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KAM tori for higher dimensional beam equations with constant potentials*

Jiansheng Geng and Jiangong You

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail: jianshenggeng@yahoo.com and jyou@nju.edu.cn

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Abstract

In this paper, we consider the higher dimensional nonlinear beam equations

$$u_{tt} + \Delta^2 u + \sigma u + f(u) = 0,$$

with periodic boundary conditions, where the nonlinearity $f(u)$ is a real–analytic function near $u = 0$ with $f(0) = f'(0) = 0$ and σ is a real parameter in an interval $\mathcal{I} \equiv [\sigma_1, \sigma_2]$. It is proved that for 'most' positive parameters σ lying in the finite interval $\mathcal{I}$, the above equations admit a family of small-amplitude, linearly stable quasi-periodic solutions corresponding to a Cantor family of finite dimensional invariant tori of an associated infinite dimensional dynamical system. The proof is based on an infinite dimensional KAM theorem, modified from (Geng and You 2006 *Commun. Math. Phys.* **262** 343–72) and (Xu J *et al* 1996 *Sci. China Ser.* A **39** 372–83, 383–94) with weaker non-degeneracy conditions.

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1. Introduction and main result

The dynamics of linear Hamiltonian partial differential equations is quite clear: in many cases, the equation has families of periodic solutions, quasi-periodic solutions and almost-periodic solutions. The stability of the solutions is also obvious. One would like to know if these solutions and the related dynamics continue to the nonlinear equations in the neighbourhood of equilibrium. There are plenty of works along this line. Below let us roughly describe these works and their methods.

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1. *Infinite dimensional KAM theory.* Motivated by the construction of quasi-periodic solutions for Hamiltonian partial differential equations, in the late 1980s, the celebrated KAM theory has been successfully extended to infinite dimensional settings by Wayne [27], Kuksin [20] and Pöschel [25]. Such generalizations are based on the KAM theorem for lower dimensional tori in finite dimensional phase space ([19,23]). These infinite dimensional KAM theorems apply to, as typical examples, one-dimensional semi-linear Schrödinger equations with parameters

$$\mathrm{i}u_t - u_{xx} + V(x, \xi)u = f(u)$$

and wave equations

$$u_{tt} - u_{xx} + V(x, \xi)u = f(u),$$

with Dirichlet boundary conditions to obtain the following result: if one carefully chooses a family of potentials $V(x, \xi)$ so that the eigenvalues of $A = -(d^2/dx^2) + V(x, \xi)$ satisfies some kind of non-degeneracy condition, then for typical ξ, the equation has an invariant torus carrying quasi-periodic solutions. In addition, Xu *et al* [29] also obtained the same results if the eigenvalues of $A = -(d^2/dx^2) + V(x, \xi)$ satisfy weaker non-degeneracy condition (see also Rüssmann [26], Cheng and Sun [9]).

Later, a KAM theorem was given by Chierchia and You [10] which applies to a one-dimensional wave equation with periodic boundary conditions. In [17], Geng and You gave a KAM theorem which applies to some types of higher dimensional Hamiltonian partial differential equations. Recently, Eliasson and Kuksin [13] gave a KAM theorem which applies to nonlinear Schrödinger equations in higher dimensional space.

2. *Craig–Wayne–Bourgain method.* One-dimensional partial differential equations with periodic boundary conditions are more complicated since the eigenvalues of A are no longer distinct, i.e.

$$\mu_0 < \mu_1 \leqslant \mu_2 < \cdots < \mu_{2n-1} \leqslant \mu_{2n} < \cdots.$$

For semi-linear partial differential equations in higher dimensional space, the eigenvalues of A are always asymptotically multiple. To overcome this difficulty, Craig and Wayne [11, 12] went to the origin of the KAM method—the Newtonian iteration method, together with Liapunov–Schmidt decomposition and techniques by Fröhlich and Spencer [14]—which involves a Green's function analysis and the control of the inverse of infinite matrices with small eigenvalues. They succeeded in constructing periodic solutions of one-dimensional semi-linear wave equations with periodic boundary conditions.

Later Bourgain further developed the Craig–Wayne method and proved the existence of quasi-periodic solutions of Hamiltonian partial differential equations in higher dimensional space with Dirichlet boundary conditions or periodic boundary conditions. More precisely, Bourgain gave the existence of quasi–periodic solutions for

$$\mathrm{i}u_t - \Delta u + M_\sigma u + f(u) = 0,$$

$$u_{tt} - \Delta u + M_\sigma u + f(u) = 0,$$

where M_σ is the real Fourier multiplier, see Bourgain [4–8] for details. We remark that the Fourier multiplier M_σ makes the spectrum of the operator $-\Delta + M_\sigma$ simple, which is crucial for the proof. However, the physical meaning of the equation is also weakened.

3. *'Natural' Hamiltonian partial differential equations.* We remark that when the potential is a constant, generally speaking, normal form techniques have to be used.

In the one-dimensional case, this has been done by Wayne [27], Kuksin and Pöschel [21] and Pöschel [24] for $V(x) = m > 0$.

For similar results of other type of one-dimensional equations, we refer to Geng and You [16] (see also Geng and Yi [18] for a simple proof), Geng and You [15,17], Liang–You [22] and references therein.

Recently, Yuan [30, 31], using the normal form technique, proved the existence of quasi-periodic solutions for complete resonant one-dimensional wave equations $u_{tt} - u_{xx} \pm u^3 = 0$ and $u_{tt} - u_{xx} + V(x)u + u^3 = 0$ for typical $V(x)$ not necessary constant.

In the case that the space dimension is greater than one and the potential is *natural*, due to resonances, the normal form techniques are very complicated and technical. So far the only result was due to Bourgain ([8]). In [8], Bourgain proved the existence of *two-frequency* quasi-periodic solutions for the two-dimensional Schrödinger equation with constant potential

$$\mathrm{i}u_t - \Delta u + mu + u|u|^2 = 0. \tag{1.1}$$

More concretely, for two fixed distinguished lattice points $i_1, i_2 \in \mathbb{Z}^2$ on a circle

$$|i_1| = |i_2| = R, \qquad i_1 \neq -i_2,$$

where $|\cdot|$ denotes Euclid–norm, Bourgain proved that (1.1) possesses quasi–periodic solutions

$$u(t, x) = \sum_{j=1}^{2} \xi_j \mathrm{e}^{\mathrm{i}(\omega_j t + \langle i_j, x\rangle)} + O(|\xi|^3)$$

with frequencies $\omega = (\omega_1, \omega_2)$ satisfying

$$\omega_j = |i_j|^2 + m + O(|\xi|^2), \qquad j = 1, 2,$$

for $\xi = (\xi_1, \xi_2)$ in a Cantor set $\mathcal{O}$ of positive measure.

In this paper, we consider dD (d-dimensional) nonlinear beam equations with periodic boundary conditions

$$u_{tt} + \Delta^2 u + \sigma u + f(u) = 0, \qquad x \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

where $0 < \sigma \in \mathcal{I} \equiv [\sigma_1, \sigma_2]$, and $f(u)$ is a real-analytic function near $u = 0$ with $f(0) = f'(0) = 0$. We will construct quasi-periodic solutions with arbitrary many frequencies for the above equations. Note that, in the higher dimensional case, for a fixed lattice point n, there will be many other lattice points m such that $|m| = |n|$, which will bring two main difficulties: one difficulty is that the first Melnikov conditions and the second Melnikov conditions are partially violated; this difficulty can be overcome by assuming that $f(u)$ *does not depend explicitly on the space variables and the time variable*; the point is that the perturbation has some special structure which makes some first and second Melnikov conditions unnecessary. The other difficulty is that there are more resonances between tangential frequencies and normal frequencies in the higher dimensional case so that the Hamiltonian cannot keep the desired form along the KAM iteration. For example, the lattice points $n_1 = (0, 1), n_2 = (0, -1), n_3 = (1, -1), n_4 = (1, 1)$ satisfy

$$n_1 \neq n_2, \qquad n_1 \neq n_4, \qquad n_3 \neq n_4,$$

$$|n_1| = |n_2|, \qquad |n_3| = |n_4|,$$

$$n_1 - n_2 + n_3 - n_4 = 0.$$

If we put two of them into tangential sites, after one KAM step the normal form will contain the non-integrable terms. This is not allowed by the KAM method. However, such a phenomenon does not appear in the one-dimension case. In the higher dimensional case, to avoid such difficulty, we have to carefully choose the tangential sites.

In the following, we formulate the main result of this paper. Let $\phi_n(x) = \sqrt{(1/(2\pi)^d)}\mathrm{e}^{\mathrm{i}\langle n,x\rangle}$ be eigenvectors of the operator $\Delta^2 + \sigma$ with periodic boundary conditions corresponding to eigenvalues

$$\lambda_n = |n|^4 + \sigma, n \in \mathbb{Z}^d,$$

where σ is parameter. For any fixed lattice points $i_1, \ldots, i_b \in \mathbb{Z}^d$, it is obvious that the linearized equations have a small-amplitude quasi-periodic solution

$$u(t, x) = \sum_{j=1}^{b} \xi_j \mathrm{e}^{\mathrm{i}\sqrt{\lambda_{i_j}}t}\phi_{i_j}(x), \qquad \xi_j > 0.$$

We will prove that for 'most' σ (in the sense of Lebesgue measure) the quasi–periodic solutions of the linearized equation continue to the nonlinear equation if one chooses the lattice points (called tangential sites) $i_1, \ldots, i_b$ in the following way: $\{i_1, \ldots, i_b\} \in \mathcal{J}$ where $\mathcal{J}$ is defined as follows:

$$\mathcal{J} = \left\{ \{i_1, \ldots, i_b\} : \begin{array}{ll} i_j = (i_{j_1}, \ldots, i_{j_d}), & 1 \leqslant j \leqslant b, \\ |i_{j_{k+1}}| > 4d(i_{j_k})^2, & 1 \leqslant k \leqslant d-1, \\ |i_{(j+1)_1}| > 4d(i_{j_d})^2, & 1 \leqslant j \leqslant b-1. \end{array} \right\}. \tag{1.2}$$

Theorem 1. *Consider dD nonlinear beam equations*

$$u_{tt} + \Delta^2 u + \sigma u + f(u) = 0, \qquad x \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

$$u(t, x_1 + 2\pi, \ldots, x_d) = \cdots = u(t, x_1, \ldots, x_d + 2\pi) = u(t, x_1, \ldots, x_d),$$

where $\sigma \in \mathcal{I} \equiv [\sigma_1, \sigma_2]$ as parameters and $f(u)$ is a real-analytic function near $u = 0$ with $f(0) = f'(0) = 0$. Then for a fixed $\{i_1, \ldots, i_b\} \in \mathcal{J}$ and any $0 < \gamma \ll 1$, there exists a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{I}$ with $\mathrm{meas}(\mathcal{I} \setminus \mathcal{O}_\gamma) = O(\gamma^\vartheta)$ (ϑ is specified in appendix B), such that for each $\sigma \in \mathcal{O}_\gamma$, the above nonlinear beam equation admits a small-amplitude, linearly stable quasi-periodic solution of the form

$$u(t, x) = \sum_{n \in \mathbb{Z}^d} u_n(\omega_1 t, \ldots, \omega_b t)\phi_n(x),$$

where $u_n : \mathbb{T}^b \to \mathbb{R}$ and Diophantine frequency $\omega = (\omega_1, \ldots, \omega_b)$ is close to the unperturbed frequency $(\sqrt{\lambda_{i_1}}, \ldots, \sqrt{\lambda_{i_b}})$.

Remark 1. We remark that our way of choosing $\mathcal{J}$ is tricky; the other ways of choosing $i_1, \ldots, i_b$ are also possible. The basic principle is to choose the distinguished lattice points $i_1, \ldots, i_b$ so that if $\lambda_{n_1} + \lambda_{n_2} - \lambda_{n_3} - \lambda_{n_4} = 0$, there will be at most one of n_1, n_2, n_3, n_4 belonging to tangential sites $\{i_1, \ldots, i_b\}$. For example, in the case of two dimension, in order to construct two-frequency torus, we may choose $\mathcal{J} = \{(0, 1), (9, 650)\}$.

Remark 2. It is plausible that the equation possesses quasi-periodic solutions for all σ, but we cannot prove this stronger result so far.

The paper is organized as follows: in section 2 we formulate an infinite dimensional KAM theorem, the proof of which is based on Geng and You [17] and Xu *et al* [29]; in section 3 the Hamiltonian is transformed into the desired form; as a consequence, theorem 1 follows by applying theorem 2 in section 2. Some technical lemmas are given in appendix A and appendix B.

2. An infinite dimensional KAM theorem for Hamiltonian partial differential equations

We start by introducing some notations. For given b vectors in $\mathbb{Z}^d$, say $\{i_1, \ldots, i_b\}$, we denote $\mathbb{Z}_1^d = \mathbb{Z}^d \setminus \{i_1, \ldots, i_b\}$. Let $z = (\ldots, z_n, \ldots)_{n\in\mathbb{Z}_1^d}$, $z_n \in \mathbb{C}$ and its complex conjugate $\bar{z} = (\ldots, \bar{z}_n, \ldots)_{n\in\mathbb{Z}_1^d}$, $\bar{z}_n \in \mathbb{C}$. We introduce the weighted norm

$$\|z\|_{a,\rho} = \sum_{n\in\mathbb{Z}_1^d} |z_n||n|^a \mathrm{e}^{|n|\rho},$$

where $|n| = \sqrt{n_1^2 + \cdots + n_d^2}$, $n = (n_1, \ldots, n_d)$ and $a \geqslant 0$, $\rho > 0$. Denote a neighbourhood of $\mathbb{T}^b \times \{I = (I_1, \ldots, I_b) = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ by

$$D(r, s) = \{(\theta, I, z, \bar{z}) : |\mathrm{Im}\theta| < r, |I| < s^2, \|z\|_{a,\rho} < s, \|\bar{z}\|_{a,\rho} < s\},$$

where $|\cdot|$ denotes the sup-norm of complex vectors. Moreover, let $\mathcal{O}$ be a compact subset of $\mathbb{R}^\nu$ of positive Lebesgue measure, and let $C^{N,1}(\mathcal{O})$ be N-order Lipschitz continuously differentiable function space; here the derivatives of function on $\mathcal{O}$ are understood in the sense of Whitney, so the space $C^{N,1}(\mathcal{O})$ is also understood in the sense of Whitney (for the related definition about the notation $C^{N,1}(\mathcal{O})$ and Whitney differentiability, see [28, 29]).

Let $\alpha \equiv (\ldots, \alpha_n, \ldots)_{n\in\mathbb{Z}_1^d}$, $\beta \equiv (\ldots, \beta_n, \ldots)_{n\in\mathbb{Z}_1^d}$, α_n and $\beta_n \in \mathbb{N}$ with finitely many non-zero components of positive integers. The product $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$. For any given function

$$F(\theta, I, z, \bar{z}) = \sum_{\alpha,\beta} F_{\alpha\beta}(\theta, I) z^\alpha \bar{z}^\beta, \tag{2.1}$$

where $F_{\alpha\beta}$ belongs to $C^{N,1}(\mathcal{O})$ in parameter ξ, we define the weighted norm of F by

$$\|F\|_{D(r,s),\mathcal{O}} \equiv \sup_{\substack{\|z\|_{a,\rho}<s\\ \|\bar{z}\|_{a,\rho}<s}} \sum_{\alpha,\beta} \|F_{\alpha\beta}\| \, |z^\alpha||\bar{z}^\beta|, \tag{2.2}$$

where, if $F_{\alpha\beta} = \sum_{k\in\mathbb{Z}^b, l\in\mathbb{N}^b} F_{kl\alpha\beta}(\xi) I^l \mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}$, ($\langle\cdot,\cdot\rangle$ being the standard inner product in $\mathbb{C}^b$), $\|F_{\alpha\beta}\|$ is short for

$$\|F_{\alpha\beta}\| \equiv \sum_{k,l} |F_{kl\alpha\beta}(\xi)|_{C^{N,1}(\mathcal{O})} s^{2|l|} \mathrm{e}^{|k|r}. \tag{2.3}$$

To function F, we associate a Hamiltonian vector field defined by

$$X_F = (F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}_{n\in\mathbb{Z}_1^d}, \{-\mathrm{i}F_{\bar{z}_n}\}_{n\in\mathbb{Z}_1^d}).$$

Its weighted norm is defined by[1]

$$\|X_F\|_{D(r,s),\mathcal{O}} \equiv \|F_I\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_\theta\|_{D(r,s),\mathcal{O}} + \frac{1}{s}\left(\sum_{n\in\mathbb{Z}_1^d} \|F_{z_n}\|_{D(r,s),\mathcal{O}} |n|^{\bar{a}} \mathrm{e}^{|n|\rho} + \sum_{n\in\mathbb{Z}_1^d} \|F_{\bar{z}_n}\|_{D(r,s),\mathcal{O}} |n|^{\bar{a}} \mathrm{e}^{|n|\rho}\right) \tag{2.4}$$

Remark. In this paper, we require that $\bar{a} > a$, i.e. the weight of vector fields is a little heavier than that of $z, \bar{z}$. The boundedness of $\|X_F\|_{D_\rho(r,s),\mathcal{O}}$ means X_F sends a decaying z-sequence to a faster decaying sequence.

[1] The norm $\|\cdot\|_{D(r,s),\mathcal{O}}$ for scalar functions is defined in (2.2). The vector function $G : D(r, s) \times \mathcal{O} \to \mathbb{C}^m$, $(m < \infty)$ is similarly defined as $\|G\|_{D(r,s),\mathcal{O}} = \sum_{i=1}^m \|G_i\|_{D(r,s),\mathcal{O}}$.

The starting point will be a family of integrable Hamiltonians of the form

$$N = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n, \tag{2.5}$$

where $\xi \in \mathcal{O}$ is a parameter, the phase space is endowed with the symplectic structure $\mathrm{d}I \wedge \mathrm{d}\theta + \mathrm{i} \sum_{n \in \mathbb{Z}_1^d} \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$.

For each $\xi \in \mathcal{O}$, the Hamiltonian equations of motion for N, i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}z_n}{\mathrm{d}t} = -\mathrm{i}\Omega_n z_n, \quad \frac{\mathrm{d}\bar{z}_n}{\mathrm{d}t} = \mathrm{i}\Omega_n \bar{z}_n, \quad n \in \mathbb{Z}_1^d, \tag{2.6}$$

admit special solutions $(\theta, 0, 0, 0) \to (\theta + \omega t, 0, 0, 0)$ that corresponds to an invariant torus in the phase space.

Consider now the perturbed Hamiltonian

$$H = N + P = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi). \tag{2.7}$$

Our goal is to prove that, for most values of parameter $\xi \in \mathcal{O}$ (in Lebesgue measure sense), the Hamiltonians $H = N + P$ still admit invariant tori provided that $\|X_P\|_{D(r,s),\mathcal{O}}$ is sufficiently small.

To this end, we need to impose some conditions on $\omega(\xi)$, $\Omega_n(\xi)$ and the perturbation P. As we already remarked, the persistence of the lower dimensional torus may not be true if one only assumes the smallness of the perturbation. This is an essential difference between infinite and finite dimensional cases.

($A1$) *Regularity of the perturbation.* The perturbation P is *regular* in the sense that $\|X_P\|_{D(r,s),\mathcal{O}} < \infty$ with $\bar{a} > a$.

($A2$) *Special form of the perturbation.* The perturbation is taken from a special class of analytic functions,

$$\mathcal{A} = \left\{ P : P = \sum_{k \in \mathbb{Z}^b, l \in \mathbb{N}^b, \alpha, \beta} P_{kl\alpha\beta}(\xi) I^l \mathrm{e}^{\mathrm{i}\langle k, \theta \rangle} z^\alpha \bar{z}^\beta \right\},$$

where k, α, β has the following relation

$$\sum_{j=1}^{b} k_j i_j + \sum_{n \in \mathbb{Z}_1^d} (\alpha_n - \beta_n) n = 0. \tag{2.8}$$

($A3$) *Non-degeneracy.* Suppose for $\forall \xi \in \mathcal{O}$,

$$\operatorname{rank} \left\{ \frac{\partial \omega_1}{\partial \xi}, \cdots, \frac{\partial \omega_b}{\partial \xi} \right\} = \kappa,$$

$$\operatorname{rank} \left\{ \frac{\partial^\beta \omega}{\partial \xi^\beta} | \forall \beta, 1 \leqslant |\beta| \leqslant b - \kappa + 1 \right\} = b, \tag{2.9}$$

where κ is a given integer with $1 \leqslant \kappa \leqslant \min\{b, \nu\}$, $\frac{\partial \omega_1}{\partial \xi}, \ldots, \frac{\partial \omega_b}{\partial \xi}$ are vectors of all 1-order partial derivatives in ξ and for a fixed β $\frac{\partial^\beta \omega}{\partial \xi^\beta} = (\frac{\partial^\beta \omega_1}{\partial \xi^\beta}, \ldots, \frac{\partial^\beta \omega_b}{\partial \xi^\beta})$. Moreover, for some $N > b - \kappa + 4$, ω belongs to $C^{N,1}(\mathcal{O})$.

$(A4)$ *Asymptotics of normal frequencies.* There exists an $\iota > 0$ such that for all $n = (n_1, \ldots, n_d) \in \mathbb{Z}_1^d$,

$$\Omega_n \neq 0, \qquad n \in \mathbb{Z}_1^d, \tag{2.10}$$

$$\Omega_n = \bar{\Omega}_n + \tilde{\Omega}_n, \qquad |\tilde{\Omega}_n|_{C^{N,1}(\mathcal{O})} = o(|n|^{-\iota}), \tag{2.11}$$

where $\bar{\Omega}_n$s are real and independent of ξ; furthermore, the asymptotic behaviour of $\bar{\Omega}_n$ is assumed to be as follows

$$\bar{\Omega}_n = |n|^p + o(|n|^p), \qquad \bar{\Omega}_n - \bar{\Omega}_m = |n|^p - |m|^p + o(|m|^{-\iota}), \quad |m| \leqslant |n|, \tag{2.12}$$

where $p \geqslant 2$ for $d > 1$ or $p \geqslant 1$ for $d = 1$.

$(A5)$ *Non-resonance conditions and admissible tangential sites.* For a fixed $\gamma > 0$ small enough and τ sufficiently large, we assume that either

$$\begin{aligned}
&|\langle k, \omega(\xi)\rangle| \geqslant \frac{\gamma}{|k|^\tau}, \qquad k \neq 0,\\
&|\langle k, \omega(\xi)\rangle + \Omega_n| \geqslant \frac{\gamma}{|k|^\tau},\\
&|\langle k, \omega(\xi)\rangle + \Omega_n + \Omega_m| \geqslant \frac{\gamma}{|k|^\tau},\\
&|\langle k, \omega(\xi)\rangle + \Omega_n - \Omega_m| \geqslant \frac{\gamma}{|k|^\tau}, |k| + ||n| - |m|| \neq 0
\end{aligned} \tag{2.13}$$

or

$$\langle k, i\rangle + n + m \neq 0, \text{for}(k, |n|, |m|) = (-e_j - e_l, |i_j|, |i_l|), \qquad 1 \leqslant j < l \leqslant b,$$

$$\langle k, i\rangle + n - m \neq 0, (k, |n|, |m|) = (-e_j + e_l, |i_j|, |i_l|), \qquad 1 \leqslant j < l \leqslant b,$$

where $i = (i_1, \ldots, i_b)$ and e_j denotes b-vector with its jth component being 1 and the other components being zero.

Now we are ready to state our KAM theorem.

Theorem 2. *Assume that the Hamiltonian $H = N + P$ satisfies $(A1)$, $(A2)$, $(A3)$, $(A4)$, $(A5)$, then there exists a positive constant $\varepsilon = \varepsilon(b, d, p, \kappa, \iota, \bar{a} - a, \gamma, \tau)$ such that if $\|X_P\|_{D(r,s),\mathcal{O}} < \varepsilon$, then the following holds true: there exists a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with* $\mathrm{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma^\vartheta)$ *(ϑ is specified in appendix B) and two maps (analytic in θ and belonging to $C^{N,1}(\mathcal{O}_\gamma)$ in ξ)*

$$\Psi : \mathbb{T}^b \times \mathcal{O}_\gamma \to D(r, s), \qquad \tilde{\omega} : \mathcal{O}_\gamma \to \mathbb{R}^b,$$

where Ψ is $(\varepsilon/\gamma^{N+1})$-close to the trivial embedding $\Psi_0 : \mathbb{T}^b \times \mathcal{O} \to \mathbb{T}^b \times \{0, 0, 0\}$ and $\tilde{\omega}$ is ε-close to the unperturbed frequency ω, such that for any $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathbb{T}^b$, the curve $t \to \Psi(\theta + \tilde{\omega}(\xi)t, \xi)$ is a quasi-periodic solution of the Hamiltonian equations governed by $H = N + P$. Moreover, the obtained solutions are linearly stable.

Remark 1. Just as commented in [17], according to the assumption $(A2)$, when $k = (k_1, \ldots, k_b) = 0$ and $n \neq m$, we get

$$P_{0lnm} = 0 \qquad \text{if } \sum_{j=1}^{b} k_j i_j + n - m = n - m \neq 0.$$

This means that there are no terms of the form $\sum_{n\neq m} P_{0lnm} I^l z_n \bar{z}_m$ in the perturbation; hence, we will not encounter small divisor $\Omega_n - \Omega_m$ in the KAM iteration. Similarly, due to assumption

$(A2)$, we will not encounter small divisor $-\omega_j + \Omega_n$ in the KAM iteration; thus, although the first Melnikov conditions are partially violated while $|n| = |i_j|$, $1 \leqslant j \leqslant b$, our KAM theorem holds true under assumption $(A2)$. It should be noted that, P and F satisfy $(A2)$, then $\{P, F\}$ also satisfies $(A2)$. The detailed proof can be found in [17].

Remark 2. Analogously, due to assumption $(A5)$, we will not encounter small divisor $-\omega_j + \omega_l + \Omega_n - \Omega_m$ and $-\omega_j - \omega_l + \Omega_n + \Omega_m$ in the KAM iteration; thus, although the corresponding second Melnikov conditions are violated, our KAM theorem still holds true.

Remark 3. In the case of $\xi \in \mathcal{O} \subset \mathbb{R}$, non-degeneracy conditions (2.9) have a particularly simple form

$$\begin{vmatrix} \frac{d\omega_1}{d\xi} & \frac{d\omega_2}{d\xi} & \cdots & \frac{d\omega_b}{d\xi} \\ \frac{d^2\omega_1}{d\xi^2} & \frac{d^2\omega_2}{d\xi^2} & \cdots & \frac{d^2\omega_b}{d\xi^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^b\omega_1}{d\xi^b} & \frac{d^b\omega_2}{d\xi^b} & \cdots & \frac{d^b\omega_b}{d\xi^b} \end{vmatrix} \neq 0, \tag{2.14}$$

this form is of special interest in the applications.

The proof of this theorem includes two parts: one is KAM iteration, which is the same as [17]; the other is the measure estimates under weaker non-degeneracy condition $(A3)$, which can be obtained by following the proof of measure estimates in Xu *et al* [29]. For the sake of completeness, we give the proof of measure estimates in appendix B.

3. Application to higher dimensional beam equations

Consider dD beam equations

$$u_{tt} + B^2 u + f(u) = 0, \qquad Bu \equiv (\Delta^2 + \sigma)^{1/2} u, \ x \in \mathbb{R}^d, \ t \in \mathbb{R}, \tag{3.1}$$
$$u(t, x_1 + 2\pi, \ldots, x_d) = \ldots = u(t, x_1, \ldots, x_d + 2\pi) = u(t, x_1, x_2, \ldots, x_d),$$

where $f(u)$ is a real-analytic function near $u = 0$ with $f(0) = f'(0) = 0$. Introducing $v = u_t$, (3.1) reads

$$\begin{aligned} u_t &= v, \\ v_t &= -B^2 u - f(u). \end{aligned} \tag{3.2}$$

Letting $w = (1/\sqrt{2})B^{1/2}u - \mathrm{i}(1/\sqrt{2})B^{-1/2}v$, we thus obtain

$$\frac{1}{\mathrm{i}} w_t = Bw + \frac{1}{\sqrt{2}} B^{-1/2} f\left(B^{-1/2}\left(\frac{w+\bar{w}}{\sqrt{2}}\right)\right). \tag{3.3}$$

Equation (3.3) can be rewritten as the Hamiltonian equation

$$w_t = \mathrm{i}\frac{\partial H}{\partial \bar{w}} \tag{3.4}$$

and the corresponding Hamiltonian is

$$H = \int_{\mathbb{T}^d} (Bw)\bar{w}\, \mathrm{d}x + \int_{\mathbb{T}^d} g\left(B^{-1/2}\left(\frac{w+\bar{w}}{\sqrt{2}}\right)\right) \mathrm{d}x, \tag{3.5}$$

where g is a primitive of f (this is similar to the cases of Schrödinger equation and wave equation, see [21, 24]). The operator B with periodic boundary conditions has an exponential basis $\phi_n(x) = \sqrt{(1/(2\pi)^d)}\mathrm{e}^{\mathrm{i}\langle n,x\rangle}$ and corresponding eigenvalues

$$\mu_n = \sqrt{|n|^4 + \sigma}, \qquad n \in \mathbb{Z}^d. \tag{3.6}$$

Let

$$w(x) = \sum_{n\in\mathbb{Z}^d} q_n\phi_n(x).$$

System (3.4) is then equivalent to the lattice Hamiltonian equations

$$\dot{q}_n = \mathrm{i}\left(\mu_n q_n + \frac{\partial G}{\partial \bar{q}_n}\right), \qquad G \equiv \int_{\mathbb{T}^d} g\left(\sum_{n\in\mathbb{Z}^d} \frac{q_n\phi_n + \bar{q}_n\bar{\phi}_n}{\sqrt{2\mu_n}}\right)\mathrm{d}x, \tag{3.7}$$

with corresponding Hamiltonian function

$$H = \Lambda + G = \sum_{n\in\mathbb{Z}^d} \mu_n q_n \bar{q}_n + \int_{\mathbb{T}^d} g\left(\sum_{n\in\mathbb{Z}^d} \frac{q_n\phi_n + \bar{q}_n\bar{\phi}_n}{\sqrt{2\mu_n}}\right)\mathrm{d}x. \tag{3.8}$$

Since $f(u)$ is real analytic in u, then $g(w,\bar{w})$ is real analytic in w, $\bar{w}$: making use of $w(x) = \sum_{n\in\mathbb{Z}^d} q_n\phi_n(x)$ again, let $q = (\ldots, q_n, \ldots)_{n\in\mathbb{Z}^d}$, $\bar{q} = (\cdots, \bar{q}_n, \cdots)_{n\in\mathbb{Z}^d}$, then we may rewrite g as follows

$$g(w,\bar{w}) = \sum_{\alpha,\beta} g_{\alpha\beta} q^\alpha \bar{q}^\beta \phi^\alpha \bar{\phi}^\beta,$$

hence

$$G(q,\bar{q}) \equiv \int_{\mathbb{T}^d} g\left(\sum_{n\in\mathbb{Z}^d} \frac{q_n\phi_n + \bar{q}_n\bar{\phi}_n}{\sqrt{2\mu_n}}\right)\mathrm{d}x = \sum_{\alpha,\beta} G_{\alpha\beta} q^\alpha \bar{q}^\beta,$$

$$G_{\alpha\beta} = 0, \qquad \text{if} \sum_{n\in\mathbb{Z}^d} (\alpha_n - \beta_n)n \neq 0. \tag{3.9}$$

Next we consider the regularity of the gradient of G. To this end, let $\ell^{a,\rho}$ be the Banach spaces of all bi-infinite, complex valued sequences $q = (\ldots, q_n, \ldots)_{n\in\mathbb{Z}^d}$ with finite weighted norm

$$\|q\|_{a,\rho} = \sum_{n\in\mathbb{Z}^d} |q_n||n|^a \mathrm{e}^{|n|\rho}.$$

The convolution $q * p$ of two such sequences is defined by $(q * p)_n = \sum_m q_{n-m} p_m$.

Lemma 3.1. *For $a \geqslant 0$, $\rho > 0$, the space $\ell^{a,\rho}$ is a Banach algebra with respect to convolution of sequences and*

$$\|q * p\|_{a,\rho} \leqslant c\|q\|_{a,\rho}\|p\|_{a,\rho}$$

with a constant c depending only on a.

For the proof of lemma 3.1, see [10, 15, 16, 21, 24].

Lemma 3.2. *For $a \geqslant 0$ and $\rho > 0$, the gradient $G_{\bar{q}}$ is real analytic as a map from some neighbourhood of the origin in $\ell^{a,\rho}$ into $\ell^{a+1,\rho}$, with*

$$\|G_{\bar{q}}\|_{a+1,\rho} \leqslant c\|q\|^2_{a,\rho}. \tag{3.10}$$

For the proof of lemma 3.2, see [10, 15, 16, 21, 24].

For any $\{i_1,\ldots,i_b\} \in \mathcal{J}$, we introduce standard action-angle variables $(\theta, I) = ((\theta_1,\ldots,\theta_b),(I_1,\ldots,I_b))$ in the $(q_{i_1},\ldots,q_{i_b},\bar q_{i_1},\ldots,\bar q_{i_b})$-space by letting

$$I_j = q_{i_j}\bar q_{i_j}, \qquad j=1,\ldots,b,$$

and $q_n = z_n, \bar q_n = \bar z_n, n \neq i_1,\ldots,i_b$. Let $\mathbb{Z}_1^d = \mathbb{Z}^d\setminus\{i_1,\ldots,i_b\}$, so that system (3.7) becomes

$$\begin{aligned} &\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \omega_j + P_{I_j}, \qquad \frac{\mathrm{d}I_j}{\mathrm{d}t} = -P_{\theta_j}, \quad j=1,\ldots,b,\\ &\frac{\mathrm{d}z_n}{\mathrm{d}t} = -\mathrm{i}(\Omega_n z_n + P_{\bar z_n}), \qquad \frac{\mathrm{d}\bar z_n}{\mathrm{d}t} = \mathrm{i}(\Omega_n \bar z_n + P_{z_n}), \quad n\in\mathbb{Z}_1^d, \end{aligned} \tag{3.11}$$

where P is just G with the $(q_{i_1},\ldots,q_{i_b},\bar q_{i_1},\ldots,\bar q_{i_b},q_n,\bar q_n)$-variables expressed in terms of the $(\theta, I, z_n, \bar z_n)$ variables. The Hamiltonian associated to (3.11) (with respect to the symplectic structure $\mathrm{d}I\wedge\mathrm{d}\theta + \mathrm{i}\sum_{n\in\mathbb{Z}_1^d}\mathrm{d}z_n\wedge\mathrm{d}\bar z_n$) is given by

$$H = \langle\omega(\sigma), I\rangle + \sum_{n\in\mathbb{Z}_1^d}\Omega_n(\sigma)z_n\bar z_n + P(\theta, I, z, \bar z, \sigma), \tag{3.12}$$

where $\omega(\sigma) = (\omega_1(\sigma),\ldots,\omega_b(\sigma)) = (\mu_{i_1}(\sigma),\ldots,\mu_{i_b}(\sigma))$, $\Omega_n(\sigma) = \mu_n(\sigma)$.

Lemma 3.3. *P has the special form defined in $(A2)$, i.e. $P(\theta, I, z, \bar z, \sigma)\in\mathcal{A}$.*

For the proof, see [17]. Moreover the regularity of P holds true.

Lemma 3.4. *For any $\varepsilon > 0$ sufficiently small and $s = \varepsilon^{1/2}$, if $|I| < s^2$ and $\|z\|_{a,\rho} < s$, then*

$$\|X_P\|_{D(r,s),\mathcal{O}} \leqslant \varepsilon, \qquad \bar a = a+1. \tag{3.13}$$

Thanks to lemma 4.1 in the appendix, one can easily get the following lemma.

Lemma 3.5.

$$\begin{vmatrix} \dfrac{d\omega_1}{d\sigma} & \dfrac{d\omega_2}{d\sigma} & \cdots & \dfrac{d\omega_b}{d\sigma}\\ \dfrac{d^2\omega_1}{d\sigma^2} & \dfrac{d^2\omega_2}{d\sigma^2} & \cdots & \dfrac{d^2\omega_b}{d\sigma^2}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{d^b\omega_1}{d\sigma^b} & \dfrac{d^b\omega_2}{d\sigma^b} & \cdots & \dfrac{d^b\omega_b}{d\sigma^b} \end{vmatrix} \neq 0.$$

To verify assumption $(A5)$ in theorem 2, we need the following lemmas; first recall the definition of $\mathcal{J}$ (see (1.2)):

$$\mathcal{J} = \left\{ \begin{array}{rl} i_j = (i_{j_1},\ldots,i_{j_d}), & 1\leqslant j\leqslant b,\\ \{i_1,\ldots,i_b\} : |i_{j_{k+1}}| > 4d(i_{j_k})^2, & 1\leqslant k\leqslant d-1,\\ |i_{(j+1)_1}| > 4d(i_{j_d})^2, & 1\leqslant j\leqslant b-1. \end{array}\right\}.$$

Lemma 3.6. *For any $\{i_1,\ldots,i_b\}\in\mathcal{J}$, one has*

$$\langle k, i\rangle + n + m \neq 0,\ (k, |n|, |m|) = (-e_j - e_l, |i_j|, |i_l|), \qquad 1\leqslant j < l\leqslant b,$$

$$\langle k, i\rangle + n - m \neq 0,\ (k, |n|, |m|) = (-e_j + e_l, |i_j|, |i_l|), \qquad 1\leqslant j < l\leqslant b.$$

Proof. If $(k, |n|, |m|) = (-e_j + e_l, |i_j|, |i_l|)$, by contradiction, we assume $\langle k, i\rangle + n - m = -i_j + i_l + n - m = 0$, then $m = n + i_l - i_j$; thus $|n + i_l - i_j| = |m| = |i_l|$, i.e. $|n + i_l - i_j|^2 = |i_l|^2$; therefore

$$\langle n + i_l - i_j, n + i_l - i_j\rangle = \langle i_l, i_l\rangle,$$

i.e.

$$|i_j|^2 + \langle n, i_l\rangle - \langle n, i_j\rangle - \langle i_j, i_l\rangle = 0;$$

thus

$$\langle n - i_j, i_l - i_j\rangle = 0,$$

i.e.

$$(n_1 - i_{j_1})(i_{l_1} - i_{j_1}) + \cdots + (n_d - i_{j_d})(i_{l_d} - i_{j_d}) = 0;$$

due to $n \neq i_j$, let $n_s - i_{j_s}$ denote the first non-vanishing component among $n_d - i_{j_d}, \ldots, n_1 - i_{j_1}$, then

$$(n_1 - i_{j_1})(i_{l_1} - i_{j_1}) + \cdots + (n_s - i_{j_s})(i_{l_s} - i_{j_s}) = 0;$$

hence

$$(n_1 - i_{j_1})(i_{l_1} - i_{j_1}) + \cdots + (n_s - i_{j_s})(-i_{j_s}) = -(n_s - i_{j_s})i_{l_s}; \tag{3.14}$$

again according to $|n| = |i_j|$ and the definition of $\mathcal{J}$, one has

$$|(n_1 - i_{j_1})(i_{l_1} - i_{j_1}) + \cdots + (n_s - i_{j_s})(-i_{j_s})| \leqslant 4s|i_{l_{s-1}}|^2 < |i_{l_s}| \leqslant | - (n_s - i_{j_s})i_{l_s}|,$$

which is contradicted by equality (3.14). In conclusion, $-i_j + i_l + n - m \neq 0$. The other case can be proved analogously and lemma 3.6 is obtained. ■

To check non-resonance conditions $(A5)$, we have the following lemma.

Lemma 3.7. *For a fixed $\gamma > 0$ small enough, there exists τ sufficiently large and a subset $\mathcal{O} \subset \mathcal{I}$ with* meas$(\mathcal{I} \setminus \mathcal{O}) = O(\gamma^{\vartheta})$ *(ϑ is specified in appendix B), such that for each $\sigma \in \mathcal{O}$, one has the following non-resonance conditions*

$$\begin{aligned}
&|\langle k, \omega(\sigma)\rangle| \geqslant \frac{\gamma}{|k|^{\tau}}, \quad k \neq 0,\\
&|\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma)| \geqslant \frac{\gamma}{|k|^{\tau}},\\
&|\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma) + \Omega_m(\sigma)| \geqslant \frac{\gamma}{|k|^{\tau}},\\
&|\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma) - \Omega_m(\sigma)| \geqslant \frac{\gamma}{|k|^{\tau}}, |k| + ||n| - |m|| \neq 0.
\end{aligned} \tag{3.15}$$

The proof of lemma 3.7 is very similar to section 8 in Bambusi [2] (also see section 6 in Bambusi [1]); however, here we deal with a higher dimensional case; hence, for the sake of completeness, we give its proof in appendix A. In addition, we simplify the proof in [2]. In [2], the author separated the measure estimates into two parts: $|k|$ sufficiently large and $|k|$ small. Here we do not need to distinguish them, which reduces a lot of computation.

In addition,

$$\Omega_n = \bar{\Omega}_n + \tilde{\Omega}_n, \qquad \bar{\Omega}_n = |n|^2, \quad |\tilde{\Omega}_n|_{C^{N,1}(\mathcal{O})} = o(|n|^{-1}).$$

Now we have verified all the assumptions of theorem 2 for (3.12) with $p = 2, \iota = 1, \bar{a} - a = 1$. Consequently, theorem 1 follows by applying theorem 2.

Appendix A

The proof of lemma 3.7.

In the first KAM step, we have to exclude the resonant set such that lemma 3.7 holds true. Clearly in (3.15), when $(k, |n|) = (-e_j, |i_j|)$, $1 \leqslant j \leqslant b$, one has $\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma) \equiv 0$; similarly, when $(k, |n|, |m|) = (-e_j + e_l, |i_j|, |i_l|)$, $1 \leqslant j < l \leqslant b$, one has $\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma) - \Omega_m(\sigma) \equiv 0$. But due to lemmas 3.3 and 3.6, we will not encounter such terms in the perturbation; thus without loss of generality, we suppose $\Omega_n \neq \sqrt{|i_j|^4 + \sigma}$, $\Omega_m \neq \sqrt{|i_l|^4 + \sigma}$. We have to throw away the following resonant set

$$\mathcal{R} = \left(\bigcup_{k\neq 0}\mathcal{R}_k\right)\bigcup\left(\bigcup_{k,n}\mathcal{R}_{kn}\right)\bigcup\left(\bigcup_{k,n,m}\mathcal{R}^1_{knm}\right)\bigcup\left(\bigcup_{k,|n|\neq|m|}\mathcal{R}^2_{knm}\right),$$

where

$$\mathcal{R}_k = \left\{\sigma \in \mathcal{I} : |\langle k, \omega(\sigma)\rangle| < \frac{\gamma}{|k|^\tau}\right\}, \tag{4.1}$$

$$\mathcal{R}_{kn} = \left\{\sigma \in \mathcal{I} : |\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma)| < \frac{\gamma}{|k|^\tau}\right\}, \tag{4.2}$$

$$\mathcal{R}^1_{knm} = \left\{\sigma \in \mathcal{I} : |\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma) + \Omega_m(\sigma)| < \frac{\gamma}{|k|^\tau}\right\}, \tag{4.3}$$

$$\mathcal{R}^2_{knm} = \left\{\sigma \in \mathcal{I} : |\langle k, \omega(\sigma)\rangle + \Omega_n(\sigma) - \Omega_m(\sigma)| < \frac{\gamma}{|k|^\tau}\right\}. \tag{4.4}$$

Here we only consider the most complicated case $\mathcal{R}^2_{knm}$; the other cases can be handled in the same way.

Lemma 4.1. *For any given $i_1, \ldots, i_L \in \mathbb{Z}^d$, $|i_1| < \cdots < |i_L|$, $L \leqslant b + 2$, one has*

$$\begin{vmatrix} \frac{d\mu_{i_1}}{d\sigma} & \frac{d\mu_{i_2}}{d\sigma} & \cdots & \frac{d\mu_{i_L}}{d\sigma} \\ \frac{d^2\mu_{i_1}}{d\sigma^2} & \frac{d^2\mu_{i_2}}{d\sigma^2} & \cdots & \frac{d^2\mu_{i_L}}{d\sigma^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^L\mu_{i_1}}{d\sigma^L} & \frac{d^L\mu_{i_2}}{d\sigma^L} & \cdots & \frac{d^L\mu_{i_L}}{d\sigma^L} \end{vmatrix} \geqslant \frac{c}{|i_1|^{4L-2}|i_2|^{4L-2}\cdots|i_L|^{4L-2}}. \tag{4.5}$$

Proof. First remark that by explicit computation one has

$$\frac{d^s\mu_{i_j}}{d\sigma^s} = \frac{(2s-3)!!}{2^s}\frac{(-1)^{s+1}}{(|i_j|^4+\sigma)^{s-(1/2)}}. \tag{4.6}$$

Substituting (4.6) into the lhs of (4.5) we get the determinant to be estimated. To obtain the estimate factorize from the jth column the term $(|i_j|^4+\sigma)^{-1/2}$ and from the lth row the term $(2l-3)!!/2^l$. Forgetting the inessential powers of -1, we obtain that the determinant to be estimated is given by

$$\left[\prod_{j=1}^{L}(|i_j|^4+\sigma)^{-1/2}\right]\left[\prod_{l=1}^{L}\frac{(2l-3)!!}{2^l}\right]\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_L \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{L-1} & x_2^{L-1} & \cdots & x_L^{L-1} \end{vmatrix}, \tag{4.7}$$

where $x_j \equiv (|i_j|^4 + \sigma)^{-1}$. The last determinant is a Vandermond determinant whose value is given by

$$\prod_{j<l\leqslant L} (x_j - x_l). \tag{4.8}$$

Now one has

$$|x_j - x_l| = |(|i_j|^4 + \sigma)^{-1} - (|i_l|^4 + \sigma)^{-1}| \geqslant (|i_j|^4 + \sigma)^{-1}(|i_l|^4 + \sigma)^{-1} = x_j x_l,$$

then (4.8) is estimated by

$$\prod_{l=2}^{L}\prod_{j=1}^{l-1} x_j x_l = \prod_{l=2}^{L}\left(x_l^{l-1}\prod_{j=1}^{l-1} x_j\right) = \prod_{l=1}^{L} x_l^{L-1} = \prod_{l=1}^{L}(|i_l|^4 + \sigma)^{-(L-1)},$$

from which, using the asymptotics of the frequencies, lemma 4.1 immediately follows. □

Lemma 4.2. *(Proposition of appendix B in [3]). Let $u^{(1)}, \dots, u^{(L)}$ be L independent vectors with $\|u^{(s)}\|_{\ell^1} \leqslant 1$. Let $w \in \mathbb{R}^L$ be an arbitrary vector, then there exists $s \in [1, \dots, L]$, such that*

$$|u^{(s)} \cdot w| \geqslant \frac{\|w\|_{\ell^1} \det(u^{(s)})}{L^{\frac{3}{2}}},$$

where $\det(u^{(s)})$ is the determinant of the matrix formed by the components of the vectors $u^{(s)}$.

For the proof see [3].

Corollary 1. *For any $\sigma \in \mathcal{I}$ and any vector $w \in \mathbb{R}^L$, $L \leqslant b + 2$, there exists $s \in [1, \dots, L]$ such that*

$$\left|w \cdot \frac{d^s u}{d\sigma^s}\right| \geqslant c \frac{\|w\|_{\ell^1}}{\prod_{l=1}^{L} |i_l|^{4b+6}},$$

where $u = (\mu_{i_1}, \dots, \mu_{i_L})$.

Proof. Consider the vector

$$u^{(s)} \equiv \begin{cases} \dfrac{\frac{d^s u}{d\sigma^s}}{\left\|\frac{d^s u}{d\sigma^s}\right\|_{\ell^1}} & \text{if } \left\|\frac{d^s u}{d\sigma^s}\right\|_{\ell^1} > 1 \\ \dfrac{d^s u}{d\sigma^s} & \text{if } \left\|\frac{d^s u}{d\sigma^s}\right\|_{\ell^1} \leqslant 1 \end{cases}$$

and apply lemma 4.2. We thus get that there exists $s \in [1, \dots, L]$ such that

$$\left|w \cdot \frac{d^s u}{d\sigma^s}\right| \geqslant c \frac{\|w\|_{\ell^1}}{\prod_{l=1}^{L} |i_l|^{4b+6}}.$$

Lemma 4.3. *(Lemma 8.4 of [2]). Let $g : \mathcal{I} \to \mathbb{R}$ be $b + 3$ times differentiable and assume that*

(1) $\forall \sigma \in \mathcal{I}$ there exists $s \leqslant b + 2$ such that $g^{(s)}(\sigma) > B$,
(2) there exists A such that $|g^{(s)}(\sigma)| \leqslant A$ for $\forall \sigma \in \mathcal{I}$ and $\forall s$ with $1 \leqslant s \leqslant b + 3$.

Define

$$\mathcal{I}_h \equiv \{\sigma \in \mathcal{I} : |g(\sigma)| \leqslant h\},$$

then

$$\frac{\text{meas}(\mathcal{I}_h)}{\text{meas}(\mathcal{I})} \leqslant \frac{A}{B} 2(2+3+\cdots+(b+3)+2B^{-1})h^{1/(b+3)}.$$

For the proof see [2] and [29].

By combining lemma 4.3 and corollary 1 we get the following lemma.

Lemma 4.4. *For any* $\{i_1, \ldots, i_b\} \in \mathcal{J}$, $\omega = (\mu_{i_1}, \ldots, \mu_{i_b})$, $\Omega_n = \mu_n$, $\Omega_m = \mu_m$, *then for fixed* k, n, m, *we have*

$$\text{meas}(\mathcal{R}^2_{knm}) < c \frac{|n|^{8b+12}|m|^{8b+12}\gamma^{1/(b+3)}}{|k|^{(\tau/(b+3))+1}}.$$

Lemma 4.5.

$$\text{meas}\left(\bigcup_{k,n,m} \mathcal{R}^2_{knm}\right) < c\gamma^{\vartheta}, \qquad \vartheta > 0.$$

Proof. Suppose that $|n|^2 - |m|^2 = l \geqslant 0$. If $l > c|k|$, $\mathcal{R}^2_{knm} = \emptyset$; if $l \leqslant c|k|$, one has

$$|\Omega_n - \Omega_m - l| \leqslant O(|m|^{-1}).$$

It follows that

$$\mathcal{R}^2_{knm} \subseteq Q^2_{klm} \stackrel{\text{def}}{=} \left\{\sigma : |\langle k, \omega(\sigma)\rangle + l| < \frac{\gamma}{|k|^{\tau}} + O(|m|^{-1})\right\}. \tag{4.9}$$

Moreover, $Q^2_{klm} \subseteq Q^2_{klm_0}$ for $|m| \geqslant |m_0|$. Due to lemmas 4.4 and 4.3, one has

$$\begin{aligned}
&\text{meas}\left(\bigcup_{l \leqslant c|k|} \bigcup_{|n|^2-|m|^2=l} \mathcal{R}^2_{knm}\right) \\
&\leqslant \sum_{l \leqslant c|k|} \sum_{|m|<|m_0|} \text{meas}(\mathcal{R}^2_{knm}) + \sum_{l \leqslant c|k|} \text{meas}(Q^2_{klm_0}) \\
&< c\left(\frac{\gamma^{1/(b+3)}|m_0|^{8(2b+3)+C(d)}}{|k|^{(\tau/(b+3))-4b-6}} + \left(\frac{\gamma}{|k|^{\tau}} + O(|m_0|^{-1})\right)^{1/(b+3)}\right) \\
&< c\left(\frac{\gamma^{1/(b+3)}|m_0|^{8(2b+3)+C(d)}}{|k|^{(\tau/(b+3))-4b-6}} + O(|m_0|^{-1/(b+3)})\right),
\end{aligned} \tag{4.10}$$

where $C(d)$ is a constant depending only on space dimension d. By choosing

$$\frac{\gamma^{1/(b+3)}|m_0|^{8(2b+3)+C(d)}}{|k|^{(\tau/(b+3))-4b-6}} = |m_0|^{-1/(b+3)},$$

i.e.

$$\|m_0| = \left(\frac{|k|^{(\tau/(b+3))-4b-6}}{\gamma^{1/(b+3)}}\right)^{1/8(2b+3)+C(d)+(1/(b+3))},$$

we arrive at

$$\text{meas}\left(\bigcup_{l \leqslant c|k|} \bigcup_{|n|^2-|m|^2=l} \mathcal{R}^2_{knm}\right) < c\frac{\gamma^{\frac{1}{(b+3)^2(8(2b+3)+C(d))+b+3}}}{|k|^{\frac{(\tau/(b+3))-4b-6}{(b+3)(8(2b+3)+C(d))+1}}}. \tag{4.11}$$

Let

$$\vartheta = \frac{1}{(b+3)^2(8(2b+3)+C(d))+b+3}$$

and

$$\tau > b(b+3)^2(8(2b+3)+C(d)) + (b+3)(5b+6),$$

then

$$\begin{aligned} &\text{meas}\left(\bigcup_{k,|n|\neq|m|} \mathcal{R}^2_{knm}\right) \\ &\leqslant c\sum_k \frac{\gamma^\vartheta}{|k|^{\frac{(\tau/(b+3))-4b-6}{(b+3)(8(2b+3)+C(d))+1}}} \\ &< c\gamma^\vartheta. \end{aligned} \tag{4.12}$$

Lemma 4.5 is obtained. As a consequence, we get

$$\text{meas}(\mathcal{R}) < c\gamma^\vartheta.$$

□

Let $\mathcal{O} = \mathcal{I}\setminus\mathcal{R}$, then $\mathcal{O}$ is positive-measure subset of $\mathcal{I}$ and for each $\sigma \in \mathcal{O}$, non-resonance conditions (3.15) hold true and lemma 3.7 is obtained. ■

Appendix B

The proof of theorem 2 includes two parts: one is KAM iteration, which is the same as [17]; the other is the measure estimates under weaker non-degeneracy condition $(A3)$, which can be obtained by following the proof of measure estimates in Xu *et al* [29]. For the sake of completeness, we give the proof of measure estimates in this appendix.

Lemma 5.1. *Suppose that $g(x)$ is an m times differentiable function on the closure $\bar{I}$ of I, where $I \subset \mathbb{R}^1$ is an interval. Let $I_h = \{x||g(x)| < h\}$, $h > 0$. If for some constant $d > 0$, $|g^{(m)}(x)| \geqslant d$ for $\forall x \in I$, then* $\text{meas}(I_h) \leqslant ch^{1/m}$*, where $c = 2(2+3+\cdots+m+d^{-1})$.*

For the proof see [2] and [29].

Next, we give the proof of measure estimates under weaker non-degeneracy conditions. Since for $|k| \leqslant K \sim |\ln\varepsilon|$, according to assumption $(A5)$ and $\|X_P\| < \varepsilon$, when $|k| \leqslant K \sim |\ln\varepsilon|$, we do not need to excise the parameter set. Thus in the following, we suppose $|k| \geqslant K$.

Since the two vector groups $\{\partial^\beta\omega/\partial\xi^\beta|\forall\beta, |\beta| = r\}$ and $\{D^r_v\omega(\xi)|v \in \mathbb{R}^b\}$ are linearly equivalent, where $r > 0$ is an integer, $D^r_v\omega(\xi) = \mathrm{d}^r\omega(\xi+tv)/\mathrm{d}t^r|_{t=0}$ are direction derivatives of ω at ξ along v. By assumption $(A3)$ in section 2, for $\xi \in \mathcal{O}$ there exist b integers, $1 \leqslant r_1, \ldots, r_b \leqslant b-\kappa+1$, and b direction vectors, $v_1, \cdots, v_b \in \mathbb{R}^b$ such that

$$\text{rank}\{D^{r_1}_{v_1}\omega(\xi), \ldots, D^{r_b}_{v_b}\omega(\xi)\} = b. \tag{5.1}$$

There exists a neighbourhood of ξ, $\mathcal{O}_\xi \subset \mathcal{O}$, such that (5.1) holds on $\mathcal{O}_\xi$. Since $\mathcal{O}$ is compact, we can choose such finite neighbourhoods to cover $\mathcal{O}$, so without loss of generality, we suppose that (5.1) holds for $\forall\xi \in \mathcal{O}$.

Let the matrix $A(\xi) = (D^{r_1}_{v_1}\omega(\xi), \ldots, D^{r_b}_{v_b}\omega(\xi))$. Since $\det(A(\xi)) \neq 0$ for $\forall\xi \in \mathcal{O}$, there exists $c_1 > 0$ such that for $\forall(\xi, y) \in \mathcal{O}\times S$, $|A(\xi)y| \geqslant c_1$, where $S = \{y|y \in$

$\mathbb{R}^b$, $|y| = \sum_{j=1}^b |y_j| = 1\}$, the norm of the vector $A(\xi)y$ is in the same way as that of y. Thus, $\forall(\xi, y) \in \mathcal{O} \times S$, there exists a neighbourhood of ξ in $\mathcal{O}$, $\mathcal{O}_\xi$, and a neighbourhood of y in S, S_y such that for some i,

$$|\langle D_{v_i}^{r_i}\omega(\xi'), y'\rangle| \geqslant \frac{c_1}{2n}, \qquad \forall(\xi', y') \in \mathcal{O}_\xi \times S_y.$$

Since $\{\mathcal{O}_\xi \times S_y | (\xi, y) \in \mathcal{O} \times S\}$ covers the compact set $\mathcal{O} \times S$, there exists a finite subcover: $\mathcal{O}_1 \times S_1, \ldots, \mathcal{O}_{\bar{N}} \times S_{\bar{N}}$, such that $\bigcup_{j=1}^{\bar{N}} \mathcal{O}_j \times S_j \supset \mathcal{O} \times S$ and for $(\xi, y) \in \mathcal{O}_j \times S_j$,

$$|\langle D_{\bar{v}_i}^{\bar{r}_i}\omega(\xi), y\rangle| \geqslant \frac{c_1}{2n},$$

where $\bar{r}_i \in \{r_1, \ldots, r_b\}$ and $\bar{v}_i \in \{v_1, \ldots, v_b\}$, $j = 1, \ldots, \bar{N}$.

Now fix $|k| \geqslant K$ and suppose $k/|k| \in S_j$, then for $\xi \in \mathcal{O}_j$,

$$\left|\left\langle D_{\bar{v}_i}^{\bar{r}_i}\omega(\xi), \frac{k}{|k|}\right\rangle\right| \geqslant \frac{c_1}{2n}. \tag{5.2}$$

Let us consider small divisor $f(\xi) = \langle k, \omega\rangle + \Omega_n - \Omega_m$, which is the most complicated case. Since $1/|k| D_{\bar{v}_i}^{\bar{r}_i} f(\xi) = \langle D_{\bar{v}_i}^{\bar{r}_i}\omega(\xi), k/|k|\rangle + (D_{\bar{v}_i}^{\bar{r}_i}(\Omega_n(\xi) - \Omega_m(\xi))/|k|)$, by (5.2) and assumption $(A4)$ in section 2,

$$\frac{1}{|k|}|D_{\bar{v}_i}^{\bar{r}_i} f(\xi)| \geqslant \frac{c_1}{2n} - \frac{o(|n|^{-\iota}) + o(|m|^{-\iota})}{|k|}.$$

Since $|k| \geqslant K$, then $\forall \xi \in \mathcal{O}_j$, $|1/|k| D_{\bar{v}_i}^{\bar{r}_i} f(\xi)| \geqslant c_1/4n$, consequently, $|D_{\bar{v}_i}^{\bar{r}_i} f(\xi)| \geqslant c_1|k|/4n$. Let

$$\mathcal{R}_{knm\bar{v}_i}^j = \left\{t : |f(\xi + \bar{v}_i t)| < \frac{\gamma}{|k|^\tau}, \quad \xi \in \mathcal{O}_j, \xi + \bar{v}_i t \in \mathcal{O}_j\right\},$$

$$\mathcal{R}_{knm}^j = \left\{\xi : |f(\xi)| < \frac{\gamma}{|k|^\tau}, \quad \xi \in \mathcal{O}_j\right\}.$$

Since for $\xi + \bar{v}_i t \in \mathcal{O}_j$,

$$\left|\frac{\mathrm{d}^{\bar{r}_i}}{\mathrm{d}t^{\bar{r}_i}} f(\xi + \bar{v}_i t)\right| = |D_{\bar{v}_i}^{\bar{r}_i} f(\xi)| \geqslant \frac{c_1|k|}{4n},$$

by lemma 5.1 it follows that $\mathrm{meas}(\mathcal{R}_{knm\bar{v}_i}^j) \leqslant c_2(\frac{\gamma}{|k|^\tau})^{1/\bar{r}_i}$; hence,

$$\mathrm{meas}(\mathcal{R}_{knm}^j) \leqslant c_2(\mathrm{diam}\mathcal{O})^{b-1}\left(\frac{\gamma}{|k|^\tau}\right)^{1/\bar{r}_i}.$$

Since $k/|k|$ belongs at most to the $\bar{N}$ sets $S_1, \ldots, S_{\bar{N}}$, it follows that for $\bar{r}_i \leqslant b - \kappa + 1$

$$\begin{aligned}\mathrm{meas}(\mathcal{R}_{knm}) &\leqslant \bar{N}c_2(\mathrm{diam}\mathcal{O})^{b-1}\left(\frac{\gamma}{|k|^\tau}\right)^{1/\bar{r}_i} \\ &\leqslant \bar{N}c_2(\mathrm{diam}\mathcal{O})^{b-1}\left(\frac{\gamma}{|k|^\tau}\right)^{1/(b-\kappa+1)}.\end{aligned} \tag{5.3}$$

The following proof is similar to the proof of lemma 4.5 in appendix A; for the sake of completeness, we formulate it again.

Suppose that $|n|^2 - |m|^2 = l \geqslant 0$. If $l > c_3|k|$, $\mathcal{R}_{knm} = \emptyset$; if $l \leqslant c_3|k|$, one has

$$|\Omega_n - \Omega_m - l| \leqslant O(|m|^{-1}).$$

It follows that

$$\mathcal{R}_{knm} \subseteq Q_{klm} \overset{\text{def}}{=} \left\{ \xi : |\langle k, \omega(\xi) \rangle + l| < \frac{\gamma}{|k|^\tau} + O(|m|^{-1}) \right\}.$$

Moreover, $Q_{klm} \subseteq Q_{klm_0}$ for $|m| \geqslant |m_0|$. Due to (5.3) and lemma 5.1, one has

$$\begin{aligned} &\text{meas}\left(\bigcup_{l \leqslant c_3|k|} \bigcup_{|n|^2 - |m|^2 = l} \mathcal{R}_{knm} \right) \\ &\leqslant \sum_{l \leqslant c_3|k|} \sum_{|m| < |m_0|} \text{meas}(\mathcal{R}_{knm}) + \sum_{l \leqslant c_3|k|} \text{meas}(Q_{klm_0}) \\ &< c_4 \left(\frac{\gamma^{1/(b-\kappa+1)} |m_0|^{C(d)}}{|k|^{\tau/(b-\kappa+1)-1}} + \left(\frac{\gamma}{|k|^\tau} + O(|m_0|^{-1}) \right)^{1/(b-\kappa+1)} \right) \\ &< c_4 \left(\frac{\gamma^{1/(b-\kappa+1)} |m_0|^{C(d)}}{|k|^{(\tau/(b-\kappa+1))-1}} + O(|m_0|^{-1/(b-\kappa+1)}) \right), \end{aligned} \tag{5.4}$$

where $C(d)$ is a constant depending only on space dimension d. By choosing

$$\frac{\gamma^{1/(b-\kappa+1)} |m_0|^{C(d)}}{|k|^{(\tau/(b-\kappa+1))-1}} = |m_0|^{-1/(b-\kappa+1)},$$

i.e.

$$|m_0| = \left(\frac{|k|^{(\tau/(b-\kappa+1))-1}}{\gamma^{1/(b-\kappa+1))}} \right)^{\frac{1}{C(d)+(1/b-\kappa+1))}},$$

we arrive at

$$\text{meas}\left(\bigcup_{l \leqslant c_3|k|} \bigcup_{|n|^2 - |m|^2 = l} \mathcal{R}_{knm} \right) < c_5 \frac{\gamma^{1/((b-\kappa+1)(1+C(d)(b-\kappa+1)))}}{|k|^{\frac{(\tau/(b-\kappa+1))-1}{(b-\kappa+1)C(d)+1}}}. \tag{5.5}$$

Combining with section appendix A, let

$$\vartheta = \min\left\{ \frac{1}{(b+3)^2(8(2b+3)+C(d))+b+3}, \frac{1}{(b-\kappa+1)(1+C(d)(b-\kappa+1))} \right\}$$

and

$$\begin{aligned} \tau > \max\{&b(b+3)^2(8(2b+3)+C(d)) + (b+3)(5b+6), (b+1)(b-\kappa+1)^2 C(d) \\ &+(2+b)(b-\kappa+1)\}, \end{aligned}$$

then

$$\begin{aligned} &\text{meas}\left(\bigcup_{|k| \geqslant K, n, m} \mathcal{R}_{knm} \right) \\ &\leqslant c_6 \sum_{|k| \geqslant K} \frac{\gamma^\vartheta}{|k|^{\frac{\frac{\tau}{b-\kappa+1}-1}{(b-\kappa+1)C(d)+1}}} \\ &< c_7 \frac{\gamma^\vartheta}{K}. \end{aligned} \tag{5.6}$$

At νth KAM step, the excluded measure is $O(\gamma^\vartheta / K_\nu)$, then after infinite KAM steps, the total excluded measure is $\sum_{\nu \geqslant 1} O(\frac{\gamma^\vartheta}{K_\nu}) = O(\gamma^\vartheta)$. As a consequence, we complete the proof of measure estimates under weaker non-degeneracy conditions. □

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