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Almost Periodic Solutions of One Dimensional Schrödinger Equation with the External Parameters

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Abstract A whitney family of almost periodic solutions for one dimensional Schrödinger equations with the external parameters are proved. It's based on a detailed analysis to the shift of frequency and an improved infinite dimension KAM theory.

Keywords Schrödinger equation · KAM theory · Almost periodic solution · Strong Töplitz–Lipschitz property

1 Introduction and Main Result

In this paper, we focus on the nonlinear Schrödinger equation (NLS):

$$iu_t - u_{xx} + M_\xi u + \frac{\partial f(|u|^2)}{\partial \bar{u}} = 0, \tag{1.1}$$

on tori $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with periodic boundary condition

$$u(t, x) = u(t, x + 2\pi), \ -\infty < t < +\infty,$$

$f(\cdot)$ is real analytic in the neighborhood of $0 \in \mathbb{C}$. Operator $-\partial_{xx} + M_\xi := A$ has eigenvalues $\lambda_n = n^2 + \xi_n$ $(n \in \mathbb{Z})$ and eigenfunctions $\phi_n(x) = \sqrt{\frac{1}{2\pi}} e^{inx}$ under periodic boundary

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condition. From now on $\xi = (\ldots\ldots, \xi_n, \ldots\ldots)_{n\in\mathbb{Z}}$ is named to be the external parameter and is varied in the set $B = \{\xi : \xi \in [0, 1]^{\mathbb{Z}}\}$.

The corresponding Hamiltonian is:

$$H = N + P = \frac{1}{2}\langle Au, u\rangle + \frac{1}{2}\int_{\mathbb{T}} f(|u|^2)dx \tag{1.2}$$

$$= \sum_{n\in\mathbb{Z}} \lambda_n |z_n|^2 + \frac{1}{2}\int_{\mathbb{T}} f\left(\left|\sum_{n\in\mathbb{Z}} \left|z_n\phi_n(x)\right|\right|^2\right) dx. \tag{1.3}$$

Our aim is to show the existence of a positive measure set of almost periodic solutions for such a family of nonlinear Schrödinger equations.

Up to now, there are two ways to obtain almost periodic solution. The first is due to Bourgain [1], using direct Newton iteration method of Craig and Wayne [3]. The second is to excite oscillators with KAM iteration, and then prove existence of full dimensional tori, this is first adopted by Pöschel [11]. In 2005, Bourgain [2] proved the existence of full dimensional tori with exponentially decay actions for one dimension schrödinger equation. Following Pöschel, Niu–Geng [9] and Wu–Geng [15] got similar result for higher dimension beam equation and semilinear quantum harmonic oscillators. Recently, Geng [5] showed the existence of almost periodic solution for Schrödinger equation without the external parameters. However, in [2], Bourgain only obtained existence of almost periodic solution without conducting the measure estimates; in [11], Pöschel required that the perturbation has more regularity; in [5], the author required an assumption of non-degenerate twist conditions.

In this paper, we will construct a positive measure set of almost periodic solutions of Eq. (1.1) with KAM iteration, which compensates the drawback of [2,5,11] mentioned above. In each KAM iteration it is crucial to prove proper non-degenerate conditions to the increased tangential frequencies. To overcome this difficulty, we should explore more structure of the frequency.

Our main result is stated as follows:

Theorem 1 *For one dimensional Schrödinger Eq. (1.1) under periodic boundary condition, there exists a positive-measure Cantor-like set $B^* \subset B$, such that for each $\xi \in B^*$, Eq. (1.1) admits real analytic almost periodic solutions*

$$u(t, x) = \sum_{k\in\mathbb{Z}_0^{\mathbb{Z}}} u_k(x)e^{i\langle k,\omega\rangle t} \tag{1.4}$$

where $\omega = (\ldots, \omega_n, \ldots)_{n\in\mathbb{Z}}$ is a rational independent sequence.

Remark 1.1 The positive-measure of $B^* \subset B$ means for any finite dimension closed subset $\Theta \in [0, 1]^{\mathbb{Z}}$, we have

$$\frac{meas(B \cap \Theta - B^* \cap \Theta)}{meas(B \cap \Theta)} \ll 1.$$

Remark 1.2 $\mathbb{Z}_0^{\mathbb{Z}}$ is the space of all integer sequences $k = (\ldots, k_{-1}, k_0, k_1, \ldots)$ with only finitely many nonzero components.

2 Hamiltonian and KAM Theorem

For b numbers in $\mathbb{Z}$, say $\mathcal{J}_b = \{i_1, \ldots, i_b\}$, we denote $\mathbb{Z}_1^b = \mathbb{Z} \setminus \{i_1, \ldots, i_b\}$. Let $z = (\ldots, z_n, \ldots)_{n\in\mathbb{Z}_1^b}$, and its complex conjugate $\bar{z} = (\ldots, \bar{z}_n, \ldots)_{n\in\mathbb{Z}_1^b}$. We introduce the weighted norm

$$\|z\|_\rho = \sum_{n\in\mathbb{Z}_1^b} |z_n|e^{|n|\rho} < \infty, \quad \rho > 0.$$

Denote a neighborhood of $\mathbb{T}^b \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ by

$$D_\rho(r, s) = \{(\theta, I, z, \bar{z}) : |\mathrm{Im}\theta| < s, |I| < r^2, \|z\|_\rho < r, \|\bar{z}\|_\rho < r\},$$

where $|\cdot|$ denotes the sup-norm of complex vectors.

Let $\alpha \equiv (\ldots, \alpha_n, \ldots)_{n\in\mathbb{Z}_1}$, $\beta \equiv (\ldots, \beta_n, \ldots)_{n\in\mathbb{Z}_1^b}$, α_n and $\beta_n \in \mathbb{N}$ with finitely many non-zero components of positive integers. The product $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$. Let

$$F(\theta, I, z, \bar{z}) = \sum_{\alpha,\beta} F_{\alpha\beta}(\theta, I) z^\alpha \bar{z}^\beta, \tag{2.1}$$

where $F_{\alpha\beta} = \sum_{k,l} F_{kl\alpha\beta} I^l e^{i\langle k,\theta\rangle}$ are C_W^1 functions in parameter ξ in the sense of Whitney. For a function F, we denote

$$\|F\|_{\mathcal{O}} = \sum_{\alpha,\beta,k,l} |F_{kl\alpha\beta}|_{\mathcal{O}}\, |I^l| e^{|k||\mathrm{Im}\theta|}\, |z^\alpha||\bar{z}^\beta|,$$

where $|F_{kl\alpha\beta}|_{\mathcal{O}}$ is short for

$$|F_{kl\alpha\beta}|_{\mathcal{O}} \equiv \sup_{\xi\in\mathcal{O}} \left(\left|F_{kl\alpha\beta}\right| + \left| \frac{\partial F_{kl\alpha\beta}}{\partial \xi} \right| \right). \tag{2.2}$$

(the derivatives with respect to ξ are in the sense of Whitney). The norm of F is given by

$$\|F\|_{D_\rho(r,s),\mathcal{O}} \equiv \sup_{D_\rho(r,s)} \|F\|_{\mathcal{O}}. \tag{2.3}$$

To a function F, we associate a Hamiltonian vector field:

$$X_F = \left(F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}_{n\in\mathbb{Z}_1^b}, \{-\mathrm{i}F_{\bar{z}_n}\}_{n\in\mathbb{Z}_1^b}\right).$$

Its weighted norm is defined by[1]

$$\begin{aligned} \|X_F\|_{D_\rho(r,s),\mathcal{O}} \equiv\ & \|F_I\|_{D_\rho(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_\theta\|_{D_\rho(r,s),\mathcal{O}} \\ & + \frac{1}{s} \sup_{D_\rho(r,s)} \sum_{n\in\mathbb{Z}_1^b} (\|F_{z_n}\|_{\mathcal{O}} e^{|n|\rho} + \|F_{\bar{z}_n}\|_{\mathcal{O}} e^{|n|\rho}). \end{aligned} \tag{2.4}$$

To describe the drift of frequency, a structure of perturbation is required. Töplitz–Lipschitz in [4,6,7] and Quasi-Töplitz in [14] can both be used. In this paper, based on Töplitz–Lipschitz, a certain decay of derivative with respect to parameters is observed which seems only holds for the case $d = 1$. We name Töplitz–Lipschitz together with this decay in parameter as Strong–Töplitz–Lipschitz.

In this paper, we work in the function class with momentum conservation and integral invariance, these properties can be preserved by Poisson bracket.

[1] The norm $\|\cdot\|_{D_\rho(r,s),\mathcal{O}}$ for scalar functions is defined in (2.3). The vector function $G : D_\rho(r, s)\times\mathcal{O} \to \mathbb{C}^m$, $(m < \infty)$ is similarly defined as $\|G\|_{D_\rho(r,s),\mathcal{O}} = \sum_{i=1}^{m} \|G_i\|_{D_\rho(r,s),\mathcal{O}}$.

Momentum conservation: A function F is said to satisfy momentum conservation if $\{F, \mathbb{M}\} = 0$ with $\mathbb{M} = \sum_{j=1}^{b} i_j I_j + \sum_{j\in\mathbb{Z}_1^b} j|z_j|^2$. This implies

$$F_{kl\alpha\beta} = 0, \quad \text{if } \sum_{j=1}^{b} i_j k_j + \sum_{j\in\mathbb{Z}_1^b} j(\alpha_j - \beta_j) \neq 0. \tag{2.5}$$

Integral invariance: A function is said to satisfy integral invariance if $\{F, \mathbb{I}\} = 0$ with $\mathbb{I} = \sum_{j=1}^{b} I_j + \sum_{j\in\mathbb{Z}_1^b} |z_j|^2$. This implies

$$F_{kl\alpha\beta} = 0\,, \quad \text{if } \sum_{j=1}^{b} k_j + \sum_{j\in\mathbb{Z}_1^b} (\alpha_j - \beta_j) \neq 0. \tag{2.6}$$

Lemma 2.1 *The momentum conservation and integral invariance are both preserved by Poisson bracket. That is to say, if F and G satisfy momentum conservation or integral invariance, then $\{F, G\}$ satisfies momentum conservation or Integral invariance.*

By Jacobi's identity the Lemma above is preserved by Poisson bracket.

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} \equiv 0.$$

Definition 2.1 We denote by $\mathcal{A}_{r,s}$ the space of regular analytic functions in $D(r, s)$ with finite norm (2.4), satisfy momentum conservation (2.5) and integral invariance (2.6).

An auxiliary norm which evaluate the derivative with respect to ξ is given for function class $\mathcal{A}_{r,s}$:

$$\|F\|^{\varsigma}_{D_\rho(r,s),\mathcal{O}} = \|F\|_{D_\rho(r,s),\mathcal{O}} + \sup_{\ell\in\mathbb{Z}_1^b} |\ell|^{\varsigma} \left\| \frac{\partial}{\partial \xi_\ell} F \right\|_{D_\rho(r,s),\mathcal{O}}. \tag{2.7}$$

Definition 2.2 For $d = 1$, an analytic function $F \in \mathcal{A}_{r,s}$ is said to satisfy Strong–Töplitz–Lipschitz (STL) condition with a fixed parameter $\varsigma \in (\frac{1}{2}, 1)$: if for any $n, m \in \mathbb{Z}_1^b$, the limits below exist,

$$\lim_{t\to\infty} \frac{\partial^2 F}{\partial q_{n+t} \partial q_{m-t}}, \quad \lim_{t\to\infty} \frac{\partial^2 F}{\partial q_{n+t} \partial \bar{q}_{m+t}}, \quad \lim_{t\to\infty} \frac{\partial^2 F}{\partial \bar{q}_{n+t} \partial \bar{q}_{m-t}}, \tag{2.8}$$

what's more, for any $\ell \in \mathbb{Z}$, there is the estimate:

$$\left\| \frac{\partial \lim_{t\to\infty} \frac{\partial^2 F}{\partial q_{n+t} \partial q_{m-t}}}{\partial \xi_\ell} \right\|_{D_\rho(r,s),\mathcal{O}}, \quad \left\| \frac{\partial \lim_{t\to\infty} \frac{\partial^2 F}{\partial \bar{q}_{n+t} \partial \bar{q}_{m-t}}}{\partial \xi_\ell} \right\|_{D_\rho(r,s),\mathcal{O}} \leq \frac{\varepsilon e^{-|n+m|\rho}}{|\ell|^{\varsigma}},$$
$$\left\| \frac{\partial \lim_{t\to\infty} \frac{\partial^2 F}{\partial q_{n+t} \partial \bar{q}_{m+t}}}{\partial \xi_\ell} \right\|_{D_\rho(r,s),\mathcal{O}} \leq \frac{\varepsilon e^{-|n-m|\rho}}{|\ell|^{\varsigma}}, \tag{2.9}$$

and

$$\left\| \frac{\partial^2 F}{\partial q_{n+t} \partial q_{m-t}} - \lim_{t\to\infty} \frac{\partial^2 F}{\partial q_{n+t} \partial q_{m-t}} \right\|^{\varsigma}_{D_\rho(r,s),\mathcal{O}} \leq \max\left\{ \frac{\varepsilon e^{-|n+m|\rho}}{|n+t|^{\varsigma}}, \frac{\varepsilon e^{-|n+m|\rho}}{|m-t|^{\varsigma}} \right\},$$
$$\left\| \frac{\partial^2 F}{\partial q_{n+t} \partial \bar{q}_{m+t}} - \lim_{t\to\infty} \frac{\partial^2 F}{\partial q_{n+t} \partial \bar{q}_{m+t}} \right\|^{\varsigma}_{D_\rho(r,s),\mathcal{O}} \leq \max\left\{ \frac{\varepsilon e^{-|n-m|\rho}}{|n+t|^{\varsigma}}, \frac{\varepsilon e^{-|n-m|\rho}}{|m+t|^{\varsigma}} \right\},$$

$$\left\| \frac{\partial^2 F}{\partial \bar{q}_{n+t} \partial \bar{q}_{m-t}} - \lim_{t\to\infty} \frac{\partial^2 F}{\partial \bar{q}_{n+t} \partial \bar{q}_{m-t}} \right\|^{\varsigma}_{D_\rho(r,s),\mathcal{O}} \leq \max\left\{ \frac{\varepsilon e^{-|n+m|\rho}}{|n+t|^{\varsigma}}, \frac{\varepsilon e^{-|n+m|\rho}}{|m-t|^{\varsigma}} \right\}.$$

Remark 2.1 The definition of STL means existence of limit when $t \to \infty$, the speed tends to the limit along t and also the decay of derivative with respect to parameters.

Lemma 2.2 *The perturbation P in (1.2) is regular with $|X_P|_{D_\rho(r,s),\mathcal{O}} < \infty$ and satisfies STL with ς, where $\frac{1}{2} < \varsigma < 1$.*

Proof The regularity is easy to see with the help of [8]. An easy calculation gives

$$\int_0^{2\pi} \frac{\partial^2 f(|u|^2)}{\partial u \partial \bar{u}} \phi_{n+t} \bar{\phi}_{m+t} dx = \int_0^{2\pi} \frac{\partial^2 f(|u|^2)}{\partial u \partial \bar{u}} e^{i(n-m)x} dx,$$

It is independent of t and decay with $\varepsilon e^{-|n-m|\rho}$. The other two parts can be obtained similarly. Perturbation P in (1.2) does not depend on parameter ξ, the condition required by STL is satisfied. □

Now we consider perturbed Hamiltonian

$$H = N + P = \langle \omega, I \rangle + \langle \Omega z, \bar{z} \rangle + P(\xi, I, \theta, z, \bar{z}; y), \tag{2.10}$$

where $\omega = (\omega_{-b}, \dots, \omega_b) = (\lambda_{-b}, \dots, \lambda_b)$, $\Omega_n = \lambda_n, |n| > b$. Our aim is to prove that, under suitable hypotheses, the Hamiltonian H admits full dimensional invariant tori.

We require some hypotheses on N and P:

(A1) **Nondegeneracy**:
$\lambda = (\dots \lambda_n, \dots, \lambda_{-n} \dots)$ has partition $\lambda_n = |n|^2 + \xi_n + \bar{\lambda}(\xi) + \breve{\lambda}_n(\xi)$ with

$$|\breve{\lambda}_n| \leq \frac{\varepsilon_0}{|n|^{\varsigma}}, \quad |\frac{\partial}{\partial \xi_\ell} \breve{\lambda}_n| \leq \frac{\varepsilon_0}{|n|^{\varsigma} |\ell|^{\varsigma}},$$

where $\bar{\lambda}(\xi) = \lim_{n\to\infty} (\lambda_n - |n|^2 - \xi_n)$ and $\frac{1}{2} < \varsigma < 1$.

(A2) **Momentum conservation and integral invariance**:
The perturbation P satisfies momentum conservation $\{P, \mathbb{M}\} = 0$, and integral invariance $\{P, \mathbb{I}\} = 0$.

(A3) **Regularity of perturbation**:
P is regular with $\|X_P\|_{D_\rho(r,s),B} < \infty$ and STL with ς

Theorem 2 *Hamilton H in (2.10) is assumed to satisfy condition $(A1)$, $(A2)$ and $(A3)$, then for any small $\gamma > 0$, there exists $\varepsilon = \varepsilon(\gamma, \varepsilon_0, r, s, \rho)$ such that if $\|X_P\|_{D_\rho(r,s),B} \leq \varepsilon$, there is a positive-measure Cantor-like set $B^* \subset B$, for each $\xi \in B^*$, Hamiltonian (2.10) admits the full dimensional KAM tori.*

Remark 2.2 The decay with derivative in $(A1)$ means that we can work in the sense of ℓ^2. Though we do not observe any information on the uniform part $\bar{\lambda}$, it's enough for measure estimate with Integral invariance. The frequencies also depend on the amplitude, which comes from exciting the oscillators, we ignore this for the notational convenience.

3 One KAM Step

At ν-th step of KAM iteration, Hamiltonian

$$H_\nu = \langle \omega^\nu, I\rangle + \langle \Omega^\nu z, \bar{z}\rangle + P_\nu(\xi, \theta, I, z, \bar{z}; y),$$

is well defined on $D_{\rho_\nu}(r_\nu, s_\nu) \times E_\nu$ with $\|X_{P_\nu}\|_{D_{\rho_\nu}(r_\nu,s_\nu),E_\nu} \le \varepsilon_\nu$, the perturbation satisfies Momentum conservation, Integral invariance and STL with ς; what's more there is partition for frequency $\lambda^\nu = (\omega^\nu, \Omega^\nu)$:

$$\lambda_n^\nu = |n|^2 + \xi_n + \bar{\lambda}^\nu(\xi) + \breve{\lambda}_n^\nu(\xi), \quad \bar{\lambda}^\nu(\xi) = \lim_{n\to\infty}(\lambda_n^\nu - |n|^2 - \xi_n), \tag{3.1}$$

with estimate for any $n, \ell \in \mathbb{Z}$.

$$|\breve{\lambda}_n^\nu| \le \frac{\varepsilon_0}{|n|^\varsigma}, \quad \left|\frac{\partial}{\partial \xi_\ell}\breve{\lambda}_n^\nu\right| \le \frac{\varepsilon_0}{|n|^\varsigma |\ell|^\varsigma}, \tag{3.2}$$

$$|\breve{\lambda}_n^\nu - \breve{\lambda}_n^{\nu-1}| \le \frac{\varepsilon_\nu}{|n|^\varsigma}, \quad \left|\frac{\partial}{\partial \xi_\ell}(\breve{\lambda}_n^\nu - \breve{\lambda}_n^{\nu-1})\right| \le \frac{\varepsilon_\nu}{|n|^\varsigma |\ell|^\varsigma}, \tag{3.3}$$

These inequalities lead to the estimate (Norm $\|\cdot\|_M$ is given in the Appendix)

$$\left\|\frac{\partial \breve{\lambda}^\nu}{\partial \xi}\right\|_M \le \varepsilon_0, \tag{3.4}$$

For notational convenience, we usually denote quantities at νth step without subscript, while quantities at $(\nu+1)$th step with $(\cdot)_+$ or $(\cdot)^+$ below.

3.1 Homological Equation and Estimate

Let

$$R := \sum_{k, 2|l|+|\alpha|+|\beta|\le 2} P_{k,l,\alpha,\beta} e^{\mathrm{i}\langle k,\theta\rangle} I^l z^\alpha \bar{z}^\beta, \quad \langle R\rangle := \sum_{i\in\mathcal{J}} P_{0,e_i,0,0} I_i + \sum_{j\in\mathbb{Z}\setminus\mathcal{J}} P_{0,0,j,j}|z_j|^2.$$

The generating function of our symplectic transformation, denoted by F, solves the "homological equation":

$$\{N, F\} = \Pi_{\le\mathcal{K}} R - \langle R\rangle, \tag{3.5}$$

where $\Pi_{\le\mathcal{K}}$ is the projection which collects all terms in R with $|k| \le \mathcal{K}$.

The aim of solving the homological equation to get a smaller perturbation in a narrow domain. A detailed proof on preservation of STL is given in the next section.

Lemma 3.1 *Let* $D_i = D_{\rho_+}\left(\frac{i}{4}r, s_+ + \frac{i}{4}(s - s_+)\right)$, $0 < i \le 4$. *Then*

$$\|X_F\|_{D_3,\mathcal{O}} \le \gamma^{-2}\mathcal{K}^{4\tau}\varepsilon, \tag{3.6}$$

Lemma 3.2 *Let* $\eta = \varepsilon^{\frac{1}{3}}$, $D_{i\eta} = D\left(\frac{i}{4}\eta r, s_+ + \frac{i}{4}(s - s_+)\right)$, $0 < i \le 4$. *If* $\varepsilon \ll \left(\frac{1}{2}\gamma^2\mathcal{K}^{-\tau}\right)^{\frac{3}{2}}$, *we have*

$$\phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \le t \le 1, \tag{3.7}$$

Moreover,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} \le \gamma^{-2}\mathcal{K}^{4\tau}\varepsilon, \tag{3.8}$$

One has the following result for the new perturbation:

Lemma 3.3 *The new perturbation P_+ satisfies the estimate*

$$\|X_{P_+}\|_{D_{\rho_+}(r_+,s_+),\mathcal{O}} \le \gamma^{-2}\mathcal{K}^{4\tau}\varepsilon^{4/3}.$$

3.2 Preservation of STL

The shift of normal frequencies are due to the average of perturbation $\langle R\rangle$, thus one can get the following proposition since P satisfies STL with ς.

Proposition 1 *There exists a partition for frequency $\lambda^{\nu+1} = (\omega^{\nu+1}, \Omega^{\nu+1})$:*

$$\lambda_n^{\nu+1} = |n|^2 + \xi_n + \bar{\lambda}^{\nu+1}(\xi) + \breve{\lambda}_n^{\nu+1}(\xi), \quad \bar{\lambda}^{\nu+1} =: \lim_{n\to\infty}(\lambda_n^{\nu+1} - |n|^2 - \xi_n),$$

with estimate for $\forall n, \ell \in \mathbb{Z}$,

$$|\breve{\lambda}_n^{\nu+1}| \le \frac{\varepsilon_0}{|n|^\varsigma}, \quad \left|\frac{\partial}{\partial\xi_\ell}\breve{\lambda}_n^{\nu+1}\right| \le \frac{\varepsilon_0}{|n|^\varsigma|\ell|^\varsigma}, \tag{3.9}$$

$$|\breve{\lambda}_n^{\nu+1} - \breve{\lambda}_n^{\nu}| \le \frac{\varepsilon_{\nu+1}}{|n|^\varsigma}, \quad \left|\frac{\partial}{\partial\xi_\ell}\left(\breve{\lambda}_n^{\nu+1} - \breve{\lambda}_n^{\nu}\right)\right| \le \frac{\varepsilon_{\nu+1}}{|n|^\varsigma|\ell|^\varsigma}, \tag{3.10}$$

More precisely,

$$\left\|\frac{\partial\breve{\lambda}^{\nu+1}}{\partial\xi}\right\|_M \le \varepsilon_0, \quad \left\|\frac{\partial(\breve{\lambda}^{\nu+1} - \breve{\lambda}^{\nu})}{\partial\xi}\right\|_M \le \varepsilon_{\nu+1}.$$

For the convenience of the next proposition, we use symbol:

$$\begin{aligned}\mathcal{G}_+ &=: D_{\rho_+}(r_+, s_+) \times \mathcal{O},\\ f_{nm}^{11} &=: \lim_{t\to\infty}\frac{\partial^2 F}{\partial z_{n+t}\partial\bar{z}_{m+t}},\ p_{nm}^{11} =: \lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{n+t}\partial\bar{z}_{m+t}},\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{nm}^{11}(t) &=: \frac{\partial^2 F}{\partial z_{n+t}\partial\bar{z}_{m+t}} - f_{nm}^{11},\ \mathcal{P}_{nm}^{11}(t)\\ &=: \frac{\partial^2 P}{\partial z_{n+t}\partial\bar{z}_{m+c}} - p_{nm}^{11},\end{aligned}$$

$f_{nm}^{20}, f_{nm}^{02}, p_{nm}^{20}, p_{nm}^{02}, \mathcal{F}_{nm}^{20}(t), \mathcal{F}_{nm}^{02}(t), \mathcal{P}_{nm}^{20}(t), \mathcal{P}_{nm}^{02}(t)$ are defined similarly.

Actually, one can find $f_{nm}^{20} = f_{nm}^{02} = 0$.

Assume g_{nm}^{**} is any function above, the regularity of F and P lead to Taylor–Fourier expansion $g_{nm}^{**} = \sum_{k,|\ell|\le 1} g_{k,\ell,n,m}^{**} e^{i\langle k,x\rangle} I^\ell$.

Proposition 2 ***(KAM Preserve Strong-Töplitz–Lipschitz)***

The solution F of the homological equations and new perturbation P_+ are STL with ς, they also satisfy momentum conservation and integral invariance.

Proof (1) We use the STL property of P and $\langle\breve{\Omega}z, \bar{z}\rangle$ to prove that F satisfies STL. Notice

$$F_{k,\ell,m+t,n+t} = \frac{\mathrm{i}P_{k,\ell,m+t,n+t}}{\langle k,\omega\rangle + \Omega_{m+t} - \Omega_{n+t}}, \tag{3.11}$$

One has

a) If $m \neq n$,

$$\left\|\frac{\partial^2 F}{\partial z_{m+t}\bar{z}_{n+t}} - 0\right\|_{\mathscr{G}_+} = \left\|\sum_{\substack{|k|\le\mathcal{K}\\|\ell|\le 1}} \frac{P_{k,\ell,m+t,n+t}e^{i\langle k,x\rangle}I^\ell}{\langle k,\omega\rangle + \Omega_{m+t} - \Omega_{n+t}}\right\|_{\mathscr{G}_+} \le \frac{\varepsilon\mathcal{K}^\tau e^{-|n-m|\rho}}{|t|}, \tag{3.12}$$

since $\Omega_{m+t} - |m+t|^2 - \xi_{m+t}$ and $\Omega_{n+t} - |n+t|^2 - \xi_{n+t}$ has same limits $\bar{\lambda}$ when $t \to \infty$, one has

$$\left\|\frac{\partial}{\partial \xi_\ell}\left(\frac{\partial^2 F}{\partial z_{m+t}\partial\bar{z}_{n+t}} - 0\right)\right\|_{\mathscr{G}_+} = \left\|\sum_{|k|\le\mathcal{K},|\ell|\le 1} \frac{P_{k,\ell,m+t,n+t}e^{i\langle k,x\rangle}I^\ell}{(\langle k,\omega\rangle + \Omega_{m+t} - \Omega_{n+t})^2}\frac{\partial}{\partial\xi_\ell}(\langle k,\omega\rangle + \Omega_{m+t} - \Omega_{n+t})\right\|_{\mathscr{G}_+} + \left\|\sum_{|k|\le\mathcal{K},|\ell|\le 1} \frac{\frac{\partial}{\partial\xi_\ell}P_{k,\ell,m+t,n+t}e^{i\langle k,x\rangle}I^\ell}{\langle k,\omega\rangle + \Omega_{m+t} - \Omega_{n+t}}\right\|_{\mathscr{G}_+} \le \frac{\varepsilon\mathcal{K}^{2\tau}e^{-|n-m|\rho}}{\min\{|n+t|^\varsigma, |m+t|^\varsigma\}\cdot|\ell|^\varsigma}, \tag{3.13}$$

b) If $m = n$, due to STL property of perturbation P and structure of F, one has $\lim_{t\to\infty}\frac{\partial^2 F}{\partial z_{n+t}\bar{z}_{n+t}} = \mathrm{i}\sum_{|k|\le\mathcal{K},|\ell|\le 1}\frac{p^{11}_{k,\ell,n,n}}{\langle k,\omega\rangle}e^{i\langle k,\theta\rangle}I^\ell$,

$$\left\|\frac{\partial^2 F}{\partial z_{n+t}\bar{z}_{n+t}} - \mathrm{i}\sum_{|k|\le\mathcal{K},|\ell|\le 1}\frac{p^{11}_{k,\ell,n,n}}{\langle k,\omega\rangle}e^{i\langle k,\theta\rangle}I^\ell\right\|_{\mathscr{G}_+} = \left\|\sum_{|k|\le\mathcal{K},|\ell|\le 1}\frac{P_{k,\ell,n+t,n+t} - p^{11}_{k,\ell,n,n}}{\langle k,\omega\rangle}e^{i\langle k,\theta\rangle}I^\ell\right\|_{\mathscr{G}_+} \le \frac{\varepsilon\mathcal{K}^\tau}{|n+t|^\varsigma},$$

The estimate on the derivative with respect to the parameter is as follows

$$\left\|\frac{\partial}{\partial\xi_\ell}\sum_{|k|\le\mathcal{K},|\ell|\le 1}\frac{p^{11}_{k,\ell,n,n}}{\langle k,\omega\rangle}e^{i\langle k,\theta\rangle}I^\ell\right\|_{\mathscr{G}_+} = \left\|\sum_{|k|\le\mathcal{K},|\ell|\le 1}\left(\frac{\frac{\partial}{\partial\xi_\ell}p^{11}_{k,\ell,n,n}}{\langle k,\omega\rangle} - \frac{p^{11}_{k,\ell,n,n}}{\langle k,\omega\rangle^2}\frac{\partial\langle k,\omega\rangle}{\partial\xi_\ell}\right)e^{i\langle k,\theta\rangle}I^\ell\right\|_{\mathscr{G}_+} \le \frac{\varepsilon\mathcal{K}^{2\tau}}{|\ell|^\varsigma}, \tag{3.14}$$

$$\left\| \frac{\partial}{\partial \xi_\ell} \left(\frac{\partial^2 F}{\partial z_{n+t} \bar{z}_{n+t}} - \mathrm{i} \sum_{|k| \le \mathcal{K}, |\ell| \le 1} \frac{p^{11}_{k,\ell,n,n}}{\langle k, \omega \rangle} e^{i\langle k,\theta\rangle} I^\ell \right) \right\|_{\mathscr{G}_+}$$
$$\le \left\| \sum_{|k| \le \mathcal{K}, |\ell| \le 1} \frac{P_{k,\ell,n+t,n+t} - p^{11}_{k,\ell,n,n}}{\langle k, \omega \rangle^2} e^{i\langle k,\theta\rangle} I^\ell \frac{\partial}{\partial \xi_\ell} \langle k, \omega \rangle \right\|_{\mathscr{G}_+}$$
$$+ \left\| \sum_{|k| \le \mathcal{K}, |\ell| \le 1} \frac{\frac{\partial}{\partial \xi_\ell} (P_{k,\ell,n+t,n+t} - p^{11}_{k,\ell,n,n})}{\langle k, \omega \rangle} e^{i\langle k,\theta\rangle} I^\ell \right\|_{\mathscr{G}_+}$$
$$\le \frac{\varepsilon \mathcal{K}^{2\tau}}{|n+t|^\varsigma \cdot |\ell|^\varsigma}, \tag{3.15}$$

There is the estimate of F^{20} and F^{02} similarly, hence F satisfies STL with ς.

(2) Note that

$$P_+ = P - R + \sum_{n \ge 1} \frac{1}{n!} \underbrace{\{\cdots\{}_{n} P, F\}, \cdots, F\}, \tag{3.16}$$

The second part is a consequence of iterative Lemma below.

(3) From structure of P and the truncation R, F satisfies Momentum conservation and Integral invariance. It also holds true for P_+ since Momentum conservation and Integral invariance are closed under poisson bracket. □

Lemma 3.4 *Assume that $P, F \in \mathcal{A}_{r,s}$ satisfy STL with ς, and F satisfies*

$$\frac{\partial^2 F}{\partial z_n \partial z_m} = 0(|n+m| > \mathcal{K}), \ \frac{\partial^2 F}{\partial z_n \partial \bar{z}_m} = 0(|n-m| > \mathcal{K}), \ \frac{\partial^2 F}{\partial \bar{z}_n \partial \bar{z}_m} = 0(|n+m| > \mathcal{K}),$$

then $\{P, F\}$ satisfies STL with ς.

Proof For $\ell \in \mathbb{Z}^b_1$, one has

$$\left| \frac{\partial p^{11}_{nm}}{\partial \xi_\ell} \right| \le \frac{\varepsilon e^{-|n-m|\rho}}{|\ell|^\varsigma}, \quad \left| \frac{\partial p^{20}_{nm}}{\partial \xi_\ell} \right| \le \frac{\varepsilon e^{-|n+m|\rho}}{|\ell|^\varsigma}, \quad \left| \frac{\partial p^{02}_{nm}}{\partial \xi_\ell} \right| \le \frac{\varepsilon e^{-|n+m|\rho}}{|\ell|^\varsigma}, \tag{3.17}$$

Recall that the operator $\Gamma_{\mathcal{K}} F$ collects all terms in F with $|k| \le \mathcal{K}$ and $\sum_{i \in \mathbb{Z}^b_1} |i|(\alpha_i + \beta_i) \le \mathcal{K}$.

For $|t| \ge \max\{|n|, |m|\}^6$, one has

$$\left\| \frac{\partial^2 \{F, P\}}{\partial z_{n+t} \partial \bar{z}_{m+t}} - \frac{\partial^2 \{f, g\}}{\partial z_n \partial \bar{z}_m} \right\|_{\mathscr{G}_+}$$
$$= \left\| \frac{\partial^2 \{F, P\}}{\partial z_{n+t} \partial \bar{z}_{m+t}} - \sum_{|j+n| \le \mathcal{K}} \left(f^{20}_{nj} p^{02}_{jm} - f^{02}_{nj} p^{20}_{jm} \right) - \sum_{|j+m| \le \mathcal{K}} \left(f^{20}_{mj} p^{02}_{jn} - f^{02}_{mj} p^{20}_{jn} \right) \right.$$
$$\left. - \sum_{|j-n| \le \mathcal{K}} f^{11}_{nj} p^{11}_{jm} + \sum_{|j-m| \le \mathcal{K}} f^{11}_{mj} p^{11}_{jn} - \frac{\partial^2 \{\Gamma_{\mathcal{K}} f, p\} + \{f, \Gamma_{\mathcal{K}} p\}}{\partial z_n \partial \bar{z}_m} \right\|_{\mathscr{G}_+}$$
$$\le \sum_j \left\| \mathcal{F}^{11}_{nj}(t) \right\|_{\mathscr{G}_+} \left\| p^{11}_{jm} \right\|_{\mathscr{G}_+} + \left\| \mathcal{F}^{11}_{mj}(t) \right\|_{\mathscr{G}_+} \left\| p^{11}_{jn} \right\|_{\mathscr{G}_+}$$

$$
\begin{aligned}
&+\sum_j \left\|\mathcal{P}^{11}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|f^{11}_{jm}\right\|_{\mathscr{G}_+}+\left\|\mathcal{P}^{11}_{mj}(t)\right\|_{\mathscr{G}_+}\left\|f^{11}_{jn}\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\mathcal{F}^{20}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|p^{02}_{jm}\right\|_{\mathscr{G}_+}+\left\|\mathcal{F}^{20}_{mj}(t)\right\|_{\mathscr{G}_+}\left\|p^{02}_{jn}\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\mathcal{F}^{02}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|p^{20}_{jm}\right\|_{\mathscr{G}_+}+\left\|\mathcal{F}^{02}_{mj}(t)\right\|_{\mathscr{G}_+}\left\|p^{20}_{jn}\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\mathcal{F}^{11}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|\mathcal{P}^{11}_{jm}(t)\right\|_{\mathscr{G}_+}+\sum_j\left\|\mathcal{P}^{11}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|\mathcal{F}^{11}_{jm}(t)\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\mathcal{F}^{11}_{mj}(t)\right\|_{\mathscr{G}_+}\left\|\mathcal{P}^{11}_{jn}(t)\right\|_{\mathscr{G}_+}+\sum_j\left\|\mathcal{P}^{11}_{mj}(t)\right\|_{\mathscr{G}_+}\left\|\mathcal{F}^{11}_{jn}(t)\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\mathcal{P}^{20}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|\mathcal{F}^{02}_{mj}(t)\right\|_{\mathscr{G}_+}+\sum_j\left\|\mathcal{P}^{02}_{nj}(t)\right\|_{\mathscr{G}_+}\left\|\mathcal{F}^{20}_{mj}(t)\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\frac{\partial\mathcal{P}^{11}_{nm}(t)}{\partial z_j}\right\|_{\mathscr{G}_+}\left\|F^{01}_{k,j}\right\|_{\mathscr{G}_+}+\left\|\frac{\partial\mathcal{P}^{11}_{nm}(t)}{\partial \bar{z}_j}\right\|_{\mathscr{G}_+}\left\|F^{10}_{k,j}\right\|_{\mathscr{G}_+}\\
&+\sum_j \left\|\frac{\partial\mathcal{P}^{11}_{nm}(t)}{\partial z_j}\right\|_{\mathscr{G}_+}\left\|F^{01}_{k,j}\right\|_{\mathscr{G}_+}+\left\|\frac{\partial\mathcal{P}^{11}_{nm}(t)}{\partial \bar{z}_j}\right\|_{\mathscr{G}_+}\left\|F^{10}_{k,j}\right\|_{\mathscr{G}_+}\\
&+\left\|\{\mathcal{P}^{11}_{nm}(t),\Gamma_{\mathcal{K}}F\}\right\|_{\mathscr{G}_+}+\left\|\{\Gamma_{\mathcal{K}}P,\mathcal{F}^{11}_{nm}(t)\}\right\|_{\mathscr{G}_+}\\
\le{}& \max\left\{\frac{\varepsilon^{\frac{5}{3}}\mathcal{K}^b}{|n+t|^{\varsigma}},\frac{\varepsilon^{\frac{5}{3}}\mathcal{K}^b}{|n+t|^{\varsigma}}\right\}e^{-|n-m|\rho}\\
\le{}& \max\left\{\frac{\varepsilon^{\frac{4}{3}}}{|n+t|^{\varsigma}},\frac{\varepsilon^{\frac{4}{3}}}{|n+t|^{\varsigma}}\right\}e^{-|n-m|\rho}.
\end{aligned}
$$

According to the formula

$$
\frac{\partial}{\partial\xi_\ell}\{P,F\}=\left\{\frac{\partial}{\partial\xi_\ell}P,F\right\}+\left\{P,\frac{\partial}{\partial\xi_\ell}F\right\}, \tag{3.18}
$$

we give estimate of the derivative, the given condition on f^{11}_{nm},p^{11}_{nm}, p^{20}_{nm},p^{02}_{nm} and (3.17) guarantee that

$$
\left\|\frac{\partial}{\partial\xi_\ell}\left(\frac{\partial^2\{F,P\}}{\partial z_{n+t}\partial\bar{z}_{m+t}}-\frac{\partial^2\{f,g\}}{\partial z_n\partial\bar{z}_m}\right)\right\|_{\mathscr{G}_+}\le\max\left\{\frac{\varepsilon^{\frac{4}{3}}}{|n+t|^{\varsigma}},\frac{\varepsilon^{\frac{4}{3}}}{|n+t|^{\varsigma}}\right\}\frac{e^{-|n-m|\rho}}{|\ell|^{\varsigma}},
$$

For the limit case, with estimate (3.17), one has

$$
\left\|\frac{\partial}{\partial\xi_\ell}\frac{\partial^2\{f,g\}}{\partial z_n\partial\bar{z}_m}\right\|_{\mathscr{G}_+}=\left\|\frac{\partial^2\left\{\frac{\partial}{\partial\xi_\ell}f,g\right\}}{\partial z_n\partial\bar{z}_m}\right\|_{\mathscr{G}_+}+\left\|\frac{\partial^2\left\{f,\frac{\partial}{\partial\xi_\ell}g\right\}}{\partial z_n\partial\bar{z}_m}\right\|_{\mathscr{G}_+}\le\frac{\varepsilon^{\frac{4}{3}}e^{-|n-m|\rho}}{|\ell|^{\varsigma}}.
$$

□

3.3 Frequency Exciting

one has

$$
H_+=H\circ\Phi=\sum_{i\in\mathcal{J}}\lambda_i^+I_i+\sum_{i\in\mathbb{Z}\setminus\mathcal{J}}\lambda_i^+|z_i|^2+P_+(\theta,I,z,\bar{z},\xi;y),
$$

where $\Phi : D_{\rho_+}(r_+, s_+) \times \mathcal{O} \to D_\rho(r, s) \times E$ with $meas(E \setminus \mathcal{O}) \leq O(\gamma)$. The perturbation satisfies STL with ς. One get new parameter set $B_+ = \{(\xi, \xi') \in B : \xi \in \mathcal{O}\}$ naturally.

At the next step of iteration construction, and likewise of all subsequent steps, we do the following. Take H_+ well defined on $D_{\rho_+}(r_+, s_+) \times \mathcal{O}$. We excite only **TWO** oscillators so that the perturbation is relatively small,

$$z_j = \sqrt{I_j + y_j} e^{i\theta_j}, \ \bar{z}_j = \sqrt{I_j + y_j} e^{-i\theta_j}, \ |j| = \nu + 1, \tag{3.19}$$

and let the remaining normal variable

$$q_n = z_n, \bar{q}_n = \bar{z}_n \text{ for } |n| > \nu + 1.$$

Then we obtain a new Hamiltonian on $D_{\rho_+}(r_+, s_+) \times \mathcal{O}$

$$H_+ = \sum_{i \in \mathcal{J}_+} \lambda_i^+ I_i + \sum_{i \in \mathbb{Z} \setminus \mathcal{J}_+} \lambda_i^+ |q_i|^2 + P_+(\theta, I, q, \bar{q}, \xi, y),$$

The new amplitude satisfy

$$s_+^2 < I_j \leq s_+^{2\chi}, \frac{2}{3} < \chi < 1.$$

To make KAM machine work smoothly in the next iteration, we choose $\gamma_+ = \varepsilon^{\varrho}$, $(1 < \varrho < \frac{4}{3})$, it plays a role of Diophantine constant; The parameter $\xi \in B_+$ is split into "essential" and "dummy" part: $\xi = (\xi^+, \xi'') \in B_+$. It will be enough when one consider the essential set

$$E_+ =: \{\xi^+ : (\xi^+, \xi'') \in B_+\}.$$

4 Iteration Lemma, Convergence and Measure Estimate

For any given $\varepsilon, r, s, \gamma$ and for all $\nu \geq 1$, we define the following sequences

$$\mathcal{J}_\nu = \{i \in \mathbb{Z} : |i| \leq \nu\}, \tag{4.1}$$

$$\begin{aligned}
s_\nu &= s\left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right),\\
\varepsilon_\nu &= \left(\gamma_{\nu-1}^{-1} \mathcal{K}_{\nu-1}^{\tau_{\nu-1}}\right)^3 \varepsilon_{\nu-1}^{\frac{4}{3}}, \quad \eta_\nu = \varepsilon_\nu^{\frac{1}{3}},\\
r_\nu &= \frac{1}{4}\eta_{\nu-1} r_{\nu-1} = 2^{-2\nu}\left(\prod_{i=0}^{\nu-1} \varepsilon_i\right)^{\frac{1}{3}},\\
\rho_\nu &= \rho\left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right), \quad \gamma_\nu = \varepsilon_\nu^{\frac{1}{24}},\\
\mathcal{K}_\nu &= (\rho_{\nu-1} - \rho_\nu)^{-1} \ln \varepsilon_\nu^{-1},\\
\tau_\nu &= 4\nu + 4,
\end{aligned} \tag{4.2}$$

where $\mathcal{J}_0 = \{i \in \mathbb{Z} : i = 0\}$ and $\varepsilon_0, r_0, s_0, \gamma_0, \tau_0, \rho_0$ are defined to be $\varepsilon, r, s, \gamma, 4, \rho$ respectively.

4.1 Iteration Lemma

The preceding analysis can be summarized as follows.

Lemma 4.1 *Let ε_ν be small enough and $\nu \geq 0$. Suppose that*

$$H_\nu = \sum_{i \in \mathcal{J}_\nu} \lambda_i^\nu I_i + \sum_{n \in \mathbb{Z} \setminus \mathcal{J}_\nu} \Omega_n^\nu |q_n|^2 + P_\nu(\theta, I, q, \bar{q}, \xi; y),$$

defined in $D_{\rho_\nu}(r_\nu, s_\nu) \times E_\nu$ with perturbation satisfies momentum conservation, integral invariance and also:

i) $\lambda^\nu \equiv (\omega^\nu, \Omega^\nu)$ is given in Proposition 1.
ii) $\|X_{P_\nu}\|_{D_{\rho_\nu}(r_\nu, s_\nu), \Theta_\nu} \leq \varepsilon_\nu$ and satisfies STL with ς

Then there is a subset $\mathcal{O}_\nu \subset E_\nu$ with $meas(E_\nu \setminus \mathcal{O}_\nu) = o(\gamma_\nu)$ and a symplectic transformation

$$\Phi_\nu : D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_\nu}(r_\nu, s_\nu), \tag{4.3}$$

such that on $D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}) \times E_{\nu+1}$, $H_{\nu+1} = H_\nu \circ \Phi_\nu$ has the form

$$H_{\nu+1} = \sum_{i \in \mathcal{J}_{\nu+1}} \lambda_i^{\nu+1} I_i + \sum_{n \in \mathbb{Z} \setminus \mathcal{J}_{\nu+1}} \Omega_n^{\nu+1} |q_n|^2 + P_{\nu+1}(\theta, I, q, \bar{q}, \xi; y),$$

with perturbation satisfies momentum conservation, integral invariance and

i) $\lambda^{\nu+1} \equiv (\omega^{\nu+1}, \Omega^{\nu+1})$ is given in Proposition 1.
ii) $\|X_{P_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}), E_{\nu+1}} \leq \varepsilon_{\nu+1}$ and STL with ς

4.2 Convergence

Assume the assumptions of Theorem 2 are satisfied. Recall $\varepsilon_0 = \varepsilon$, $r_0 = r$, $s_0 = s$, $\rho_0 = \rho$, $L_0 = L$, $H_0 = H$, and the tangential site is taken to be $\mathcal{J}_0$. The small divisor is assumed to be satisfied by setting $B_{-1} = B$, $B_0 = \{(\xi^{b_0}, \xi^0) : \xi^{b_0} \in \mathcal{O}_0\}$ and $\mathcal{O}_0 = E_0 \setminus (\bigcup_{kl} R_{kl}^0(\gamma))$ when $\nu = 0$. Thus our iteration will work inductively.

We obtain a sequence for any $\nu \geq 1$:

$$\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_{\rho_\nu}(r_\nu, s_\nu) \times E_\nu \to D_{\rho_0}(s_0, r_0),$$

such that $H_\nu = H \circ \Psi^\nu = \langle \omega^\nu, y \rangle + \langle \Omega^\nu q, \bar{q} \rangle + P_\nu(\theta, I, q, \bar{q}, \xi; y)$.

Since $\gamma_\nu = \varepsilon_\nu^{\frac{1}{24}}$, the number of tangential frequency $b_\nu = 2\nu + 1$ and $\tau_\nu = 4\nu + 4$, one has

$$\varepsilon_{\nu+1} = (\gamma_\nu^{-1} \mathcal{K}_\nu^{\tau_\nu})^3 \varepsilon_\nu^{\frac{4}{3}} \leq \left\{ \left(\frac{29}{24} \right)^\nu \ln \frac{1}{\varepsilon_0} \right\}^{12\nu+12} (\varepsilon_0)^{\left(\frac{29}{24}\right)^\nu}, \tag{4.4}$$

it follows that $\varepsilon_{\nu+1} \to 0$ provided ε_0 is sufficiently small. The linear growth of b_ν guarantee iteration work smoothly and one finally has $\sum_{\nu=0}^{\infty} \varepsilon_\nu \leq c\varepsilon_0$.

Let $B^* = \cap_{\nu=0}^{\infty} B_\nu$. As in [12,13], thanks to Lemma 3.2, it concludes that N_ν, Ψ^ν, $D\Psi^\nu$, ω_ν converge uniformly on $D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times B^*$.

Let ϕ_H^t be the flow of X_H. Since $H \circ \Psi^\nu = H_{\nu+1}$, there is

$$\phi_H^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{H_{\nu+1}}^t. \tag{4.5}$$

The uniform convergence of Ψ^ν, $D\Psi^\nu$, ω_ν and X_{H_ν} implies that the limits can be taken on both sides of (4.5). Hence on $D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times B^*$ one gets

$$\phi_H^t \circ \Psi^\infty = \Psi^\infty \circ \phi_{H_\infty}^t, \tag{4.6}$$

and

$$\Psi^\infty : D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times B^* \to D_\rho(r, s) \times B,$$

It follows from (4.6) that

$$\phi_H^t(\Psi^\infty(\mathbb{T}^\infty \times \{\xi\})) = \Psi^\infty(\mathbb{T}^\infty \times \{\xi\})$$

for $\xi \in B^*$. This means that $\Psi^\infty(\mathbb{T}^\infty \times \{\xi\})$ is an embedded torus which is invariant for the original perturbed Hamiltonian system at $\xi \in B^*$, and this give the existence of almost periodic solutions.

4.3 Measure Estimate

A function $f : U \to V$ is said to be Lipschitz if for any $x, y \in U$, there is constant K such that

$$\frac{d_V(f(x), f(y))}{d_U(x, y)} \le K,$$

the lipschitz norm $\|f\|_U^{\mathcal{L}}$ is defined to be the infimum of these K.

Like before, $\lambda^\nu =: (\omega^\nu, \Omega^\nu)$, we use notation

$$\begin{aligned}
\bar{\lambda}^\nu &=: \lim_{n\to\infty}(\lambda_n^\nu - |n|^2 - \xi_n),\\
\hat{\lambda}_n^\nu &=: \lambda_n^\nu - \bar{\lambda}_n^\nu = |n|^2 + \xi_n + \check{\lambda}_n^\nu\\
\hat{\omega}^\nu &=: (\hat{\lambda}_{-\nu}^\nu, \cdots, \hat{\lambda}_\nu^\nu)
\end{aligned}$$

At ν-step of iteration, one need to exclude parameter set

$$X_\nu = \{(\xi^\nu, \xi') \in B_\nu : \xi^\nu \in \mathcal{R}^\nu\},$$

with

$$\mathcal{R}^\nu = \bigcup_{\substack{|k|\le\mathcal{K}_\nu\\ |l|\le 2}} \mathcal{R}_{kl}^\nu, \tag{4.7}$$

Lemma 4.2 *Since perturbation satisfies Integral invariance, one has:*

$$\mathcal{R}_{kl}^\nu = \left\{\xi \in \Theta_\nu : |\langle k, \omega^\nu\rangle + l \cdot \Omega^\nu| < \frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}}\right\} \tag{4.8}$$

$$\equiv \left\{\xi \in \Theta_\nu : |\langle k, \widehat{\omega}^\nu\rangle + l \cdot \widehat{\Omega}^\nu| < \frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}}\right\}, \tag{4.9}$$

where $\Theta_\nu =: \{\xi^\nu : (\xi^\nu, \xi'') \in B_\nu\}$, *and* $|\ell| = \sum_i |\ell_i| \le 2$ *with* $\ell \in \mathbb{Z}^{\mathbb{Z}\setminus\mathcal{J}_\nu}$.

This Lemma is obvious when one notice Integral invariance. It says that the uniform (or limit) part will not affect measure estimate.

Lemma 4.3 *For* $\max_i sign(\ell_i)i \geq \nu^3\mathcal{K}_\nu^2$, *one has* $\mathcal{R}_{k\ell}^\nu = \emptyset$. *It follows:*

$$meas\left(\bigcup_{k,\ell}\mathcal{R}_{k\ell}^\nu\right) = meas\left(\bigcup_{\substack{k,\\ \max_i sign(\ell_i)i\leq\nu^3\mathcal{K}_\nu^2}}\mathcal{R}_{k\ell}^\nu\right). \tag{4.10}$$

This proof is clear.

Lemma 4.4 *For fixed* ν, k, l

$$meas(\mathcal{R}_{kl}^\nu) \leq \zeta^{2\nu}\frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}},$$

here ζ *is the diameter of* B_ν.

Proof For the convenience of notation, let $\chi^\nu = \left(\widehat{\lambda}_{-\nu+1}^\nu, \ldots, \widehat{\lambda}_{\nu-1}^\nu\right)$, and divide the modified frequency $\hat{\omega}^\nu$ into two parts

$$\begin{aligned}\widehat{\omega}^\nu &= \left(\widehat{\lambda}_{-\nu}^\nu, \ldots, \widehat{\lambda}_i^\nu, \ldots, \widehat{\lambda}_\nu^\nu\right)\\ &= \left(\xi_{-\nu}, \widehat{\omega}^{\nu-1}, \xi_\nu\right) + \left(\widehat{\lambda}_{-\nu} - \xi_{-\nu}, \chi^\nu - \widehat{\omega}^{\nu-1}, \widehat{\lambda}_\nu - \xi_\nu\right)\\ &=: \zeta + \varpi^\nu(\zeta),\end{aligned}$$

ζ is the parameters over domain $\Delta = (\xi_{-\nu}, \widehat{\omega}^{\nu-1}, \xi_\nu)(B_\nu)$, $\varpi^\nu(\cdot)$ is a bilipschitz map and defined on Δ, $\mathcal{R}_{kl}^\nu(\Delta) \equiv (\xi_{-\nu}, \widehat{\omega}^{\nu-1}, \xi_\nu)(\mathcal{R}_{kl}^\nu)$ is the corresponding resonance set in Δ.

From analysis about the shift of frequency in Proposition 1, one has

$$\left\|(\varpi_{-\nu+1}^\nu, \cdots, \varpi_{\nu-1}^\nu)\right\|_\Delta^{\mathcal{L}} \leq \varepsilon_\nu, \tag{4.11}$$

$$\left\|\widehat{\Omega}\right\|_{\varsigma,\Delta}^{\mathcal{L}} =: \max_{\ell\in\mathbb{Z}\setminus\mathcal{J}_\nu} |\ell|^\varsigma \left\|\widehat{\Omega}_\ell\right\|_\Delta^{\mathcal{L}} \leq \varepsilon_\nu, \tag{4.12}$$

Following Pöschel [10], we define

$$f(\zeta) = \langle k, \widehat{\omega}^\nu\rangle + \ell\cdot\widehat{\Omega}^\nu.$$

$v \in \{-1, 1\}^{2\nu+1}$ is taken such that $\langle k, v\rangle = |k|$ and let $\zeta = xv + \eta$ with $x \in \mathbb{R}$, $\eta \in v^\perp$, and $f(xv + \eta)$ is considered to be a function of x.

For any x, η and $t > s$, one has

$$\begin{aligned}\langle k, \widehat{\omega}^\nu\rangle|_s^t &= \langle k, \zeta\rangle|_s^t + \langle(0, k_{-\nu+1}, \ldots, k_{\nu-1}, 0), \varpi\rangle|_s^t\\ &\quad+ k_{-\nu}(\widehat{\lambda}_{-\nu} - \xi_{-\nu})|_s^t + k_\nu(\widehat{\lambda}_\nu - \xi_\nu)|_s^t\\ &\geq |k|(t-s) - (2\nu+1)\|k\|^2(\|(\varpi_{-\nu+1}^\nu, \ldots, \varpi_{\nu-1}^\nu)\|_\Delta^{\mathcal{L}})^2(t-s)\\ &\quad-\varepsilon_0(|k_{-\nu}| + |k_\nu|)(t-s)\\ &\geq |k|(t-s) - (2\nu+1)\varepsilon_\nu^2\|k\|^2(t-s) - \varepsilon_0(|k_{-\nu}| + |k_\nu|)(t-s)\\ &\geq \frac{1}{2}|k|(t-s),\end{aligned} \tag{4.13}$$

$$\langle\ell, \widehat{\Omega}\rangle|_s^t \leq \varepsilon_0(t-s), \tag{4.14}$$

All these inequality lead to:

$$f(xv+\eta)|_s^t \geq \frac{1}{2}|k|(t-s) - \varepsilon_0(t-s) \geq \frac{1}{4}|k|(t-s).$$

It follows

$$\left\{x : xv + \eta \in \Delta, |f(xv+\eta)| < \frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}}\right\} \subseteq \left\{x : |x - x_0| < \frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}}|k|^{-1}\right\}. \tag{4.15}$$

making use of Fubini theory, one has

$$|\mathcal{R}_{kl}^\nu(\Delta)| \le \zeta^{2\nu}\frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}}. \tag{4.16}$$

Go back to the original parameter domain B_ν by inverse map $(\xi_{-\nu}, \widehat{\omega}^{\nu-1}, \xi_\nu)^{-1}$ and observe that $diam\Delta \le 2 diam B_\nu$, we get the estimate of the resonant set. □

Then by counting the number of the nonempty resonant set, we obtain finite dimension lebesgue measure of the resonant set $\mathcal{R}^\nu$,

$$\begin{aligned} meas(\mathcal{R}^\nu) &\le \zeta^{2\nu}((\nu^3\mathcal{K}_\nu^2)^2 \cdot \mathcal{K}_\nu^{2\nu} + \nu^3\mathcal{K}_\nu^2 \cdot \mathcal{K}_\nu^{2\nu} + \mathcal{K}_\nu^{2\nu}) \cdot \frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}} \\ &\le \zeta^{2\nu}(\nu^6\mathcal{K}_\nu^{4+2\nu}) \cdot \frac{\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu}} \\ &= \frac{\zeta^{2\nu}\nu^6\gamma_\nu}{\mathcal{K}_\nu^{\tau_\nu-2\nu-4}} \le \frac{\gamma_\nu}{\mathcal{K}_\nu^{\nu}}. \end{aligned}$$

One has $meas(\mathcal{R}^\nu) \le O(\gamma_\nu) \to 0$ as $\nu \to \infty$. It follows that the total measure of excluded parameter is as small as we wish, and we finally get a Cantor like parameter set $B^* = \bigcap_{\nu=0}^\infty B_\nu$.

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5 Appendix

A class of infinite dimensional matrices is given:

Definition 5.1 We denote $\mathscr{B}$ to be the set of infinite dimension matrix A with $\sum_{i,j\in\mathbb{Z}} a_{ij}^2 < \infty$, its norm is defined $\|A\|_M = \sqrt{\sum_{i,j\in\mathbb{Z}} a_{ij}^2}$.

The set $\mathscr{B}$ is algebra closed. An easy calculation gives lemma below.

Lemma 5.1 *If $T_1, T_2 \in \mathscr{B}$, then $T_1 + T_2, T_1 - T_2, T_1T_2 \in \mathscr{B}$.*

Lemma 5.2 *One has $\|FG\|_{D_\rho(r,s)} \le \|F\|_{D_\rho(r,s)} \cdot \|G\|_{D_\rho(r,s)}$.*

Lemma 5.3 *(Cauchy inequalities) For $\eta \ll 1$,*

$$\|F_{\theta_i}\|_{D_\rho(r-\sigma,s)} \le \frac{c}{\sigma}\|F\|_{D_\rho(r,s)}, \ \|F_{I_i}\|_{D_\rho(r,\frac{s}{2})} \le \frac{c}{r^2}\|F\|_{D_\rho(r,s)},$$

and

$$\|F_z\|_{D_\rho(r,\frac{s}{2})} \le \frac{c}{s}\|F\|_{D_\rho(r,s)}, \quad \|F_{\bar{z}}\|_{D_\rho(r,\frac{s}{2})} \le \frac{c}{s}\|F\|_{D_\rho(r,s)},$$

Lemma 5.4 *If $\|X_P\|_{D_\rho(r,s)} < \infty$ and $\|X_G\|_{D_\rho(r,s)} < \infty$, one has*

$$\|X_{\{P,G\}}\|_{D_\rho(r-\sigma,\eta s)} \le c\sigma^{-1}\eta^{-1}\|X_P\|_{D_\rho(r,s)} \cdot \|X_G\|_{D_\rho(r,s)}.$$

For the proof see [7].

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