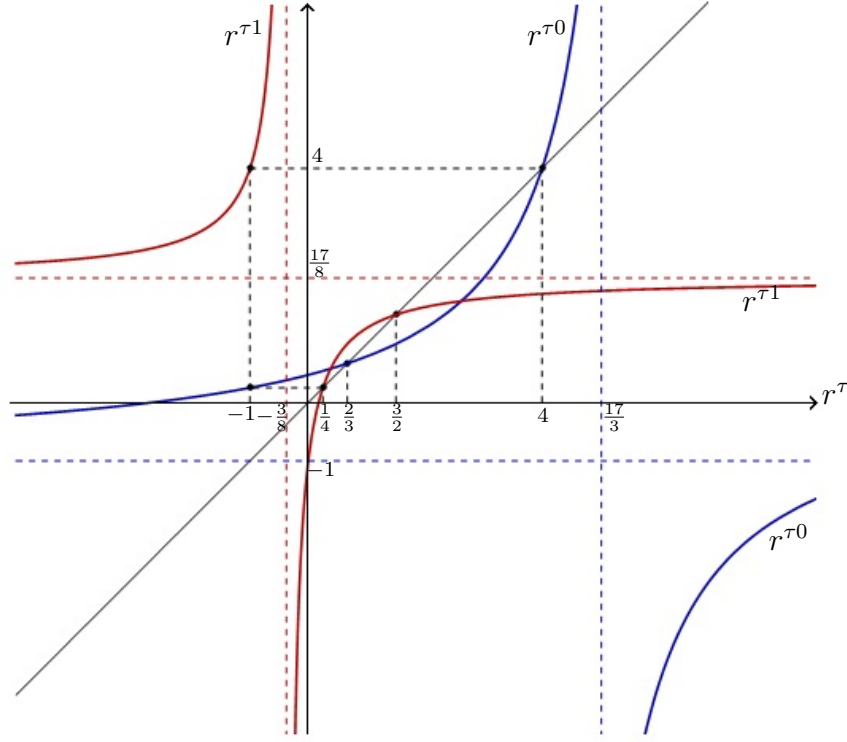
Figure 5. Graphs of $r^{\tau 0}$ and $r^{\tau 1}$ of r^{τ}

Remark 2.2. *It is easy to check from Fig. 5 that the mapping $f(x) = \frac{8+3x}{17-3x}$ from $\mathbb{R} \cup \{\infty\}$ onto $\mathbb{R} \cup \{\infty\}$ has a single stable fixed point $x = \frac{2}{3}$ and a single unstable fixed point $x = 4$, which means that for any $x \neq 4$, the iterated sequence $x, f(x), f(f(x)), \cdots$ will go to $\frac{2}{3}$. Similarly, the mapping $f(x) = \frac{17x-3}{8x+3}$ has a single stable fixed point $x = \frac{3}{2}$ and a single unstable fixed point $x = \frac{1}{4}$.*

Lemma 2.3. *For any word τ, the restriction of h to E can not attain an extremum at p_{01}^{τ} if $r^{\tau} \neq -1$.*

Proof. Without loss of generality, we only need to prove that h can not attain an extremum at p_{01} if $r \neq -1$. Suppose $r \neq -1$, then $h(p_0) \neq h(p_1)$. We divide the restriction of h to E into its symmetric part h_S and antisymmetric part h_A. That is,

$$h_S(p_0) = h_S(p_1) = \frac{1}{2}\big(h(p_0) + h(p_1)\big), \quad h_S(p_{01}) = h(p_{01}),$$

$$h_A(p_0) = -h_A(p_1) = \frac{1}{2}\big(h(p_0) - h(p_1)\big), \quad h_A(p_{01}) = 0,$$

and both h_S and h_A satisfy (2.1). Then we have $h = h_S + h_A$ on E, and $r(h_S) = -1$(without loss of generality, assuming that h_S is not a constant function), $r(h_A) = 1$. Notice that h_A can not attain an extremum at p_{01}^{τ} from its antisymmetry.

Now we consider the asymptotic behavior of h_S and h_A near the point p_{01}. By Lemma 2.1 and Remark 2.2, $\lim_{l\to\infty} r^{10^l}(h_S) = 4$, $\lim_{l\to\infty} r^{01^l}(h_S) = \frac{1}{4}$, $\lim_{l\to\infty} r^{10^l}(h_A) = \frac{2}{3}$,

and $\lim_{l\to\infty} r^{01^l}(h_A) = \frac{3}{2}$. Thus,

$$
\begin{aligned}
\beta^{E_0}(h_S) &= \lim_{l\to\infty} \frac{|h_S(p_0^{01^l}) - h_S(p_{01})|}{|p_0^{01^l} - p_{01}|} \\
&= \frac{|h_S(p_0) - h_S(p_{01})|}{|p_0 - p_{01}|} \lim_{l\to\infty} 2^l \Big| \frac{r^0(h_S)}{1 + r^0(h_S)} \cdot \frac{r^{01}(h_S)}{1 + r^{01}(h_S)} \cdots \frac{r^{01^{l-1}}(h_S)}{1 + r^{01^{l-1}}(h_S)} \Big| \\
&= 0, \\
\alpha^{E_1}(h_S) &= \lim_{l\to\infty} \frac{|h_S(p_1^{10^l}) - h_S(p_{01})|}{|p_1^{10^l} - p_{01}|} \\
&= \frac{|h_S(p_1) - h_S(p_{01})|}{|p_1 - p_{01}|} \lim_{l\to\infty} 2^l \Big| \frac{1}{1 + r^1(h_S)} \cdot \frac{1}{1 + r^{10}(h_S)} \cdots \frac{1}{1 + r^{10^{l-1}}(h_S)} \Big| \\
&= 0.
\end{aligned}
$$

And for h_A, similar argument yields that,

$$
\begin{aligned}
\beta^{E_0}(h_A) &= \lim_{l\to\infty} \frac{|h_A(p_0^{01^l}) - h_A(p_{01})|}{|p_0^{01^l} - p_{01}|} = \infty, \\
\alpha^{E_1}(h_A) &= \lim_{l\to\infty} \frac{|h_A(p_1^{10^l}) - h_A(p_{01})|}{|p_1^{10^l} - p_{01}|} = \infty.
\end{aligned}
$$

On the other hand, by symmetry, $h_S(p_0^{01^l}) - h_S(p_{01})$ and $h_S(p_1^{10^l}) - h_S(p_{01})$ have same signs, and $h_A(p_0^{01^l}) - h_A(p_{01})$ and $h_A(p_1^{10^l}) - h_A(p_{01})$ have opposite signs for l large enough.

Since $h = h_A + h_S$ on E, we then obtain that $\beta^{E_0}(h) = \infty$, $\alpha^{E_1}(h) = \infty$, and $h(p_0^{01^l}) - h(p_{01})$ and $h(p_1^{10^l}) - h(p_{01})$ have opposite signs for l large enough. It follows that the restriction of h to E can not attain an extremum at p_{01}. $\qquad\square$

Proof of Theorem 1.1. From the definition of r^τ and the continuity of $h|_E$ we know that if $r^\tau \in (0, +\infty)$ for any word τ, then $h|_E$ is strictly monotone. Conversely, if $r^\tau \in (-\infty, 0)$ for some τ, then $h|_E$ has at least one extremum.

It is easy to check that if $r^\tau \in (\frac{1}{4}, 4)$ for some τ, then $r^{\tau 0} \in (\frac{1}{4}, 4)$ and $r^{\tau 1} \in (\frac{1}{4}, 4)$. So if we assume that $r \in (\frac{1}{4}, 4)$, then by iteration, we always have $r^\tau \in (\frac{1}{4}, 4)$ for any word τ, and hence $h|_E$ is strictly monotone. Moreover, similar to the proof of Lemma 2.3, we have $\lim_{l\to\infty} r^{0^l} = \frac{2}{3}$, $\lim_{l\to\infty} r^{1^l} = \frac{3}{2}$, and thus

$$
\begin{aligned}
\alpha &= \lim_{l\to\infty} \frac{|h(p_1^{0^l}) - h(p_0)|}{|p_1^{0^l} - p_0|} \\
&= \frac{|h(p_1) - h(p_0)|}{|p_1 - p_0|} \lim_{l\to\infty} 2^l \Big| \frac{1}{1 + r} \cdot \frac{1}{1 + r^0} \cdots \frac{1}{1 + r^{0^{l-1}}} \Big| = \infty,
\end{aligned}
$$

$$\beta = \lim_{l \to \infty} \frac{|h(p_0^{1^l}) - h(p_1)|}{|p_0^{1^l} - p_1|}$$

$$= \frac{|h(p_1) - h(p_0)|}{|p_1 - p_0|} \lim_{l \to \infty} 2^l \Big| \frac{r}{1+r} \cdot \frac{r^1}{1+r^1} \cdot \ldots \cdot \frac{r^{1^{l-1}}}{1+r^{1^{l-1}}} \Big| = \infty,$$

where $\alpha = \alpha^E(h)$, $\beta = \beta^E(h)$ for short.

If we assume $r = 4$, then $r^0 = 4$ and $r^1 \in (\frac{1}{4}, 4)$ by Lemma 2.1. By iteration, we always have $r^\tau \in (\frac{1}{4}, 4]$ for any τ, so $h|_E$ is still strictly monotone. However, since $r^{0^l} = 4$ for any $l \geq 0$, we have $\alpha = \lim_{l \to \infty} \frac{|h(p_1^{0^l}) - h(p_0)|}{|p_1^{0^l} - p_0|} = 0$ instead. And there is a similar argument if we assume $r = \frac{1}{4}$. Thus (1) of the theorem follows.

Now we consider the case that $r \in (-\infty, \frac{1}{4}) \cup (4, +\infty) \cup \{\infty\}$. If in particular $r = -1$, then $h|_E$ has at least one extremum. But in this case, $r^0 = \frac{1}{4}$ and $r^1 = 4$, so both $h|_{E_0}$ and $h|_{E_1}$ are strictly monotone as we discussed before. Thus $h|_E$ has a single extremum at p_{01}. If in addition $r \neq -1$, then by Lemma 2.3, p_{01} cannot be an extreme point, and we can easily check from Fig. 5 that either $r^0 \in (\frac{1}{4}, 4)$ and $r^1 \notin [\frac{1}{4}, 4]$ when $r \in (-1, \frac{1}{4})$, or $r^0 \notin [\frac{1}{4}, 4]$ and $r^1 \in (\frac{1}{4}, 4)$ when $r \in (-\infty, -1) \cup (4, +\infty) \cup \{\infty\}$. So that either $h|_{E_0}$ is strictly monotone and $r^1 \notin [\frac{1}{4}, 4]$, or $h|_{E_1}$ is strictly monotone and $r^0 \notin [\frac{1}{4}, 4]$. Thus we need continue to consider the local behavior of h restricted to E_1 or E_0. Repeat this process recursively, by Lemma 2.3, using a binary search, one can find that there is no more than one extremum for $h|_E$.

Next we show that there does exist an extremum in this case. If $r \in (\frac{17}{3}, +\infty) \cup \{\infty\}$, then $r^0 < 0$, so h attains an extremum in E_0. If $r = \frac{17}{3}$, then $r^0 = \infty$ and $r^{00} = -1$ and thus h attains an extremum at p_{01}^{00}. If $r \in (4, \frac{17}{3})$, then since $x = 4$ is the single unstable fixed point of the mapping $f(x) = \frac{8+3x}{17-3x}$, there exists an number $k \in \mathbb{N}$ such that $r^{0^k} \in [\frac{17}{3}, +\infty) \cup \{\infty\}$. Then the case is similar to the case that $r \in [\frac{17}{3}, +\infty) \cup \{\infty\}$. In summary, we have proved that h attains at least one extremum in E when $r \in (4, +\infty) \cup \{\infty\}$. For the case $r \in [0, \frac{1}{4})$, it is similar by symmetric consideration. For the case $r \in (-\infty, 0)$, the existence of extrema is obvious.

Combining the binary search as we described before, we have proved that there indeed exists only one extremum for $h|_E$ when $r \notin [\frac{1}{4}, 4]$, and thus (2) of the theorem holds. The boundary behavior of the restriction is similar to the case for $r \in (\frac{1}{4}, 4)$.

Thus we have complete the proof of the theorem. □

Remark 2.4. *In case of $r \notin [\frac{1}{4}, 4]$, the above proof actually yields a binary searching algorithm for the unique extreme point of $h|_E$. In fact, we can recursively define a parameter $\theta(r)$ as a function of r as*

$$\theta(r) = \begin{cases} \frac{1}{2}, & \text{if } r = -1, \\ \frac{1}{2} + \frac{1}{2}\theta(r^1), & \text{if } r \in (-1, \frac{1}{4}), \\ \frac{1}{2}\theta(r^0), & \text{otherwise.} \end{cases}$$

Then $p_0 + \theta(r)(p_1 - p_0)$ is the unique extreme point of $h|_E$.

3. RESTRICTIONS OF EIGENFUNCTIONS TO EDGES

Now we come to the restrictions of Laplacian eigenfunctions to edges. We will prove Theorem 1.2 and Theorem 1.3 in this section.

Let u be a Dirichlet or Neumann eigenfunction associated with an eigenvalue λ, i.e., $-\Delta u = \lambda u$. We use m_0 and m_1 to denote its level of birth and level of fixation respectively. For each $m \geq m_0$, we write λ_m the graph eigenvalue such that $-\Delta_m u = \lambda_m u$ holds. There is a *local extension algorithm* similar to that of the harmonic functions, which extends a graph eigenfunction u on V_m associated with λ_m to a graph eigenfunction on V_{m+1} associated with λ_{m+1}, provided that $\lambda_{m+1} \neq 2, 5$. For any m-cell whose boundary vertices are p_0, p_1, p_2, we have

$$(3.1) \qquad u(p_{ij}) = \frac{(4 - \lambda_{m+1})\big(u(p_i) + u(p_j)\big) + 2u(p_k)}{(2 - \lambda_{m+1})(5 - \lambda_{m+1})}$$

with p_{ij} being the vertex in V_{m+1} in that cell, opposite to p_k, for distinct $i, j, k \in \{0, 1, 2\}$.

We summarize this spectral decimation property into the following proposition, see also Lemma 3.2.1 in [19].

Proposition 3.1. *Suppose* $\lambda_{m+1} \neq 2, 5, 6$ *and* λ_m *is given by (1.2). If* $u|_{V_m}$ *is a* λ_m*-eigenfunction of* $-\Delta_m$ *and is extended to* V_{m+1} *by (3.1), then* $u|_{V_{m+1}}$ *is a* λ_{m+1}*-eigenfunction of* $-\Delta_{m+1}$. *Conversely, if* $u|_{V_{m+1}}$ *is a* λ_{m+1}*-eigenfunction of* $-\Delta_{m+1}$, *then* $u|_{V_m}$ *is a* λ_m*-eigenfunction of* $-\Delta_m$.

This spectral decimation property was proved by Fukushima and Shima [5, 17] mathematically. It is not true for general Laplacians on p.c.f. self-similar sets, even for other non-symmetric Laplacians on $\mathcal{SG}$. See [18] for the extension of this property to some fully symmetric fractals.

Let E be a m-edge in $\mathcal{SG}$. Throughout this section, we always assume that u is non-constant on E. We still use the notations p_0, p_1, p_{01}, p_0^τ, p_1^τ, p_{01}^τ, r, r^τ as we did for the harmonic functions, where τ is any word $\tau = \tau_1 \cdots \tau_l$ with $\tau_i = 0$ or 1 and $|\tau| = l \geq 0$. In particular,

$$r = \frac{u(p_1) - u(p_{01})}{u(p_{01}) - u(p_0)} \text{ and } r^\tau = \frac{u(p_1^\tau) - u(p_{01}^\tau)}{u(p_{01}^\tau) - u(p_0^\tau)}.$$

By using the extension algorithm (3.1), we could write

$$(3.2) \qquad r = \frac{(6 - 6\lambda_{m+1} + \lambda_{m+1}^2)u(p_1) - (4 - \lambda_{m+1})u(p_0) - 2u(p_2)}{(4 - \lambda_{m+1})u(p_1) + 2u(p_2) - (6 - 6\lambda_{m+1} + \lambda_{m+1}^2)u(p_0)},$$

where p_2 is the other vertex in the m-cell containing E. There is a similar formula for r^τ with $m + 1$ replaced by $m + l + 1$ and p_i replaced by p_i^τ.

Analogous to (2.1), there is also an algorithm which enables us to calculate the restriction of a Laplaician eigenfunctions to any edge in $\mathcal{SG}$. Let p_{001}^τ be the midpoint

of the edge connecting p_0^τ and p_{01}^τ. Then

$$u(p_{001}^\tau) \;=\; \frac{4 - \lambda_{m+l+2}}{5 - \lambda_{m+l+2}} u(p_{01}^\tau)$$

$$(3.3) \qquad + \Big(\frac{3 - \lambda_{m+l+2}}{(2 - \lambda_{m+l+2})(5 - \lambda_{m+l+2})} + \frac{1}{(2 - \lambda_{m+l+2})(5 - \lambda_{m+l+2})(5 - \lambda_{m+l+1})} \Big) u(p_0^\tau)$$

$$- \Big(\frac{1}{(2 - \lambda_{m+l+2})(5 - \lambda_{m+l+2})} + \frac{1}{(2 - \lambda_{m+l+2})(5 - \lambda_{m+l+2})(5 - \lambda_{m+l+1})} \Big) u(p_1^\tau).$$

See Algorithm 2.5 in [4] for a proof of (3.3) by using the extension rule (3.1).

Notice that the coefficients in (3.3) sum to 1, which also indicates that u is constant on E if and only if $u(p_0) = u(p_1) = u(p_{01})$.

For an integer $l \geq 0$, we write $a_l = (2 - \lambda_{m+l+2})(4 - \lambda_{m+l+2})$, $b_l = 3 - \lambda_{m+l+2}$ and $c_l = (2 - \lambda_{m+l+2})^2(5 - \lambda_{m+l+2}) - 3$. We list some lemmas.

Lemma 3.2. *Let $l \geq 0$. For any word τ with $|\tau| = l$, we have*

$$(3.4) \qquad\qquad r^{\tau 0} = \frac{a_l + b_l r^\tau}{c_l - b_l r^\tau} \ \text{ and } \ r^{\tau 1} = \frac{c_l r^\tau - b_l}{a_l r^\tau + b_l}.$$

Furthermore, the two mappings in (3.4) are both bijections: $\mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$. In particular, $r^{\tau 0} = -1$, $r^{\tau 1} = \frac{c_l}{a_l}$ when $r^\tau = \infty$.

Proof. The proof is similar to that of Lemma 2.1. Noticing that $m \geq m_0$, we have $\lambda_{m+k} \in (0, 2) \cup (3, 5) \setminus \{4\}$ for any $k \geq 2$, since $\lambda_{m_0} = 2, 5$ or 6. Hence we always have $a_l, b_l \neq 0$ for any $l \geq 0$. By (3.3) and using the definition of r^τ and $r^{\tau 0}$, we get the first equality in (3.4). The second equality in (3.4) follows by a symmetric consideration. It can be directly checked that the two mappings are bijections. $\qquad\square$

Lemma 3.3. *For any word τ with $|\tau| = l \geq 0$, we have*

  (1) if $r^\tau = 4 - \lambda_{m+l+1}$, then $r^{\tau 0} = 4 - \lambda_{m+l+2}$;
  (2) if $r^\tau = \frac{1}{4 - \lambda_{m+l+1}}$, then $r^{\tau 1} = \frac{1}{4 - \lambda_{m+l+2}}$;
  (3) if $r^\tau = -1$, then $r^{\tau 0} = \frac{1}{4 - \lambda_{m+l+2}}$ and $r^{\tau 1} = 4 - \lambda_{m+l+2}$.

Proof. Noticing that from (1.2), $\lambda_{m+l+1} = \lambda_{m+l+2}(5 - \lambda_{m+l+2})$, we then have

$$a_l = (4 - \lambda_{m+l+2}) + (4 - \lambda_{m+l+1}),$$
$$b_l = (4 - \lambda_{m+l+2}) - 1,$$
$$c_l = (4 - \lambda_{m+l+2})(4 - \lambda_{m+l+1}) + 1.$$

Then by Lemma 3.2,

$$
\begin{aligned}
r^{\tau 0} &= \frac{a_l + b_l r^{\tau}}{c_l - b_l r^{\tau}} \\
&= \frac{(4 - \lambda_{m+l+2}) + (4 - \lambda_{m+l+1}) + (4 - \lambda_{m+l+2} - 1)r^{\tau}}{(4 - \lambda_{m+l+2})(4 - \lambda_{m+l+1}) + 1 - (4 - \lambda_{m+l+2} - 1)r^{\tau}} \\
&= \frac{(4 - \lambda_{m+l+2})(1 + r^{\tau}) + 4 - \lambda_{m+l+1} - r^{\tau}}{(4 - \lambda_{m+l+2})(4 - \lambda_{m+l+1} - r^{\tau}) + 1 + r^{\tau}}.
\end{aligned}
$$

So it is easy to check that $r^{\tau 0} = 4 - \lambda_{m+l+2}$ when $r^{\tau} = 4 - \lambda_{m+l+1}$; and $r^{\tau 0} = \frac{1}{4 - \lambda_{m+l+2}}$ when $r^{\tau} = -1$. On the other hand, since $r^{\tau 1} = \left(\frac{a_l + b_l (r^{\tau})^{-1}}{c_l - b_l (r^{\tau})^{-1}}\right)^{-1}$, we have $r^{\tau 1} = \frac{1}{4 - \lambda_{m+l+2}}$ when $r^{\tau} = \frac{1}{4 - \lambda_{m+l+1}}$; and $r^{\tau 1} = 4 - \lambda_{m+l+2}$ when $r^{\tau} = -1$. $\qquad \square$

Similar to Lemma 2.3, we have

Lemma 3.4. *For any word τ, the restriction of u to E can not attain an extremum at p_{01}^{τ} if $r^{\tau} \neq -1$.*

Proof. As in Lemma 2.3, we only need to prove that u can not attain an extremum at p_{01} if $r \neq -1$. Suppose $r \neq -1$, then we divide the restriction of u to E into its symmetric part u_S and antisymmetric part u_A. Then $r(u_S) = -1$ and $r(u_A) = 1$.

Since $r(u_S) = -1$, by Lemma 3.3, we have $r^1(u_S) = 4 - \lambda_{m+2}$ and $r^0(u_S) = \frac{1}{4 - \lambda_{m+2}}$, and then $r^{10^l}(u_S) = 4 - \lambda_{m+l+2}$ and $r^{01^l}(u_S) = \frac{1}{4 - \lambda_{m+l+2}}$ for any $l \geq 0$. This gives that $\lim_{l \to \infty} r^{10^l}(u_S) = 4$ and $\lim_{l \to \infty} r^{01^l}(u_S) = \frac{1}{4}$.

As for u_A, since $r(u_A) = 1$, we have $r^1(u_A) \neq 4 - \lambda_{m+2}$, then since the mapping $f_l(x) = \frac{a_{l+1} + b_{l+1} x}{c_{l+1} - b_{l+1} x}$ is bijective for any $l \geq 0$, we have $r^{10^l}(u_A) \neq 4 - \lambda_{m+l+2}$ for any $l \geq 0$. Moreover, noticing that $|\frac{df_l}{dx}|_{4 - \lambda_{m+l+2}}| > 1$ for any $l \geq 0$, we could not have $\lim_{l \to \infty} r^{10^l}(u_A) = 4$. On the other hand, since $\lim_{l \to \infty} a_l = 8$, $\lim_{l \to \infty} b_l = 3$, $\lim_{l \to \infty} c_l = 17$, and $x = \frac{2}{3}$ is the unique stable fixed point of $f(x) = \frac{8 + 3x}{17 - 3x}$, we have $\lim_{l \to \infty} r^{10^l}(u_A) = \frac{2}{3}$. A similar argument yields that $\lim_{l \to \infty} r^{01^l}(u_A) = \frac{3}{2}$.

The remaining proof is same as that of Lemma 2.3. $\qquad \square$

We separate the proof of Theorem 1.2 and Theorem 1.3 into the following two subsections.

3.1. **Proof of Theorem 1.2.** In this subsection, we consider the case that $m \geq m_1 - 1$. Then $\varepsilon_{m+l+1} = -1$ for any $l \geq 0$.

Proof of Theorem 1.2. Since $m \geq m_1 - 1$, for any word τ with $|\tau| = l \geq 0$, we have $\lambda_{m+l+1} \leq \frac{5 - \sqrt{5}}{2}$ and $\lambda_{m+l+2} \leq \frac{5 - \sqrt{15 + 2\sqrt{5}}}{2}$. So it is easy to check that

$$
-1 < -\frac{b_l}{a_l} < 0 < \frac{1}{4 - \lambda_{m+l+1}} < 4 - \lambda_{m+l+1} < \frac{c_l}{b_l}
$$

and $a_l, b_l, c_l > 0$. Combining this with Lemma 3.2 and Lemma 3.3, we get that the graphs of $r^{\tau 0}$ and $r^{\tau 1}$ in terms of r^τ look like what Fig. 6 presents.
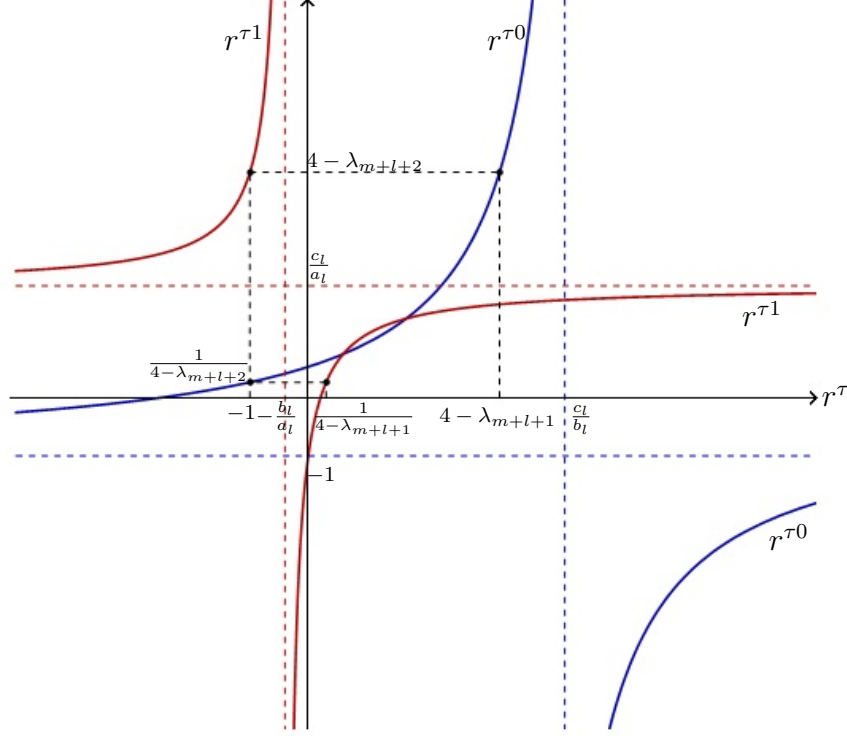
Figure 6. Graphs of $r^{\tau 0}$ and $r^{\tau 1}$ of r^τ

The proof is similar to that of Theorem 1.1. If $r \in [\frac{1}{4-\lambda_{m+1}}, 4-\lambda_{m+1}]$, by iteration, we have $r^\tau \in [\frac{1}{4-\lambda_{m+l+1}}, 4-\lambda_{m+l+1}]$ for any word τ with $|\tau| = l \geq 0$. Thus in this case u is strictly monotone along the line segment E. Otherwise if $r \notin [\frac{1}{4-\lambda_{m+1}}, 4-\lambda_{m+1}]$, then there must exist some τ such that $r^\tau < 0$ and there is a binary searching algorithm so that u has exactly one extremum along E.

As for the boundary behavior of u along E, we need to consider $\lim_{l\to\infty} r^{0^l}$ and $\lim_{l\to\infty} r^{1^l}$. Similar to the proof of Lemma 3.4, if $r = 4 - \lambda_{m+1}$, then we have

$$\lim_{l\to\infty} r^{0^l} = \lim_{l\to\infty} (4 - \lambda_{m+l+1}) = 4,$$

which leads to

$$\begin{aligned}
\alpha &= \lim_{l\to\infty} \frac{|u(p_1^{0^l}) - u(p_0)|}{|p_1^{0^l} - p_0|} \\
&= \frac{|u(p_1) - u(p_0)|}{|p_1 - p_0|} \lim_{l\to\infty} 2^l \frac{1}{1+r} \cdot \frac{1}{1+r^0} \cdots\cdots \frac{1}{1+r^{0^{l-1}}} = 0,
\end{aligned}$$

where $\alpha = \alpha^E(u)$ for short. Otherwise if $r \neq 4 - \lambda_{m+1}$, then $\lim_{l\to\infty} r^{0^l} = \frac{2}{3}$, which leads to that $\alpha = \lim_{l\to\infty} \frac{|u(p_1^{0^l}) - u(p_0)|}{|p_1^{0^l} - p_0|} = \infty$. There is a similar argument for $\beta = \beta^E(u)$ by considering $\lim_{l\to\infty} r^{1^l}$ instead.

Thus we have proved the theorem. $\qquad\square$

We have a binary searching algorithm for the unique extreme point of $u|_E$ in case of $r \notin [\frac{1}{4-\lambda_{m+1}}, 4 - \lambda_{m+1}]$, analogous to the case for harmonic functions.

Remark 3.5. *When $r \notin [\frac{1}{4-\lambda_{m+1}}, 4 - \lambda_{m+1}]$, then $p_0 + \theta(r)p_1$ is the unique extreme point of $u|_E$, where $\theta(r)$ is a function of r defined recursively as*

$$\theta(r^\tau) = \begin{cases} \frac{1}{2}, & \text{if } r^\tau = -1, \\ \frac{1}{2} + \frac{1}{2}\theta(r^{\tau 1}), & \text{if } r^\tau \in (-1, \frac{1}{4-\lambda_{m+|\tau|+1}}), \\ \frac{1}{2}\theta(r^{\tau 0}), & \text{otherwise,} \end{cases}$$

for any word τ.

Corollary 3.6. *Assume $m \geq m_1 - 1$. Let τ be a word with $|\tau| = l \geq 0$. Then*

(1) p_{01}^τ is an extreme point of u along E if and only if $r^\tau = -1$;

(2) if $\tau \neq 0^l$ (write $\tau = \tau'10^k$ with $k \geq 0$), then $p_0^\tau = p_{01}^{\tau'} = p_1^{\tau'01^k}$ is an extreme point of u along E if and only if $r^\tau = 4 - \lambda_{m+l+1}$; and if so, $r^{\tau'} = -1$ and $r^{\tau'01^k} = \frac{1}{4-\lambda_{m+l+1}}$;

(3) if $\tau \neq 1^l$ (write $\tau = \tau'01^k$ with $k \geq 0$), then $p_1^\tau = p_{01}^{\tau'} = p_0^{\tau'10^k}$ is an extreme point of u along E if and only if $r^\tau = \frac{1}{4-\lambda_{m+l+1}}$; and if so, $r^{\tau'} = -1$ and $r^{\tau'10^k} = 4 - \lambda_{m+l+1}$.

Proof. For (1), if $r^\tau = -1$, then from Lemma 3.3, we have $r^{\tau 0} = \frac{1}{4-\lambda_{m+l+2}}$ and $r^{\tau 1} = 4 - \lambda_{m+l+2}$. By using Theorem 1.2, both $u|_{E_{\tau 0}}$ and $u|_{E_{\tau 1}}$ are monotone. This gives that p_{01}^τ is an extreme point of u since $r^\tau = -1$. Combing this with Lemma 3.4, we have proved (1).

For (2), by Lemma 3.2 and Lemma 3.3, it is easy to check that

$$r^\tau = 4 - \lambda_{m+l+1} \iff r^{\tau'} = -1 \iff r^{\tau'01^k} = \frac{1}{4 - \lambda_{m+l+1}}.$$

Then (2) follows from (1).

The proof of (3) is similar to that of (2). $\qquad\square$

3.2. **Proof of Theorem 1.3.**

Now we turn to the general case $m \geq m_0$. In particular, we mainly focus on the case that $m_0 \leq m < m_1 - 1$ (so we naturally assume $m_0 < m_1 - 1$).

Write $\{\varepsilon_{m_0+i}\}_{i\geq 1}$ the associated ε-sequence of u and λ. Obviously, $\varepsilon_{m_1-1} = 1$. In this subsection, we always denote by $\tilde{u}$ a new eigenfunction such that $\tilde{u}|_{V_m} = u|_{V_m}$ whose

associated ε-sequence $\{\tilde{\varepsilon}_{m_0+i}\}_{i\geq 1}$ satisfies

$$\tilde{\varepsilon}_{m_0+i} = \begin{cases} \varepsilon_{m_0+i}, & \text{if } i \neq m_1 - m_0 - 1, \\ -1, & \text{if } i = m_1 - m_0 - 1. \end{cases}$$

We write $\{\tilde{\lambda}_{m_0+i}\}_{i\geq 0}$ and $\tilde{\lambda}$ the associated graph eigenvalues and eigenvalue of $\tilde{u}$. Of cause, $\lambda_{m_0+i} = \tilde{\lambda}_{m_0+i}$ for $i \leq m_1 - m_0 - 2$. We additional require that $m > m_0$ if $\lambda_{m_0} = 6$ to avoid the possibility that $m_1 - 1 = m_0 + 1$ since it is forbidden that $\tilde{\varepsilon}_{m_0+1} = -1$ and $\tilde{\lambda}_{m_0+1} = 2$.

Write $\tilde{r}$, $\tilde{r}^\tau$, $\tilde{a}_l$, $\tilde{b}_l$, $\tilde{c}_l$ of $\tilde{u}$ on E, analogous to those of u on E. Then we have the following lemma.

Lemma 3.7. *Assume $m_0 \leq m < m_1 - 1$ ($m_0 < m < m_1 - 1$ if $\lambda_{m_0} = 6$). Then for any $(m_1 - 1)$-edge E_τ contained in E, we have*

(1) if $\tilde{r}^\tau \in (\frac{1}{4-\tilde{\lambda}_{m_1}}, 4 - \tilde{\lambda}_{m_1})$, then $r^\tau \notin [\frac{1}{4-\lambda_{m_1}}, 4 - \lambda_{m_1}]$;

(2) if $\tilde{r}^\tau = \frac{1}{4-\tilde{\lambda}_{m_1}}$, then $r^\tau = \frac{1}{4-\lambda_{m_1}}$; if $\tilde{r}^\tau = 4 - \tilde{\lambda}_{m_1}$, then $r^\tau = 4 - \lambda_{m_1}$;

(3) if $\tilde{r}^\tau \notin [\frac{1}{4-\tilde{\lambda}_{m_1}}, 4 - \tilde{\lambda}_{m_1}]$, then $r^\tau \in (\frac{1}{4-\lambda_{m_1}}, 4 - \lambda_{m_1})$.

Proof. Without loss of generality, we assume $m = m_1 - 2$, i.e., E is a $(m_1 - 2)$-edge, otherwise we just need to consider every $(m_1 - 2)$-edge contained in E instead of E separately. Then $|\tau| = 1$, i.e., $\tau = 0$ or 1. From the definition of $\tilde{u}$, since E is a (m_1-2)-edge, we have $\tilde{u}(p_i) = u(p_i)$, $i = 0, 1, 2$, and $\tilde{\lambda}_{m_0+i} = \lambda_{m_0+i}$ for $0 \leq i \leq m_1 - m_0 - 2$.

First we assume that $m_0 < m(= m_1 - 2)$. We make a "virtual" $(m-1)$-cell such that E is a half of one of its edge, as shown in Fig 7. Note that such cell may not exist as a "real" cell contained in $\mathcal{SG}$. Let p_0, q_1, q_2 be its boundary vertices, p_1 be the midpoint of the $(m-1)$-edge joining p_0 and q_1.
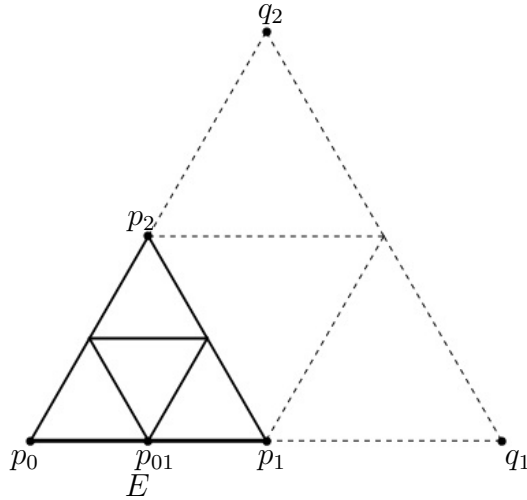

Figure 7. A "virtual" $(m-1)$-cell containing E

Comparing to (3.1), we assign the values of u at the virtual vertices q_i to be

$$u(q_i) = -\frac{4 - \lambda_m}{6 - \lambda_m} u(p_0) + \frac{(4 - \lambda_m)(5 - \lambda_m)}{6 - \lambda_m} u(p_i) - \frac{2(5 - \lambda_m)}{6 - \lambda_m} u(p_j),$$

for $i, j \in \{1, 2\}$, in order that the extension relation

$$u(p_i) = \frac{(4 - \lambda_m)\big(u(p_0) + u(q_i)\big) + 2u(q_j)}{(2 - \lambda_m)(5 - \lambda_m)}$$

to hold. Define $r^* = \frac{u(q_1) - u(p_1)}{u(p_1) - u(p_0)}$, then by Lemma 3.2, we have

$$r = \frac{a_{-1} + b_{-1} r^*}{c_{-1} - b_{-1} r^*} \quad \text{and} \quad \tilde{r} = \frac{\tilde{a}_{-1} + \tilde{b}_{-1} r^*}{\tilde{c}_{-1} - \tilde{b}_{-1} r^*}$$

where a_{-1}, b_{-1}, c_{-1} are functions of λ_{m+1}, and $\tilde{a}_{-1}, \tilde{b}_{-1}, \tilde{c}_{-1}$ are functions of $\tilde{\lambda}_{m+1}$, which are same as a_l, b_l, c_l with $l = -1$ and $m = m_1 - 2$.

Analogously, $r^0 = \frac{a_0 + b_0 r}{c_0 - b_0 r}$, $\tilde{r}^0 = \frac{\tilde{a}_0 + \tilde{b}_0 \tilde{r}}{\tilde{c}_0 - \tilde{b}_0 \tilde{r}}$, $r^1 = \frac{c_0 r - b_0}{a_0 r + b_0}$, $\tilde{r}^1 = \frac{\tilde{c}_0 \tilde{r} - \tilde{b}_0}{\tilde{a}_0 \tilde{r} + \tilde{b}_0}$, where $a_0, b_0, c_0, \tilde{a}_0, \tilde{b}_0, \tilde{c}_0$ are functions of λ_{m+2} or $\tilde{\lambda}_{m+2}$.

Since $m = m_1 - 2 > m_0$, it is easy to check that $\lambda_m \in (0, \frac{5 - \sqrt{5}}{2}] \cup [\frac{5 + \sqrt{5}}{2}, 5) \cup \{3\} \setminus \{1, 4\}$, and then $\lambda_{m+1} = \frac{5 + \sqrt{25 - 4\lambda_m}}{2} \in (\frac{5 + \sqrt{5}}{2}, 5) \setminus \{1, 4\}$, $\tilde{\lambda}_{m+1} = \frac{5 - \sqrt{25 - 4\lambda_m}}{2} \in (0, \frac{5 - \sqrt{5}}{2}) \setminus \{1, 4\}$, and similarly $\lambda_{m+2}, \tilde{\lambda}_{m+2} \in (0, \frac{5 - \sqrt{5}}{2}) \setminus \{1, 4\}$.

Then we have $\tilde{b}_{-1} > 0$, $-1 < 4 - \lambda_m < \frac{\tilde{c}_{-1}}{\tilde{b}_{-1}}$, so the graph of the relation $\tilde{r} = \frac{\tilde{a}_{-1} + \tilde{b}_{-1} r^*}{\tilde{c}_{-1} - \tilde{b}_{-1} r^*}$ looks like what Fig. 8-1 presents. Similarly, we have $b_{-1} < 0$, $\frac{c_{-1}}{b_{-1}} < -1 < 4 - \lambda_m$ when $\lambda_{m+1} < 4$; and $-1 < \frac{c_{-1}}{b_{-1}} < 4 - \lambda_m$ when $\lambda_{m+1} > 4$, see Fig. 8-2, 8-3 for the two possible cases of graphs of $r = \frac{a_{-1} + b_{-1} r^*}{c_{-1} - b_{-1} r^*}$.

For $\tilde{r}^0$, we have $\tilde{b}_0 > 0$ and $-1 < 4 - \tilde{\lambda}_{m+1} < \frac{\tilde{c}_0}{\tilde{b}_0}$, see Fig. 8-4 for the graph of $\tilde{r}^0 = \frac{\tilde{a}_0 + \tilde{b}_0 \tilde{r}}{\tilde{c}_0 - \tilde{b}_0 \tilde{r}}$. Similarly for r^0, we have $b_0 > 0$ and $-1 < 4 - \lambda_{m+1} < \frac{c_0}{b_0}$, see Fig. 8-5 for the graph of $r^0 = \frac{a_0 + b_0 r}{c_0 - b_0 r}$.

For $\tilde{r}^1$, we have $\tilde{b}_0 > 0$ and $-1 < -\frac{\tilde{b}_0}{\tilde{a}_0} < \frac{1}{4 - \tilde{\lambda}_{m+1}}$, see Fig. 8-6 for the graph of $\tilde{r}^1 = \frac{\tilde{c}_0 \tilde{r} - \tilde{b}_0}{\tilde{a}_0 \tilde{r} + \tilde{b}_0}$. Similarly for r^1, we have $b_0 > 0$ and $-1 < -\frac{b_0}{a_0} < \frac{1}{4 - \lambda_{m+1}}$ when $\lambda_{m+1} < 4$; $\frac{1}{4 - \lambda_{m+1}} < -1 < -\frac{b_0}{a_0}$ when $\lambda_{m+1} > 4$, see Fig. 8-7, 8-8 for the two possible graphs of $r^1 = \frac{c_0 r - b_0}{a_0 r + b_0}$.

Looking at the graphs in Fig. 8, we can find that:

(i) if $r^* \in (-\infty, 4 - \lambda_m)$, then $r \in (-\infty, -1) \cup (4 - \lambda_{m+1}, +\infty) \cup \{\infty\}$, $\tilde{r} \in (-1, 4 - \tilde{\lambda}_{m+1})$, and then $r^0 \notin [\frac{1}{4 - \lambda_{m+2}}, 4 - \lambda_{m+2}]$, $\tilde{r}^0 \in (\frac{1}{4 - \tilde{\lambda}_{m+2}}, 4 - \tilde{\lambda}_{m+2})$;

(ii) if $r^* \in (4 - \lambda_m, +\infty)$, then $r \in (-1, 4 - \lambda_{m+1})$, $\tilde{r} \in (-\infty, -1) \cup (4 - \tilde{\lambda}_{m+1}, +\infty) \cup \{\infty\}$, and then $r^0 \in (\frac{1}{4 - \lambda_{m+2}}, 4 - \lambda_{m+2})$, $\tilde{r}^0 \notin [\frac{1}{4 - \tilde{\lambda}_{m+2}}, 4 - \tilde{\lambda}_{m+2}]$;
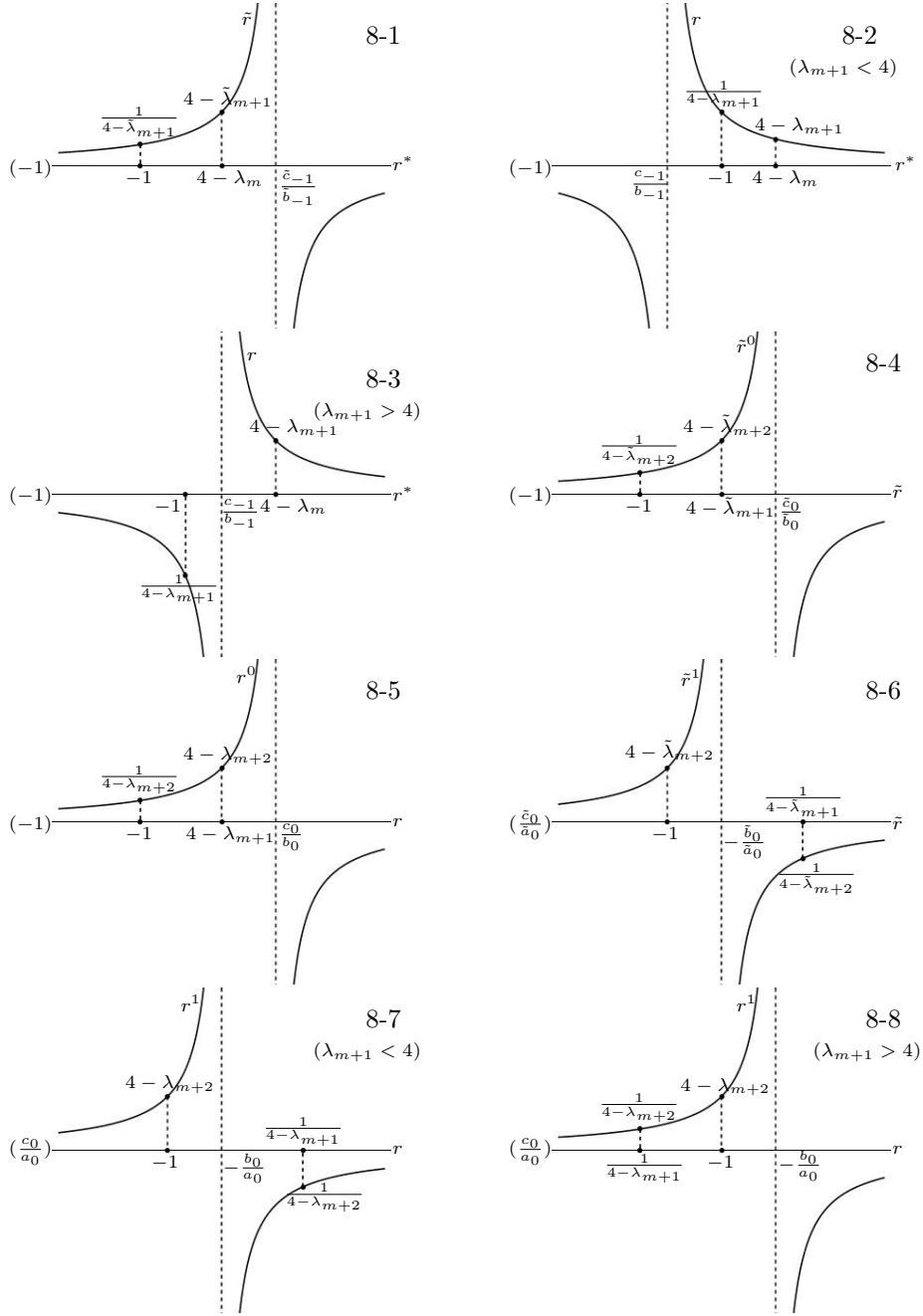
Figure 8. Graphs of $r, \tilde{r}, r^0, \tilde{r}^0, r^1, \tilde{r}^1$

(iii) if $r^* = 4 - \lambda_m$, then $r = 4 - \lambda_{m+1}$, $\tilde{r} = 4 - \tilde{\lambda}_{m+1}$, and then $r^0 = 4 - \lambda_{m+2}$, $\tilde{r}^0 = 4 - \tilde{\lambda}_{m+2}$;

(iv) if $r^* = \infty$, then $r = \tilde{r} = -1$, and then $r^0 = \frac{1}{4 - \lambda_{m+2}}$, $\tilde{r}^0 = \frac{1}{4 - \tilde{\lambda}_{m+2}}$.

Thus we have verified that the conclusion of the lemma is true for $\tau = 0$.

As for r^1 and $\tilde{r}^1$, using a similar argument, we have:

(i) if $r^* \in (-\infty, -1)$, then $r^1 \in (\frac{1}{4-\lambda_{m+2}}, 4 - \lambda_{m+2})$ and $\tilde{r}^1 \notin [\frac{1}{4-\tilde{\lambda}_{m+2}}, 4 - \tilde{\lambda}_{m+2}]$;

(ii) if $r^* \in (-1, +\infty)$, then $r^1 \notin [\frac{1}{4-\lambda_{m+2}}, 4 - \lambda_{m+2}]$ and $\tilde{r}^1 \in (\frac{1}{4-\tilde{\lambda}_{m+2}}, 4 - \tilde{\lambda}_{m+2})$;

(iii) if $r^* = -1$, then $r^1 = \frac{1}{4-\lambda_{m+2}}$ and $\tilde{r}^1 = \frac{1}{4-\tilde{\lambda}_{m+2}}$;

(iv) if $r^* = \infty$, then $r^1 = 4 - \lambda_{m+2}$ and $\tilde{r}^1 = 4 - \tilde{\lambda}_{m+2}$.

So the conclusion of the lemma is also true for $\tau = 1$.

Thus we have proved the lemma under the assumption that $m_0 < m$.

It remains to consider the case that $m_0 = m(= m_1 - 2)$. In this case, E is a m_0-edge, $\lambda_{m_0} = 2$ or 5, and $\varepsilon_{m_0+1} = 1$, $\tilde{\varepsilon}_{m_0+1} = -1$ since $m_0 + 1 = m_1 - 1$. We now verify the conclusion according to the value of λ_{m_0}.

Case 1: $\lambda_{m_0} = 2$. It is easy to check that $6 - 6\lambda_{m_0+1} + \lambda_{m_0+1}^2 = 4 - \lambda_{m_0+1}$, then by (3.2) we have

$$r = \frac{(4 - \lambda_{m_0+1})u(p_1) - (4 - \lambda_{m_0+1})u(p_0) - 2u(p_2)}{(4 - \lambda_{m_0+1})u(p_1) - (4 - \lambda_{m_0+1})u(p_0) + 2u(p_2)} = \frac{4 - \lambda_{m_0+1} - 2\theta}{4 - \lambda_{m_0+1} + 2\theta}$$

with $\theta = \frac{u(p_2)}{u(p_1)-u(p_0)}$ $\left(\text{allowing } \theta = \infty \text{ when } u(p_1) = u(p_0)\right)$. Similarly, $\tilde{r} = \frac{4-\tilde{\lambda}_{m_0+1}-2\theta}{4-\tilde{\lambda}_{m_0+1}+2\theta}$.

Notice that if $\theta = -1$, then $r = \frac{6-\lambda_{m_0+1}}{2-\lambda_{m_0+1}} = 4 - \lambda_{m_0+1}$ and similarly $\tilde{r} = 4 - \tilde{\lambda}_{m_0+1}$; and if $\theta = 1$, then $r = \frac{2-\lambda_{m_0+1}}{6-\lambda_{m_0+1}} = \frac{1}{4-\lambda_{m_0+1}}$ and similarly $\tilde{r} = \frac{1}{4-\tilde{\lambda}_{m_0+1}}$.

Now we look at r^0 and $\tilde{r}^0$. By using Lemma 3.2 and Lemma 3.3, noticing that $\lambda_{m_0+1} = \frac{5+\sqrt{17}}{2} \approx 4.562$, $\tilde{\lambda}_{m_0+1} = \frac{5-\sqrt{17}}{2} \approx 0.438$, $\lambda_{m_0+2} = \frac{5-\sqrt{25-4\lambda_{m_0+1}}}{2} \approx 1.201$ and $\tilde{\lambda}_{m_0+2} = \frac{5-\sqrt{25-4\tilde{\lambda}_{m_0+1}}}{2} \approx 0.089$, it is easy to check that:

(i) if $\theta \in (-1, +\infty)$, then $r \in (-\infty, -1) \cup (4-\lambda_{m_0+1}, +\infty) \cup \{\infty\}$, $\tilde{r} \in (-1, 4-\tilde{\lambda}_{m_0+1})$, and then $r^0 \notin [\frac{1}{4-\lambda_{m_0+2}}, 4 - \lambda_{m_0+2}]$, $\tilde{r}^0 \in (\frac{1}{4-\tilde{\lambda}_{m_0+2}}, 4 - \tilde{\lambda}_{m_0+2})$;

(ii) if $\theta \in (-\infty, -1)$, then $r \in (-1, 4-\lambda_{m_0+1})$, $\tilde{r} \in (-\infty, -1) \cup (4-\tilde{\lambda}_{m_0+1}, +\infty) \cup \{\infty\}$, and then $r^0 \in (\frac{1}{4-\lambda_{m_0+2}}, 4 - \lambda_{m_0+2})$, $\tilde{r}^0 \notin [\frac{1}{4-\tilde{\lambda}_{m_0+2}}, 4 - \tilde{\lambda}_{m_0+2}]$;

(iii) if $\theta = -1$, then $r = 4 - \lambda_{m_0+1}$, $\tilde{r} = 4 - \tilde{\lambda}_{m_0+1}$, and then $r^0 = 4 - \lambda_{m_0+2}$, $\tilde{r}^0 = 4 - \tilde{\lambda}_{m_0+2}$;

(iv) if $\theta = \infty$, then $r = \tilde{r} = -1$, and then $r^0 = \frac{1}{4-\lambda_{m_0+2}}$, $\tilde{r}^0 = \frac{1}{4-\tilde{\lambda}_{m_0+2}}$.

The argument is similar for r^1 and $\tilde{r}^1$ by instead considering $\theta \in (-\infty, 1)$, $\theta \in (1, +\infty)$, $\theta = 1$ and $\theta = \infty$ separately. Thus the conclusion is true for $\lambda_{m_0} = 2$.

Case 2: $\lambda_{m_0} = 5$. In this case, $6 - 6\lambda_{m_0+1} + \lambda_{m_0+1}^2 = 1 - \lambda_{m_0+1}$, then by (3.2),

$$r = \frac{(1 - \lambda_{m_0+1})u(p_1) - (4 - \lambda_{m_0+1})u(p_0) - 2u(p_2)}{(4 - \lambda_{m_0+1})u(p_1) - (1 - \lambda_{m_0+1})u(p_0) + 2u(p_2)}$$

and
$$\tilde{r} = \frac{(\lambda_{m_0+1} - 4)u(p_1) - (\lambda_{m_0+1} - 1)u(p_0) - 2u(p_2)}{(\lambda_{m_0+1} - 1)u(p_1) - (\lambda_{m_0+1} - 4)u(p_0) + 2u(p_2)} = \frac{1}{r}.$$

Now we look at r^0 and $\tilde{r}^0$. Noticing that $\lambda_{m_0+1} = \frac{5+\sqrt{5}}{2} \approx 3.618$, $\tilde{\lambda}_{m_0+1} = \frac{5-\sqrt{5}}{2} \approx 1.382$, $\lambda_{m_0+2} \approx 0.878$ and $\tilde{\lambda}_{m_0+2} \approx 0.294$, by using Lemma 3.2, Lemma 3.3 and the fact that $4 - \lambda_{m_0+1} = \frac{1}{\lambda_{m_0+1}-1} = \frac{1}{4-\tilde{\lambda}_{m_0+1}}$, it is easy to check that:

(i) if $r \in (-1, 4 - \lambda_{m_0+1})$, then $\tilde{r} = \frac{1}{r} \in (-\infty, -1) \cup (4 - \tilde{\lambda}_{m_0+1}, +\infty) \cup \{\infty\}$, and then $r^0 \in (\frac{1}{4-\lambda_{m_0+2}}, 4 - \lambda_{m_0+2})$, $\tilde{r}^0 \notin [\frac{1}{4-\tilde{\lambda}_{m_0+2}}, 4 - \tilde{\lambda}_{m_0+2}]$;

(ii) if $r \in (-\infty, -1) \cup (4 - \lambda_{m_0+1}, +\infty) \cup \{\infty\}$, then $\tilde{r} \in (-1, 4 - \tilde{\lambda}_{m_0+1})$, and then $r^0 \notin [\frac{1}{4-\lambda_{m_0+2}}, 4 - \lambda_{m_0+2}]$, $\tilde{r}^0 \in (\frac{1}{4-\tilde{\lambda}_{m_0+2}}, 4 - \tilde{\lambda}_{m_0+2})$;

(iii) if $r = -1$, then $\tilde{r} = -1$, and then $r^0 = \frac{1}{4-\lambda_{m_0+2}}$, $\tilde{r}^0 = \frac{1}{4-\tilde{\lambda}_{m_0+2}}$;

(iv) if $r = 4 - \lambda_{m_0+1}$, then $\tilde{r} = 4 - \tilde{\lambda}_{m_0+1}$, and then $r^0 = 4 - \lambda_{m_0+2}$, $\tilde{r}^0 = 4 - \tilde{\lambda}_{m_0+2}$.

The argument is similar for r^1 and $\tilde{r}^1$ by instead considering $r \in (-1, \frac{1}{4-\lambda_{m_0+1}})$, $r \in (-\infty, -1) \cup (\frac{1}{4-\lambda_{m_0+1}}, +\infty) \cup \{\infty\}$, $r = -1$ and $r = \frac{1}{4-\lambda_{m_0+1}}$ separately. So the conclusion is also true for $\lambda_{m_0} = 5$.

Thus we have proved the lemma. □

As in Section 1, we use $N^E(u)$ and $N^E(\tilde{u})$ to denote the number of local extrema of u and $\tilde{u}$ restricted to E, respectively.

Lemma 3.8. *Assume $m_0 \le m < m_1 - 1$ ($m_0 < m < m_1 - 1$ if $\lambda_{m_0} = 6$). We have*

(1) *if $r \ne 4 - \lambda_{m+1}$ and $\frac{1}{4-\lambda_{m+1}}$, then $N^E(u) + N^E(\tilde{u}) = 2^{m_1-m-1}$;*

(2) *if $r = 4 - \lambda_{m+1}$ or $\frac{1}{4-\lambda_{m+1}}$, then $N^E(u) + N^E(\tilde{u}) = 2^{m_1-m-1} - 1$.*

Proof. (1) Suppose $r \ne 4 - \lambda_{m+1}$ and $\frac{1}{4-\lambda_{m+1}}$. Let τ be any word with length $|\tau| = m_1 - m - 1$. Then E_τ is a $(m_1 - 1)$-edge contained in E. If $r^\tau \ne 4 - \lambda_{m_1}$ and $\frac{1}{4-\lambda_{m_1}}$, then by Lemma 3.7 and Theorem 1.2, there must be one of $u|_{E^\tau}$ and $\tilde{u}|_{E^\tau}$ having a local extremum. If $r^\tau = 4 - \lambda_{m_1}$, then by Lemma 3.2 and Lemma 3.3, $\tau \ne 0^{m_1-m-1}$, and thus we could write $\tau = \tau'10^k$ for some word τ' and integer $k \ge 0$. Then by Corollary 3.6, p_0^τ is an extreme point of u in E, and $r^{\tau'01^k} = \frac{1}{4-\lambda_{m_1}}$. Furthermore, by Lemma 3.7, we also have $\tilde{r}^\tau = 4 - \tilde{\lambda}_{m_1}$, $\tilde{r}^{\tau'01^k} = \frac{1}{4-\tilde{\lambda}_{m_1}}$, and p_0^τ is also an extreme point of $\tilde{u}$ in E. The situation is similar if $r^\tau = \frac{1}{4-\lambda_{m_1}}$. Sum up the extreme points provided by all words τ with $|\tau| = m_1 - m - 1$, we then have $N^E(u) + N^E(\tilde{u}) = 2^{m_1-m-1}$.

(2) When $r = 4 - \lambda_{m+1}$ or $\frac{1}{4-\lambda_{m+1}}$, we only need to notice that there exists an exception of $\tau = 0^{m_1-m-1}$ when $r = 4 - \lambda_{m+1}$, or $\tau = 1^{m_1-m-1}$ when $r = \frac{1}{4-\lambda_{m+1}}$, which provides no extreme point. So we have $N^E(u) + N^E(\tilde{u}) = 2^{m_1-m-1} - 1$ in this case. □

Now we come to the proof of Theorem 1.3.

Proof of Theorem 1.3. For $n \geq 0$, we write n in binary form,

$$n = \sum_{j=0}^{\infty} \delta_j^{(n)} 2^j, \quad \delta_j^{(n)} = 0 \text{ or } 1,$$

with all but a finite number of $\delta_j^{(n)}$ equal to 0. Let $\{\varepsilon_{m_0+i}^{(n)}\}_{i \geq 1}$ be an ε-sequence defined as

(3.5)
$$\varepsilon_{m_0+i}^{(n)} = \begin{cases} \varepsilon_{m_0+i}, & \text{if } m_0 + i \leq m, \\ (-1)^{1+\delta_{m_0+i-m-1}^{(n)}+\delta_{m_0+i-m}^{(n)}}, & \text{if } m_0 + i > m, \end{cases}$$

and $\{\lambda_{m_0+i}^{(n)}\}_{i \geq 0}$ be the associated sequence of graph eigenvalues initialed from λ_{m_0}. From (1.3) we write $\lambda^{(n)} = \frac{3}{2} \lim_{i \to \infty} 5^{m_0+i} \lambda_{m_0+i}^{(n)}$. For any positive integer i, from (1.2) we have

$$\lambda_{m_0+i}^{(n)} = \varphi(\varepsilon_{m_0+i}^{(n)}, \lambda_{m_0+i-1}^{(n)}),$$

where $\varphi(-1, \cdot)$ and $\varphi(1, \cdot)$ are single variable functions defined by $\varphi(\pm 1, x) = \frac{5 \pm \sqrt{25-4x}}{2}$. Notice that $\varphi(-1, \cdot)$ is increasing on the interval $[0, 6]$ with $\varphi(-1, 0) = 0$, $\varphi(-1, 6) = 2$; and $\varphi(1, \cdot)$ is decreasing on $[0, 6]$ with $\varphi(1, 0) = 5$, $\varphi(1, 6) = 3$.

Then it is easy to check that $\lambda^{(n)} < \lambda^{(n')}$ whenever $n < n'$, and $\{\lambda^{(n)}\}_{n \geq 0}$ consists of all eigenvalues initialed from λ_m. Hence, $\forall n \geq 0$, ψ_n is the associated eigenfunction of $\lambda^{(n)}$ satisfying $\psi_n|_{V_m} = u|_{V_m}$. See Tab. 1 for the relations of n, $\{\delta_j^{(n)}\}_{j \geq 0}$ and $\{\varepsilon_{m_0+i}^{(n)}\}_{i \geq m-m_0+1}$.

n	$\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \cdots$	$\varepsilon_{m+1}, \varepsilon_{m+2}, \varepsilon_{m+3}, \varepsilon_{m+4}, \varepsilon_{m+5}, \cdots$
0	$0, 0, 0, 0, 0, \cdots$	$-1, -1, -1, -1, -1, \cdots$
1	$1, 0, 0, 0, 0, \cdots$	$1, -1, -1, -1, -1, \cdots$
2	$0, 1, 0, 0, 0, \cdots$	$1, 1, -1, -1, -1, \cdots$
3	$1, 1, 0, 0, 0, \cdots$	$-1, 1, -1, -1, -1, \cdots$
4	$0, 0, 1, 0, 0, \cdots$	$-1, 1, 1, -1, -1, \cdots$
5	$1, 0, 1, 0, 0, \cdots$	$1, 1, 1, -1, -1, \cdots$
6	$0, 1, 1, 0, 0, \cdots$	$1, -1, 1, -1, -1, \cdots$
7	$1, 1, 1, 0, 0, \cdots$	$-1, -1, 1, -1, -1, \cdots$
8	$0, 0, 0, 1, 0, \cdots$	$-1, -1, 1, 1, -1, \cdots$
$\cdots$	$\cdots$	$\cdots$

Table 1. The list of n, $\delta_j^{(n)}$ and $\varepsilon_{m_0+i}^{(n)}$

For $n \neq 0$, denote $t^{(n)} = \max\{i | \varepsilon_{m_0+i}^{(n)} = 1, i \geq 1\}$, then $m_0 + t^{(n)} + 1$ is the level of fixation of ψ_n. Furthermore, from (3.5), we know that $\delta_{t^{(n)}+m_0-m-1}^{(n)} = 1$ and $2^{t^{(n)}+m_0-m-1} \leq n < 2^{t^{(n)}+m_0-m}$. Since $m < m_0 + t^{(n)}$, there exists an integer $n' \geq 0$ such that $\psi_{n'} = \tilde{\psi}_n$, and the associated ε-sequence of $\psi_{n'}$ should be

$$\varepsilon_{m_0+i}^{(n')} = \begin{cases} \varepsilon_{m_0+i}^{(n)}, & \text{if } i \neq t^{(n)}, \\ -1, & \text{if } i = t^{(n)}. \end{cases}$$

This gives that

$$\delta_j^{(n')} = \begin{cases} 1 - \delta_j^{(n)}, & \text{if } j \leq t^{(n)} + m_0 - m - 1, \\ \delta_j^{(n)} = 0, & \text{if } j > t^{(n)} + m_0 - m - 1, \end{cases}$$

and so $n' + n = 2^{t^{(n)} + m_0 - m} - 1$.

So for any $n \neq 0$, by letting $q \geq 1$ be the unique integer such that $2^{q-1} \leq n < 2^q$, we have $n' = 2^q - 1 - n$.

On the other hand, for $n \geq 0$, denote by $r_n = r^E(\psi_n)$, then since $\lambda_{m+1}^{(n)} = \varphi(\varepsilon_{m+1}^{(n)}, \lambda_m)$ is either $\lambda_{m+1}^{(0)}$ $\left(= \varphi(-1, \lambda_m) \right)$ or $\lambda_{m+1}^{(1)}$ $\left(= \varphi(1, \lambda_m) \right)$, and $\psi_n|_{V_m} = u|_m$, it is easy to check that r_n equals either r_0 or r_1. Then since $\psi_0 = \tilde{\psi}_1$ $\left(\text{if } n = 1, \text{ then } n' = 0\right)$ and the level of fixation of ψ_1 is $m + 2$, by using Lemma 3.3 and Lemma 3.7, we have that for any $n \neq 0$, $r_n = \frac{1}{4 - \lambda_{m+1}^{(n)}}$ or $4 - \lambda_{m+1}^{(n)}$ if and only if $r_0 = \frac{1}{4 - \lambda_{m+1}^{(0)}}$ or $4 - \lambda_{m+1}^{(0)}$.

Thus if $\frac{1}{4 - \lambda_{m+1}^{(0)}} < r_0 < 4 - \lambda_{m+1}^{(0)}$, then for any $n \neq 0$, $r_n \neq \frac{1}{4 - \lambda_{m+1}^{(n)}}$ and $4 - \lambda_{m+1}^{(n)}$. By Theorem 1.2, we have $N^E(\psi_0) = 0$, and by Lemma 3.8, for any $n \neq 0$, $N^E(\psi_n) + N^E(\psi_{2^q-1-n}) = 2^q$, with q being the positive integer such that $2^{q-1} \leq n < 2^q$. Then by induction, we have $N^E(\psi_n) = 2[\frac{n+1}{2}]$.

If $r_0 = \frac{1}{4 - \lambda_{m+1}^{(0)}}$ or $4 - \lambda_{m+1}^{(0)}$, then for $n \neq 0$, $r_n = \frac{1}{4 - \lambda_{m+1}^{(n)}}$ or $4 - \lambda_{m+1}^{(n)}$, and furthermore, $N^E(\psi_0) = 0$, and $N^E(\psi_n) + N^E(\psi_{2^q-1-n}) = 2^q - 1$ for n satisfying $2^{q-1} \leq n < 2^q$. This gives that $N^E(\psi_n) = n$ by induction.

Similarly, if $r_0 \notin [\frac{1}{4 - \lambda_{m+1}^{(0)}}, 4 - \lambda_{m+1}^{(0)}]$, then $N^E(\psi_0) = 1$ and $N^E(\psi_n) + N^E(\psi_{2^q-1-n}) = 2^q$ for n satisfying $2^{q-1} \leq n < 2^q$. This gives that $N^E(\psi_n) = 2[\frac{n}{2}] + 1$.

Thus we have proved the theorem. $\qquad \square$

4. Proof of Theorem 1.4

We deal with $\{\psi_n^{(2)}\}_{n \geq 0}$, $\{\psi_n^{(5)}\}_{n \geq 0}$ and $\{\psi_n^{(6)}\}_{n \geq 0}$ respectively.

1. The $\psi_n^{(2)}$ case. We use x_0, x_1, x_2 to denote the boundary vertices of $\mathcal{SG}$, and y_0, y_1, y_2 to denote the vertices in $V_1 \setminus V_0$ with y_i opposite x_i. See the values of $\psi_n^{(2)}$ at x_i, y_i in Fig. 9. If we let $E = \overline{y_1 y_0}$ (or $\overline{y_2 y_1}$, $\overline{y_0 y_2}$ by symmetry), by using (3.3), it is easy to check that $\forall n \geq 0$, $\psi_n^{(2)}$ takes constant 1 on E as $\lambda_1^{(n)} = 2$ and $\frac{2(4 - \lambda_2^{(n)})}{(2 - \lambda_2^{(n)})(5 - \lambda_2^{(n)})} = 1$.

If we let $E = \overline{x_0 y_2}$ (or $\overline{x_1 y_2}$, etc. by symmetry), then from (3.2) we have

$$r^E(\psi_0^{(2)}) = \frac{6 - 6\lambda_2^{(0)} + (\lambda_2^{(0)})^2 - 2}{4 - \lambda_2^{(0)} + 2} = \frac{1}{4 - \lambda_2^{(0)}}, \quad \left(\lambda_2^{(0)} = \frac{1}{2}(5 - \sqrt{17})\right).$$

Then by Theorem 1.3, $N^E(\psi_n^{(2)}) = n$. Moreover, by symmetry, $\psi_n^{(2)}(y_2)$ must be an extremum, so the number of extrema of $\psi_n^{(2)}$ on $\overline{x_0 x_1}$ is $2n + 1$.

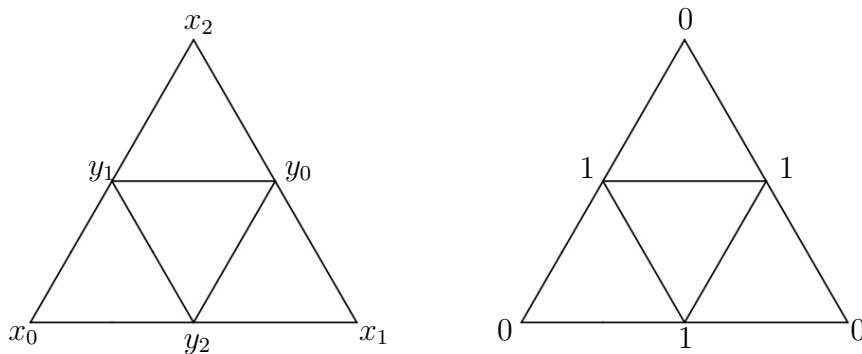See Fig. 10 for the restrictions of $\psi_n^{(2)}$'s to the edge $\overline{x_0 x_1}$.

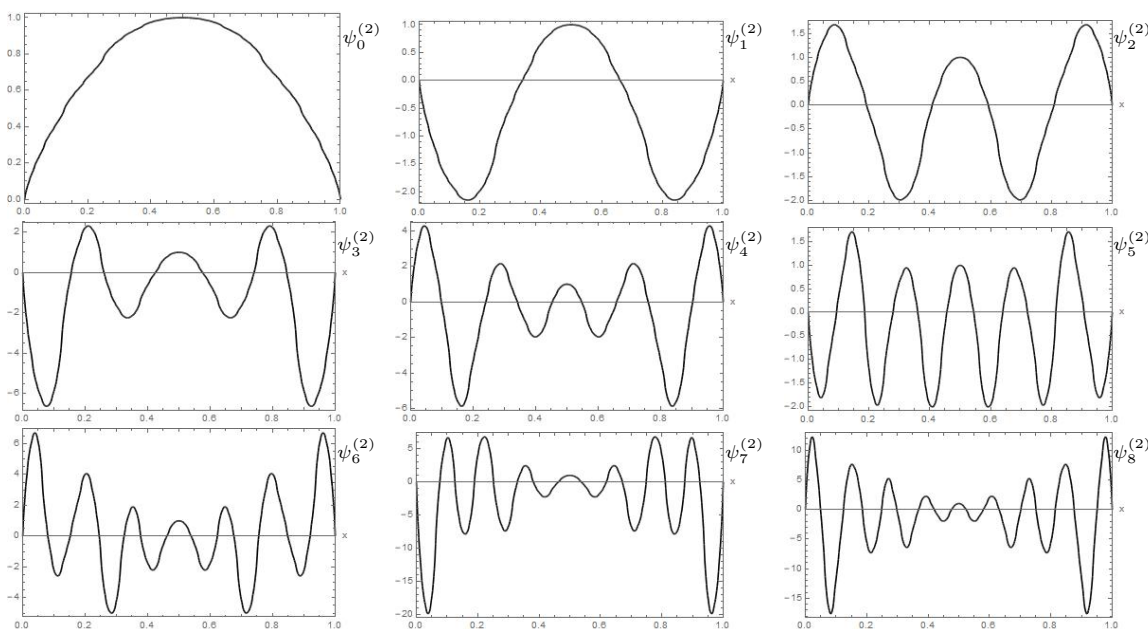**Figure 9. The family $\psi_n^{(2)}$ with $\lambda_1^{(n)} = 2$**

Figure 10. Restriction of $\psi_n^{(2)}$ to the edge $\overline{x_0 x_1}$ for $n = 0, 1, \cdots, 8$

2. The $\psi_n^{(5)}$ case. See the values of $\psi_n^{(5)}$ at x_i, y_i in Fig. 11.

For $E = \overline{x_0 y_2}$ (or $\overline{x_1 y_2}$), by (3.2) we have $r^E(\psi_0^{(5)}) = -1 < \frac{1}{4 - \lambda_2^{(0)}}$ $\left(\lambda_2^{(0)} = \frac{1}{2}(5 - \sqrt{5})\right)$. From Theorem 1.3, $N^E(\psi_n^{(5)}) = 2[\frac{n}{2}] + 1$. Moreover, as $\psi_n^{(5)}$ is antisymmetric on $\overline{x_0 x_1}$, $\psi_n^{(5)}(y_2)$ is not an extremum. So the total number of extrema of $\psi_n^{(5)}$ on $\overline{x_0 x_1}$ is $4[\frac{n}{2}] + 2$.

For $E = \overline{x_0 y_1}$ (or $\overline{x_1 y_0}$), $r^E(\psi_0^{(5)}) = \frac{1 - \lambda_2^{(0)}}{4 - \lambda_2^{(0)}} < \frac{1}{4 - \lambda_2^{(0)}}$, $N^E(\psi_n^{(5)}) = 2[\frac{n}{2}] + 1$.

For $E = \overline{y_1 x_2}$ (or $\overline{y_0 x_2}$), $r^E(\psi_0^{(5)}) = \frac{1}{4 - \lambda_2^{(0)}}$, $N^E(\psi_n^{(5)}) = n$.

Consider the point y_1. Like we did in Lemma 2.3, we write $\psi_n^{(5)}$ into the summation of its symmetric part $\psi_{n,S}^{(5)}$ and antisymmetric part $\psi_{n,A}^{(5)}$ with respect to the line connecting
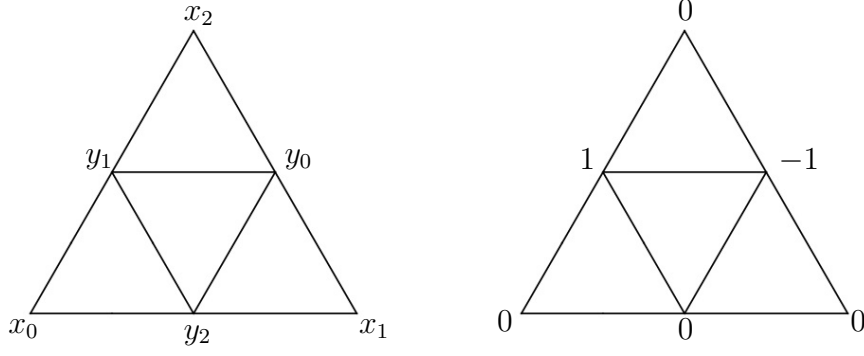
Figure 11. The family $\psi_n^{(5)}$ with $\lambda_1^{(n)} = 5$

y_1 and x_1, that is

$$\psi_{n,S}^{(5)}(x_0) = \psi_{n,S}^{(5)}(x_1) = \psi_{n,S}^{(5)}(x_2) = 0$$

$$\psi_{n,S}^{(5)}(y_2) = \psi_{n,S}^{(5)}(y_0) = -\frac{1}{2}, \quad \psi_{n,S}^{(5)}(y_1) = 1,$$

$$\psi_{n,A}^{(5)}(x_0) = \psi_{n,A}^{(5)}(x_1) = \psi_{n,A}^{(5)}(x_2) = \psi_{n,A}^{(5)}(y_1) = 0,$$

$$\psi_{n,A}^{(5)}(y_2) = -\psi_{n,A}^{(5)}(y_0) = \frac{1}{2}.$$

Then using the same argument, it is easy to check that $\psi_n^{(5)}(y_1)$ is not an extremum for any n, and thus the total number of extrema of $\psi_n^{(5)}$ on $\overline{x_0 x_2}$ is $2[\frac{n}{2}] + n + 1$.

In addition, for $E = \overline{y_2 y_1}$ (or $\overline{y_2 y_0}$), $r^E(\psi_0^{(5)}) = \frac{1 - \lambda_2^{(0)}}{4 - \lambda_2^{(0)}} < \frac{1}{4 - \lambda_2^{(0)}}$, $N^E(\psi_n^{(5)}) = 2[\frac{n}{2}] + 1$. And for $E = \overline{y_1 y_0}$, $\frac{1}{4 - \lambda_2^{(0)}} < r^E(\psi_0^{(5)}) = 1 < 4 - \lambda_2^{(0)}$, $N^E(\psi_n^{(5)}) = 2[\frac{n+1}{2}]$.

See Fig. 12 for the restrictions of $\psi_n^{(5)}$'s to the edge $\overline{x_0 x_1}$ and $\overline{x_0 x_2}$.

3. The $\psi_n^{(6)}$ case. To apply Theorem 1.3, we need to consider smaller edges in $\mathcal{SG}$. See the values of $\psi_n^{(6)}$ on V_3 in Fig. 13, noticing that $\lambda_3^{(n)} = 3$, $\forall n \geq 0$.

It is easy to check that $\forall n \geq 0$, $\psi_n^{(6)}$ takes constant -1 on $\overline{z_2 z_0}$ as $\psi_n^{(6)}(z_2) = \psi_n^{(6)}(z_0) = \psi_n^{(6)}(w_4) = -1$.

For $E = \overline{x_0 w_2}$ (or $\overline{z_2 w_2}$, $\overline{z_2 w_5}$, $\overline{y_2 w_5}$, $\overline{z_1 w_0}$, $\overline{z_2 w_0}$, etc. by symmetry, noticing the direction of the segments), we have $r^E(\psi_0^{(6)}) = 4 - \lambda_4^{(0)}$ $\left(\lambda_4^{(0)} = \frac{1}{2}(5 - \sqrt{13})\right)$, and thus $N^E(\psi_n^{(6)}) = n$.

On the other hand, noticing that $\psi_n^{(6)}(z_2) < \psi_n^{(6)}(w_5) < \psi_n^{(6)}(y_2)$, $\psi_n^{(6)}(z_2) < \psi_n^{(6)}(w_2) < \psi_n^{(6)}(x_0)$, and $\frac{\psi_n^{(6)}(z_2) - \psi_n^{(6)}(w_5)}{\psi_n^{(6)}(w_5) - \psi_n^{(6)}(y_2)} = \frac{\psi_n^{(6)}(z_2) - \psi_n^{(6)}(w_2)}{\psi_n^{(6)}(w_2) - \psi_n^{(6)}(x_0)}$, we could find that $\psi_n^{(6)}$ attains an extremum at z_2, but not at w_2 or w_5. And from the symmetry $\psi_n^{(6)}(y_2)$ must be an extremum. So the total number of extrema of $\psi_n^{(6)}$ on $\overline{x_0 x_1}$ is $2(4n + 1) + 1 = 8n + 3$.

In addition, one could also verify that for $\psi_n^{(6)}$ on $\overline{x_0 y_1}$ (or $\overline{y_1 y_2}$), the number is $4n + 1$; and for $\psi_n^{(6)}$ on $\overline{z_1 z_2}$ (or $\overline{z_1 z_0}$), the number is $2n$.
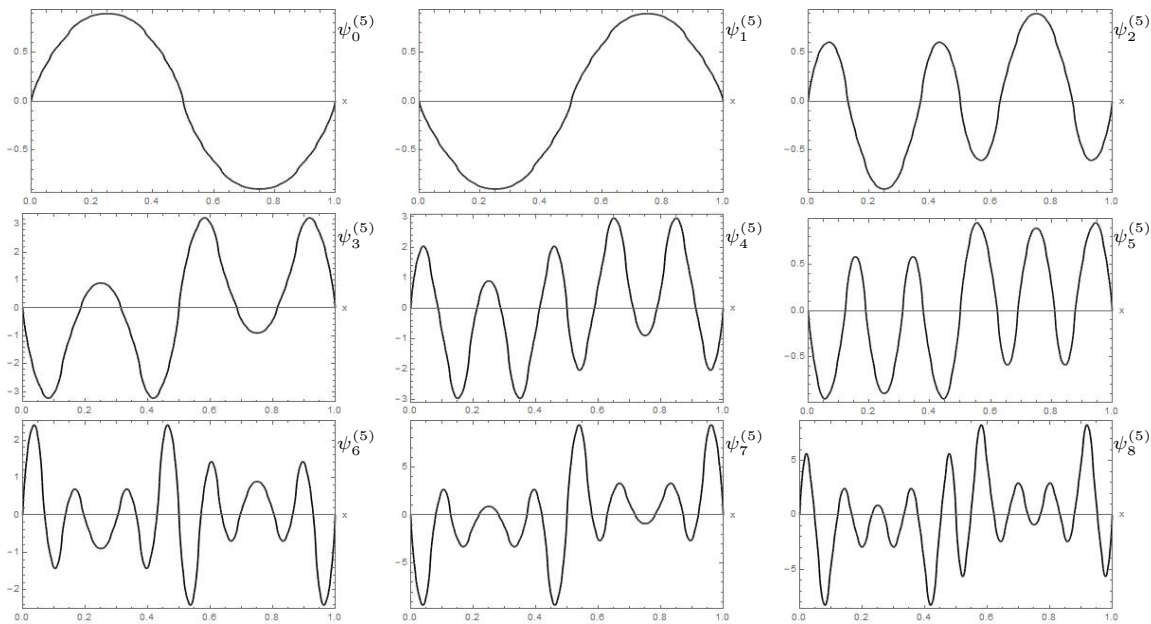
Figure 12-a. Restriction of $\psi_n^{(5)}$ to the edge $\overline{x_0 x_1}$ for $n = 0, 1, \cdots, 8$
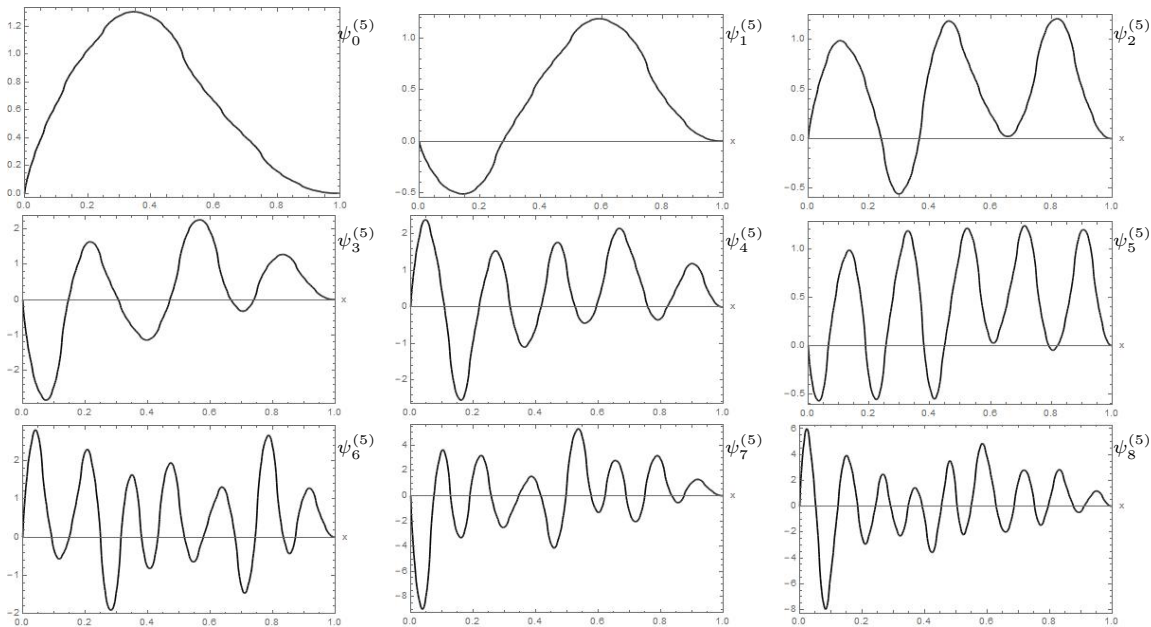

Figure 12-b. Restriction of $\psi_n^{(5)}$ to the edge $\overline{x_0 x_2}$ for $n = 0, 1, \cdots, 8$

See Fig. 14 for the restrictions of $\psi_n^{(6)}$'s to the edge $\overline{x_0 x_1}$.
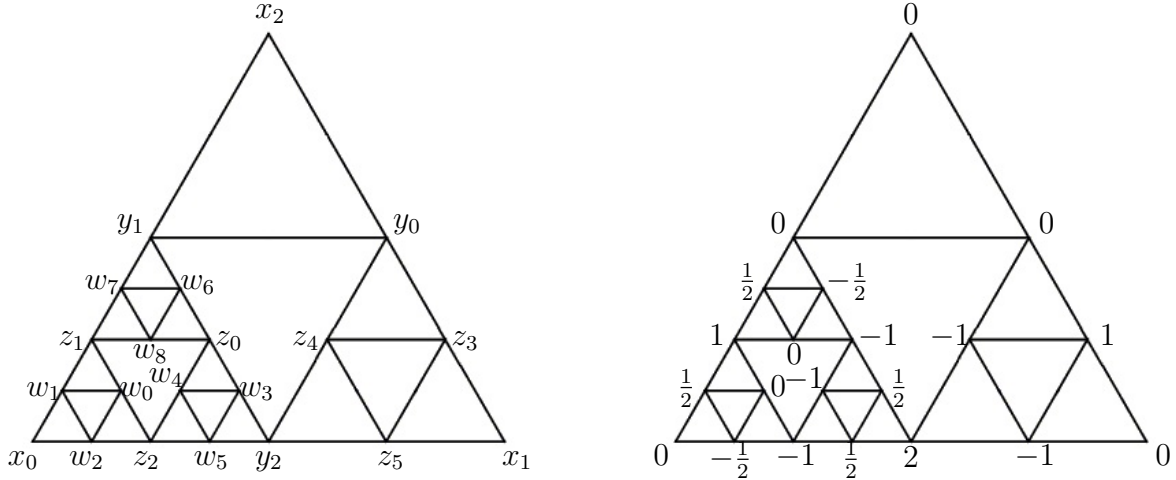
Thus we have proved Theorem 1.4.

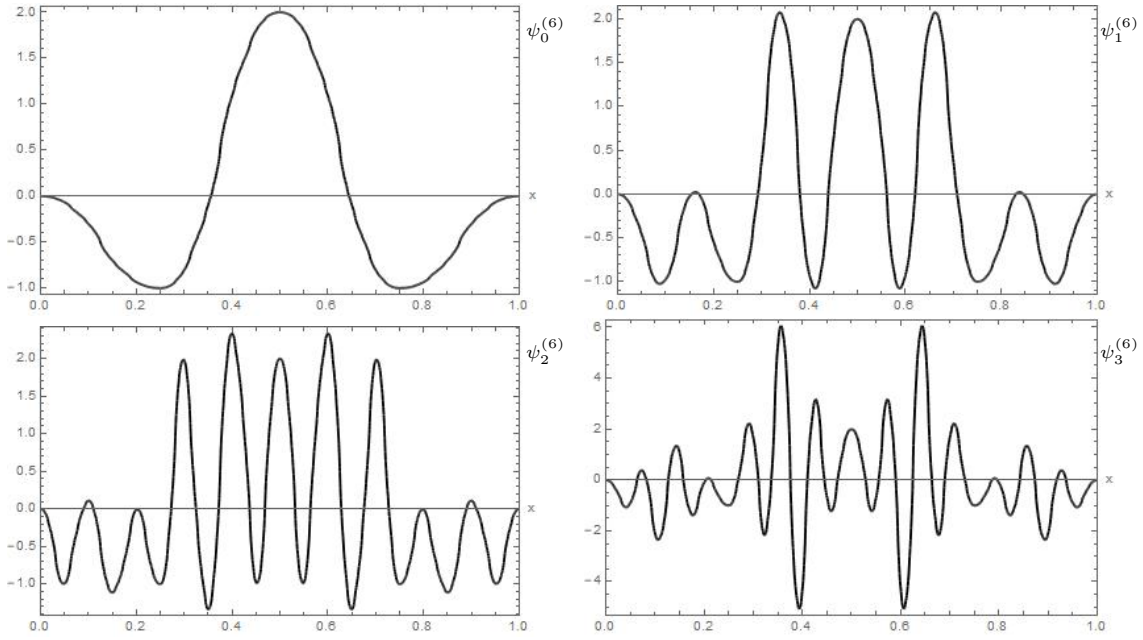**Figure 13.** The family $\psi_n^{(6)}$ with $\lambda_2 = 6$

Figure 14. Restriction of $\psi_n^{(6)}$ to the edge $\overline{x_0 x_1}$ for $n = 0, 1, 2, 3$

ACKNOWLEDGMENTS

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