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Quasi-periodic solutions in a nonlinear Schrödinger equation

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Abstract

In this paper, one-dimensional (1D) nonlinear Schrödinger equation

$$\mathrm{i}u_t - u_{xx} + mu + |u|^4 u = 0$$

with the periodic boundary condition is considered. It is proved that for each given constant potential m and each prescribed integer $N > 1$, the equation admits a Whitney smooth family of small amplitude, time quasi-periodic solutions with N Diophantine frequencies. The proof is based on a partial Birkhoff normal form reduction and an improved KAM method.

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Keywords: Schrödinger equation; Hamiltonian systems; KAM theory; Normal form; Quasi-periodic solution

1. Introduction and main result

Consider a nonlinear Schrödinger equation

$$\mathrm{i}u_t + Au + \frac{\partial F}{\partial \bar{u}}(u, \bar{u}) = 0 \tag{1.1}$$

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with either the homogeneous Dirichlet boundary condition on a domain $\Omega \subset R^d$ or the periodic boundary condition on R^d, where

$$A : -\Delta + V$$

is a self-adjoint operator on $X=H_0^1(\Omega)$ or $H^1(T^d)$ and F consists of higher order and perturbative terms. The equation defines an infinite-dimensional Hamiltonian system

$$u_t = \mathrm{i}\frac{\partial \widetilde{H}}{\partial \bar{u}}$$

associated with the Hamiltonian

$$\widetilde{H} = \langle Au, u\rangle + \widetilde{F}(u, \bar{u}), \quad \text{where}$$
$$\widetilde{F} = \int F(u, \bar{u})\, dx$$

integrating either over Ω in the case of the Dirichlet boundary condition or over T^d in the case of the periodic boundary condition.

Let μ_n, ϕ_n, $n \in Z^d$, denote the eigenvalues, eigenfunctions of A, respectively. The problem of the existence of (time-) quasi-periodic solution for (1.1) is to find, for a given integer $N > 1$, a solution of the form

$$u(t, x) = \sum_{n \in Z^d} q_n(t)\phi_n(x)$$

such that all q_n, $n \in Z^d$, are quasi-periodic with the same N-frequencies. With the symplectic structure

$$\mathrm{i}\sum_n dq_n \wedge d\bar{q}_n,$$

it is clear that $q = (q_n)_{n \in Z^d}$ satisfies the Hamiltonian lattice equations

$$\begin{cases} \dot{q}_n = -\mathrm{i}\frac{\partial H}{\partial \bar{q}_n}, \\ \dot{\bar{q}}_n = \mathrm{i}\frac{\partial H}{\partial q_n}, \quad n \in Z^d, \end{cases} \tag{1.2}$$

where

$$H = H(q, \bar{q}) = \widetilde{H}\left(\sum_n q_n\phi_n, \sum_n \bar{q}_n\bar{\phi}_n\right). \tag{1.3}$$

Motivated by the classical KAM (Kolmogorov–Arnold–Moser) theory in finite-dimensional Hamiltonian systems, a natural way to obtain quasi-periodic solutions for (1.2) is to re-formulate the system into a perturbation of a nondegenerate, partially integrable system, for which parameters need to be introduced in order to adjust frequencies to encounter small divisors problem. More precisely, given an integer $N > 1$, one finds a parameter space $\Lambda \subset R^N$ and (symplectic) action-angle-normal coordinates $I = (I_1, \ldots, I_N)$, $\theta = (\theta_1, \ldots, \theta_N)$, $z = (z_n)_{n \in Z^d}$ such that the Hamiltonian (1.3) can be transformed into a parametrized Hamiltonian normal form

$$H = \langle \omega(\xi), I \rangle + \sum_n \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi), \quad \xi \in \Lambda, \tag{1.4}$$

where $\omega : \Lambda \to R^N$ is a local diffeomorphism and P is viewed as a perturbation. The existence problem for quasi-periodic solutions then becomes finding a positive measure subset $\tilde{\Lambda}$ of Λ such that each $\xi \in \tilde{\Lambda}$ corresponds to a quasi-periodic, invariant N-torus of (1.4) which is a small perturbation of the unperturbed, quasi-periodic N-torus $T_\xi = \{0\} \times \{\theta_0 + \omega(\xi)t\} \times \{0\}$ corresponding to $P = 0$.

There are several ways of introducing parameters into (1.1) or (1.2). For perturbations of linear Schrödinger equations, one way is to consider parametrized potentials, i.e.,

$$V = V(\cdot, \xi),$$

where ξ is an N-dimensional parameter, for which the unperturbed nondegeneracy is a generic condition on potentials. In this setting, one deals with a family of nonlinear Schrödinger equations with generic potentials. Another way is to consider Floquet operator potentials

$$V = M_\sigma,$$

where M_σ is a Floquet multiplier defined by

$$M_\sigma e^{\mathrm{i}\langle n,x\rangle} = \sigma_n e^{\mathrm{i}\langle n,x\rangle}, \quad n \in Z^d,$$

for a set of real numbers $\{\sigma_n,\ n \in Z^d\}$ such that N of them can be treated as parameters. In this setting, one considers nonlinear Schrödinger equations in operator forms. To deal with a single nonlinear Schrödinger equation with the potential

$$V = V(x),$$

a natural way of introducing parameters, as what have been done in finite dimensions, is to reduce the lattice Hamiltonian (1.3) to a partial Birkhoff normal form then further to a parametrized normal form (1.4) having N nondegenerate, integrable directions whose amplitudes become natural parameters.

With the availability of a parametrized normal form (1.4), both KAM (Kolmogorov–Arnold–Moser) and CWB (Craig–Wayne–Bourgain) methods have been developed in studying the existence of quasi-periodic solutions for nonlinear Schrödinger equations as well as for other infinite-dimensional Hamiltonian systems like nonlinear wave, KdV, beam equations, and Hamiltonian lattices.

The KAM method concerns the construction of a sequence of symplectic transformations to (1.4) so that at each KAM step resonant terms in the perturbation are removed, quadratic terms in the perturbation are averaged and added to the new normal form, and angular-dependent terms in the perturbation are pushed into higher order. Based on original works of Melnikov [24], Eliasson [12], Kuksin [18], and Pöschel [25], the KAM method has been extensively developed in finite dimensions concerning the persistence of lower-dimensional tori in Hamiltonian systems (see [2,3,16,23,29,31,32] and references therein). Recently, the KAM method was extended to infinite dimensions in works of Kuksin [19], Wayne [30], and Pöschel [27] in studying quasi-periodic solutions for 1D nonlinear Schrödinger and wave equations with the Dirichlet boundary condition and parametrized potentials.

The CWB method was first introduced by Craig and Wayne [10,11] in studying periodic solutions for 1D nonlinear wave equations with the periodic boundary condition and later developed by Bourgain [4–8] in studying quasi-periodic solutions for nonlinear Schrödinger and wave equations with periodic boundary conditions and either parametrized potentials in 1D or Floquet operator potentials in any space dimension. Unlike 1D Schrödinger and wave equations with the Dirichlet boundary condition, multiple eigenvalues always occur in cases with the periodic boundary condition even in one space dimension, which causes additional difficulties in studying the existence of quasi-periodic solutions. To overcome the difficulties, the CWB method traces back to the origin of the KAM method by using Newton iteration for solving infinitely many homological equations and uses Lyapunov–Schmidt decomposition of amplitude–frequency equations together with techniques of Fröhlich and Spencer [13] concerning analysis of Green's functions. With respect to the normal form (1.4), the CWB method differs from the KAM method by only considering elimination of first order resonant terms in each iteration step.

The KAM method is more restrictive than the CWB one, that is, if the existence of quasi-periodic solutions in a Hamiltonian system can be shown by the KAM method, then it can be also shown by the CWB method. However, the KAM method, if applicable, can capture more properties of quasi-periodic solutions such as their continuous (in fact, Whitney smooth) dependence on parameters, their Floquet forms and linear stability, while CWB method mainly yields the existence. Both methods share some common difficulties in infinite dimension, for instance, estimates on the inverse of an infinite-dimensional matrix which has small divisors on the diagonal, measure estimate involving infinitely many small divisor conditions for a fixed Fourier mode, multiplicity of eigenvalues under the periodic boundary condition, and non-improvement of regularities (decay rate of Fourier coefficients) after iterations.

Besides nonlinear Schrödinger and wave equations with parameterized and Floquet operator potentials, there have been also studies on quasi-periodic solutions of a single nonlinear Schrödinger or wave equation with a constant potential using either the CWB or the KAM method. By deriving a partial Birkhoff normal form of order four, Kuksin and Pöschel [21], Pöschel [26] further developed the KAM method to study the existence of small amplitude quasi-periodic solutions corresponding to any finite number of Fourier modes for perturbations of the 1D nonlinear Schrödinger and wave equations

$$\mathrm{i}u_t - u_{xx} + mu + |u|^2u = 0, \quad m \in \mathbb{R}, \tag{1.5}$$

$$u_{tt} - u_{xx} + mu + u^3 = 0, \quad m > 0, \tag{1.6}$$

with the Dirichlet boundary condition. Using the CWB method, similar existence results of quasi-periodic solutions were shown by Bourgain [6,8] with respect to the periodic boundary condition. A KAM type of theorem was later given by Chierchia and You [9] for the 1D wave equation (1.6) with the periodic boundary condition, which, due to a spectral gap condition and a regularity condition assumed, does not hold for the 1D Schrödinger equation (1.5) with the periodic boundary condition. Recently, a general KAM type of result is obtained by Geng and You [14] which is particularly applicable to the 1D Schrödinger equation (1.5) with the periodic boundary condition. We note that the 1D Schrödinger equation (1.5) with cubic nonlinearities are completely integrable, hence one can perturb any finite number of Fourier modes to obtain small amplitude quasi-periodic solutions, which is however not the case if the nonlinearities are of higher order. In a recent work, Liang and You [22] made use of some complicated Birkhoff normal form reductions and KAM techniques to obtain small amplitude quasi-periodic solutions corresponding to any finite number of admissible Fourier modes for the nonlinear Schrödinger equation

$$\mathrm{i}u_t - u_{xx} + mu + |u|^4 u = 0, \quad m \in \mathbb{R}, \tag{1.7}$$

with the Dirichlet boundary condition.

There are also some works on the existence of quasi-periodic solutions in higher dimensional nonlinear Schrödinger equations and other type of infinite-dimensional Hamiltonian systems such as quasi-linear KdV equations, nonlinear beam equations and Hamiltonian lattices. For instance, the existence of two-frequency, quasi-periodic solutions was shown by Bourgain [6,8] for the 2D nonlinear Schrödinger equation

$$\mathrm{i}u_t - \triangle u + mu + |u|^2 u = 0,$$

by using CWB method and normal form reductions. Geng and You [15] proved a KAM type of theorem which is applicable to certain Hamiltonian partial differential equations in higher space dimension including beam equations and Schrödinger equations with nonlocal nonlinearity. Recently, Kuksin [20], Kappeler and Pöschel [17] showed the existence of quasi-periodic solutions for quasi-linear KdV equations.

In this paper, we will study the 1D Schrödinger equation (1.7) with the periodic boundary condition

$$u(t,x) = u(t, x + 2\pi) \tag{1.8}$$

and show the existence of small amplitude quasi-periodic solutions corresponding to any finite number of admissible Fourier modes. More precisely, let

$$\mu_n = n^2 + m, \quad n \in \mathbb{Z},$$

be the eigenvalues of $A = -\partial_{xx} + m$ with the periodic boundary condition (1.8). For any fixed integer $N > 1$ and any ordered N-index $\{n_1, n_2, \dots, n_N\}$, where $0 < n_1 < n_2 < \cdots < n_N$, it is clear that the linear equation associated with (1.7) with the same periodic boundary condition (1.8) has quasi-periodic solutions

$$u(t,x) = \sum_{i=1}^{N} \sqrt{\xi_i} e^{\mathrm{i}(\omega_i t + n_i x)}, \quad \omega_i = n_i^2 + m, \ \xi_i > 0.$$

An ordered N-index $\{n_1, n_2, \dots, n_N\}$ is said to be *admissible* if whenever i, j, k, l, m, n are integers such that $i + j + k = l + m + n$, $(i, j, k) \neq (l, m, n)$, and at least four of them lying in $\{n_1, n_2, \dots, n_N\}$, then

$$\mu_i + \mu_j + \mu_k - \mu_l - \mu_m - \mu_n \neq 0.$$

We let $\mathcal{J}$ denote the set of all admissible N-indexes. It is known that for any given $N > 1$, $\mathcal{J}$ is an infinite set (see [22, Appendix]). In particular, when $N = 2$, $\mathcal{J}$ is simply the set

$$\mathcal{J} = \bigl\{\{n_1, n_2\}\colon\ n_1 - n_2 \text{ is odd}\bigr\}.$$

Our main result states as follows.

Theorem. *Consider the* 1*D nonlinear Schrödinger equation* (1.7) *with the periodic boundary condition* (1.8)*. For given $N > 1$, let $\{n_1, \dots, n_N\}$ be a fixed admissible N-index. Then there exists a Cantor subset $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}(n_1, \dots, n_N) \subset \mathbb{R}^N_+$ of positive Lebesgue measure, such that each $\xi \in \widetilde{\mathcal{O}}$ corresponds to a real analytic, quasi-periodic solution*

$$u(t, x) = \sum_{i=1}^{N} \sqrt{\xi_i}\, e^{\mathrm{i}(\tilde{\omega}_i t + n_i x)} + O\bigl(|\xi|^{\frac{5}{2}}\bigr)$$

of (1.7), (1.8) *with Diophantine frequencies*

$$\tilde{\omega}_i = \omega_i + O\bigl(|\xi|^2\bigr), \quad 1 \leqslant i \leqslant N.$$

Moreover, the quasi-periodic solutions u are linearly stable and depend on ξ Whitney smoothly.

The proof of our result uses the KAM method for which a partial Birkhoff normal form needs to be derived. By taking advantage of the special form of the nonlinearity, the normal form will be derived using arguments of Liang and You [22] for eliminating lower-order, non-integrable terms. It turns out that not only is the normal form in the present situation as simple as the case with the Dirichlet boundary condition, but also its perturbation term has a so-called *compact form* (see Section 2 for details) which is preserved under the KAM iterations. Such a compact form significantly simplifies the KAM iteration. In particular, at each KAM step, the small divisor conditions are similar to those in finite-dimensional cases and we do not need to shrink the decay weight which is usually necessary in infinite-dimensional KAM problems in order to put the less regular terms in the perturbation into the next KAM cycle and to preserve decay properties of the coefficient matrices of the homological equations. This is also the reason for obtaining Diophantine tori in the present case, which does not seem to be known in most Hamiltonian PDEs previously studied.

With similar partial Birkhoff normal form reductions, our method equally applies to the 1D beam equation

$$u_{tt} + u_{xxxx} + u^5 = 0$$

with the periodic boundary condition to yield the existence of quasi-periodic solutions. But with respect to finding quasi-periodic solutions it applies neither to the 1D nonlinear Schrödinger equations of higher order nonlinearities:

$$\mathrm{i}u_t - u_{xx} + mu + |u|^{2p} u = 0, \quad p \geqslant 3, \tag{1.9}$$

with either the Dirichlet or the periodic boundary condition, nor to the completely resonant wave equation

$$u_{tt} - u_{xx} + u^3 = 0 \tag{1.10}$$

with either the Dirichlet or the periodic boundary condition. For (1.9), the failure of our method is due to the unavailability of similar Birkhoff normal forms (see Section 2 for details). However, our method works in finding periodic solutions of (1.9) for any $p \geqslant 1$ due to the availability of partial Birkhoff normal forms in the case of $N = 1$ and the unnecessity of small divisor conditions for the existence of periodic conditions. For (1.10), the failure of our method is due to the lack of super-linear growth of eigenvalues because our method crucially depend on the spectral asymptotics $\mu_n \sim n^2$. We note that by avoiding Birkhoff normal form reductions, special quasi-periodic solutions were discovered by Bambusi [1] for wave equations

$$u_{tt} - u_{xx} + mu + u^{2p-1} = 0, \quad p \geqslant 2,$$

with the Dirichlet boundary condition, for typical $m > 0$. In general, it is certainly interesting to know whether quasi-periodic solutions can exist for Eqs. (1.9) with either the Dirichlet or the periodic boundary condition if $p \geqslant 3$.

The rest of the paper is devoted to the proof of the main result. For simplicity, we only treat the case $N = 2$. The general case can be treated similarly. Section 2 is a preliminary section in which we define the weighted norms and compact forms and study their basic properties. In Section 3, we derive a partial Birkhoff normal form of order six for the lattice Hamiltonian associated with (1.7), (1.8) then transform it into a parameterized Hamiltonian normal form. In Sections 4, we give details for one step of KAM iteration. Proof of the theorem is completed in Section 5 by showing an iteration lemma, convergence, and conducting measure estimate.

2. Preliminary

2.1. Weighted norms

For a given $\rho > 0$, we let ℓ^ρ be the Banach space of bi-infinite, complex valued sequences $q = (\{q_n\})$, endowed with the weighted norm

$$\|q\|_\rho = \sum_{n \in \mathbb{Z}} |q_n| e^{|n|\rho}.$$

Similarly, let $\mathcal{L}^\rho$ be the Banach space of functions $u(x) = \sum_{n \in \mathbb{Z}} q_n \phi_n(x)$ for $(\{q_n\}) \in \ell^\rho$, endowed with the norm $\|u\|_\rho = \|q\|_\rho$. Then $\mathcal{L}^\rho$ and ℓ^ρ are isometric, and the product of two functions $u(x) = \sum_{n \in \mathbb{Z}} p_n \phi_n(x)$, $v(x) = \sum_{n \in \mathbb{Z}} q_n \phi_n(x)$ in $\mathcal{L}^\rho$ defines the convolution $q * p$: $(q * p)_n = \sum_m q_{n-m} p_m$, $n \in \mathbb{Z}$, in ℓ^ρ, under which ℓ^ρ becomes a Banach algebra. In particular,

$$\|q * p\|_\rho \leqslant \|q\|_\rho \|p\|_\rho,$$

for any $p, q \in \ell^\rho$.

Let $|\cdot|$ denote the sup-norm of complex vectors. For given $r, s > 0$, we let $D(r, s)$ be the complex neighborhood

$$D(r, s) = \left\{(\theta, I, z, \bar{z}):\ |\mathrm{Im}\,\theta| < r, |I| < s^2, \|z\|_\rho < s, \|\bar{z}\|_\rho < s\right\}$$

of $\mathbb{T}^2 \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ in $\mathbb{T}^2 \times \mathbb{R}^2 \times \ell^\rho$. Also let $\mathcal{O}$ be a bounded set in $\mathbb{R}^2_+$.

Let $F(\theta, I, z, \bar{z})$ be a real analytic function on $D(r,s)$ which depends on a parameter $\xi \in \mathcal{O}$ Whitney smoothly (i.e., C^1 in the sense of Whitney). We expand F into the Taylor–Fourier series with respect to $\theta, I, z, \bar{z}$:

$$F(\theta, I, z, \bar{z}) = \sum_{\alpha,\beta} F_{\alpha\beta} z^{\alpha} \bar{z}^{\beta},$$

where, for multi-indices $\alpha \equiv (\dots, \alpha_n, \dots)$, $\beta \equiv (\dots, \beta_n, \dots)$, $\alpha_n, \beta_n \in \mathbb{N}$ with finitely many non-vanishing components,

$$F_{\alpha\beta} = \sum_{k \in \mathbb{Z}^2, l \in \mathbb{N}^2} F_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k, \theta\rangle}.$$

We define the weighted norm of F as

$$\|F\|_{D(r,s),\mathcal{O}} \equiv \sup_{\substack{\|z\|_\rho < s \\ \|\bar{z}\|_\rho < s}} \sum_{\alpha,\beta} \|F_{\alpha\beta}\| \left|z^{\alpha}\right| \left|\bar{z}^{\beta}\right|, \quad \text{where}$$

$$\|F_{\alpha\beta}\| \equiv \sum_{k,l} |F_{kl\alpha\beta}|_{\mathcal{O}} s^{2|l|} e^{|k|r}, \quad |F_{kl\alpha\beta}|_{\mathcal{O}} \equiv \sup_{\xi \in \mathcal{O}} \left(|F_{kl\alpha\beta}| + \left| \frac{\partial F_{kl\alpha\beta}}{\partial \xi} \right| \right).$$

In the above and for the rest of the paper, derivatives in $\xi \in \mathcal{O}$ are in the sense of Whitney.

For a vector-valued function $G : D(r,s) \times \mathcal{O} \to \mathbb{C}^m$, $m < \infty$, we simply define its weighed norm by

$$\|G\|_{D(r,s),\mathcal{O}} \equiv \sum_{i=1}^{m} \|G_i\|_{D(r,s),\mathcal{O}}.$$

For the Hamiltonian vector field

$$X_F = \left(F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}, \{-\mathrm{i}F_{\bar{z}_n}\}\right)$$

associated with a function F on $D(r,s) \times \mathcal{O}$, we define its weighted norm by

$$\|X_F\|_{D(r,s),\mathcal{O}} \equiv \|F_I\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2} \|F_\theta\|_{D(r,s),\mathcal{O}} + \frac{1}{s} \left(\sum_n \|F_{z_n}\|_{D(r,s),\mathcal{O}} e^{|n|\rho} + \sum_n \|F_{\bar{z}_n}\|_{D(r,s),\mathcal{O}} e^{|n|\rho} \right).$$

Let F, G be two real analytic functions on $D(r,s)$ which depend on a parameter $\xi \in \mathcal{O}$ Whitney smoothly.

Lemma 2.1.

$$\|FG\|_{D(r,s),\mathcal{O}} \leqslant \|F\|_{D(r,s),\mathcal{O}} \|G\|_{D(r,s),\mathcal{O}}.$$

Proof. Since $(FG)_{kl\alpha\beta} = \sum_{k',l',\alpha',\beta'} F_{k-k',l-l',\alpha-\alpha',\beta-\beta'} G_{k'l'\alpha'\beta'}$, we have

$$\begin{aligned}\|FG\|_{D(r,s),\mathcal{O}} &= \sup_{\substack{\|z\|_\rho<s\\ \|\bar z\|_\rho<s}} \sum_{k,l,\alpha,\beta} \left|(FG)_{kl\alpha\beta}\right|_{\mathcal{O}} s^{2l}\left|z^\alpha\right|\left|\bar z^\beta\right| e^{|k|r} \\ &\leqslant \sup_{\substack{\|z\|_\rho<s\\ \|\bar z\|_\rho<s}} \sum_{k,l,\alpha,\beta}\sum_{k',l',\alpha',\beta'} |F_{k-k',l-l',\alpha-\alpha',\beta-\beta'} G_{k'l'\alpha'\beta'}|_{\mathcal{O}} s^{2l}\left|z^\alpha\right|\left|\bar z^\beta\right| e^{|k|r} \\ &\leqslant \|F\|_{D(r,s),\mathcal{O}}\|G\|_{D(r,s),\mathcal{O}}. \qquad \square\end{aligned}$$

Lemma 2.2 *(Cauchy inequalities).*

$$\|F_\theta\|_{D(r-\sigma,s)} \leqslant \frac{1}{\sigma}\|F\|_{D(r,s)}, \qquad \|F_I\|_{D(r,\frac{1}{2}s)} \leqslant \frac{4}{s^2}\|F\|_{D(r,s)}, \quad \textit{and}$$
$$\|F_{z_n}\|_{D(r,\frac{1}{2}s)} \leqslant \frac{2}{s}\|F\|_{D(r,s)} e^{|n|\rho}, \qquad \|F_{\bar z_n}\|_{D(r,\frac{1}{2}s)} \leqslant \frac{2}{s}\|F\|_{D(r,s)} e^{|n|\rho}.$$

Proof. It follows from the standard Cauchy estimate. □

Consider the Poisson bracket

$$\{F,G\} = \left\langle \frac{\partial F}{\partial I}, \frac{\partial G}{\partial \theta}\right\rangle - \left\langle \frac{\partial F}{\partial \theta}, \frac{\partial G}{\partial I}\right\rangle + \mathrm{i}\sum_n \left(\frac{\partial F}{\partial z_n}\frac{\partial G}{\partial \bar z_n} - \frac{\partial F}{\partial \bar z_n}\frac{\partial G}{\partial z_n}\right).$$

Lemma 2.3. *There exists a constant $c > 0$ such that if*

$$\|X_F\|_{D(r,s),\mathcal{O}} < \varepsilon', \qquad \|X_G\|_{D(r,s),\mathcal{O}} < \varepsilon''$$

for some $\varepsilon', \varepsilon'' > 0$, then

$$\|X_{\{F,G\}}\|_{D(r-\sigma,\eta s),\mathcal{O}} < c\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon'', \quad \textit{for any } 0<\sigma<r \textit{ and } 0<\eta\ll 1.$$

In particular, if $\eta \sim \varepsilon^{\frac{1}{3}}$, $\varepsilon', \varepsilon'' \sim \varepsilon$, then $\|X_{\{F,G\}}\|_{D(r-\sigma,\eta s),\mathcal{O}} \sim \varepsilon^{\frac{4}{3}}$.

Proof. See [14]. □

2.2. Compact form

Given $n_1, n_2 \in \mathbb{Z}$. A real analytic function

$$F = F(\theta, I, z, \bar z) = \sum_{k,\alpha,\beta} F_{k\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar z^\beta$$

on $D(r,s)$ is said to admit a *compact form with respect to n_1, n_2* if

$$F_{k\alpha\beta} = 0, \quad \text{whenever } k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n) n \neq 0,$$

where $k = (k_1, k_2) \in \mathbb{Z}^2$, and, $\alpha \equiv (\dots, \alpha_n, \dots)$, $\beta \equiv (\dots, \beta_n, \dots)$, $\alpha_n, \beta_n \in \mathbb{N}$, with finitely many nonvanishing components. In the case that $n_1 = n_2 = 0$, we simply say that F has a *compact form*.

Let F has a compact form with respect to n_1, n_2. We consider the term $F^{k11}_{n(-n)} e^{\mathrm{i}\langle k,\theta\rangle} z_n \bar{z}_{-n}$ in the Taylor–Fourier expansion of F. It is clear that $F^{011}_{n(-n)} = 0$, and even when $k \neq 0$, $F^{k11}_{n(-n)} = 0$ if $|n| > \frac{1}{2}\max\{|n_1|, |n_2|\}|k|$ (because $k_1 n_1 + k_2 n_2 + 2n \neq 0$ in this case). Hence, for each k, there are only finitely many coupled terms $z_n \bar{z}_{-n}$ in $F_k = \sum_{\alpha,\beta} F_{k\alpha\beta} z^\alpha \bar{z}^\beta$. A crucial idea in the proof of our main theorem is to show that compact forms will be preserved by KAM iterations. This property, enabling the consideration of essentially finite small divisors in each KAM step, will play an important role in the measure estimate later on.

Lemma 2.4. *Given $n_1, n_2 \in \mathbb{Z}$ and consider two real analytic functions $F(\theta, I, z, \bar{z})$, $G(\theta, I, z, \bar{z})$ on $D(r, s)$. If both F and G have compact forms with respect to n_1, n_2, then so does $\{F, G\}$.*

Proof. Let

$$F = \sum_{A_1} F_{k_1\alpha_1\beta_1}(I) e^{\mathrm{i}\langle k_1,\theta\rangle} z^{\alpha_1} \bar{z}^{\beta_1}, \qquad G = \sum_{A_2} G_{k_2\alpha_2\beta_2}(I) e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_2} \bar{z}^{\beta_2},$$

where

$$A_i = \Big\{(k_i, \alpha_i, \beta_i)\colon\ k_{i1} n_1 + k_{i2} n_2 + \sum_n (\alpha_{in} - \beta_{in}) n = 0\Big\},$$
$$k_i = (k_{i1}, k_{i2}), \qquad \alpha_i = \big(\{\alpha_{in}\}_{n\in\mathbb{Z}}\big), \qquad \beta_i = \big(\{\beta_{in}\}_{n\in\mathbb{Z}}\big),$$

for $i = 1, 2$, respectively. A straightforward calculation yields that

$$\begin{aligned}
\{F, G\} = {} & \sum_{A_1}\sum_{A_2} \Big\langle \frac{\partial F_{k_1\alpha_1\beta_1}(I)}{\partial I}, \mathrm{i}k_2 \Big\rangle G_{k_2\alpha_2\beta_2}(I) e^{\mathrm{i}\langle k_1,\theta\rangle} z^{\alpha_1} \bar{z}^{\beta_1} e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_2} \bar{z}^{\beta_2} \\
& - \sum_{A_1}\sum_{A_2} \Big\langle \mathrm{i}k_1, \frac{\partial G_{k_2\alpha_2\beta_2}(I)}{\partial I} \Big\rangle F_{k_1\alpha_1\beta_1}(I) e^{\mathrm{i}\langle k_1,\theta\rangle} z^{\alpha_1} \bar{z}^{\beta_1} e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_2} \bar{z}^{\beta_2} \\
& + \mathrm{i} \sum_m \sum_{\tilde{A}_1 \cup \tilde{A}_2} F_{k_1\alpha_1\beta_1}(I) G_{k_2\alpha_2\beta_2}(I) e^{\mathrm{i}\langle k_1,\theta\rangle} e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_1 - e_m} \bar{z}^{\beta_1} z^{\alpha_2} \bar{z}^{\beta_2 - e_m} \\
& - \mathrm{i} \sum_m \sum_{\tilde{A}_1 \cup \tilde{A}_2} F_{k_1\alpha_1\beta_1}(I) G_{k_2\alpha_2\beta_2}(I) e^{\mathrm{i}\langle k_1,\theta\rangle} e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_1} \bar{z}^{\beta_1 - e_m} z^{\alpha_2 - e_m} \bar{z}^{\beta_2},
\end{aligned}$$

where for each $i = 1, 2$, $m \in \mathbb{Z}$, e_m is the multi-index whose mth component is 1 and other components are all 0,

$$\tilde{A}_i = \tilde{A}_i(m) = \big\{(k_i, \alpha_i, \beta_i)\colon\ k_{i1} n_1 + k_{i2} n_2 + (\alpha_{im} - \beta_{im}) m + \sum_{n\in\mathbb{Z}\setminus\{m\}} (\alpha_{in} - \beta_{in}) n = 0\big\},$$
$$k_i = (k_{i1}, k_{i2}), \qquad \alpha_i = \big(\{\alpha_{in}\}_{n\in\mathbb{Z}\setminus\{m\}}\big), \qquad \beta_i = \big(\{\beta_{in}\}_{n\in\mathbb{Z}\setminus\{m\}}\big).$$

Since all terms above have compact forms with respect to n_1, n_2, so does $\{F, G\}$. $\square$

All the above notions and properties on weighted norms and compact forms can be similarly extended to the case $(\theta, I) \in T^N \times R^N$ for any $N \geqslant 2$. In particular, compact forms can be similarly defined with respect to any N integers $n_1, n_2, \ldots, n_N$. We treated the case $N = 2$ only because our main theorem will be proved for an admissible 2-index.

3. Normal form

Using the Hamiltonian formulation, we rewrite Eq. (1.7) with the periodic boundary condition (1.8) as the Hamiltonian system

$$u_t = \mathrm{i}\frac{\partial H}{\partial \bar{u}}, \quad \text{where}$$
$$H = \int_0^{2\pi} |u_x|^2 + m|u|^2\,dx + \frac{1}{3}\int_0^{2\pi} |u|^6\,dx.$$

Note that the operator $A = -\partial_{xx} + m$ with the periodic boundary condition has an orthonormal basis $\{\phi_n(x) = \sqrt{(1/2\pi)}e^{\mathrm{i}nx}\}$ and corresponding eigenvalues

$$\mu_n = n^2 + m.$$

Let

$$u(x,t) = \sum_{n\in\mathbb{Z}} q_n(t)\phi_n(x).$$

Then associated with the symplectic structure $\mathrm{i}\sum_n dq_n \wedge d\bar{q}_n$, $\{q_n\}_{n\in\mathbb{Z}}$ satisfies the Hamiltonian equations

$$\begin{aligned} &\dot{q}_n = \mathrm{i}\frac{\partial H}{\partial \bar{q}_n}, \quad n \in \mathbb{Z}, \quad \text{where} \\ &H = \Lambda + G \quad \text{with} \\ &\Lambda = \sum_{n\in\mathbb{Z}} \mu_n |q_n|^2, \qquad G = \frac{1}{3}\int_0^{2\pi}\left|\sum_{n\in\mathbb{Z}} q_n\phi_n\right|^6 dx. \end{aligned} \tag{3.1}$$

Lemma 3.1. *The gradient $G_{\bar{q}}$ is a real analytic map from a neighborhood of the origin of ℓ^ρ into ℓ^ρ, with*

$$\|G_{\bar{q}}\|_\rho = O\big(\|q\|_\rho^5\big).$$

Proof. Let $G_{\bar{q}} = (\{\frac{\partial G}{\partial \bar{q}_n}\})$, where

$$\frac{\partial G}{\partial \bar{q}_n} = \int_0^{2\pi} |u|^4 u\bar{\phi}_n\,dx$$

for $u=\sum_{n\in\mathbb{Z}} q_n\phi_n$, i.e., $\frac{\partial G}{\partial \bar{q}_n}=(|u|^4u)_n$. Hence,

$$\|G_{\bar{q}}\|_\rho=\big\||u|^4u\big\|_\rho\leqslant\|u\|_\rho^5=\|q\|_\rho^5.$$

The analyticity of $G_{\bar{q}}$ follows from the regularity of its components and its local boundedness [28, Appendix A]. □

Note that

$$G=\frac{1}{3}\sum_{i,j,k,l,m,n} G_{ijklmn}q_iq_jq_k\bar{q}_l\bar{q}_m\bar{q}_n,\quad\text{where}$$
$$G_{ijklmn}=\int_0^{2\pi}\phi_i\phi_j\phi_k\bar{\phi}_l\bar{\phi}_m\bar{\phi}_n\,dx.$$

We immediately have the following.

Lemma 3.2. *G has a compact form, i.e.,*

$$G_{ijklmn}=0\quad\textit{whenever } i+j+k-l-m-n\neq 0.$$

Moreover,

$$G_{ijkijk}=\frac{1}{4\pi^2}.$$

To transform the Hamiltonian (3.1) into a partial Birkhoff normal form, we fix $\{n_1,n_2\}\in\mathcal{J}$, i.e., n_1-n_2 is odd, and consider the index sets Δ_*, $*=0,1,2,3$, as follows. For each $*=0,1,2$, Δ_* is the set of indices (i,j,k,l,m,n) which have exactly $*$ components not in $\{n_1,n_2\}$. Δ_3 is the set of the indices (i,j,k,l,m,n) which have at least three components not in $\{n_1,n_2\}$. We also consider the resonance sets $\mathcal{N}=\{(i,j,k,i,j,k)\}\cap\Delta_0$, $\mathcal{M}=\{(i,j,k,i,j,k)\}\cap\Delta_2$.

Lemma 3.3. *Let $(i,j,k,l,m,n)\in(\Delta_0\setminus\mathcal{N})\cup\Delta_1\cup(\Delta_2\setminus\mathcal{M})$. If*

$$i+j+k-l-m-n=0, \tag{3.2}$$

then

$$\mu_i+\mu_j+\mu_k-\mu_l-\mu_m-\mu_n=i^2+j^2+k^2-l^2-m^2-n^2\neq 0. \tag{3.3}$$

Proof. We assume without loss of generality that $\{i,j,k\}\cap\{l,m,n\}=\emptyset$.

In the case that $(i,j,k,l,m,n)\in\Delta_0\setminus\mathcal{N}$, if (3.3) fails, then $i=j=k=n_1$ and $l=m=n=n_2$, or vice versa. This is a contradiction to (3.2).

In the case that $(i,j,k,l,m,n)\in\Delta_1$, we have $i=j=k=n_1$, $l=m=n_2$ and $n\neq n_1,n_2$, or vice versa. Without loss of generality, we assume that $i=j=k=n_1$, $l=m=n_2$ and $n\neq n_1,n_2$. Since

$$i+j+k-l-m-n=3n_1-2n_2-n=0,$$

we have $n = 3n_1 - 2n_2$. Consequently,

$$\begin{aligned}\mu_i + \mu_j + \mu_k - \mu_l - \mu_m - \mu_n &= 3n_1^2 - 2n_2^2 - (3n_1 - 2n_2)^2 \\ &= -6n_1^2 + 12n_1n_2 - 6n_2^2 = -6(n_1 - n_2)^2 \neq 0.\end{aligned}$$

In the case that $(i, j, k, l, m, n) \in \Delta_2 \setminus \mathcal{M}$, we either have (a) $i = j = k = n_1$, $l = n_2$, and $m, n \neq n_1, n_2$; or (b) $i = j = n_1$, $l = m = n_2$, and $k, n \neq n_1, n_2$.

In case (a), since

$$i + j + k - l - m - n = 3n_1 - n_2 - m - n = 0,$$

we have $n = 3n_1 - n_2 - m$. Consequently,

$$\begin{aligned}\mu_i + \mu_j + \mu_k - \mu_l - \mu_m - \mu_n &= 3n_1^2 - n_2^2 - m^2 - (3n_1 - n_2 - m)^2 \\ &= -6n_1^2 - 2n_2^2 + 6n_1n_2 + 6n_1m - 2n_2m - 2m^2 \\ &= -2\left(m - \frac{3n_1 - n_2}{2}\right)^2 - \frac{3}{2}(n_1 - n_2)^2 \neq 0.\end{aligned}$$

In case (b), since

$$i + j + k - l - m - n = 2n_1 - 2n_2 + k - n = 0,$$

we have $n = 2n_1 - 2n_2 + k$. Consequently,

$$\begin{aligned}\mu_i + \mu_j + \mu_k - \mu_l - \mu_m - \mu_n &= 2n_1^2 - 2n_2^2 + k^2 - (2n_1 - 2n_2 + k)^2 \\ &= -2n_1^2 - 6n_2^2 - 4n_1k + 8n_1n_2 + 4n_2k \\ &= 4k(n_2 - n_1) - 2(3n_2 - n_1)(n_2 - n_1) \\ &= 2(n_2 - n_1)\big(2k - (3n_2 - n_1)\big) \neq 0,\end{aligned}$$

as $3n_2 - n_1$ is odd. $\square$

Using these index sets, it is clear that G has the following decomposition

$$\begin{aligned}&G = G^0 + G^1 + G^2 + \widehat{G}, \quad \text{where} \\ &G^* = \frac{1}{3} \sum_{\substack{i+j+k-l-m-n=0,\\ (i,j,k,l,m,n)\in\Delta_*}} G_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n, \quad * = 0, 1, 2, \quad \text{and} \\ &\widehat{G} = \frac{1}{3} \sum_{\substack{i+j+k-l-m-n=0,\\ (i,j,k,l,m,n)\in\Delta_3}} G_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n.\end{aligned}$$

Proposition 3.1. *Given $\{n_1, n_2\} \in \mathcal{J}$, there exists a real analytic, symplectic change of coordinates Γ in a neighborhood of the origin of ℓ^ρ which transforms the Hamiltonian* (3.1) *into the partial Birkhoff normal form*

$$H \circ \Gamma = \Lambda + \overline{G} + \widehat{G} + K \tag{3.4}$$

such that the corresponding Hamiltonian vector fields $X_{\overline{G}}$, $X_{\widehat{G}}$ and X_K are real analytic in a neighborhood of the origin in ℓ^ρ, where

$$\begin{aligned}
\overline{G} &= \frac{1}{12\pi^2}\big(|q_{n_1}|^6 + |q_{n_2}|^6\big) + \frac{9}{12\pi^2}\big(|q_{n_1}|^4|q_{n_2}|^2 + |q_{n_2}|^4|q_{n_1}|^2\big) \\
&\quad + \frac{9}{12\pi^2}\sum_{n\neq n_1,n_2}\big(|q_{n_1}|^4 + |q_{n_2}|^4 + 4|q_{n_1}|^2|q_{n_2}|^2\big)|q_n|^2, \\
\widehat{G} &= \frac{1}{3}\sum_{\substack{i+j+k-l-m-n=0,\\ (i,j,k,l,m,n)\in\Delta_3}} G_{ijklmn}q_iq_jq_k\bar{q}_l\bar{q}_m\bar{q}_n, \\
|K| &= O\big(\|q\|_\rho^{10}\big).
\end{aligned}$$

Moreover, $K(q,\bar{q})$ has a compact form.

Proof. We want to construct the symplectic transformation Γ as the time-1 map of the Hamiltonian flow Φ_F^t associated with a Hamiltonian

$$F = \frac{1}{3}\sum_{i,j,k,l,m,n} F_{ijklmn}q_iq_jq_k\bar{q}_l\bar{q}_m\bar{q}_n$$

which eliminates all terms in G^0, G^1, G^2 that are not of the form $|q_i|^2|q_j|^2|q_k|^2$. To do so, let

$$\mathrm{i}F_{ijklmn} = \begin{cases}
\frac{G_{ijklmn}}{\mu_i+\mu_j+\mu_k-\mu_l-\mu_m-\mu_n}, & i+j+k-l-m-n=0,\ (i,j,k,l,m,n)\in\Delta_0\setminus\mathcal{N}, \\
\frac{G_{ijklmn}}{\mu_i+\mu_j+\mu_k-\mu_l-\mu_m-\mu_n}, & i+j+k-l-m-n=0,\ (i,j,k,l,m,n)\in\Delta_1, \\
\frac{G_{ijklmn}}{\mu_i+\mu_j+\mu_k-\mu_l-\mu_m-\mu_n}, & i+j+k-l-m-n=0,\ (i,j,k,l,m,n)\in\Delta_2\setminus\mathcal{M}, \\
0, & \text{otherwise.}
\end{cases}$$

It follows from Lemma 3.3 that F is well defined.

To show the analyticity of the transformation, we note that there exists a constant $c>0$ such that for each $n\in\mathbb{Z}$

$$\left|\frac{\partial F}{\partial \bar{q}_n}\right| \leqslant c\sum_{i+j+k-l-m=n}|q_i||q_j||q_k||\bar{q}_l||\bar{q}_m| = c\big(|q|*|q|*|q|*|\bar{q}|*|\bar{q}|\big)_n,$$

where $|q| = (\{|q_j|\})$.

Hence

$$\|F_{\bar{q}}\|_\rho \leqslant c\big\||q|*|q|*|q|*|\bar{q}|*|\bar{q}|\big\|_\rho \leqslant c\|q\|_\rho^5.$$

The analyticity of $F_{\bar{q}}$ then follows from that of each of its component and its local boundedness (see [28, Appendix A]).

Let $\Gamma = \Phi_F^1$. Then

$$\begin{aligned} H \circ \Gamma &= \Lambda + \overline{G} + \widehat{G} + K, \quad \text{where} \\ \overline{G} &= \left(G^0 + G^1 + G^2 + \{\Lambda, F\}\right) \\ &= \frac{1}{12\pi^2}\left(|q_{n_1}|^6 + |q_{n_2}|^6\right) + \frac{9}{12\pi^2}\left(|q_{n_1}|^4|q_{n_2}|^2 + |q_{n_2}|^4|q_{n_1}|^2\right) \\ &\quad + \frac{9}{12\pi^2}\sum_{n\neq n_1,n_2}\left(|q_{n_1}|^4 + |q_{n_2}|^4 + 4|q_{n_1}|^2|q_{n_2}|^2\right)|q_n|^2, \\ \widehat{G} &= \frac{1}{3}\sum_{\substack{i+j+k-l-m-n=0,\\ (i,j,k,l,m,n)\in\Delta_3}} G_{ijklmn}q_iq_jq_k\bar{q}_l\bar{q}_m\bar{q}_n, \\ K &= \{G, F\} + \frac{1}{2!}\left\{\{\Lambda, F\}, F\right\} + \frac{1}{2!}\left\{\{G, F\}, F\right\} \\ &\quad + \cdots + \frac{1}{n!}\{\ldots\{\Lambda, \underbrace{F\}\ldots, F}_{n}\} + \frac{1}{n!}\{\ldots\{G, \underbrace{F\}\ldots, F}_{n}\} + \cdots. \end{aligned}$$

It is clear that $|K| = O(\|q\|_\rho^{10})$. To show that K has a compact form, we note that since G has a compact form, so does F. Hence by Lemma 2.4, $\{G, F\}$ has a compact form. Note that Λ is already in a compact form. Repeated applications of Lemma 2.4 show that all terms in K have compact forms, so does K. □

Remark 3.1. (1) The above partial Birkhoff normal form reduction does not hold if the 7th order nonlinearity $|u|^6u$ is considered, simply because Lemma 3.3 is not valid. To see this, let $i = j = k = n_1$, $m = n = p = n_2$, $l = 2n_2 - n_1$, $q = 2n_1 - n_2$. Then $i + j + k + l - m - n - p - q = 0$ and $(i, j, k, l, m, n, p, q) \in \Delta_2 \setminus \mathcal{M}$. But we also have $i^2 + j^2 + k^2 + l^2 - m^2 - n^2 - p^2 - q^2 = 0$. Thus when transforming the Hamiltonian associated with the 7th order nonlinearity into a partial Birkhoff normal form, many non-integrable terms have to be included into the normal form. This makes the normal form very complicated.

(2) If we look for periodic solutions for a nonlinear Schrödinger equation of form (1.9) with a higher order nonlinearity, then a normal form reduction similar to the above can be carried out regardless the order of the nonlinearity. Therefore, it seems that there are essential difference between finding quasi-periodic solutions and periodic ones in problem of this nature.

(3) As we will show later, the compact form of K leads to a compact form of the perturbation at each KAM step, in particular, no term of the form $|q_{n_1}|^2 \cdots |q_{n_d}|^2 q_n\bar{q}_{-n}$ will be involved in the perturbation at each KAM step. This property enables us to actually consider a finite small divisor problem at each KAM step, hence to make the measure estimate work.

Next, we introduce action-angle-normal variables and parameters to the partial Birkhoff normal form (3.4). Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}_+^2$ be a parameter and $(I, \theta) \in \mathbb{R}^2 \times \mathbb{T}^2$ be the standard action-angle variables in the $(q_{n_1}, q_{n_2}, \bar{q}_{n_1}, \bar{q}_{n_2})$-space around ξ. Then

$$q_{n_i}\bar{q}_{n_i} = I_i + \xi_i, \quad i = 1, 2.$$

Also let $z_n = q_n$ for $n \neq n_1, n_2$. Then the partial Birkhoff normal form (3.4) becomes

$$\widetilde{H} = \langle \tilde{\omega}(\xi), I \rangle + \sum_n \widetilde{\Omega}_n(\xi) z_n \bar{z}_n + \widetilde{P}(\theta, I, z, \bar{z}, \xi),$$

where $\tilde{\omega}(\xi) = (\tilde{\omega}_1(\xi), \tilde{\omega}_2(\xi))$ with

$$\tilde{\omega}_i(\xi) = n_i^2 + m + \frac{3}{4\pi^2}(\xi_1 + \xi_2)^2 - \frac{2}{4\pi^2}\xi_i^2, \quad i = 1, 2, \quad \text{and}$$

$$\widetilde{\Omega}_n(\xi) = n^2 + m + \frac{6}{4\pi^2}(\xi_1 + \xi_2)^2 - \frac{3}{4\pi^2}\xi_1^2 - \frac{3}{4\pi^2}\xi_2^2, \quad n \neq n_1, n_2,$$

$$\widetilde{P} = K + O\big(|I|^3\big) + O\big(|\xi||I|^2\big) + O\big(|\xi||I|\|z\|_\rho^2\big) + O\big(|I|^2\|z\|_\rho^2\big) + O\big(|\xi|^{\frac{3}{2}}\|z\|_\rho^3\big) + O\big(|\xi|\|z\|_\rho^4\big)$$

with the variables q_{n_1}, q_{n_2} in K expressed in terms of I, θ.

Consider the Taylor–Fourier expansion of $\widetilde{P}$:

$$\widetilde{P} = \sum_{k,\alpha,\beta} \widetilde{P}_{k\alpha\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta.$$

It follows from the compact forms of $\widehat{G}$ and K that $\widetilde{P}$ has a compact form with respect to n_1, n_2, i.e.,

$$\widetilde{P}_{k\alpha\beta}(I) = 0, \quad \text{whenever } k_1 n_1 + k_2 n_2 + \sum_{n \in \mathbb{Z}\setminus\{n_1,n_2\}} (\alpha_n - \beta_n) n \neq 0.$$

This particularly implies that $\widetilde{P}$ contains no terms of the form $z_n \bar{z}_{-n}$. As to be seen below, such a compact form of the perturbation will be preserved under the KAM iteration.

Now, let $\varepsilon > 0$ be sufficiently small. By considering the re-scalings: $\xi_j \to \varepsilon^2 \sqrt{\xi_j}$, $j = 1, 2$, $z \to \varepsilon^2 z$, and $I \to \varepsilon^4 I$, we obtain the rescaled Hamiltonian

$$\begin{aligned} H(I, \theta, z, \bar{z}, \xi) &= \varepsilon^{-8} \tilde{H}\big(\varepsilon^4 I, \theta, \varepsilon^2 z, \varepsilon^2 \bar{z}, \varepsilon^2\sqrt{\xi_1}, \varepsilon^2\sqrt{\xi_2}\big) \\ &= \langle \omega^*(\xi), I \rangle + \sum_n \Omega_n^*(\xi) z_n \bar{z}_n + \varepsilon P^*(I, \theta, z, \bar{z}, \xi, \varepsilon), \end{aligned} \tag{3.5}$$

where $\omega^*(\xi) = (\omega_1^*(\xi), \omega_2^*(\xi))$ with

$$\omega_1^*(\xi) = \varepsilon^{-4}\big(n_1^2 + m\big) + \frac{3}{4\pi^2}\big(\sqrt{\xi_1} + \sqrt{\xi_2}\big)^2 - \frac{2}{4\pi^2}\xi_1,$$

$$\omega_2^*(\xi) = \varepsilon^{-4}\big(n_2^2 + m\big) + \frac{3}{4\pi^2}\big(\sqrt{\xi_1} + \sqrt{\xi_2}\big)^2 - \frac{2}{4\pi^2}\xi_2, \quad \text{and}$$

$$\Omega_n^*(\xi) = \varepsilon^{-4}\big(n^2 + m\big) + \frac{6}{4\pi^2}\big(\sqrt{\xi_1} + \sqrt{\xi_2}\big)^2 - \frac{3}{4\pi^2}\xi_1 - \frac{3}{4\pi^2}\xi_2, \quad n \neq n_1, n_2,$$

$$P^* = \varepsilon^{-7}\widetilde{P}\big(\varepsilon^4 I, \theta, \varepsilon^2 z, \varepsilon^2 \bar{z}, \varepsilon^2\sqrt{\xi_1}, \varepsilon^2\sqrt{\xi_2}\big).$$

Note that the nonlinear Schrödinger equation (1.7) has another conserved quantity $\int_0^{2\pi} |u|^2\,dx = \sum_n |q_n|^2 = \delta$, i.e.,

$$|q_{n_1}|^2 + |q_{n_2}|^2 + \sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |q_n|^2 = \delta.$$

The above rescalings yields that

$$\varepsilon^2 I_1 + \sqrt{\xi_1} + \varepsilon^2 I_2 + \sqrt{\xi_2} + \varepsilon^2 \sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |z_n|^2 = \delta, \quad \text{i.e.,}$$

$$\sqrt{\xi_1} + \sqrt{\xi_2} = \delta - \varepsilon^2\Big(I_1 + I_2 + \sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |z_n|^2\Big) = \delta + O\big(\varepsilon^2\big).$$

Let $\omega(\xi) = (\omega_1(\xi), \omega_2(\xi))$, $\Omega_n(\xi) = \Omega_n(\xi)$, $n \neq n_1, n_2$, where

$$\begin{aligned}
\omega_1(\xi) &= \varepsilon^{-4}\big(n_1^2 + m\big) + \frac{3}{4\pi^2}\delta^2 - \frac{2}{4\pi^2}\xi_1,\\
\omega_2(\xi) &= \varepsilon^{-4}\big(n_2^2 + m\big) + \frac{3}{4\pi^2}\delta^2 - \frac{2}{4\pi^2}\xi_2,\\
\Omega_n(\xi) &= \varepsilon^{-4}\big(n^2 + m\big) + \frac{6}{4\pi^2}\delta^2 - \frac{3}{4\pi^2}\xi_1 - \frac{3}{4\pi^2}\xi_2, \quad n \neq n_1, n_2.
\end{aligned}$$

We can rewrite (3.5) as

$$H(I,\theta,z,\bar z,\xi) = \big\langle \omega(\xi), I\big\rangle + \sum_n \Omega_n(\xi) z_n \bar z_n + P(I,\theta,z,\bar z,\xi,\varepsilon), \quad \text{where} \tag{3.6}$$

$$\begin{aligned}
P = \varepsilon P^* &- \frac{3}{2\pi^2}\varepsilon^2\delta\Big(I_1 + I_2 + \sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |z_n|^2\Big)\Big(I_1 + I_2 + 2\sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |z_n|^2\Big)\\
&+ \frac{3}{4\pi^2}\varepsilon^4\Big(I_1 + I_2 + \sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |z_n|^2\Big)^2\Big(I_1 + I_2 + 2\sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} |z_n|^2\Big).
\end{aligned}$$

Let $\mathcal{O}$ be a neighborhood of the origin in $\mathbb{R}_+^2$, $\gamma = \varepsilon^{\frac{1}{8}}$, and $\tau > 4$ be fixed. We consider the set $\mathcal{O}_0$ consisting of all $\xi \in \mathcal{O}$ such that

$$\begin{aligned}
&\big|\big\langle k, \omega(\xi)\big\rangle\big| \geqslant \frac{\gamma}{|k|^\tau}, && k \neq 0,\\
&\big|\big\langle k, \omega(\xi)\big\rangle + \Omega_n(\xi)\big| \geqslant \frac{\gamma}{|k|^\tau}, && k \neq 0,\ |n| = |k_1 n_1 + k_2 n_2|,\\
&\big|\big\langle k, \omega(\xi)\big\rangle + \Omega_n(\xi) + \Omega_m(\xi)\big| \geqslant \frac{\gamma}{|k|^\tau}, && k \neq 0,\ |n+m| = |k_1 n_1 + k_2 n_2|,\\
&\big|\big\langle k, \omega(\xi)\big\rangle + \Omega_n(\xi) - \Omega_m(\xi)\big| \geqslant \frac{\gamma}{|k|^\tau}, && k \neq 0,\ |n-m| = |k_1 n_1 + k_2 n_2|.
\end{aligned}$$

Proposition 3.2. $\text{meas}(\mathcal{O} \setminus \mathcal{O}_0) = O(\gamma)$.

Proof. We first consider the following nonresonance conditions:

$$\begin{aligned}
&\big\langle k, \omega(\xi)\big\rangle \not\equiv 0, \quad k \neq 0,\\
&\big\langle k, \omega(\xi)\big\rangle + \Omega_n(\xi) \not\equiv 0, \quad \forall k \in \mathbb{Z},\\
&\big\langle k, \omega(\xi)\big\rangle + \Omega_n(\xi) + \Omega_m(\xi) \not\equiv 0, \quad \forall k \in \mathbb{Z},\\
&\big\langle k, \omega(\xi)\big\rangle + \Omega_n(\xi) - \Omega_m(\xi) \not\equiv 0, \quad \forall k \in \mathbb{Z},\ |n| \neq |m|.
\end{aligned}$$

Rewrite $\omega(\xi)$, $\Omega(\xi) = (\{\Omega_n(\xi)\}_{n\in\mathbb{Z}\setminus\{n_1,n_2\}})$ as

$$\omega(\xi) = \alpha + A\xi, \qquad \Omega(\xi) = \beta + B\xi,$$

where $\alpha = (\varepsilon^{-4}\mu_{n_1} + \frac{3}{4\pi^2}\delta^2, \varepsilon^{-4}\mu_{n_2} + \frac{3}{4\pi^2}\delta^2)$, $\beta = (\varepsilon^{-4}\mu_n + \frac{6}{4\pi^2}\delta^2)_{n\neq n_1,n_2}$,

$$A = \begin{pmatrix} -\frac{2}{4\pi^2} & 0 \\ 0 & -\frac{2}{4\pi^2} \end{pmatrix}, \qquad B = \begin{pmatrix} -\frac{3}{4\pi^2} & -\frac{3}{4\pi^2} \\ \vdots & \vdots \end{pmatrix}_{n\neq n_1,n_2}.$$

Since $\det(A) = \frac{1}{4\pi^4} \neq 0$, we have that $\langle k, \omega(\xi)\rangle \not\equiv 0$ for $k \neq 0$.

To show the validity of the remaining three non-resonance conditions, we need to check that for all $k \in \mathbb{Z}$ and $1 \leqslant |l| \leqslant 2$, $\langle \alpha, k\rangle + \langle \beta, l\rangle$ and $Ak + B^T l$ do not vanish simultaneously. Suppose $Ak + B^T l = 0$ for some $k \in \mathbb{Z}$ and $1 \leqslant |l| \leqslant 2$. We let d be the sum of at most two nonzero components of l. Then

$$-\frac{2}{4\pi^2}k_i - \frac{3}{4\pi^2}d = 0, \quad i = 1, 2, \quad \text{i.e.,} \quad k_i = -\frac{3}{2}d, \quad i = 1, 2,$$

which have the following integer solutions:

$$\begin{aligned}
&\text{(i)} \ \ d = 0, \quad k_1 = k_2 = 0;\\
&\text{(ii)} \ \ d = \pm 2, \quad k_1 = k_2 = \mp 3.
\end{aligned}$$

In the case (i), $k = 0$, l has one "1" and one "-1," but then $\langle \alpha, k\rangle + \langle \beta, l\rangle = \varepsilon^{-4}(\mu_n - \mu_m) = \varepsilon^{-4}(n^2 - m^2) \neq 0$. In the case (ii), $\mp\frac{18}{4\pi^2}\delta^2 \pm \frac{12}{4\pi^2}\delta^2 \neq 0$, hence $\langle \alpha, k\rangle + \langle \beta, l\rangle \neq 0$. Similarly, if $\langle \alpha, k\rangle + \langle \beta, l\rangle = 0$ for some $k \in \mathbb{Z}$ and $1 \leqslant |l| \leqslant 2$, then one can show that $Ak + B^T l \neq 0$.

The validity of all these nonresonance conditions implies that these functions are nontrivial affine functions of $\xi \in \mathcal{O}$. The desired measure estimate of $\text{meas}(\mathcal{O} \setminus \mathcal{O}_0)$ then follows from the same argument as that in Section 5. □

4. KAM step

In what follows, we will perform KAM iterations to (3.6) which involves infinite many successive steps, called KAM steps, of iterations, to eliminate lower order θ-dependent terms in P. Each KAM step will make the perturbation smaller than the previous one at a cost of excluding a small measure set of parameters. At the end, the KAM iterations will be convergent and the measure of the total excluding set will remain to be small.

To begin with the KAM iteration, we fix $r, s, \rho > 0$ and restrict the Hamiltonian (3.6) to the domain $D(r,s)$ and restrict the parameter to the set $\mathcal{O}_0$. Initially, we set $e_0 = 0$, $\omega_0 = \omega$, $\Omega_n^0 = \Omega_n$, $P_0 = P$, $r_0 = r$, $s_0 = s$, $\gamma_0 = \gamma$, and

$$N_0 = e_0 + \langle \omega_0(\xi), I \rangle + \sum_n \Omega_n^0(\xi) z_n \bar{z}_n, \qquad H_0 = N_0 + P_0.$$

Hence, H_0 is real analytic on $D(r_0, s_0)$ and also depend on $\xi \in \mathcal{O}_0$ Whitney smoothly. It is clear that there is a constant $c_0 > 0$ such that

$$\|X_{P_0}\|_{D(r_0,s_0),\mathcal{O}_0} \leqslant c_0 \varepsilon \equiv \varepsilon_0.$$

We recall that

$$\begin{aligned}\mathcal{O}_0 = \Bigg\{ \xi\colon\ & \big|\langle k, \omega_0(\xi)\rangle\big| \geqslant \frac{\gamma_0}{|k|^\tau},\ k \neq 0,\ \big|\langle k, \omega_0(\xi)\rangle + \Omega_n^0(\xi)\big| \geqslant \frac{\gamma_0}{|k|^\tau},\ k \neq 0,\ |n| = |k_1 n_1 + k_2 n_2|, \\ & \big|\langle k, \omega_0(\xi)\rangle + \Omega_n^0(\xi) + \Omega_m^0(\xi)\big| \geqslant \frac{\gamma_0}{|k|^\tau},\ k \neq 0,\ |n+m| = |k_1 n_1 + k_2 n_2|, \\ & \big|\langle k, \omega_0(\xi)\rangle + \Omega_n^0(\xi) - \Omega_m^0(\xi)\big| \geqslant \frac{\gamma_0}{|k|^\tau},\ k \neq 0,\ |n-m| = |k_1 n_1 + k_2 n_2| \Bigg\},\end{aligned}$$

and $P_0 = \sum_{k,\alpha,\beta} P^0_{k\alpha\beta}(I) e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta$ has a compact form with respect to n_1, n_2, i.e.,

$$P^0_{k\alpha\beta} = 0, \quad \text{whenever } k_1 n_1 + k_2 n_2 + \sum_{n \in \mathbb{Z} \setminus \{n_1, n_2\}} (\alpha_n - \beta_n) n \neq 0.$$

Suppose that after a νth KAM step, we arrive at a Hamiltonian

$$H = H_\nu = N + P(\theta, I, z, \bar{z}), \qquad N = N_\nu = e(\xi) + \langle \omega(\xi), I \rangle + \sum_n \Omega_n(\xi) z_n \bar{z}_n,$$

which is real analytic in $(\theta, I, z, \bar{z}) \in D = D_\nu = D(r,s)$ for some $r = r_\nu \leqslant r_0$, $s = s_\nu \leqslant s_0$, and depends on $\xi \in \mathcal{O} = \mathcal{O}_\nu \subset \mathcal{O}_0$ Whitney smoothly, where

$$\begin{aligned}\mathcal{O} = \Bigg\{ \xi\colon\ & \big|\langle k, \omega(\xi)\rangle\big| \geqslant \frac{\gamma}{|k|^\tau},\ k \neq 0,\ \big|\langle k, \omega(\xi)\rangle + \Omega_n(\xi)\big| \geqslant \frac{\gamma}{|k|^\tau},\ k \neq 0,\ |n| = |k_1 n_1 + k_2 n_2|, \\ & \big|\langle k, \omega(\xi)\rangle + \Omega_n(\xi) + \Omega_m(\xi)\big| \geqslant \frac{\gamma}{|k|^\tau},\ k \neq 0,\ |n+m| = |k_1 n_1 + k_2 n_2|, \\ & \big|\langle k, \omega(\xi)\rangle + \Omega_n(\xi) - \Omega_m(\xi)\big| \geqslant \frac{\gamma}{|k|^\tau},\ k \neq 0,\ |n-m| = |k_1 n_1 + k_2 n_2| \Bigg\},\end{aligned}$$

for some $\gamma = \gamma_\nu \leqslant \gamma_0$. We also assume that

$$\|X_P\|_{D,\mathcal{O}} \leqslant \varepsilon,$$

for some $0 < \varepsilon = \varepsilon_\nu \leqslant \varepsilon_0$, and that $P = \sum_{k,\alpha,\beta} P_{k\alpha\beta}(I)e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta$ has a compact form with respect to n_1, n_2, i.e.,

$$P_{k\alpha\beta} = 0, \quad \text{whenever } k_1 n_1 + k_2 n_2 + \sum_{n\in\mathbb{Z}\setminus\{n_1,n_2\}} (\alpha_n - \beta_n)n \neq 0.$$

We will construct a symplectic transformation $\Phi = \Phi_\nu$, which, in smaller frequency and phase domains, carries the above Hamiltonian into the next KAM cycle. Thereafter, quantities (domains, normal form, perturbation, etc.) in the next KAM cycle will be simply indexed by $+$ $(= \nu + 1)$. Also, all constants $c_1 - c_5$ below are positive and independent of the iteration process.

4.1. Truncation

We expand P into the Fourier–Taylor series

$$P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle} I^l z^\alpha \bar{z}^\beta,$$

where $k \in \mathbb{Z}^2, l \in \mathbb{N}^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$, $\beta = (\beta_1, \ldots, \beta_n, \ldots)$, $\alpha_n, \beta_n \in \mathbb{N}$, are multi-indices with finitely many non-vanishing components. We denote by 0 the multi-index whose components are all zeros and by e_n the multi-index whose nth component is 1 and other components are all zeros.

Let R be the following truncation of P:

$$\begin{aligned} R(\theta, I, z, \bar{z}) &= \sum_{k,|l|\leqslant 1} P_{kl00} e^{\mathrm{i}\langle k,\theta\rangle} I^l \\ &\quad + \sum_{k,n} \big(P_n^{k10} z_n + P_n^{k01} \bar{z}_n\big) e^{\mathrm{i}\langle k,\theta\rangle} \\ &\quad + \sum_{k,n,m} \big(P_{nm}^{k20} z_n z_m + P_{nm}^{k11} z_n \bar{z}_m + P_{nm}^{k02} \bar{z}_n \bar{z}_m\big) e^{\mathrm{i}\langle k,\theta\rangle}, \end{aligned}$$

where $P_n^{k10} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = 0$; $P_n^{k01} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n$; $P_{nm}^{k20} = P_{kl\alpha\beta}$ with $\alpha = e_n + e_m, \beta = 0$; $P_{nm}^{k11} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = e_m$; $P_{nm}^{k02} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n + e_m$.

Since P has a compact form with respect to n_1, n_2,

$$\begin{aligned} P_{kl00} &= 0, \quad \text{if } k_1 n_1 + k_2 n_2 \neq 0, \\ P_n^{k10} &= 0, \quad \text{if } k_1 n_1 + k_2 n_2 + n \neq 0, \\ P_n^{k01} &= 0, \quad \text{if } k_1 n_1 + k_2 n_2 - n \neq 0, \\ P_{nm}^{k20} &= 0, \quad \text{if } k_1 n_1 + k_2 n_2 + n + m \neq 0, \\ P_{nm}^{k11} &= 0, \quad \text{if } k_1 n_1 + k_2 n_2 + n - m \neq 0, \\ P_{nm}^{k02} &= 0, \quad \text{if } k_1 n_1 + k_2 n_2 - n - m \neq 0. \end{aligned}$$

In particular $P_{nm}^{k11}=0$ if $|k|=0$ and $n\neq m$.

By definition of the weighted-norms, we clearly have

$$\|X_R\|_{D(r,s),\mathcal{O}}\leqslant\|X_P\|_{D(r,s),\mathcal{O}}\leqslant\varepsilon.$$

Let $\eta=\varepsilon^{\frac{1}{3}}$. It follows from Cauchy estimate that

$$\|X_{(P-R)}\|_{D(r,\eta s)}\leqslant c_1\eta\varepsilon.$$

4.2. The homological equation

Let $r_+=\frac{r}{2}+\frac{r_0}{4}$. We now look for a real analytic function F, defined in the smaller domain $D(r_+,s)$ such that the time-1 map $\Phi=\Phi_F^1: D(r_+,s)\to D$ of the Hamiltonian flow Φ_F^t associated with F transforms H into the Hamiltonian H_+ in the next KAM cycle. Let F have the form

$$\begin{aligned}F(\theta,I,z,\bar z)&=F_0+F_1+F_2\\&\equiv\sum_{k\neq0,|l|\leqslant1}F_{kl00}e^{\mathrm{i}\langle k,\theta\rangle}I^l+\sum_{k,n}\big(F_n^{k10}z_n+F_n^{k01}\bar z_n\big)e^{\mathrm{i}\langle k,\theta\rangle}\\&\quad+\sum_{k,n,m}\big(F_{nm}^{k20}z_nz_m+F_{nm}^{k02}\bar z_n\bar z_m\big)e^{\mathrm{i}\langle k,\theta\rangle}+\sum_{|k|+||n|-|m||\neq0}F_{nm}^{k11}z_n\bar z_me^{\mathrm{i}\langle k,\theta\rangle}\end{aligned}$$

which satisfies

$$\begin{aligned}F_{kl00}&=0,\quad\text{if }k_1n_1+k_2n_2\neq0,\\F_n^{k10}&=0,\quad\text{if }k_1n_1+k_2n_2+n\neq0,\\F_n^{k01}&=0,\quad\text{if }k_1n_1+k_2n_2-n\neq0,\\F_{nm}^{k20}&=0,\quad\text{if }k_1n_1+k_2n_2+n+m\neq0,\\F_{nm}^{k11}&=0,\quad\text{if }k_1n_1+k_2n_2+n-m\neq0,\\F_{nm}^{k02}&=0,\quad\text{if }k_1n_1+k_2n_2-n-m\neq0,\end{aligned}$$

and the homological equation

$$\{N,F\}+R-P_{0000}-\langle\omega',I\rangle-\sum_nP_{nn}^{011}z_n\bar z_n=0,\quad\text{where}\tag{4.1}$$

$$\omega'=\int_{\mathbb{T}^2}\frac{\partial P}{\partial I}\,d\theta\Big|_{z=\bar z=0,I=0}.$$

By comparing coefficients, it is easy to see that the homological equation (4.1) is equivalent to

$$\langle k,\omega\rangle F_{kl00}=\mathrm{i}P_{kl00},\quad k\neq 0,\ |l|\leqslant 1,$$
$$\big(\langle k,\omega\rangle-\Omega_n\big)F_n^{k10}=\mathrm{i}P_n^{k10},\qquad \big(\langle k,\omega\rangle+\Omega_n\big)F_n^{k01}=\mathrm{i}P_n^{k01},$$
$$\big(\langle k,\omega\rangle-\Omega_n-\Omega_m\big)F_{nm}^{k20}=\mathrm{i}P_{nm}^{k20},$$
$$\big(\langle k,\omega\rangle-\Omega_n+\Omega_m\big)F_{nm}^{k11}=\mathrm{i}P_{nm}^{k11},\quad |k|+\big||n|-|m|\big|\neq 0,$$
$$\big(\langle k,\omega\rangle+\Omega_n+\Omega_m\big)F_{nm}^{k02}=\mathrm{i}P_{nm}^{k02}.$$

Hence the homological equation (4.1) is uniquely solvable on $\mathcal{O}$ to yield the function F which is real analytic in $(\theta, I, z, \bar{z})$ and Whitney smooth in $\omega\in\mathcal{O}$.

The following two lemmas follow from standard arguments using Cauchy estimate. We refer the readers to [9,14,19,27] for details.

Lemma 4.1. *Let $D_i=D(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}s)$, $0<i\leqslant 4$. Then there is a constant $c_2>0$ such that*

$$\|X_F\|_{D_3,\mathcal{O}}\leqslant c_2\gamma^{-2}(r-r_+)^{-(2+2\tau)}\varepsilon.$$

Lemma 4.2. *Let $D_{i\eta}=D(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}\eta s)$, $0<i\leqslant 4$. If*

(C1) $\varepsilon<(\frac{1}{c_2}\gamma^2(r-r_+)^{2+2\tau})^{\frac{3}{2}}$,

then

$$\Phi_F^t: D_{2\eta}\to D_{3\eta},\quad |t|\leqslant 1,$$

and moreover,

$$\big\|D\Phi_F^t-I\big\|_{D_{1\eta}}<c_3\gamma^{-2}(r-r_+)^{-(2+2\tau)}\varepsilon.$$

Now let $\Phi=\Phi_F^1$, $s_+=\frac{1}{8}\eta s$, $D_+=D(r_+,s_+)$, and

$$N_+=N+P_{0000}+\langle\omega',I\rangle+\sum_n P_{nn}^{011}z_n\bar{z}_n,$$
$$P_+=\int_0^1(1-t)\big\{\{N,F\},F\big\}\circ\Phi_F^t\,dt+\int_0^1\{R,F\}\circ\Phi_F^t\,dt+(P-R)\circ\Phi_F^1.$$

Then $\Phi: D_+\times\mathcal{O}\to D$, and, by the second order Taylor formula,

$$\begin{aligned}H_+\equiv H\circ\Phi&=(N+R)\circ\Phi_F^1+(P-R)\circ\Phi_F^1\\&=N+\{N,F\}+R+\int_0^1(1-t)\big\{\{N,F\},F\big\}\circ\Phi_F^t\,dt+\int_0^1\{R,F\}\circ\Phi_F^t\,dt+(P-R)\circ\Phi_F^1\end{aligned}$$

$$= N_+ + P_+ + \{N, F\} + R - P_{0000} - \langle \omega', I\rangle - \sum_n P^{011}_{nn} z_n \bar{z}_n$$

$$= N_+ + P_+.$$

4.3. The new Hamiltonian

Below, we show that the new Hamiltonian H_+ enjoys similar properties as H.

Due to the compact form of P with respect to n_1, n_2, z_n and z_{-n} are not coupled in P for any n. This leads to the following simple new normal form

$$N_+ = N + P_{0000} + \langle \omega', I\rangle + \sum_n P^{011}_{nn} z_n \bar{z}_n = e_+ + \langle \omega_+, I\rangle + \sum_n \Omega^+_n z_n \bar{z}_n, \quad \text{where}$$

$$e_+ = e + P_{0000}, \omega_+ = \omega + \big(\{P_{0l00}\}_{|l|=1}\big), \qquad \Omega^+_n = \Omega_n + P^{011}_{nn}.$$

By the assumptions on P, we have that there is a constant $c_4 > 0$ such that

$$|\omega_+ - \omega|_{\mathcal{O}} < c_4\varepsilon, \quad |\Omega^+_n - \Omega_n|_{\mathcal{O}} < c_4\varepsilon.$$

Let $\gamma_+ = \frac{\gamma}{2} + \frac{\gamma_0}{4}$ and $K > 0$ be such that

(C2) $c_4\varepsilon K^{\tau+1} \leqslant \gamma - \gamma_+$.

We have that

$$\big|\langle k, \omega_+\rangle\big| \geqslant \big|\langle k, \omega\rangle\big| - \big|\big\langle k, \big(\{P_{0l00}\}_{|l|=1}\big)\big\rangle\big| \geqslant \frac{\gamma}{|k|^\tau} - c_4\varepsilon|k| \geqslant \frac{\gamma_+}{|k|^\tau},$$

$$\big|\langle k, \omega_+\rangle + \Omega^+_n\big| \geqslant \big|\langle k, \omega\rangle + \Omega_n\big| - \big|\big\langle k, \big(\{P_{0l00}\}_{|l|=1}\big)\big\rangle + P^{011}_{nn}\big| \geqslant \frac{\gamma_+}{|k|^\tau}, \quad |n| = |k_1 n_1 + k_2 n_2|,$$

for all $0 < |k| \leqslant K$. Similarly,

$$\big|\langle k, \omega_+\rangle + \Omega^+_n + \Omega^+_m\big| \geqslant \frac{\gamma_+}{|k|^\tau}, \quad 0 < |k| \leqslant K, |n+m| = |k_1 n_1 + k_2 n_2|,$$

$$\big|\langle k, \omega_+\rangle + \Omega^+_n - \Omega^+_m\big| \geqslant \frac{\gamma_+}{|k|^\tau}, \quad 0 < |k| \leqslant K, |n-m| = |k_1 n_1 + k_2 n_2|.$$

This means that in the next KAM step, small divisor conditions are automatically satisfied for $|k| \leqslant K$. Let

$$\begin{aligned}\mathcal{O}_+ = \bigg\{\xi\colon\ &\big|\big\langle k, \omega_+(\xi)\big\rangle\big| \geqslant \frac{\gamma_+}{|k|^\tau},\ k \neq 0,\ \big|\big\langle k, \omega_+(\xi)\big\rangle + \Omega^+_n(\xi)\big| \geqslant \frac{\gamma_+}{|k|^\tau},\ k \neq 0,\\ &|n| = |k_1 n_1 + k_2 n_2|, \big|\big\langle k, \omega_+(\xi)\big\rangle + \Omega^+_n(\xi) + \Omega^+_m(\xi)\big| \geqslant \frac{\gamma_+}{|k|^\tau},\ k \neq 0,\\ &|n+m| = |k_1 n_1 + k_2 n_2|\ \big|\big\langle k, \omega_+(\xi)\big\rangle + \Omega^+_n(\xi) - \Omega^+_m(\xi)\big| \geqslant \frac{\gamma_+}{|k|^\tau},\ k \neq 0,\\ &|n-m| = |k_1 n_1 + k_2 n_2|\bigg\}.\end{aligned}$$

Then

$$\mathcal{O}_+ = \mathcal{O} \Big\backslash \Big(\bigcup_{|k|>K} \mathcal{R}_k^+(\gamma_+) \Big),$$

where,

$$\mathcal{R}_k^+(\gamma_+) = \Big\{ \xi \in \mathcal{O}\colon \ \big|\langle k, \omega_+\rangle\big| < \frac{\gamma_+}{|k|^\tau}, \text{ or } \big|\langle k, \omega_+\rangle + \Omega_n^+\big| < \frac{\gamma_+}{|k|^\tau}, \ |n| = |k_1 n_1 + k_2 n_2|, \text{ or}$$
$$\big|\langle k, \omega_+\rangle + \Omega_n^+ + \Omega_m^+\big| < \frac{\gamma_+}{|k|^\tau}, \ |n+m| = |k_1 n_1 + k_2 n_2|, \text{ or}$$
$$\big|\langle k, \omega_+\rangle + \Omega_n^+ - \Omega_m^+\big| < \frac{\gamma_+}{|k|^\tau}, \ |n-m| = |k_1 n_1 + k_2 n_2| \Big\}.$$

We rewrite P_+ as

$$P_+ = \int_0^1 \big\{R(t), F\big\} \circ \Phi_F^t \, dt + (P - R) \circ \Phi_F^1,$$

where $R(t) = (1-t)(N_+ - N) + tR$. Hence

$$X_{P_+} = \int_0^1 \big(\Phi_F^t\big)^* X_{\{R(t),F\}} \, dt + \big(\Phi_F^1\big)^* X_{(P-R)}.$$

By Lemma 4.2, if

(C3) $c_3 \gamma^{-2} (r - r_+)^{-(2+2\tau)} \varepsilon \leqslant 1$,

then

$$\big\| D\Phi_F^t \big\|_{D_{1\eta}} \leqslant 1 + \big\| D\Phi_F^t - Id \big\|_{D_{1\eta}} \leqslant 2, \quad |t| \leqslant 1.$$

It follows from Lemma 2.3 that

$$\|X_{\{R(t),F\}}\|_{D_{2\eta}} \leqslant c_5 \gamma^{-2} (r - r_+)^{-(3+2\tau)} \eta^{-2} \varepsilon^2 \quad \text{and} \quad \|X_{(P-R)}\|_{D_{2\eta}} \leqslant c_1 \eta \varepsilon.$$

Let $c_0 = \max\{c_1, \ldots, c_5\}$ and $\varepsilon_+ = 2 c_0 \gamma^{-2} (r - r_+)^{-(3+2\tau)} \varepsilon^{\frac{4}{3}}$. We then have

$$\|X_{P_+}\|_{D_+, \mathcal{O}_+} \leqslant c_1 \eta \varepsilon + c_5 \gamma^{-2} (r - r_+)^{-(3+2\tau)} \eta^{-2} \varepsilon^2 \leqslant \varepsilon_+.$$

Lemma 4.3. *P_+ has a compact form with respect to n_1, n_2.*

Proof. Note that

$$P_+ = P - R + \{P, F\} + \frac{1}{2!}\big\{\{N, F\}, F\big\} + \frac{1}{2!}\big\{\{P, F\}, F\big\} + \cdots + \frac{1}{n!}\big\{\ldots\{N, \underbrace{F\}\ldots, F}_{n}\big\} + \frac{1}{n!}\big\{\ldots\{P, \underbrace{F\}\ldots, F}_{n}\big\} + \cdots.$$

Since P has a compact form with respect to n_1, n_2, so do $P - R$ and $\{N, F\} = P_{0000} + \langle\omega', I\rangle + \sum_n P_{nn}^{011} z_n \bar{z}_n - R$. The lemma then follows from Lemma 2.4. □

This completes one step of KAM iterations.

5. Iteration lemma and convergence

For any given $s_0, r_0, \varepsilon_0, \gamma_0$, we define, for all $\nu \geqslant 1$, the following sequences

$$r_\nu = r_0\left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right), \qquad \varepsilon_\nu = 2c_0 \gamma_{\nu-1}^{-2} (r_{\nu-1} - r_\nu)^{-(3+2\tau)} \varepsilon_{\nu-1}^{\frac{4}{3}},$$

$$\gamma_\nu = \gamma_0\left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right), \qquad \eta_\nu = \varepsilon_\nu^{\frac{1}{3}},$$

$$s_\nu = \frac{1}{8}\eta_{\nu-1} s_{\nu-1} = 2^{-3\nu}\left(\prod_{i=0}^{\nu-1} \varepsilon_i\right)^{\frac{1}{3}} s_0, \qquad K_\nu = \left(c_0^{-1} \varepsilon_\nu^{-1} (\gamma_\nu - \gamma_{\nu+1})\right)^{\frac{1}{\tau+1}},$$

$$D_\nu = D(r_\nu, s_\nu), \qquad \widetilde{D}_\nu = D\left(r_{\nu+1} + \frac{1}{4}(r_\nu - r_{\nu+1}), \frac{1}{4}\eta_\nu s_\nu\right),$$

$$\begin{aligned} \mathcal{O}_\nu = \Bigg\{\xi\colon\ & \big|\langle k, \omega_\nu\rangle\big| \geqslant \frac{\gamma_\nu}{|k|^\tau},\ k \neq 0,\ \big|\langle k, \omega_\nu\rangle + \Omega_n^\nu\big| \geqslant \frac{\gamma_\nu}{|k|^\tau},\ k \neq 0,\ |n| = |k_1 n_1 + k_2 n_2|, \\ & \big|\langle k, \omega_\nu\rangle + \Omega_n^\nu + \Omega_m^\nu\big| \geqslant \frac{\gamma_\nu}{|k|^\tau},\ k \neq 0,\ |n+m| = |k_1 n_1 + k_2 n_2|, \\ & \big|\langle k, \omega_\nu\rangle + \Omega_n^\nu - \Omega_m^\nu\big| \geqslant \frac{\gamma_\nu}{|k|^\tau},\ k \neq 0,\ |n-m| = |k_1 n_1 + k_2 n_2|\Bigg\}. \end{aligned}$$

5.1. Iteration lemma

The preceding analysis may be summarized as follows.

Lemma 5.1. *The following holds if ε_0 is sufficiently small. Suppose for any $\nu \geqslant 0$, $H_\nu = N_\nu + P_\nu$ is given on $D_\nu \times \mathcal{O}_\nu$ which is real analytic in $(\theta, I, z, \bar{z}) \in D_\nu$ and Whitney smooth in $\xi \in \mathcal{O}_\nu$, where*

$$N_\nu = e_\nu + \big\langle\omega_\nu(\xi), I\big\rangle + \sum_n \Omega_n^\nu(\xi) z_n \bar{z}_n,$$

P_ν has a compact form with respect to n_1, n_2, and

$$\|X_{P_\nu}\|_{D(r_\nu,s_\nu),\mathcal{O}_\nu} \leqslant \varepsilon_\nu.$$

Then there is a symplectic transformation

$$\Phi_\nu : \widetilde{D}_\nu \times \mathcal{O}_\nu \to D_\nu,$$

which is real analytic in $(\theta, I, z, \bar{z}) \in \widetilde{D}_\nu$ and Whitney smooth in $\xi \in \mathcal{O}_\nu$, such that $H_{\nu+1} = H_\nu \circ \Phi_\nu = N_{\nu+1} + P_{\nu+1}$ is defined on $D_{\nu+1} \times \mathcal{O}_{\nu+1}$ and enjoys similar properties as H_ν, i.e., $N_{\nu+1}$ has the form

$$N_{\nu+1} = e_{\nu+1} + \langle \omega_{\nu+1}, I \rangle + \sum_n \Omega_n^{\nu+1} z_n \bar{z}_n \quad \textit{with}$$

$$|\omega_{\nu+1} - \omega_\nu|_{\mathcal{O}_\nu} \leqslant c_0 \varepsilon_\nu, \qquad \left|\Omega_n^{\nu+1} - \Omega_n^\nu\right|_{\mathcal{O}_\nu} \leqslant c_0 \varepsilon_\nu,$$

$P_{\nu+1}$ has a compact form with respect to n_1, n_2, and

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}} \leqslant \varepsilon_{\nu+1}.$$

Moreover,

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \Big\backslash \Big(\bigcup_{|k|>K_\nu} \mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) \Big),$$

where,

$$\begin{aligned} \mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) = \Big\{ & \xi \in \mathcal{O}_\nu \colon \left|\langle k, \omega_{\nu+1}\rangle\right| < \frac{\gamma_{\nu+1}}{|k|^\tau}, \textit{ or } \left|\langle k, \omega_{\nu+1}\rangle + \Omega_n^{\nu+1}\right| < \frac{\gamma_{\nu+1}}{|k|^\tau}, \\ & |n| = |k_1 n_1 + k_2 n_2| \textit{ or } \left|\langle k, \omega_{\nu+1}\rangle + \Omega_n^{\nu+1} + \Omega_m^{\nu+1}\right| < \frac{\gamma_{\nu+1}}{|k|^\tau}, \\ & |n+m| = |k_1 n_1 + k_2 n_2| \textit{ or } \left|\langle k, \omega_{\nu+1}\rangle + \Omega_n^{\nu+1} + \Omega_m^{\nu+1}\right| < \frac{\gamma_{\nu+1}}{|k|^\tau}, \\ & |n-m| = |k_1 n_1 + k_2 n_2| \Big\}. \end{aligned}$$

Proof. It is sufficient to verify the conditions (C1)–(C3) for all $\nu = 0, 1, \ldots.$ The condition (C2) is automatically satisfied by the choice of K_ν. The condition (C3) easily follows from the condition (C1). To verify the condition (C1), we first choose

$$\varepsilon_0 < \left(\frac{1}{c_0}\right)^{\frac{20}{3}} \left(\frac{1}{\Psi(r_0)}\right)^{\frac{8}{3}} \left(\frac{r_0}{4}\right)^{4(3+2\tau)}, \quad \text{where}$$

$$\Psi(r_0) = \prod_{i=1}^\infty \left[(r_{i-1} - r_i)^{-(3+2\tau)}\right]^{(\frac{3}{4})^i}$$

is easily seen to be well-defined. Then

$$\varepsilon_0^{\frac{5}{8}} < \left(\frac{1}{c_0}\right)^{\frac{3}{2}} \left(\frac{r_0}{4}\right)^{\frac{15(2+2\tau)}{8}} \leqslant \left(\frac{1}{c_2}\right)^{\frac{3}{2}} \left(\frac{r_0}{4}\right)^{\frac{3(2+2\tau)}{2}}.$$

Hence

$$\varepsilon_0 < \varepsilon_0^{\frac{3}{8}} \left(\frac{1}{c_2}\left(\frac{r_0}{4}\right)^{2+2\tau}\right)^{\frac{3}{2}} = \left(\frac{1}{c_2}\gamma_0^2 (r_0 - r_1)^{2+2\tau}\right)^{\frac{3}{2}},$$

i.e., (C1) holds for $\nu = 0$. Now, for any $\nu \geqslant 1$, we have by induction that

$$\begin{aligned}
\varepsilon_\nu &= 2c_0\gamma_{\nu-1}^{-2}(r_{\nu-1} - r_\nu)^{-(3+2\tau)}\varepsilon_{\nu-1}^{\frac{4}{3}} \leqslant \left(2c_0\gamma_{\nu-1}^{-2}\Psi(r_0)\varepsilon_0\right)^{(\frac{4}{3})^{\nu-1}} \\
&\leqslant \left(2c_0\gamma_0^{-2}\Psi(r_0)\varepsilon_0\right)^{(\frac{4}{3})^{\nu-1}} \leqslant \left(2c_0\gamma_0^3\Psi(r_0)\varepsilon_0^{\frac{3}{8}}\right)^{(\frac{4}{3})^{\nu-1}} \\
&\leqslant \left(2c_0^{-\frac{3}{2}}\gamma_0^3\left(\frac{r_0}{4}\right)^{\frac{3}{2}(2+2\tau)}\right)^{(\frac{4}{3})^{\nu-1}} \leqslant \left(\frac{1}{c_2}\gamma_\nu^2(r_\nu - r_{\nu+1})^{2+2\tau}\right)^{\frac{3}{2}},
\end{aligned}$$

i.e., (C1) holds. □

5.2. Convergence

Let $\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{\nu-1}$, $\nu = 1, 2, \ldots$. Inductively, we have that $\Psi^\nu : \tilde{D}_\nu \times \mathcal{O}_\nu \to \widetilde{D}_0$ and

$$H_0 \circ \Psi^\nu = H_\nu = N_\nu + P_\nu$$

for all $\nu \geqslant 1$.

Let $\widetilde{\mathcal{O}} = \bigcap_{\nu=0}^{\infty} \mathcal{O}_\nu$. We apply Lemma 5.1 and standard arguments (e.g., [9,26]) to conclude that $H_\nu, e_\nu, N_\nu, P_\nu, \Psi^\nu, D\Psi^\nu, \omega_\nu, \Omega_n^\nu$ converge uniformly on $D(\frac{1}{2}r_0, 0) \times \widetilde{\mathcal{O}}$, say to, $H_\infty, e_\infty, N_\infty, P_\infty, \Psi^\infty, D\Psi^\infty, \omega_\infty, \Omega_n^\infty$, respectively. It is clear that

$$N_\infty = e_\infty + \langle \omega_\infty, I \rangle + \sum_n \Omega_n^\infty z_n \bar{z}_n.$$

Since

$$\varepsilon_\nu = 2c_0\gamma_{\nu-1}^{-2}(r_{\nu-1} - r_\nu)^{-(3+2\tau)}\varepsilon_{\nu-1}^{\frac{4}{3}} \leqslant \left(2c_0\gamma_0^{-2}\Psi(r_0)\varepsilon_0\right)^{(\frac{4}{3})^{\nu-1}},$$

we have by Lemma 5.1 that

$$X_{P_\infty}|_{D(\frac{1}{2}r_0,0)\times\widetilde{\mathcal{O}}} \equiv 0.$$

Let ϕ_H^t denote the flow of any Hamiltonian vector field X_H. Since $H_0 \circ \Psi^\nu = H_\nu$, we have that

$$\phi_{H_0}^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{H_\nu}^t.$$

The uniform convergence of Ψ^ν, $D\Psi^\nu$, X_{H_ν} imply that one can pass the limit in the above to conclude that

$$\phi^t_{H_0} \circ \Psi^\infty = \Psi^\infty \circ \phi^t_{H_\infty}$$

on $D(\frac{1}{2}r_0, 0) \times \widetilde{\mathcal{O}}$. It follows that

$$\phi^t_{H_0}\big(\Psi^\infty\big(\mathbb{T}^2 \times \{\xi\}\big)\big) = \Psi^\infty \phi^t_{N_\infty}\big(\mathbb{T}^2 \times \{\xi\}\big) = \Psi^\infty\big(\mathbb{T}^2 \times \{\xi\}\big)$$

for all $\xi \in \widetilde{\mathcal{O}}$. Hence $\Psi^\infty(\mathbb{T}^2 \times \{\xi\})$ is an embedded invariant torus of the original perturbed Hamiltonian system at $\xi \in \widetilde{\mathcal{O}}$. We remark that the frequencies $\omega_\infty(\xi)$ associated with $\Psi^\infty(\mathbb{T}^2 \times \{\xi\})$ are slightly deformed from the unperturbed ones $\omega(\xi)$. The normal behaviors of the invariant tori $\Psi^\infty(\mathbb{T}^2 \times \{\xi\})$ are governed by their respective normal frequencies $\Omega^\infty_n(\xi)$.

5.3. Measure estimate

For each $k \in \mathbb{Z} \setminus \{0\}$, denote

$$\mathcal{R}^{\nu+1}_k = \left\{\xi \in \mathcal{O}_\nu\colon \big|\langle k, \omega_{\nu+1}(\xi)\rangle\big| < \frac{\gamma_{\nu+1}}{|k|^\tau}\right\},$$

$$\mathcal{R}^{\nu+1}_{kn} = \left\{\xi \in \mathcal{O}_\nu\colon \big|\langle k, \omega_{\nu+1}(\xi)\rangle + \Omega^{\nu+1}_n\big| < \frac{\gamma_{\nu+1}}{|k|^\tau}, |n| = |k_1 n_1 + k_2 n_2|\right\},$$

$$\mathcal{R}^{\nu+1}_{knm} = \left\{\xi \in \mathcal{O}_\nu\colon \big|\langle k, \omega_{\nu+1}(\xi)\rangle + \Omega^{\nu+1}_n + \Omega^{\nu+1}_m\big| < \frac{\gamma_{\nu+1}}{|k|^\tau}, |n+m| = |k_1 n_1 + k_2 n_2|\right\},$$

$$\widetilde{\mathcal{R}}^{\nu+1}_{knm} = \left\{\xi \in \mathcal{O}_\nu\colon \big|\langle k, \omega_{\nu+1}(\xi)\rangle + \Omega^{\nu+1}_n - \Omega^{\nu+1}_m\big| < \frac{\gamma_{\nu+1}}{|k|^\tau}, |n-m| = |k_1 n_1 + k_2 n_2|\right\}.$$

Then

$$R^{\nu+1}_k(\gamma_{\nu+1}) = \mathcal{R}^{\nu+1}_k \cup \bigcup_{n,m}\big(\mathcal{R}^{\nu+1}_{kn} \cup \mathcal{R}^{\nu+1}_{knm} \cup \tilde{\mathcal{R}}^{\nu+1}_{knm}\big).$$

Let

$$\mathcal{R}^{\nu+1} = \bigcup_{|k|>K_\nu} R^{\nu+1}_k(\gamma_{\nu+1}).$$

Then

$$\mathcal{O}_0 \setminus \widetilde{\mathcal{O}} = \bigcup_{\nu \geqslant 0} \mathcal{R}^{\nu+1}.$$

Lemma 5.2. *There is a constant $C_1 > 0$ such that*

$$\operatorname{meas}\big(\mathcal{R}^{\nu+1}_k \cup \mathcal{R}^{\nu+1}_{kn} \cup \mathcal{R}^{\nu+1}_{knm}\big) \leqslant C_1 \frac{\gamma_{\nu+1}}{|k|^{\tau+1}}$$

for all $|k| > K_\nu$, n, and m.

Proof. Let $|\cdot|$ denote the ℓ^1-norm. By Lemma 5.1 and the definitions of ω_0, $\{\Omega_n^0\}$, we have that

$$\left|\frac{\partial(\langle k, \omega_{\nu+1}(\xi)\rangle + l_1\Omega_n^{\nu+1} + l_2\Omega_m^{\nu+1})}{\partial\xi}\right| \geqslant \frac{1}{2}|k|,$$

as $\varepsilon_0 \ll 1$, for all $|k| > K_\nu$, n, m, and $l_1, l_2 = 0, \pm 1$. The lemma then follows from the standard measure estimate using Fubini's theorem (see, e.g., [19,23,27]). □

Lemma 5.3.

$$\operatorname{meas}(\mathcal{O}_0 \setminus \widetilde{\mathcal{O}}) = \operatorname{meas}\left(\bigcup_{\nu\geqslant 0} \mathcal{R}^{\nu+1}\right) = O(\gamma_0).$$

Proof. By Lemma 5.2, we immediately have that $\operatorname{meas}(\bigcup_{|k|>K_\nu} \mathcal{R}_k^{\nu+1}) \leqslant C_2 \sum_{|k|>K_\nu} \frac{\gamma_0}{|k|^\tau}$ for some constant $C_2 > 0$.

Now, for each $l_1, l_2 = 0, \pm 1$, $|k| > K_\nu$, n, and m, with $|l_1| + |l_2| \neq 0$ and $|l_1 n + l_2 m| = |k_1 n_1 + k_2 n_2|$, we consider

$$\mathcal{R}_{knm}^{l_1,l_2,\nu+1} = \left\{\xi \in \mathcal{O}_\nu\colon \left|\langle k, \omega_{\nu+1}(\xi)\rangle + l_1\Omega_n^{\nu+1} + l_2\Omega_m^{\nu+1}\right| < \frac{\gamma_{\nu+1}}{|k|^\tau}\right\}.$$

First, consider the cases that $|n| \neq |m|$. It is easy to see from the definitions of $\omega_{\nu+1}$, $\Omega_{n,m}^{\nu+1}$, K_ν that there is a constant $C_3 > 0$ such that $\mathcal{R}_{knm}^{l_1,l_2,\nu+1} = \emptyset$ if $\max\{|n|, |m|\} > C_3|k|$. By Lemma 5.2, there exists a constant $C_4 > 0$ such that

$$\begin{aligned}
&\operatorname{meas}\left(\bigcup_{|k|>K_\nu, |n|\neq|m|, l_1, l_2} \mathcal{R}_{knm}^{l_1,l_2,\nu+1}\right) \\
&\quad = \operatorname{meas}\left(\bigcup_{\substack{|k|>K_\nu \\ |n|\neq|m|;\, |n|,|m|\leqslant C_3|k|}} \left(\mathcal{R}_{kn}^{\nu+1} \cup \mathcal{R}_{knm}^{\nu+1} \cup \widetilde{\mathcal{R}}_{knm}^{\nu+1}\right)\right) \leqslant C_4 \sum_{|k|>K_\nu} \frac{\gamma_0}{|k|^{\tau-1}}.
\end{aligned}$$

Next, consider the cases that $|n| = |m|$. If $l_1 n + l_2 m = 0$, then either (a) $l_1 = l_2 = \pm 1$, $n = -m$; or (b) $l_1 = -l_2 = \pm 1$, $n = m$. In case (a), there is a constant $C_5 > 0$ such that $\mathcal{R}_{knm}^{l_1,l_2,\nu+1} = \emptyset$ if $|n| = |m| > C_5|k|$. By Lemma 5.2, there exists a constant $C_6 > 0$ such that

$$\operatorname{meas}\left(\bigcup_{|k|>K_\nu, n=-m, l_1=l_2=\pm 1} \mathcal{R}_{knm}^{l_1,l_2,\nu+1}\right) \leqslant \operatorname{meas}\left(\bigcup_{|k|>K_\nu, |n|=|m|\leqslant C_5|k|} \mathcal{R}_{knm}^{\nu+1}\right) \leqslant C_6 \sum_{|k|>K_\nu} \frac{\gamma_0}{|k|^\tau}.$$

In case (b), we have $l_1\Omega_n^{\nu+1} + l_2\Omega_m^{\nu+1} = 0$, hence $\mathcal{R}_{knm}^{l_1,l_2,\nu+1} = \mathcal{R}_k^{\nu+1}$. It follows that

$$\operatorname{meas}\left(\bigcup_{|k|>K_\nu, n=m, l_1=-l_2=\pm 1} \mathcal{R}_{knm}^{l_1,l_2,\nu+1}\right) = \operatorname{meas}\left(\bigcup_{|k|>K_\nu} \mathcal{R}_k^{\nu+1}\right) = O\left(\sum_{|k|>K_\nu} \frac{\gamma_0}{|k|^\tau}\right).$$

If $l_1 n + l_2 m \neq 0$, then it follows from the identity $|l_1 n + l_2 m| = |k_1 n_1 + k_2 n_2|$ that $\mathcal{R}_{knm}^{l_1,l_2,\nu+1} = \emptyset$ if $|n| = |m| > \max\{|n_1|, |n_2|\}|k|$. Hence by Lemma 5.2, there exists a constant $C_7 > 0$ such that

$$\operatorname{meas}\Bigg(\bigcup_{\substack{|k|>K_\nu, |n|=|m| \\ l_1 n + l_2 m \neq 0, l_1, l_2}} \mathcal{R}_{knm}^{l_1,l_2,\nu+1}\Bigg)$$

$$\leqslant \operatorname{meas}\Bigg(\bigcup_{\substack{|k|>K_\nu \\ |n|=|m|\leqslant \max\{|n_1|,|n_2|\}|k|}} \mathcal{R}_{kn}^{\nu+1} \cup \mathcal{R}_{knm}^{\nu+1} \cup \widetilde{\mathcal{R}}_{knm}^{\nu+1}\Bigg) \leqslant C_7 \sum_{|k|>K_\nu} \frac{\gamma_0}{|k|^\tau}.$$

Note that $\tau > 4$ and

$$\mathcal{R}^{\nu+1} = \Bigg(\bigcup_{|k|>K_\nu} \mathcal{R}_k^{\nu+1}\Bigg) \cup \Bigg(\bigcup_{\substack{|k|>K_\nu, |l_1 n + l_2 m| = |k_1 n_1 + k_2 n_2| \\ l_1, l_2 = 0, \pm 1, |l_1|+|l_2| \neq 0}} \mathcal{R}_{knm}^{l_1,l_2,\nu+1}\Bigg).$$

We have by the above analysis that

$$\operatorname{meas}(\mathcal{O}_0 \setminus \widetilde{\mathcal{O}}) = \operatorname{meas}\Bigg(\bigcup_{\nu \geqslant 0} \mathcal{R}^{\nu+1}\Bigg) = O\Bigg(\sum_{\nu \geqslant 0} \frac{\gamma_0}{K_\nu}\Bigg) = O(\gamma_0). \qquad \square$$

Finally, since $\mathcal{O} \setminus \widetilde{\mathcal{O}} = (\mathcal{O} \setminus \mathcal{O}_0) \cup (\mathcal{O}_0 \setminus \tilde{\mathcal{O}})$, we have by Proposition 3.2 and Lemma 5.3 that $\operatorname{meas}(\mathcal{O} \setminus \widetilde{\mathcal{O}}) = O(\gamma_0)$. This completes the measure estimate.

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