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A KAM Theorem for Higher Dimensional Nonlinear Schrödinger Equations

Jiansheng Geng · Jiangong You

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Abstract We prove an infinite dimensional KAM theorem. As an application, we use the theorem to study the higher dimensional nonlinear Schrödinger equation

$$iu_t - \triangle u + M_\xi u + f(|u|^2)u = 0, \quad t \in \mathbb{R}, x \in \mathbb{T}^d$$

with periodic boundary conditions, where M_ξ is a real Fourier multiplier and $f(|u|^2)$ is a real analytic function near $u = 0$ with $f(0) = 0$. We obtain for the equation a Whitney smooth family of real-analytic small-amplitude linearly-stable quasi-periodic solutions with a nice linear normal form.

Keywords Schrödinger equation · KAM tori · Quasi-periodic solutions

Mathematics Subject Classification Primary 37K55 · 35B10

1 Introduction and Main Result

There have been many remarkable results in KAM (Kolmogorov–Arnold–Moser) theory of Hamiltonian PDEs achieved either by methods from the finite dimensional KAM theory [1,9,11–29], or by a Newtonian scheme developed by Craig, Wayne, Bourgain [2–8,10]. The advantage of the method from the finite dimensional KAM theory is the construction of a local normal form in a neighborhood of the obtained solutions in addition to the existence of quasi-periodic solutions. The normal form is helpful to understand the dynamics. For example, one sees the linear stability and zero Lyapunov exponents. The scheme of Craig–Wayne–Bourgain avoids the cumbersome second Melnikov conditions by solving angle dependent homological equations. All those methods are well developed for one dimensional Hamiltonian PDEs.

J. Geng (✉) · J. You
Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
e-mail: jgeng@nju.edu.cn

J. You
e-mail: jyou@nju.edu.cn

However, they meet difficulties in higher dimensional Hamiltonian PDEs. Bourgain [2] made the first breakthrough by proving that the two dimensional nonlinear Schrödinger equations admit small-amplitude quasi-periodic solutions. Later he improved in [5] his method and proved that the higher dimensional nonlinear Schrödinger and wave equations admit small-amplitude quasi-periodic solutions.

Constructing quasi-periodic solutions of higher dimensional Hamiltonian PDEs by method from the finite dimensional KAM theory appeared later. Geng and You [15,16] proved that the higher dimensional nonlinear beam equations and nonlocal Schrödinger equations admit small-amplitude linearly-stable quasi-periodic solutions. The breakthrough of constructing quasi-periodic solutions for more interesting higher dimensional Schrödinger equation by modified KAM method was made recently by Eliasson and Kuksin [12]. They proved that the higher dimensional nonlinear Schrödinger equations admit small-amplitude linearly-stable quasi-periodic solutions. Very recently, quasi-periodic solutions of two dimensional cubic Schrödinger equation

$$\mathrm{i}u_t - \Delta u + |u|^2 u = 0, \qquad x \in \mathbb{T}^2,\ t \in \mathbb{R},$$

with periodic boundary conditions are obtained by Geng et al. [13]. By carefully choosing tangential sites $\{i_1, \cdots, i_b\} \in \mathbb{Z}^2$, the authors proved that the above nonlinear Schrödinger equation admits a family of small-amplitude quasi-periodic solutions.

In this paper, we will further develop the methods of [13] and prove the existence and the linear stability of quasi-periodic solutions for higher dimensional nonlinear Schrödinger equations. This result has been obtained by Eliasson and Kuksin [12]. In this paper, we will give a simple proof for Eliasson–Kuksin's result. Since the equation we consider is x-independent, we can give a much more nice linear normal form of the obtained quasi-periodic solutions. The main novelty lies at the measure estimate part. In [12], Eliasson–Kuksin introduced the conception of the Lipschitz domain such that in the Lipschitz domain, the corresponding normal frequencies satisfy Töplitz–Lipschitz property, thus the measure estimates are feasible. While in our paper, to conduct the measure estimates, we use the elementary repeated limit to substitute more complicated Lipschitz domain, thus our measure estimates are more easy to understand and the whole proof is more KAM-like. More concretely, we consider the d-dimensional nonlinear Schrödinger equation

$$\mathrm{i}u_t + Au + f(|u|^2)u = 0, \quad Au = -\Delta u + M_\xi u, \qquad x \in \mathbb{T}^d,\ t \in \mathbb{R} \tag{1.1}$$

with periodic boundary conditions

$$u(t, x_1 + 2\pi, x_2, \cdots, x_d) = \cdots = u(t, x_1, x_2, \cdots, x_{d-1}, x_d + 2\pi) = u(t, x_1, x_2, \cdots, x_d),$$

where $f(|u|^2)$ is a real-analytic function near $u = 0$ with $f(0) = 0$.

Here we assume that the operator $A = -\Delta + M_\xi$ with periodic boundary conditions has eigenvalues $\{\mu_n\}$ satisfying

$$\begin{aligned} \omega_j &= \mu_{i_j} = |i_j|^2 + \xi_j, 1 \le j \le b \\ \Omega_n &= \mu_n = |n|^2, n \neq i_1, \cdots, i_b, \end{aligned} \tag{1.2}$$

and the corresponding eigenfunctions $\phi_n(x) = \sqrt{\frac{1}{(2\pi)^d}} e^{\mathrm{i}\langle n, x\rangle}$ form a basis in the domain of the operator. Assume that $i_1, \cdots, i_b \in \mathbb{Z}^d$ are the distinguished sites of Fourier modes (assume $0 \in \{i_1, \cdots, i_b\}$ in order to take care of $(\mu_n, k) = (0, 0)$), and the parameter $\xi = (\xi_1, \cdots, \xi_b) \in \mathcal{O} = [0, 1]^b \subset \mathbb{R}^b$.

Now we state the main theorem as follows.

Theorem 1 *For any* $0 < \gamma \ll 1$, *there is a Cantor subset* $\mathcal{O}_\gamma \subset \mathcal{O}$ *with* $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$ *such that for any* $\xi \in \mathcal{O}_\gamma$, *the equation (1.1) admits a small-amplitude, quasi-periodic solution of the form*

$$u(t,x) = \sum_{n \in \mathbb{Z}^d} u_n(\omega_1' t, \cdots, \omega_b' t) e^{\mathrm{i}\langle n, x\rangle}$$

where $u_n : \mathbb{T}^b \to \mathbb{R}$ *and* $\omega_1', \cdots, \omega_b'$ *are close to the unperturbed frequencies* $\omega_1, \cdots, \omega_b$. *Moreover, the quasi-periodic solutions obtained here are* **real analytic** *and* **linearly stable**.

This paper is organized as follows: In Sect. 2 we give an infinite dimensional KAM theorem; in Sect. 3, we give its applications to higher dimensional Schrödinger equations. The proof of the KAM theorem is given in Sects. 4, 5 and 6. Some technical lemmas are proved in the Appendix.

2 An Infinite Dimensional KAM Theorem for Hamiltonian Partial Differential Equations

In this section, we will formulate an infinite dimensional KAM theorem that can be applied to higher dimensional Schrödinger equations under periodic boundary conditions.

We start by introducing some notations. For given b vectors in $\mathbb{Z}^d$, say $\{i_1, \cdots, i_b\}$, we denote $\mathbb{Z}_1^d = \mathbb{Z}^d \setminus \{i_1, \cdots, i_b\}$. Let $z = (\cdots, z_n, \cdots)_{n \in \mathbb{Z}_1^d}$, and its complex conjugate $\bar{z} = (\cdots, \bar{z}_n, \cdots)_{n \in \mathbb{Z}_1^d}$. We introduce the weighted norm

$$\|z\|_\rho = \sum_{n \in \mathbb{Z}_1^d} |z_n| e^{|n|\rho},$$

where $|n| = \sqrt{n_1^2 + \cdots + n_d^2}$, $n = (n_1, \cdots, n_d)$ and $\rho > 0$. Denote a neighborhood of $\mathbb{T}^b \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ by

$$D_\rho(r,s) = \{(\theta, I, z, \bar{z}) : |\text{Im}\theta| < r, |I| < s^2, \|z\|_\rho < s, \|\bar{z}\|_\rho < s\},$$

where $|\cdot|$ denotes the sup-norm of complex vectors. Moreover, we denote by $\mathcal{O}$ a positive-measure parameter set in $\mathbb{R}^b$.

Let $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n \in \mathbb{Z}_1^d}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n \in \mathbb{Z}_1^d}$, α_n and $\beta_n \in \mathbb{N}$ with finitely many non-zero components of positive integers. The product $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$. For any given function

$$F(\theta, I, z, \bar{z}) = \sum_{\alpha, \beta} F_{\alpha\beta}(\theta, I) z^\alpha \bar{z}^\beta, \tag{2.1}$$

where $F_{\alpha\beta} = \sum_{k \in \mathbb{Z}^b, l \in \mathbb{N}^b} F_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k, \theta\rangle}$ is C_W^1 functions in parameter ξ in the sense of Whitney, we denote

$$\|F\|_{\mathcal{O}} = \sum_{\alpha,\beta,k,l} |F_{kl\alpha\beta}|_{\mathcal{O}} \, |I^l| e^{|k||\text{Im}\theta|} \, |z^\alpha||\bar{z}^\beta| \tag{2.2}$$

where $|F_{kl\alpha\beta}|_{\mathcal{O}}$ is short for

$$|F_{kl\alpha\beta}|_{\mathcal{O}} \equiv \sup_{\xi \in \mathcal{O}} (|F_{kl\alpha\beta}| + |\frac{\partial F_{kl\alpha\beta}}{\partial \xi}|).$$

(the derivatives with respect to ξ are in the sense of Whitney). We define the weighted norm of F by

$$\|F\|_{D_\rho(r,s),\mathcal{O}} \equiv \sup_{D_\rho(r,s)} \|F\|_{\mathcal{O}}, \tag{2.3}$$

To a function F, we associate a Hamiltonian vector field defined by

$$X_F = (F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}_{n\in\mathbb{Z}_1^d}, \{-\mathrm{i}F_{\bar{z}_n}\}_{n\in\mathbb{Z}_1^d}).$$

Its weighted norm is defined by [1]

$$\begin{aligned}\|X_F\|_{D_\rho(r,s),\mathcal{O}} &\equiv \|F_I\|_{D_\rho(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_\theta\|_{D_\rho(r,s),\mathcal{O}} \\ &\quad + \sup_{D_\rho(r,s)} [\frac{1}{s}\sum_{n\in\mathbb{Z}_1^d} \|F_{z_n}\|_{\mathcal{O}} e^{|n|\rho} + \frac{1}{s}\sum_{n\in\mathbb{Z}_1^d} \|F_{\bar{z}_n}\|_{\mathcal{O}} e^{|n|\rho}]\end{aligned} \tag{2.4}$$

The starting point will be a family of integrable Hamiltonians of the form

$$N = \langle \omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n, \tag{2.5}$$

where $\xi \in \mathcal{O}$ is a parameter, the phase space is endowed with the symplectic structure $dI \wedge d\theta + \mathrm{i}\sum_{n\in\mathbb{Z}_1^d} dz_n \wedge d\bar{z}_n$.

For each $\xi \in \mathcal{O}$, the Hamiltonian equations of motion for N, i.e.,

$$\frac{d\theta}{dt} = \omega, \quad \frac{dI}{dt} = 0, \quad \frac{dz_n}{dt} = -\mathrm{i}\Omega_n z_n, \quad \frac{d\bar{z}_n}{dt} = \mathrm{i}\Omega_n \bar{z}_n, \quad n \in \mathbb{Z}_1^d, \tag{2.6}$$

admit special solutions $(\theta, 0, 0, 0) \to (\theta + \omega t, 0, 0, 0)$ that corresponds to an invariant torus in the phase space.

Consider now the perturbed Hamiltonian

$$H = N + P = \langle \omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi). \tag{2.7}$$

Our goal is to prove that, for most values of parameter $\xi \in \mathcal{O}$ (in Lebesgue measure sense), the Hamiltonians $H = N + P$ still admit invariant tori provided that $\|X_P\|_{D_\rho(r,s),\mathcal{O}}$ is sufficiently small.

To this end, we need to impose some conditions on $\omega(\xi)$, $\Omega_n(\xi)$ and the perturbation P.

$(A1)$ *Nondegeneracy:* The map $\xi \to \omega(\xi)$ is a C_W^1 diffeomorphism between $\mathcal{O}$ and its image.

$(A2)$ *Asymptotics of normal frequencies:*

$$\Omega_n = |n|^2 \tag{2.8}$$

[1] The norm $\|\cdot\|_{D_\rho(r,s),\mathcal{O}}$ for scalar functions is defined in (2.3). The vector function $G : D_\rho(r,s)\times\mathcal{O} \to \mathbb{C}^m$, $(m < \infty)$ is similarly defined as $\|G\|_{D_\rho(r,s),\mathcal{O}} = \sum_{i=1}^m \|G_i\|_{D_\rho(r,s),\mathcal{O}}$.

$(A3)$ *Melnikov's non-resonance conditions:* There exist $\gamma, \tau > 0$ such that

$$|\langle k,\omega\rangle| \geq \frac{\gamma}{|k|^\tau}, k \neq 0,$$
$$|\langle k,\omega\rangle + \Omega_n| \geq \frac{\gamma}{|k|^\tau},$$
$$|\langle k,\omega\rangle + \Omega_n + \Omega_m| \geq \frac{\gamma}{|k|^\tau},$$
$$|\langle k,\omega\rangle + \Omega_n - \Omega_m| \geq \frac{\gamma}{|k|^\tau}, |k| + ||n| - |m|| \neq 0. \tag{2.9}$$

$(A4)$ *Regularity of the perturbation:* The perturbation P is *regular* in the sense that $\|X_P\|_{D_\rho(r,s),\mathcal{O}} < \infty$.

$(A5)$ *Special form of the perturbation:* The perturbation is taken from a special class of analytic functions,

$$\mathcal{A} = \{P : P = \sum_{k\in\mathbb{Z}^b, l\in\mathbb{N}^b,\alpha,\beta} P_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta\}$$

where k, α, β has the following relation

$$\sum_{j=1}^{b} k_j i_j + \sum_{n\in\mathbb{Z}_1^d} (\alpha_n - \beta_n) n = 0. \tag{2.10}$$

$(A6)$ *Töplitz-Lipschitz property:* For any fixed $n, m \in \mathbb{Z}_1^d,\ c \in \mathbb{Z}^d \setminus \{0\}$, the limits

$$\lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc}\partial z_{m-tc}}, \quad \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc}\partial \bar{z}_{m+tc}}, \quad \lim_{t\to\infty} \frac{\partial^2 P}{\partial \bar{z}_{n+tc}\partial \bar{z}_{m-tc}}$$

exist. Moreover, there exists $K > 0$, such that when $|t| > K,\ P$ satisfies

$$\|\frac{\partial^2 P}{\partial z_{n+tc}\partial z_{m-tc}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc}\partial z_{m-tc}}\|_{D_\rho(r,s),\mathcal{O}} \leq \frac{\varepsilon}{|t|} e^{-|n+m|\rho},$$
$$\|\frac{\partial^2 P}{\partial z_{n+tc}\partial \bar{z}_{m+tc}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc}\partial \bar{z}_{m+tc}}\|_{D_\rho(r,s),\mathcal{O}} \leq \frac{\varepsilon}{|t|} e^{-|n-m|\rho},$$
$$\|\frac{\partial^2 P}{\partial \bar{z}_{n+tc}\partial \bar{z}_{m-tc}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial \bar{z}_{n+tc}\partial \bar{z}_{m-tc}}\|_{D_\rho(r,s),\mathcal{O}} \leq \frac{\varepsilon}{|t|} e^{-|n+m|\rho}.$$

Now we are ready to state an infinite dimensional KAM Theorem.

Theorem 2 *Assume that the Hamiltonian $N + P$ in (2.7) satisfies* $(A1)$–$(A6)$. *Let $\gamma > 0$ be small enough, there exists a positive constant $\varepsilon = \varepsilon(b, d, K, \tau, \gamma, r, s, \rho)$. Such that if $\|X_P\|_{D_\rho(r,s),\mathcal{O}} < \varepsilon$, then the following holds true: There exist a Cantor set $\mathcal{O}_\gamma \subset \mathcal{O}$ with* $\mathrm{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$ *and two maps (analytic in θ and C_W^1 in ξ)*

$$\Psi : \mathbb{T}^b \times \mathcal{O}_\gamma \to D_\rho(r,s), \quad \tilde{\omega} : \mathcal{O}_\gamma \to \mathbb{R}^b,$$

where Ψ is $\frac{\varepsilon}{\gamma^2}$-close to the trivial embedding $\Psi_0 : \mathbb{T}^b \times \mathcal{O} \to \mathbb{T}^b \times \{0,0,0\}$ and $\tilde{\omega}$ is ε-close to the unperturbed frequency ω. Then for any $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathbb{T}^b$, the curve $t \to \Psi(\theta + \tilde{\omega}(\xi)t, \xi)$ is a quasi-periodic solution of the Hamiltonian equations governed by $H = N + P$. Moreover, the obtained solutions are real analytic and linearly stable.

3 Application to the Higher Dimensional Schrödinger Equations

We consider the d-dimensional nonlinear Schrödinger equations

$$\mathrm{i}u_t + Au + f(|u|^2)u = 0, \quad Au = -\triangle u + M_\xi u, \qquad x \in \mathbb{T}^d,\ t \in \mathbb{R} \tag{3.1}$$

with periodic boundary conditions

$$u(t, x_1 + 2\pi, x_2, \cdots, x_d) = \cdots = u(t, x_1, x_2, \cdots, x_{d-1}, x_d + 2\pi) = u(t, x_1, x_2, \cdots, x_d),$$

where $f(|u|^2)$ is a real-analytic function near $u = 0$ with $f(0) = 0$.

Here we assume that the operator $A = -\triangle + M_\xi$ with periodic boundary conditions has eigenvalues $\{\mu_n\}$ satisfying

$$\begin{aligned} \omega_j &= \mu_{i_j} = |i_j|^2 + \xi_j, 1 \le j \le b \\ \Omega_n &= \mu_n = |n|^2, n \ne i_1, \cdots, i_b, \end{aligned} \tag{3.2}$$

and the corresponding eigenfunctions $\phi_n(x) = \sqrt{\frac{1}{(2\pi)^d}} e^{\mathrm{i}\langle n,x\rangle}$ form a basis in the domain of the operator. Assume that $i_1, \cdots, i_b \in \mathbb{Z}^d$ are the distinguished sites of Fourier modes (assume $0 \in \{i_1, \cdots, i_b\}$ in order to take care of $(\mu_n, k) = (0, 0)$), and the parameter $\xi = (\xi_1, \cdots, \xi_b) \in \mathcal{O} = [0, 1]^b \subset \mathbb{R}^b$.

Equation (3.1) can be rewritten as a Hamiltonian equation

$$u_t = \mathrm{i}\frac{\partial H}{\partial \bar{u}} \tag{3.3}$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}(Au, u) + \int_{\mathbb{T}^d} g(|u|^2)dx, \tag{3.4}$$

where $(\cdot, \cdot)$ denotes the inner product in L^2 and g is a primitive of f.

Let

$$u(x) = \sum_{n\in\mathbb{Z}^d} q_n\phi_n(x),$$

System (3.3) is then equivalent to the lattice Hamiltonian equations

$$\dot{q}_n = \mathrm{i}(\mu_n q_n + \frac{\partial G}{\partial \bar{q}_n}), \quad G \equiv \int_{\mathbb{T}^d} g(|u|^2)dx\ , \tag{3.5}$$

with corresponding Hamiltonian function $H = \sum_{n\in\mathbb{Z}^d} \mu_n q_n \bar{q}_n + G$.

Since $f(|u|^2)u$ is real analytic in $u, \bar{u}$, then $g(|u|^2)$ is real analytic in $u, \bar{u}$, making use of $u(t, x) = \sum_{n\in\mathbb{Z}^d} q_n(t)\phi_n(x)$, we may rewrite $g(|u|^2)$ as follows

$$g(|u|^2) = \sum_{\alpha,\beta} g_{\alpha\beta} q^\alpha \bar{q}^\beta \phi^\alpha \bar{\phi}^\beta,$$

hence

$$\begin{aligned} G &\equiv \int_{\mathbb{T}^d} g(|u|^2)dx = \sum_{\alpha,\beta} G_{\alpha\beta} q^\alpha \bar{q}^\beta, \\ G_{\alpha\beta} &= 0, \qquad \text{if} \sum_{n\in\mathbb{Z}^d} (\alpha_n - \beta_n)n \ne 0. \end{aligned} \tag{3.6}$$

As in [21,22,14], the perturbation G in (3.5) has the following regularity property.

Lemma 3.1 *For any fixed $\rho > 0$, the gradient $G_{\bar{q}}$ is real analytic as a map in a neighborhood of the origin with*

$$\|G_{\bar{q}}\|_\rho \le c\|q\|_\rho^3. \tag{3.7}$$

Proof

$$\begin{aligned}\|G_{\bar{q}}\|_\rho &= \sum_{n\in\mathbb{Z}^d} |G_{\bar{q}_n}|e^{|n|\rho}\\ &\le c \sum_{\substack{n,\alpha,\beta-e_n,|\alpha|+|\beta-e_n|\ge 3\\ \sum_n(\alpha_n-\beta_n)n=0}} |q^\alpha \bar{q}^{\beta-e_n}|e^{|n|\rho}\\ &\le c \sum_{\alpha,\beta-e_n,|\alpha|+|\beta-e_n|\ge 3} |q^\alpha \bar{q}^{\beta-e_n}|e^{|\alpha|\rho}e^{|\beta-e_n|\rho}\\ &\le c\|q\|_\rho^3.\end{aligned}$$

□

Next we introduce standard action-angle variables $(\theta, I) = ((\theta_1, \cdots, \theta_b), (I_1, \cdots, I_b))$ in the $(q_{i_1}, \cdots, q_{i_b}, \bar{q}_{i_1}, \cdots, \bar{q}_{i_b})$-space by letting,

$$q_{i_j} = \sqrt{I_j}e^{\mathrm{i}\theta_j}, \bar{q}_{i_j} = \sqrt{I_j}e^{-\mathrm{i}\theta_j}, \quad j = 1, \cdots, b,$$

and $q_n = z_n$, $\bar{q}_n = \bar{z}_n$, $n \ne i_1, \cdots, i_b$. We arrive at a Hamiltonian systems with the Hamiltonian (with respect to the symplectic structure $dI \wedge d\theta + \mathrm{i}\sum_{n\in\mathbb{Z}_1^d} dz_n \wedge d\bar{z}_n$)

$$H = \langle\omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}_1^d} \Omega_n(\xi)z_n\bar{z}_n + P(\theta, I, z, \bar{z}, \xi), \tag{3.8}$$

where $\omega(\xi) = (\omega_1(\xi), \cdots, \omega_b(\xi)) = (|i_1|^2+\xi_1, \cdots, |i_b|^2+\xi_b)$, $\Omega_n(\xi) = |n|^2$, P is just G with the $(q_{i_1}, \cdots, q_{i_b}, \bar{q}_{i_1}, \cdots, \bar{q}_{i_b}, q_n, \bar{q}_n)$-variables expressed in terms of the $(\theta, I, z_n, \bar{z}_n)$ variables.

Next let us verify that $H = N + P$ satisfies the assumptions $(A1)$–$(A6)$.

Verification of $(A1)$ *and* $(A2)$: They are obvious.

Verification of $(A3)$: It suffices to prove that

$$|\langle k, \omega\rangle + l| \ge \frac{\gamma}{|k|^\tau}, |k| + |l| \ne 0, k \in \mathbb{Z}^b, l \in \mathbb{Z}.$$

Let $\gamma > 0$ be small enough, and $\tau > b$, there exists a subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with meas$(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$, such that for any $\xi \in \mathcal{O}_\gamma$,

$$|\langle k, \omega\rangle + l| \ge \frac{\gamma}{|k|^\tau}, |k| + |l| \ne 0, k \in \mathbb{Z}^b, l \in \mathbb{Z}.$$

Hence for any $\xi \in \mathcal{O}_\gamma$,

$$
\begin{aligned}
&|\langle k,\omega\rangle| \geq \frac{\gamma}{|k|^\tau}, k \neq 0,\\
&|\langle k,\omega\rangle + \Omega_n| \geq \frac{\gamma}{|k|^\tau},\\
&|\langle k,\omega\rangle + \Omega_n + \Omega_m| \geq \frac{\gamma}{|k|^\tau},\\
&|\langle k,\omega\rangle + \Omega_n - \Omega_m| \geq \frac{\gamma}{|k|^\tau}, |k| + ||n| - |m|| \neq 0.
\end{aligned}
$$

Thus $(A3)$ is verified.

Verification of $(A4)$: For a given $0 < r < 1$ and $s = \varepsilon^{\frac{1}{2}}$, according to Lemma 3.1,$\|G_{\bar{q}}\|_\rho \leq c\|q\|_\rho^3$, then

$$
\sum_{n\in\mathbb{Z}_1^d} \|P_{z_n}\|_{\mathcal{O}} e^{|n|\rho} + \sum_{n\in\mathbb{Z}_1^d} \|P_{\bar{z}_n}\|_{\mathcal{O}} e^{|n|\rho} = \|P_z\|_\rho + \|P_{\bar{z}}\|_\rho \leq c\|q\|_\rho^3 \leq c(|I|^{\frac{3}{2}} + \|z\|_\rho^3).
$$

In addition,

$$
\sup_{\|q\|_\rho<2s} \|G\|_{\mathcal{O}} \leq c \sup_{\|q\|_\rho<2s} \|q\|_\rho^4 \leq cs^4,
$$

thus

$$
\|P\|_{D_\rho(2r,2s),\mathcal{O}} = \sup_{D_\rho(2r,2s)} \|P\|_{\mathcal{O}} \leq cs^4.
$$

According to Cauchy estimates,

$$
\|P_I\|_{D_\rho(r,s),\mathcal{O}} \leq cs^2, \|P_\theta\|_{D_\rho(r,s),\mathcal{O}} \leq cs^4.
$$

Hence

$$
\begin{aligned}
\|X_P\|_{D_\rho(r,s),\mathcal{O}} &= \|P_I\|_{D_\rho(r,s),\mathcal{O}} + \frac{1}{s^2}\|P_\theta\|_{D_\rho(r,s),\mathcal{O}}\\
&\quad + \sup_{D_\rho(r,s)} [\frac{1}{s} \sum_{n\in\mathbb{Z}_1^d} \|P_{z_n}\|_{\mathcal{O}} e^{|n|\rho} + \frac{1}{s} \sum_{n\in\mathbb{Z}_1^d} \|P_{\bar{z}_n}\|_{\mathcal{O}} e^{|n|\rho}]\\
&\leq cs^2 + \frac{cs^4}{s^2} + c \sup_{D_\rho(r,s)} \frac{1}{s}(|I|^{\frac{3}{2}} + \|z\|_\rho^3)\\
&\leq cs^2 \leq c\varepsilon.
\end{aligned}
$$

Thus $(A4)$ is verified.

Verification of $(A5)$: See [15].

Verification of $(A6)$: Because $P \in \mathcal{A}$, i.e.,

$$
P = \sum_{k\in\mathbb{Z}^b, l\in\mathbb{N}^b, \alpha,\beta} P_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta
$$

satisfies

$$
P_{kl\alpha\beta}(\xi) = 0, \quad \text{if} \sum_{j=1}^b k_j i_j + \sum_{n\in\mathbb{Z}_1^d} (\alpha_n - \beta_n) n \neq 0.
$$

And $(n+tc)+(m-tc)=n+m,\ (n+tc)-(m+tc)=n-m$, hence

$$\frac{\partial^2 P}{\partial z_{n+tc}\partial z_{m-tc}} = \frac{\partial^2 P}{\partial z_n \partial z_m} = \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc}\partial z_{m-tc}},$$
$$\frac{\partial^2 P}{\partial z_{n+tc}\partial \bar z_{m+tc}} = \frac{\partial^2 P}{\partial z_n \partial \bar z_m} = \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc}\partial \bar z_{m+tc}},$$
$$\frac{\partial^2 P}{\partial \bar z_{n+tc}\partial \bar z_{m-tc}} = \frac{\partial^2 P}{\partial \bar z_n \partial \bar z_m} = \lim_{t\to\infty} \frac{\partial^2 P}{\partial \bar z_{n+tc}\partial \bar z_{m-tc}}.$$

Thus $(A6)$ is verified.

So we have verified all the assumptions of Theorem 2 for (3.8). By applying Theorem 2, we get Theorem 1.

4 KAM Step

Theorem 2 will be proved by a KAM iteration which involves an infinite sequence of change of variables. Each step of KAM iteration makes the perturbation smaller than in the previous step at the cost of excluding a small set of parameters and contraction of weight. We have to prove the convergence of the iteration and estimate the measure of the excluded set after infinite KAM steps.

At the ν-step of the KAM iteration, we consider a Hamiltonian vector field with

$$H_\nu = N_\nu + P_\nu,$$

where N_ν is an "integrable normal form" and $P_\nu \in \mathcal{A}$ is defined in $D_{\rho_\nu}(r_\nu, s_\nu)\times \mathcal{O}_\nu$ with satisfying $(A1)$–$(A6)$.

We then construct a map

$$\Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1})\times \mathcal{O}_\nu \to D_{\rho_\nu}(r_\nu, s_\nu)\times \mathcal{O}_\nu$$

so that the vector field $X_{H_\nu\circ\Phi_\nu}$ defined on $D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1})$ satisfies

$$\|X_{P_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_\nu} = \|X_{H_\nu\circ\Phi_\nu} - X_{N_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_\nu} \le \varepsilon_\nu^\kappa, \quad \kappa > 1$$

with some new normal form $N_{\nu+1}$. Moreover, the new Hamiltonian still satisfies $(A1)-(A6)$.

To simplify notations, in what follows, the quantities without subscripts and superscripts refer to quantities at the ν^{th} step, while the quantities with subscript + or superscript + denote the corresponding quantities at the $(\nu+1)^{\text{th}}$ step. Let us then consider the Hamiltonian

$$H = N + P \equiv e + \langle \omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar z_n + P(\theta, I, z, \bar z, \xi, \varepsilon) \tag{4.1}$$

defined in $D_\rho(r,s)\times\mathcal{O}$. We assume that $\xi\in\mathcal{O},\ |k|\le K$,

$$\begin{aligned} &|\langle k, \omega(\xi)\rangle| \ge \frac{\gamma}{K^\tau}, \quad k\ne 0\\ &|\langle k, \omega(\xi)\rangle + \Omega_n(\xi)| \ge \frac{\gamma}{K^\tau},\\ &|\langle k, \omega(\xi)\rangle + \Omega_n(\xi) + \Omega_m(\xi)| \ge \frac{\gamma}{K^\tau},\\ &|\langle k, \omega(\xi)\rangle + \Omega_n(\xi) - \Omega_m(\xi)| \ge \frac{\gamma}{K^\tau}, \quad |k| + ||n| - |m|| \ne 0. \end{aligned} \tag{4.2}$$

Moreover,

$$\|X_P\|_{D_\rho(r,s),\mathcal{O}} \le \varepsilon, \tag{4.3}$$

and $P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta$ is in the class $\mathcal{A}$ defined in $(A5)$, i.e.,

$$P_{kl\alpha\beta} = 0 \quad \text{if} \ \sum_{j=1}^{b} k_j i_j + \sum_{n\in\mathbb{Z}_1^d} (\alpha_n - \beta_n) n \ne 0. \tag{4.4}$$

For any fixed $n, m \in \mathbb{Z}_1^d$, $c \in \mathbb{Z}^d \setminus \{0\}$, set $\Omega_n = |n|^2 + \tilde{\Omega}_n$, the limits

$$\lim_{t\to\infty} \tilde{\Omega}_{n+tc}, \quad \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc} \partial z_{m-tc}}, \quad \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc} \partial \bar{z}_{m+tc}}, \quad \lim_{t\to\infty} \frac{\partial^2 P}{\partial \bar{z}_{n+tc} \partial \bar{z}_{m-tc}}$$

exist. And there exists $K > 0$, such that when $|t| > K$, $N + P$ satisfies

$$|\tilde{\Omega}_{n+tc} - \lim_{t\to\infty} \tilde{\Omega}_{n+tc}|_{\mathcal{O}} \le \frac{\varepsilon_0}{|t|},$$

$$\left\| \frac{\partial^2 P}{\partial z_{n+tc} \partial z_{m-tc}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc} \partial z_{m-tc}} \right\|_{D_\rho(r,s),\mathcal{O}} \le \frac{\varepsilon}{|t|} e^{-|n+m|\rho},$$

$$\left\| \frac{\partial^2 P}{\partial z_{n+tc} \partial \bar{z}_{m+tc}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{n+tc} \partial \bar{z}_{m+tc}} \right\|_{D_\rho(r,s),\mathcal{O}} \le \frac{\varepsilon}{|t|} e^{-|n-m|\rho},$$

$$\left\| \frac{\partial^2 P}{\partial \bar{z}_{n+tc} \partial \bar{z}_{m-tc}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial \bar{z}_{n+tc} \partial \bar{z}_{m-tc}} \right\|_{D_\rho(r,s),\mathcal{O}} \le \frac{\varepsilon}{|t|} e^{-|n+m|\rho}.$$

Remark 1 According to (4.4), when $k = (k_1, \cdots, k_b) = 0$ and $\alpha = e_n$, $\beta = e_m$, we get

$$P_{0le_ne_m} = 0 \quad \text{if} \ \sum_{j=1}^{b} k_j i_j + \sum_{n\in\mathbb{Z}_1^d} (\alpha_n - \beta_n) n = n - m \ne 0.$$

This means that there are not the terms of the form $\sum_{n\ne m} P_{0le_ne_m} I^l z_n \bar{z}_m$ in the perturbation. As a result, the normal variables z_n, $\bar{z}_m$ with $n \ne m$ in the new normal form N_+ will not be coupled.

Remark 2 Compared with the KAM step in the existent KAM theorems in the literature, we make an additional assumption $(A6)$. The assumption $(A6)$ makes the measure estimate available at each KAM step.

We now let $0 < r_+ < r$ and define

$$s_+ = \frac{1}{4} s \varepsilon^{\frac{1}{3}}, \quad K_+ = 3K, \quad \varepsilon_+ = c\gamma^{-4} K^{4\tau} \varepsilon^{\frac{4}{3}}. \tag{4.5}$$

Here and later, the letter c denotes suitable (possibly different) constants that do not depend on the iteration steps.

We now describe how to construct a set $\mathcal{O}_+ \subset \mathcal{O}$ and a change of variables $\Phi : D_+ \times \mathcal{O}_+ = D_\rho(r_+, s_+) \times \mathcal{O}_+ \to D_\rho(r, s) \times \mathcal{O}$ such that the transformed Hamiltonian $H_+ = N_+ + P_+ \equiv H \circ \Phi$ satisfies all the above iterative assumptions with new parameters s_+, ε_+, r_+ and with $\xi \in \mathcal{O}_+$.

4.1 Solving the Linearized Equations

Expand P into the Fourier-Taylor series

$$P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle} I^l z^\alpha \bar{z}^\beta$$

where $k \in \mathbb{Z}^b, l \in \mathbb{N}^b$ and the multi-indices α and β run over the set of all infinite dimensional vectors $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n\in\mathbb{Z}_1^d}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n\in\mathbb{Z}_1^d}$ with finitely many nonzero components of positive integers.

Let R be the truncation of P given by

$$\begin{aligned} R(\theta, I, z, \bar{z}) = R_0 + R_1 + R_2 = &\sum_{|k|\le K, |l|\le 1} P_{kl00} e^{\mathrm{i}\langle k,\theta\rangle} I^l \\ &+ \sum_{|k|\le K, n} (P_n^{k10} z_n + P_n^{k01} \bar{z}_n) e^{\mathrm{i}\langle k,\theta\rangle} \\ &+ \sum_{|k|\le K, n, m} (P_{nm}^{k20} z_n z_m + P_{nm}^{k11} z_n \bar{z}_m + P_{nm}^{k02} \bar{z}_n \bar{z}_m) e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned} \tag{4.6}$$

where $P_n^{k10} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = 0$, here e_n denotes the vector with the n^{th} component being 1 and the other components being zero; $P_n^{k01} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n$; $P_{nm}^{k20} = P_{kl\alpha\beta}$ with $\alpha = e_n + e_m, \beta = 0$; $P_{nm}^{k11} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = e_m$; $P_{nm}^{k02} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n + e_m$. Due to the assumption $(A5)$, $P \in \mathcal{A}$ implies that

$$\begin{aligned} P_{kl00} &= 0, \quad \text{if} \sum_{j=1}^b k_j i_j \neq 0 \\ P_n^{k10} &= 0, \quad \text{if} \sum_{j=1}^b k_j i_j + n \neq 0 \\ P_n^{k01} &= 0, \quad \text{if} \sum_{j=1}^b k_j i_j - n \neq 0 \\ P_{nm}^{k20} &= 0, \quad \text{if} \sum_{j=1}^b k_j i_j + n + m \neq 0 \\ P_{nm}^{k11} &= 0, \quad \text{if} \sum_{j=1}^b k_j i_j + n - m \neq 0 \\ P_{nm}^{k02} &= 0, \quad \text{if} \sum_{j=1}^b k_j i_j - n - m \neq 0 \end{aligned} \tag{4.7}$$

Remark The special form of P defined in $(A5)$, i.e., $P \in \mathcal{A}$, is crucial in this paper. With P of such special form, one knows that $P_{nm}^{k11} = 0$ if $k = 0$ and $n \neq m$, then the terms $P_{nm}^{011} z_n \bar{z}_m$ with $n \neq m$ are absent, i.e., z_n, z_m with $n \neq m$ are uncoupled in the new normal form.

Rewrite H as $H = N + R + (P - R)$. By the choice of s_+ in (4.5) and the definition of the norms, it follows immediately that

$$\|X_R\|_{D_\rho(r,s),\mathcal{O}} \le \|X_P\|_{D_\rho(r,s),\mathcal{O}} \le \varepsilon. \tag{4.8}$$

Moreover, we take $s_+ \ll s$ such that in a domain $D_\rho(r, s_+)$,

$$\|X_{(P-R)}\|_{D_\rho(r,s_+)} < c\,\varepsilon_+. \tag{4.9}$$

In the following, we will look for an F in the class $\mathcal{A}$, defined in a domain $D_+ = D_\rho(r_+, s_+)$, such that the time one map ϕ_F^1 of the Hamiltonian vector field X_F defines a map from $D_+ \to D$ and transforms H into H_+. More precisely, by second order Taylor formula, we have

$$\begin{aligned} H\circ\phi_F^1 &= (N+R)\circ\phi_F^1 + (P-R)\circ\phi_F^1 \\ &= N + \{N,F\} + R + \int_0^1 (1-t)\{\{N,F\},F\}\circ\phi_F^t dt \\ &\quad + \int_0^1 \{R,F\}\circ\phi_F^t dt + (P-R)\circ\phi_F^1 \\ &= N_+ + P_+ + \{N,F\} + R - P_{0000} - \langle\omega', I\rangle - \sum_n P_{nn}^{011} z_n\bar{z}_n, \end{aligned} \tag{4.10}$$

where

$$\omega' = \int \frac{\partial P}{\partial I} d\theta|_{z=\bar{z}=0,\ I=0},$$

$$N_+ = N + P_{0000} + \langle\omega', I\rangle + \sum_n P_{nn}^{011} z_n\bar{z}_n, \tag{4.11}$$

$$P_+ = \int_0^1 (1-t)\{\{N,F\},F\}\circ\phi_F^t dt + \int_0^1 \{R,F\}\circ\phi_F^t dt + (P-R)\circ\phi_F^1. \tag{4.12}$$

Remark Generally speaking, $\sum_n P_{nn}^{011} z_n\bar{z}_n$ should be replaced in (4.11) by $\sum_{|n|=|m|} P_{nm}^{011} z_n \bar{z}_m$, but in terms of (4.7), $P_{nm}^{011} = 0$ if $n \neq m$. Hence the terms $\sum_{n\neq m} P_{nm}^{011} z_n\bar{z}_m$ are absent. Thus N_+ has the same form as that in the first step.

We shall find a function F in $\mathcal{A}$ of the form

$$\begin{aligned} F(\theta, I, z, \bar{z}) &= F_0 + F_1 + F_2 \\ &= \sum_{0<|k|\le K, |l|\le 1} F_{kl00} e^{\mathrm{i}\langle k,\theta\rangle} I^l + \sum_{|k|\le K, n} (F_n^{k10} z_n + F_n^{k01}\bar{z}_n) e^{\mathrm{i}\langle k,\theta\rangle} \\ &\quad + \sum_{|k|\le K,n,m} (F_{nm}^{k20} z_n z_m + F_{nm}^{k02}\bar{z}_n\bar{z}_m) e^{\mathrm{i}\langle k,\theta\rangle} + \sum_{\substack{|k|\le K,n,m \\ |k|+||n|-|m||\neq 0}} F_{nm}^{k11} z_n\bar{z}_m e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned} \tag{4.13}$$

satisfying the equation

$$\{N, F\} + R - P_{0000} - \langle\omega', I\rangle - \sum_n P_{nn}^{011} z_n\bar{z}_n = 0. \tag{4.14}$$

Lemma 4.1 *F satisfies (4.14) if the Fourier coefficients of F are defined by the following equations*

$$\begin{array}{lll} (\langle k,\omega\rangle) F_{kl00} & = -\mathrm{i}P_{kl00}, & |l|\le 1, 0<|k|\le K, \\ (\langle k,\omega\rangle + \Omega_n) F_n^{k10} & = -\mathrm{i}P_n^{k10}, & |k|\le K, \\ (\langle k,\omega\rangle - \Omega_n) F_n^{k01} & = -\mathrm{i}P_n^{k01}, & |k|\le K, \\ (\langle k,\omega\rangle + \Omega_n + \Omega_m) F_{nm}^{k20} & = -\mathrm{i}P_{nm}^{k20}, & |k|\le K, \\ (\langle k,\omega\rangle + \Omega_n - \Omega_m) F_{nm}^{k11} & = -\mathrm{i}P_{nm}^{k11}, & |k|\le K,\ |k| + ||n|-|m||\neq 0, \\ (\langle k,\omega\rangle - \Omega_n - \Omega_m) F_{nm}^{k02} & = -\mathrm{i}P_{nm}^{k02}, & |k|\le K. \end{array} \tag{4.15}$$

4.2 Estimation on the Coordinate Transformation

We proceed to estimate X_F and ϕ_F^1. We start with the following

Lemma 4.2 *Let* $D_i = D(r_+ + \frac{i}{4}(r - r_+), \frac{i}{4}s),\ 0 < i \leq 4$. *Then*

$$\|X_F\|_{D_3,\mathcal{O}} \leq c\gamma^{-2}K^{2\tau}\varepsilon. \tag{4.16}$$

In the next lemma, we give some estimates for ϕ_F^t. The formula (4.17) will be used to prove our coordinate transformation is well defined. Inequality (4.18) will be used to check the convergence of the iteration.

Lemma 4.3 *Let* $\eta = \varepsilon^{\frac{1}{3}}$, $D_{i\eta} = D(r_+ + \frac{i}{4}(r - r_+), \frac{i}{4}\eta s), 0 < i \leq 4$. *If* $\varepsilon \ll \frac{1}{2}\gamma^3 K^{-3\tau}$, *we then have*

$$\phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \leq t \leq 1, \tag{4.17}$$

Moreover,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} < c\gamma^{-2}K^{2\tau}\varepsilon. \tag{4.18}$$

Proof Let

$$\|D^m F\|_{D,\mathcal{O}} = \max\{\|\frac{\partial^{|i|+|l|+|\alpha|+|\beta|}}{\partial\theta^i \partial I^l \partial z^\alpha \partial \bar{z}^\beta} F\|_{D,\mathcal{O}}, |i| + |l| + |\alpha| + |\beta| = m \geq 2\}.$$

Notice that F is a polynomial of degree 1 in I and degree 2 in z, $\bar{z}$. From (2.4), (4.16) and the Cauchy inequality, it follows that

$$\|D^m F\|_{D_2,\mathcal{O}} < c\gamma^{-2}K^{2\tau}\varepsilon, \tag{4.19}$$

for any $m \geq 2$.

To get the estimates for ϕ_F^t, we start from the integral equation,

$$\phi_F^t = id + \int_0^t X_F \circ \phi_F^s \, ds$$

so that $\phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \leq t \leq 1$, which follows directly from (4.19). Since

$$D\phi_F^t = Id + \int_0^t (DX_F) D\phi_F^s \, ds = Id + \int_0^t J(D^2F) D\phi_F^s \, ds,$$

where J denotes the standard symplectic matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, it follows that

$$\|D\phi_F^t - Id\| \leq 2\|D^2F\| < c\gamma^{-2}K^{2\tau}\varepsilon. \tag{4.20}$$

Consequently Lemma 4.3 follows. □

4.3 Estimation for the New Normal Form

The map ϕ_F^1 defined above transforms H into $H_+ = N_+ + P_+$(see (4.10) and (4.14)). Due to the special form of P defined in $(A5)$, the terms in $\sum_{n,m} P_{nm}^{011} z_n \bar{z}_m$ with $n \neq m$ are absent, i.e., $z_n,\ z_m$ with $n \neq m$ are uncoupled. Hence compared with the normal form in [14], here the normal form N_+ is simpler

$$\begin{aligned} N_+ &= N + P_{0000} + \langle \omega', I \rangle + \sum_n P^{011}_{nn} z_n \bar{z}_n \\ &= e_+ + \langle \omega_+, I \rangle + \sum_n \Omega^+_n z_n \bar{z}_n, \end{aligned} \tag{4.21}$$

where

$$e_+ = e + P_{0000},\ \omega_+ = \omega + P_{0l00}(|l| = 1),\ \Omega^+_n = \Omega_n + P^{011}_{nn} \tag{4.22}$$

Now we prove that N_+ shares the same properties as N. By the regularity of X_P and by Cauchy estimates, then we have

$$|\omega_+ - \omega| < \varepsilon, \quad |P^{011}_{nn}| < \varepsilon. \tag{4.23}$$

It follows that for $|k| \leq K$,

$$\begin{aligned} &|\langle k, \omega + P_{0l00}\rangle| \geq |\langle k, \omega\rangle| - \varepsilon K \geq \frac{\gamma}{K^\tau} - \varepsilon K \geq \frac{\gamma}{K^\tau_+}, \\ &|\langle k, \omega + P_{0l00}\rangle + \Omega^+_n| \geq |\langle k, \omega\rangle + \Omega_n| - \varepsilon K \geq \frac{\gamma}{K^\tau} - \varepsilon K \geq \frac{\gamma}{K^\tau_+}, \end{aligned}$$

Similarly, we have

$$\begin{aligned} &|\langle k, \omega + P_{0l00}\rangle + \Omega^+_n + \Omega^+_m| \geq \frac{\gamma}{K^\tau_+}, \\ &|\langle k, \omega + P_{0l00}\rangle + \Omega^+_n - \Omega^+_m| \geq \frac{\gamma}{K^\tau_+},\ |k| + ||n| - |m|| \neq 0. \end{aligned} \tag{4.24}$$

This means that in the next KAM step, small denominator conditions are automatically satisfied for $|k| \leq K$.

4.4 Estimation for the New Perturbation

Since

$$\begin{aligned} P_+ &= \int_0^1 (1-t)\{\{N, F\}, F\} \circ \phi^t_F dt + \int_0^1 \{R, F\} \circ \phi^t_F dt + (P - R) \circ \phi^1_F \\ &= \int_0^1 \{R(t), F\} \circ \phi^t_F dt + (P - R) \circ \phi^1_F, \end{aligned}$$

where $R(t) = (1-t)(N_+ - N) + tR$. Hence

$$X_{P_+} = \int_0^1 (\phi^t_F)^* X_{\{R(t), F\}} dt + (\phi^1_F)^* X_{(P-R)}.$$

According to Lemma 4.3,

$$\|D\phi^t_F - Id\|_{D_{1\eta}} < c\gamma^{-2} K^{2\tau} \varepsilon, \quad -1 \leq t \leq 1,$$

thus

$$\|D\phi^t_F\|_{D_{1\eta}} \leq 1 + \|D\phi^t_F - Id\|_{D_{1\eta}} \leq 2, \quad -1 \leq t \leq 1.$$

Due to Lemma 7.3,

$$\|X_{\{R(t),F\}}\|_{D_{2\eta}} \le c\gamma^{-2}K^{2\tau}\eta^{-2}\varepsilon^2,$$

and

$$\|X_{(P-R)}\|_{D_{2\eta}} \le c\eta\varepsilon,$$

we have

$$\|X_{P_+}\|_{D_\rho(r_+,s_+)} \le c\eta\varepsilon + c\gamma^{-2}K^{2\tau}\eta^{-2}\varepsilon^2 \le c\varepsilon_+.$$

4.5 Verification of $(A5)$ and $(A6)$ After One Step of KAM Iteration

$(A5)$ after one step of KAM iteration is proved by Geng–You in Lemma 4.4 of [15]. In the following, we have to check that the new error term P_+ satisfies $(A6)$ with $K_+, \varepsilon_+, \rho_+$ in place of K, ε, ρ. Since

$$\begin{aligned} P_+ = P - R + \{P, F\} + \frac{1}{2!}\{\{N, F\}, F\} + \frac{1}{2!}\{\{P, F\}, F\} \\ + \cdots + \frac{1}{n!}\{\cdots\{N, \underbrace{F\}\cdots, F\}}_{n} + \frac{1}{n!}\{\cdots\{P, \underbrace{F\}\cdots, F\}}_{n} + \cdots \end{aligned}$$

then for a fixed $c \in \mathbb{Z}^2 \setminus \{0\}$, and $|n-m| > K$ with $K \ge \frac{1}{\rho-\rho_+}\ln(\frac{\varepsilon}{\varepsilon_+})$,

$$\left\|\frac{\partial^2(P-R)}{\partial z_{n+tc}\partial \bar{z}_{m+tc}} - \lim_{t\to\infty}\frac{\partial^2(P-R)}{\partial z_{n+tc}\partial \bar{z}_{m+tc}}\right\| \le \frac{\varepsilon}{|t|}e^{-|n-m|\rho} \le \frac{\varepsilon_+}{|t|}e^{-|n-m|\rho_+}.$$

That is to say, $P-R$ satisfies $(A6)$ with $K_+, \varepsilon_+, \rho_+$ in place of K, ε, ρ. The proof of the remaining terms satisfying $(A6)$ is composed by the following two lemmas.

Lemma 4.4 *F satisfies $(A6)$ with $\varepsilon^{\frac{2}{3}}$ in place of ε.*

For the proof see [13].

Lemma 4.5 *Assume that P satisfies $(A6)$, F satisfies $(A6)$ with $\varepsilon^{\frac{2}{3}}$ in place of ε and*

$$\frac{\partial^2 F}{\partial z_n \partial z_m} = 0(|n+m| > K), \quad \frac{\partial^2 F}{\partial z_n \partial \bar{z}_m} = 0(|n-m| > K), \quad \frac{\partial^2 F}{\partial \bar{z}_n \partial \bar{z}_m} = 0(|n+m| > K),$$

then $\{P, F\}$ satisfies $(A6)$ with ε_+ in place of ε.

For the proof see [13].

5 Iteration Lemma and Convergence

For any given $s, \varepsilon, r, \gamma$ and for all $\nu \geq 1$, we define the following sequences

$$\begin{aligned} r_\nu &= r(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \\ \varepsilon_\nu &= c\gamma^{-4} K_{\nu-1}^{4\tau} \varepsilon_{\nu-1}^{\frac{4}{3}}, \\ \eta_\nu &= \varepsilon_\nu^{\frac{1}{3}}, \\ s_\nu &= \frac{1}{4}\eta_{\nu-1} s_{\nu-1} = 2^{-2\nu}(\prod_{i=0}^{\nu-1} \varepsilon_i)^{\frac{1}{3}} s_0, \\ \rho_\nu &= \rho(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \\ K_\nu &= 3^\nu K_0, \end{aligned} \tag{5.1}$$

where c is a constant, and the parameters r_0, ε_0, s_0 and K_0 are defined to be r, ε, s and ln $\frac{1}{\varepsilon}$ respectively.

5.1 Iteration Lemma

The preceding analysis can be summarized as follows.

Lemma 5.1 *Let ε is small enough and $\nu \geq 0$. Suppose that*

(1). $N_\nu = e_\nu + \langle \omega_\nu(\xi), I \rangle + \sum_n \Omega_n^\nu(\xi) z_n \bar{z}_n$ is a normal form with parameters ξ satisfying

$$\begin{aligned} &|\langle k, \omega_\nu \rangle| \geq \frac{\gamma}{K_\nu^\tau}, \qquad 0 < |k| \leq K_\nu, \\ &|\langle k, \omega_\nu \rangle + \Omega_n^\nu| \geq \frac{\gamma}{K_\nu^\tau}, |k| \leq K_\nu \\ &|\langle k, \omega_\nu \rangle + \Omega_n^\nu + \Omega_m^\nu| \geq \frac{\gamma}{K_\nu^\tau}, |k| \leq K_\nu \\ &|\langle k, \omega_\nu \rangle + \Omega_n^\nu - \Omega_m^\nu| \geq \frac{\gamma}{K_\nu^\tau}, \quad |k| \leq K_\nu, |k| + ||n| - |m|| \neq 0 \end{aligned}$$

on a closed set $\mathcal{O}_\nu$ of $\mathbb{R}^b$;

(2). $\omega_\nu(\xi)$, $\Omega_n^\nu(\xi)$ are C_W^1 smooth and satisfy

$$|\omega_\nu - \omega_{\nu-1}|_{\mathcal{O}_\nu} \leq \varepsilon_{\nu-1}, \quad |\Omega_n^\nu - \Omega_n^{\nu-1}|_{\mathcal{O}_\nu} \leq \varepsilon_{\nu-1};$$

(3). $N_\nu + P_\nu$ satisfies $(A5), (A6)$ with $K_\nu, \varepsilon_\nu, \rho_\nu$ and

$$\|X_{P_\nu}\|_{D(r_\nu, s_\nu), \mathcal{O}_\nu} \leq \varepsilon_\nu.$$

Then there is a subset $\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu$,

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus (\bigcup_{K_\nu < |k| \leq K_{\nu+1}} \mathcal{R}_k^{\nu+1}(\gamma)),$$

where

$$\mathcal{R}_k^{\nu+1}(\gamma) = \left\{ \xi \in \mathcal{O}_\nu : \begin{array}{l} |\langle k, \omega_{\nu+1}\rangle| < \frac{\gamma}{K_{\nu+1}^\tau} \; |\langle k, \omega_{\nu+1}\rangle + \Omega_n^{\nu+1}| < \frac{\gamma}{K_{\nu+1}^\tau}, \text{ or} \\ |\langle k, \omega_{\nu+1}\rangle \pm \Omega_n^{\nu+1} \pm \Omega_m^{\nu+1}| < \frac{\gamma}{K_{\nu+1}^\tau}, \end{array} \right\}$$

with $\omega_{\nu+1} = \omega_\nu + P_{0l00}^\nu$, *and a symplectic transformation of variables*

$$\Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_\nu}(r_\nu, s_\nu), \tag{5.2}$$

such that on $D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$, $H_{\nu+1} = H_\nu \circ \Phi_\nu$ *has the form*

$$H_{\nu+1} = e_{\nu+1} + \langle \omega_{\nu+1}, I\rangle + \sum_n \Omega_n^{\nu+1} z_n \bar{z}_n + P_{\nu+1}, \tag{5.3}$$

with

$$|\omega_{\nu+1} - \omega_\nu|_{\mathcal{O}_{\nu+1}} \le \varepsilon_\nu, \quad |\Omega_n^{\nu+1} - \Omega_n^\nu|_{\mathcal{O}_{\nu+1}} \le \varepsilon_\nu. \tag{5.4}$$

And also $N_{\nu+1} + P_{\nu+1}$ *has the special form defined in* (*A*5), (*A*6) *with* $K_{\nu+1}, \varepsilon_{\nu+1}, \rho_{\nu+1}$ *in place of* $K_\nu, \varepsilon_\nu, \rho_\nu$ *and*

$$\|X_{P_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}} \le \varepsilon_{\nu+1}. \tag{5.5}$$

5.2 Convergence

Suppose that the assumptions of Theorem 2 are satisfied. Recall that

$$\varepsilon_0 = \varepsilon, r_0 = r, s_0 = s, N_0 = N, P_0 = P,$$
$$\mathcal{O}_0 = \left\{ \xi \in \mathcal{O} : \begin{array}{ll} |\langle k, \omega(\xi)\rangle| \ge \frac{\gamma}{|k|^\tau}, \quad k \ne 0, & \\ |\langle k, \omega(\xi)\rangle + \Omega_n| \ge \frac{\gamma}{|k|^\tau}, & \\ |\langle k, \omega(\xi)\rangle + \Omega_n + \Omega_m| \ge \frac{\gamma}{|k|^\tau}, & \\ |\langle k, \omega(\xi)\rangle + \Omega_n - \Omega_m| \ge \frac{\gamma}{|k|^\tau}, & |k| + ||n| - |m|| \ne 0 \end{array} \right\},$$

the assumptions of the iteration lemma are satisfied when $\nu = 0$ if ε_0 and γ are sufficiently small. Inductively, we obtain the following sequences:

$$\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu,$$
$$\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_0}(r_0, s_0), \nu \ge 0,$$
$$H \circ \Psi^\nu = H_{\nu+1} = N_{\nu+1} + P_{\nu+1}.$$

Let $\tilde{\mathcal{O}} = \cap_{\nu=0}^\infty \mathcal{O}_\nu$. As in [22,23], thanks to Lemma 4.3, it concludes that $N_\nu, \Psi^\nu, D\Psi^\nu$, ω_ν converge uniformly on $D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ with

$$N_\infty = e_\infty + \langle \omega_\infty, I\rangle + \sum_n \Omega_n^\infty z_n \bar{z}_n.$$

Since

$$\varepsilon_{\nu+1} = c\gamma^{-4} K_\nu^{4\tau} \varepsilon_\nu^{\frac{4}{3}},$$

it follows that $\varepsilon_{\nu+1} \to 0$ provided that ε is sufficiently small.

Let ϕ_H^t be the flow of X_H. Since $H \circ \Psi^\nu = H_{\nu+1}$, we have

$$\phi_H^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{H_{\nu+1}}^t. \tag{5.6}$$

The uniform convergence of Ψ^{ν}, $D\Psi^{\nu}$, ω_{ν} and $X_{H_{\nu}}$ implies that the limits can be taken on both sides of (5.6). Hence, on $D_{\frac{1}{2}\rho}(\frac{1}{2}r,0)\times\tilde{\mathcal{O}}$ we get

$$\phi_H^t\circ\Psi^{\infty}=\Psi^{\infty}\circ\phi_{H_{\infty}}^t \tag{5.7}$$

and

$$\Psi^{\infty}:D_{\frac{1}{2}\rho}(\frac{1}{2}r,0)\times\tilde{\mathcal{O}}\to D_{\rho}(r,s)\times\mathcal{O}.$$

It follows from (5.7) that

$$\phi_H^t(\Psi^{\infty}(\mathbb{T}^b\times\{\xi\}))=\Psi^{\infty}\phi_{N_{\infty}}^t(\mathbb{T}^b\times\{\xi\})=\Psi^{\infty}(\mathbb{T}^b\times\{\xi\})$$

for $\xi\in\tilde{\mathcal{O}}$. This means that $\Psi^{\infty}(\mathbb{T}^b\times\{\xi\})$ is an embedded torus which is invariant for the original perturbed Hamiltonian system at $\xi\in\tilde{\mathcal{O}}$. We remark here that the frequencies $\omega_{\infty}(\xi)$ associated to $\Psi^{\infty}(\mathbb{T}^b\times\{\xi\})$ are slightly different from $\omega(\xi)$. The normal behavior of the invariant torus is governed by normal frequencies Ω_n^{∞}. □

6 Measure Estimates

This section is the essential part for this paper. For notational convenience, let $\mathcal{O}_{-1}=\mathcal{O}$, $K_{-1}=0$. Then at ν^{th} step of KAM iteration, we have to exclude the following resonant set

$$\mathcal{R}^{\nu}=\bigcup_{K_{\nu-1}<|k|\le K_{\nu},n,m}(\mathcal{R}_k^{\nu}\bigcup\mathcal{R}_{kn}^{\nu}\bigcup\mathcal{R}_{knm}^{\nu}),$$

where

$$\mathcal{R}_k^{\nu}=\{\xi\in\mathcal{O}_{\nu-1}:|\langle k,\omega_{\nu}(\xi)\rangle|<\frac{\gamma}{K_{\nu}^{\tau}}\}, \tag{6.1}$$

$$\mathcal{R}_{kn}^{\nu}=\{\xi\in\mathcal{O}_{\nu-1}:|\langle k,\omega_{\nu}(\xi)\rangle+\Omega_n^{\nu}|<\frac{\gamma}{K_{\nu}^{\tau}}\}, \tag{6.2}$$

$$\mathcal{R}_{knm}^{\nu}=\{\xi\in\mathcal{O}_{\nu-1}:|\langle k,\omega_{\nu}(\xi)\rangle\pm\Omega_n^{\nu}\pm\Omega_m^{\nu}|<\frac{\gamma}{K_{\nu}^{\tau}}\} \tag{6.3}$$

Remark From the Sect. 4.3, one has that at ν^{th} step, small divisor conditions are automatically satisfied for $|k|\le K_{\nu-1}$. Hence, we only need to excise the above resonant set $\mathcal{R}^{\nu}$.

In the following, we only give the proof for the most complicated case: $\{\xi\in\mathcal{O}_{\nu-1}:|\langle k,\omega_{\nu}\rangle+\Omega_n^{\nu}-\Omega_m^{\nu}|<\frac{\gamma}{K_{\nu}^{\tau}}\}$. For simplicity, set $M^{\nu}=\langle k,\omega_{\nu}\rangle+\Omega_n^{\nu}-\Omega_m^{\nu}$.

Lemma 6.1 *For any given $n,m\in\mathbb{Z}_1^d$ with $|n-m|\le K_{\nu}$, either $|\langle k,\omega_{\nu}\rangle+\Omega_n^{\nu}-\Omega_m^{\nu}|>1$ or there are $n_0,m_0,c_1,c_2,\cdots,c_{d-1}\in\mathbb{Z}^d$ with $|n_0|,|m_0|,|c_1|,|c_2|,\cdots,|c_{d-1}|\le 3K_{\nu}^2$ and $t_1,t_2,\cdots,t_{d-1}\in\mathbb{Z}$, such that $n=n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}$, $m=m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}$.*

Proof Since $|n-m|\le K_{\nu}$, with an elementary calculation

$$|n|^2-|m|^2=|n-m|^2+2\langle n-m,m\rangle$$

If $|\langle n-m,m\rangle|>K_{\nu}^2$, we have $|\langle k,\omega_{\nu}\rangle+\Omega_n^{\nu}-\Omega_m^{\nu}|>1$, there will be no small divisor.

In the case that $|\langle n-m, m\rangle| \le K_\nu^2$, clearly $n-m=0$ is trivial. Assume $n-m\neq 0$, without loss of generality, we assume that the first component $(n-m)_1$ of $n-m$ is not zero. Let

$$\begin{aligned} c_1 &= (-(n-m)_2, (n-m)_1, 0, \cdots, 0),\\ c_2 &= (-(n-m)_3, 0, (n-m)_1, 0, \cdots, 0),\\ &\vdots\\ c_{d-1} &= (-(n-m)_d, 0, \cdots, 0, (n-m)_1). \end{aligned}$$

Then

$$c_1 \perp n-m, c_2 \perp n-m, \cdots, c_{d-1} \perp n-m,$$

and $c_1, c_2, \cdots, c_{d-1} \in \mathbb{Z}^d \setminus \{0\}$ with $|c_1|, |c_2|, \cdots, |c_{d-1}| \le |n-m| \le K_\nu$. Clearly, $c_1, c_2, \cdots, c_{d-1}, n-m$ are linearly independent, hence there exist $x_1, x_2, \cdots, x_{d-1}, x_d \in \mathbb{R}$ such that

$$m = x_1c_1 + x_2c_2 + \cdots + x_{d-1}c_{d-1} + x_d(n-m).$$

Set (here $[\cdot]$ denotes the integer part of $\cdot$)

$$t_1 = [x_1], t_2 = [x_2], \cdots, t_{d-1} = [x_{d-1}],$$

then $t_1, t_2, \cdots, t_{d-1} \in \mathbb{Z}$ and $|m-(t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1})| \le 2K_\nu^2$. Take $m_0 = m-(t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}) \in \mathbb{Z}^d$ and $n_0 = m_0+n-m \in \mathbb{Z}^d$. We have $|m_0| \le 2K_\nu^2$ and

$$|n_0| \le |m_0| + |n-m| \le 3K_\nu^2.$$

□

Lemma 6.2

$$\cup_{n,m\in\mathbb{Z}_1^d}\mathcal{R}^\nu_{knm} \subset \cup_{n_0,m_0,c_1,c_2,\cdots,c_{d-1}\in\mathbb{Z}^d,t_1,t_2,\cdots,t_{d-1}\in\mathbb{Z}}\mathcal{R}^\nu_{k,n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1},m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}}$$

where $|n_0|, |m_0|, |c_1|, |c_2|, \cdots, |c_{d-1}| \le 3K_\nu^2$.

Proof If $|\langle n-m, m\rangle| > K_\nu^2$, $\mathcal{R}^\nu_{knm} = \emptyset$. If $|\langle n-m, m\rangle| \le K_\nu^2$, there exist $n_0, m_0, c_1, c_2, \cdots, c_{d-1} \in \mathbb{Z}^d$, $t_1, t_2, \cdots, t_{d-1} \in \mathbb{Z}$ with $|n_0|, |m_0|, |c_1|, |c_2|, \cdots, |c_{d-1}| \le 3K_\nu^2$ such that $n = n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}$, $m = m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}$. Hence

$$\cup_{n,m\in\mathbb{Z}_1^d}\mathcal{R}^\nu_{knm} \subset \cup_{n_0,m_0,c_1,c_2,\cdots,c_{d-1}\in\mathbb{Z}^d,t_1,t_2,\cdots,t_{d-1}\in\mathbb{Z}}\mathcal{R}^\nu_{k,n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1},m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}}$$

where $|n_0|, |m_0|, |c_1|, |c_2|, \cdots, |c_{d-1}| \le 3K_\nu^2$. □

Lemma 6.3 *For fixed* $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}$, *one has*

$$\mathrm{meas}(\cup_{t_1,t_2,\cdots,t_{d-1}\in\mathbb{Z}}\mathcal{R}^\nu_{k,n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1},m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}}) < c\frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Proof Without loss of generality, we assume $|t_1| \le |t_2| \le \cdots \le |t_{d-1}|$. Otherwise, we can rearrange $t_1, t_2, \cdots, t_{d-1}$ such that $|t_1| \le |t_2| \le \cdots \le |t_{d-1}|$. Due to Töplitz–Lipschitz property of $N_\nu + P_\nu$, set $\Omega_n^\nu = |n|^2 + \tilde{\Omega}_n^\nu$ and $\Omega_m^\nu = |m|^2 + \tilde{\Omega}_m^\nu$, then

$$|\tilde{\Omega}^\nu_{n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}} - \lim_{t_j\to\infty} \tilde{\Omega}^\nu_{n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}}| < \frac{\varepsilon_0}{|t_j|}, 1 \le j \le d-1.$$

$$|\tilde{\Omega}^\nu_{m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}} - \lim_{t_j\to\infty} \tilde{\Omega}^\nu_{m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}}| < \frac{\varepsilon_0}{|t_j|}, 1 \le j \le d-1.$$

Hence

$$|M^\nu(t_1, t_2, \cdots, t_{d-1}) - \lim_{t_j\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1})| < \frac{\varepsilon_0}{|t_j|}, 1 \le j \le d-1.$$

□

We define resonant set

$$\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}\infty^{d-1}} = \{\xi \in \mathcal{O}_{\nu-1} : |\lim_{t_1\to\infty}(\lim_{t_2,\cdots,t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1}))| < \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}\} \tag{6.4}$$

For fixed $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}$,

$$\text{meas}(\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}\infty^{d-1}}) < \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Then for $\xi \in \mathcal{O}_{\nu-1} \backslash \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}\infty^{d-1}}$, we have

$$|\lim_{t_1\to\infty}(\lim_{t_2,\cdots,t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1}))| \ge \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Case 1: When $|t_1| > K_\nu^{\frac{\tau}{d!}}$, for $\xi \in \mathcal{O}_{\nu-1} \backslash \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}\infty^{d-1}}$, we have

$$\begin{aligned} &|M^\nu(t_1, t_2, \cdots, t_{d-1})| \\ &\ge |\lim_{t_1,t_2,\cdots,t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1})| - (\frac{\varepsilon_0}{|t_1|} + \frac{\varepsilon_0}{|t_2|} + \cdots + \frac{\varepsilon_0}{|t_{d-1}|}) \\ &\ge \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}} - (d-1)\frac{\varepsilon_0}{K_\nu^{\frac{\tau}{d!}}} \\ &\ge \frac{\gamma}{2K_\nu^{\frac{\tau}{d!}}}. \end{aligned}$$

Case 2: When $|t_1| \le K_\nu^{\frac{\tau}{d!}}$, $|t_2| \ge K_\nu^{\frac{2\tau}{d!}}$, we define resonant set

$$\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1\infty^{d-2}} = \{\xi \in \mathcal{O}_{\nu-1} : |\lim_{t_2,\cdots,t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1})| < \frac{\gamma}{K_\nu^{\frac{2\tau}{d!}}}\} \tag{6.5}$$

For fixed $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}, t_1$,

$$\text{meas}(\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1\infty^{d-2}}) < \frac{\gamma}{K_\nu^{\frac{2\tau}{d!}}},$$

then

$$\text{meas}\{\cup_{|t_1|\le K_\nu^{\frac{\tau}{d!}}} \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1\infty^{d-2}}\} < K_\nu^{\frac{\tau}{d!}} \frac{\gamma}{K_\nu^{\frac{2\tau}{d!}}} \le \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Hence when $|t_1| \le K_\nu^{\frac{\tau}{d!}}$, $|t_2| \ge K_\nu^{\frac{2\tau}{d!}}$, for $\xi \in \mathcal{O}_{\nu-1}\backslash\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1\infty^{d-2}}$, we have

$$\begin{aligned}&|M^\nu(t_1,t_2,\cdots,t_{d-1})|\\ &\ge |\lim_{t_2,\cdots,t_{d-1}\to\infty} M^\nu(t_1,t_2,\cdots,t_{d-1})| - (\frac{\varepsilon_0}{|t_2|}+\cdots+\frac{\varepsilon_0}{|t_{d-1}|})\\ &\ge \frac{\gamma}{K_\nu^{\frac{2\tau}{d!}}} - (d-2)\frac{\varepsilon_0}{K_\nu^{\frac{2\tau}{d!}}}\\ &\ge \frac{\gamma}{2K_\nu^{\frac{2\tau}{d!}}}.\end{aligned}$$

Case 3: When $|t_1| \le K_\nu^{\frac{\tau}{d!}}$, $|t_2| \le K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \ge K_\nu^{\frac{3!\tau}{d!}}$, we define resonant set

$$\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2\infty^{d-3}} = \{\xi \in \mathcal{O}_{\nu-1} : |\lim_{t_3,\cdots,t_{d-1}\to\infty} M^\nu(t_1,t_2,\cdots,t_{d-1})| < \frac{\gamma}{K_\nu^{\frac{3!\tau}{d!}}}\} \tag{6.6}$$

For fixed $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}, t_1, t_2$,

$$\text{meas}(\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2\infty^{d-3}}) < \frac{\gamma}{K_\nu^{\frac{3!\tau}{d!}}},$$

then

$$\text{meas}\{\cup_{|t_1|,|t_2|\le K_\nu^{\frac{2\tau}{d!}}} \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2\infty^{d-3}}\} < K_\nu^{\frac{4\tau}{d!}} \frac{\gamma}{K_\nu^{\frac{3!\tau}{d!}}} \le \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Hence when $|t_1| \le K_\nu^{\frac{\tau}{d!}}$, $|t_2| \le K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \ge K_\nu^{\frac{3!\tau}{d!}}$, for $\xi \in \mathcal{O}_{\nu-1}\backslash\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2\infty^{d-3}}$, we have

$$\begin{aligned}&|M^\nu(t_1,t_2,\cdots,t_{d-1})|\\ &\ge |\lim_{t_3,\cdots,t_{d-1}\to\infty} M^\nu(t_1,t_2,\cdots,t_{d-1})| - (\frac{\varepsilon_0}{|t_3|}+\cdots+\frac{\varepsilon_0}{|t_{d-1}|})\\ &\ge \frac{\gamma}{K_\nu^{\frac{3!\tau}{d!}}} - (d-3)\frac{\varepsilon_0}{K_\nu^{\frac{3!\tau}{d!}}}\\ &\ge \frac{\gamma}{2K_\nu^{\frac{3!\tau}{d!}}}.\end{aligned}$$

Case 4: When $|t_1| \le K_\nu^{\frac{\tau}{d!}}$, $|t_2| \le K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \le K_\nu^{\frac{3!\tau}{d!}}$, $|t_4| > K_\nu^{\frac{4!\tau}{d!}}$, we define resonant set

$$\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\infty^{d-4}} = \{\xi \in \mathcal{O}_{\nu-1} : |\lim_{t_4,\cdots,t_{d-1}\to\infty} M^\nu(t_1,t_2,\cdots,t_{d-1})| < \frac{\gamma}{K_\nu^{\frac{4!\tau}{d!}}}\} \tag{6.7}$$

For fixed $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}, t_1, t_2, t_3$,

$$\text{meas}(\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\infty^{d-4}}) < \frac{\gamma}{K_\nu^{\frac{4!\tau}{d!}}},$$

then

$$\text{meas}\{\cup_{|t_1|,|t_2|,|t_3|\le K_\nu^{\frac{3!\tau}{d!}}} \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\infty^{d-4}}\} < K_\nu^{\frac{3!\times 3\tau}{d!}} \frac{\gamma}{K_\nu^{\frac{4!\tau}{d!}}} \le \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Hence when $|t_1| \leq K_\nu^{\frac{\tau}{d!}}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \leq K_\nu^{\frac{3!\tau}{d!}}$, $|t_4| > K_\nu^{\frac{4!\tau}{d!}}$, for $\xi \in \mathcal{O}_{\nu-1} \backslash \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\infty^{d-4}}$, we have

$$\begin{aligned}
&|M^\nu(t_1, t_2, \cdots, t_{d-1})| \\
&\geq |\lim_{t_4,\cdots,t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1})| - \left(\frac{\varepsilon_0}{|t_4|} + \cdots + \frac{\varepsilon_0}{|t_{d-1}|}\right) \\
&\geq \frac{\gamma}{K_\nu^{\frac{4!\tau}{d!}}} - (d-4)\frac{\varepsilon_0}{K_\nu^{\frac{4!\tau}{d!}}} \\
&\geq \frac{\gamma}{2K_\nu^{\frac{4!\tau}{d!}}}.
\end{aligned}$$

One can continue the above process, $\cdots$,

Case (d-1): When $|t_1| \leq K_\nu^{\frac{\tau}{d!}}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \leq K_\nu^{\frac{3!\tau}{d!}}$, $\cdots$, $|t_{d-2}| \leq K_\nu^{\frac{(d-2)!\tau}{d!}}$, $|t_{d-1}| > K_\nu^{\frac{(d-1)!\tau}{d!}}$, we define resonant set

$$\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}\infty} = \{\xi \in \mathcal{O}_{\nu-1} : |\lim_{t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1})| < \frac{\gamma}{K_\nu^{\frac{(d-1)!\tau}{d!}}}\} \tag{6.8}$$

For fixed $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}, t_1, t_2, t_3, \cdots, t_{d-2}$,

$$\text{meas}(\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}\infty}) < \frac{\gamma}{K_\nu^{\frac{(d-1)!\tau}{d!}}},$$

then

$$\text{meas}\{\cup_{|t_1|,|t_2|,|t_3|,\cdots,|t_{d-2}|\leq K_\nu^{\frac{(d-2)!\tau}{d!}}} \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}\infty}\} < K_\nu^{\frac{(d-2)!\times(d-2)\tau}{d!}} \frac{\gamma}{K_\nu^{\frac{(d-1)!\tau}{d!}}} \leq \frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Hence when $|t_1| \leq K_\nu^{\frac{\tau}{d!}}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \leq K_\nu^{\frac{3!\tau}{d!}}$, $\cdots$, $|t_{d-2}| \leq K_\nu^{\frac{(d-2)!\tau}{d!}}$, $|t_{d-1}| > K_\nu^{\frac{(d-1)!\tau}{d!}}$, for $\xi \in \mathcal{O}_{\nu-1} \backslash \mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}\infty}$, we have

$$\begin{aligned}
&|M^\nu(t_1, t_2, \cdots, t_{d-1})| \\
&\geq |\lim_{t_{d-1}\to\infty} M^\nu(t_1, t_2, \cdots, t_{d-1})| - \frac{\varepsilon_0}{|t_{d-1}|} \\
&\geq \frac{\gamma}{K_\nu^{\frac{(d-1)!\tau}{d!}}} - \frac{\varepsilon_0}{K_\nu^{\frac{(d-1)!\tau}{d!}}} \\
&\geq \frac{\gamma}{2K_\nu^{\frac{(d-1)!\tau}{d!}}}.
\end{aligned}$$

Case d: When $|t_1| \leq K_\nu^{\frac{\tau}{d!}}$, $|t_2| \leq K_\nu^{\frac{2\tau}{d!}}$, $|t_3| \leq K_\nu^{\frac{3!\tau}{d!}}$, $\cdots$, $|t_{d-2}| \leq K_\nu^{\frac{(d-2)!\tau}{d!}}$, $|t_{d-1}| \leq K_\nu^{\frac{(d-1)!\tau}{d!}}$, we define resonant set

$$\mathcal{R}^\nu_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}t_{d-1}} = \{\xi \in \mathcal{O}_{\nu-1} : |M^\nu(t_1, t_2, \cdots, t_{d-1})| < \frac{\gamma}{K_\nu^\tau}\} \tag{6.9}$$

For fixed $k, n_0, m_0, c_1, c_2, \cdots, c_{d-1}, t_1, t_2, t_3, \cdots, t_{d-2}, t_{d-1}$,

$$\text{meas}(\mathcal{R}^{\nu}_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}t_{d-1}}) < \frac{\gamma}{K_\nu^{\tau}},$$

then

$$\text{meas}\{\cup_{|t_1|,|t_2|,|t_3|,\cdots,|t_{d-2}|,|t_{d-1}|\le K_\nu^{\frac{(d-1)!\tau}{d!}}} \mathcal{R}^{\nu}_{kn_0m_0c_1c_2\cdots c_{d-1}t_1t_2t_3\cdots t_{d-2}t_{d-1}}\} < K_\nu^{\frac{(d-1)!\times(d-1)\tau}{d!}} \frac{\gamma}{K_\nu^{\tau}} \le \frac{\gamma}{K_\nu^{\frac{\tau}{d}}}.$$

As a consequence,

$$\text{meas}(\cup_{t_1,t_2,\cdots,t_{d-1}\in\mathbb{Z}}\mathcal{R}^{\nu}_{k,n_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1},m_0+t_1c_1+t_2c_2+\cdots+t_{d-1}c_{d-1}}) < c\frac{\gamma}{K_\nu^{\frac{\tau}{d!}}}.$$

Lemma 6.4

$$\text{meas}(\bigcup_{K_{\nu-1}<|k|\le K_\nu} R^{\nu}_{k}) \le cK_\nu^{b}\frac{\gamma}{K_\nu^{\tau}} = c\frac{\gamma}{K_\nu^{\tau-b}}$$

$$\text{meas}(\bigcup_{K_{\nu-1}<|k|\le K_\nu,n} R^{\nu}_{kn}) \le cK_\nu^{d+b}\frac{\gamma}{K_\nu^{\tau}} = c\frac{\gamma}{K_\nu^{\tau-d-b}}$$

$$\text{meas}(\bigcup_{K_{\nu-1}<|k|\le K_\nu,n,m} R^{\nu}_{knm}) \le c\frac{\gamma}{K_\nu^{\frac{\tau}{d!}-2d(d+1)-b}}$$

Lemma 6.5 *Let $\tau > d!(2d(d+1)+b+1)$, then the total measure need to exclude along the KAM iteration is*

$$\begin{aligned}&\text{meas}(\bigcup_{\nu\ge 0}\mathcal{R}^{\nu})\\ &= \text{meas}[\bigcup_{\nu\ge 0}(\bigcup_{K_{\nu-1}<|k|\le K_\nu,n,m} \mathcal{R}^{\nu}_{k}\bigcup\mathcal{R}^{\nu}_{kn}\bigcup\mathcal{R}^{\nu}_{knm})]\\ &\le c\sum_{\nu\ge 0}\frac{\gamma}{K_\nu} \le c\gamma.\end{aligned}$$

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7 Appendix

Lemma 7.1

$$\|FG\|_{D_\rho(r,s),\mathcal{O}} \le \|F\|_{D_\rho(r,s),\mathcal{O}}\|G\|_{D_\rho(r,s),\mathcal{O}}.$$

Proof Since $(FG)_{kl\alpha\beta} = \sum_{k',l',\alpha',\beta'} F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'}$, we have

$$\|FG\|_{D_\rho(r,s),\mathcal{O}} = \sup_{D_\rho(r,s)}\sum_{k,l,\alpha,\beta}|(FG)_{kl\alpha\beta}|_{\mathcal{O}}|I^l||z^\alpha||\bar{z}^\beta|e^{|k||\text{Im}\theta|}$$

$$\leq \sup_{D_\rho(r,s)} \sum_{k,l,\alpha,\beta} \sum_{k',l',\alpha',\beta'} |F_{k-k',l-l',\alpha-\alpha',\beta-\beta'} G_{k'l'\alpha'\beta'}|_{\mathcal{O}} |I^l||z^\alpha||\bar{z}^\beta| e^{|k||\mathrm{Im}\theta|}$$
$$\leq \|F\|_{D_\rho(r,s),\mathcal{O}} \|G\|_{D_\rho(r,s),\mathcal{O}}$$

and the proof is finished. □

Lemma 7.2 *(Generalized Cauchy inequalities)*

$$\|F_\theta\|_{D_\rho(r-\sigma,s),\mathcal{O}} \leq \frac{c}{\sigma}\|F\|_{D_\rho(r,s),\mathcal{O}},$$
$$\|F_I\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}} \leq \frac{c}{s^2}\|F\|_{D_\rho(r,s),\mathcal{O}},$$

and

$$\|F_z\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}} \leq \frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}},$$
$$\|F_{\bar{z}}\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}} \leq \frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}}.$$

Proof We only prove the third inequality, the others can be proved similarly. Let $w \neq 0$, then $f(t) = F(z + tw)$ is an analytic map from the complex disc $|t| < \frac{s}{\|w\|_\rho}$ in $\mathbb{C}$ into $D_\rho(r, s)$. Hence

$$\|f'(0)\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}} = \|F_z(w)\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}} \leq \frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}} \cdot \|w\|_\rho,$$

by the usual Cauchy inequality. Since $w \neq 0$, so

$$\frac{\|F_z(w)\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}}}{\|w\|_\rho} \leq \frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}},$$

thus

$$\|F_z\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}} = \sup_{w\neq 0} \frac{\|F_z(w)\|_{D_\rho(r,\frac{1}{2}s),\mathcal{O}}}{\|w\|_\rho} \leq \frac{c}{s}\|F\|_{D_\rho(r,s),\mathcal{O}}.$$

□

Let $\{\cdot,\cdot\}$ denote the Poisson bracket of smooth functions, i.e.,

$$\{F, G\} = \langle\frac{\partial F}{\partial I}, \frac{\partial G}{\partial \theta}\rangle - \langle\frac{\partial F}{\partial \theta}, \frac{\partial G}{\partial I}\rangle + \mathrm{i}(\langle\frac{\partial F}{\partial z}, \frac{\partial G}{\partial \bar{z}}\rangle - \langle\frac{\partial F}{\partial \bar{z}}, \frac{\partial G}{\partial z}\rangle),$$

then we have the following lemma:

Lemma 7.3 *If*

$$\|X_F\|_{D_\rho(r,s),\mathcal{O}} < \varepsilon', \ \|X_G\|_{D_\rho(r,s),\mathcal{O}} < \varepsilon'',$$

then

$$\|X_{\{F,G\}}\|_{D_\rho(r-\sigma,\eta s),\mathcal{O}} < c\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon'', \ \eta \ll 1.$$

In particular, if $\eta \sim \varepsilon^{\frac{1}{3}}$, $\varepsilon', \varepsilon'' \sim \varepsilon$, *we have* $\|X_{\{F,G\}}\|_{D_\rho(r-\sigma,\eta s),\mathcal{O}} \sim \varepsilon^{\frac{4}{3}}$.

Proof By Lemma 7.1 and Lemma 7.2,

$$\begin{aligned}
&\|\frac{\partial^2 F}{\partial I\partial I}\frac{\partial G}{\partial\theta}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-2}\|\frac{\partial F}{\partial I}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\theta}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial I\partial\theta}\frac{\partial G}{\partial\theta}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}\|\frac{\partial F}{\partial I}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\theta}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial I\partial z}\frac{\partial G}{\partial\theta}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-1}\|\frac{\partial F}{\partial I}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\theta}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial I\partial\bar z}\frac{\partial G}{\partial\theta}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-1}\|\frac{\partial F}{\partial I}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\theta}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial z\partial I}\frac{\partial G}{\partial\bar z}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-2}\|\frac{\partial F}{\partial z}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\bar z}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial z\partial\theta}\frac{\partial G}{\partial\bar z}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}\|\frac{\partial F}{\partial z}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\bar z}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial z\partial z}\frac{\partial G}{\partial\bar z}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-1}\|\frac{\partial F}{\partial z}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\bar z}\|_{D_\rho(r,s)},\\
&\|\frac{\partial^2 F}{\partial z\partial\bar z}\frac{\partial G}{\partial\bar z}\|_{D_\rho(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-1}\|\frac{\partial F}{\partial z}\|_{D_\rho(r,s)}\cdot\|\frac{\partial G}{\partial\bar z}\|_{D_\rho(r,s)}.
\end{aligned}$$

The other cases can be obtained analogously, hence

$$\|X_{\{F,G\}}\|_{D_\rho(r-\sigma,\eta s),\mathcal{O}} < c\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon''.$$

□

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