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KAM tori of Hamiltonian perturbations of 1D linear beam equations ☆

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Abstract

In this paper, one-dimensional (1D) nonlinear beam equations

$$u_{tt} + u_{xxxx} + mu = f(u),$$

with hinged boundary conditions are considered; the nonlinearity f is an analytic, odd function and $f(u) = \mathrm{O}(u^3)$. It is proved that for all real parameters $m > 0$ but a set of small Lebesgue measure, the above equation admits small-amplitude quasi-periodic solutions corresponding to finite-dimensional invariant tori of an associated infinite-dimensional dynamical system. The proof is based on infinite-dimensional KAM theory developed by Kuksin [Lecture Notes in Mathematics, Vol. 1556, Springer, Berlin, 1993], Wayne [Commun. Math. Phys. 127 (1990) 479–528], Pöschel [Ann. Sc. Norm. Sup. Pisa, Cl. Sci. 23 (1996) 119–148].

1. Introduction and main result

Presently there are many significant results with respect to the beam equations, e.g., Edent and Milani [8] established a global fast dynamics by energy methods; Wang and Chen [16] derived explicit asymptotic expressions for the eigenfrequencies of the nonhomogeneous damping beam equations by the method of Birkhoff; Drabek and Lupo [7] studied the existence of generalized periodic solutions of a nonlinear beam equation

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basing on the homotopy invariance property of the Leray–Schauder degree and the shooting method of ordinary differential equations; furthermore, Feckan [9] showed the existence of periodic solutions of beams on bearings with nonlinear elastic responses by using variational methods; recently Lee [12] gave the existence of periodic solutions to the nonautonomous case. In this paper, not basing on the above mentioned methods, but basing on another method—infinite dimensional KAM theory, we will prove that the 1D nonlinear beam equation

$$u_{tt} + u_{xxxx} + mu = f(u), \tag{1.1}$$

subject to hinged boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(\pi,t) = u_{xx}(\pi,t) = 0, \tag{1.2}$$

admits small-amplitude quasi-periodic solutions for majority of $m > 0$ (in Lebesgue measure sense). Where f is a real analytic, odd function of u of the form

$$f(u) = au^3 + \sum_{k \geqslant 5} f_k u^k, \quad a \neq 0. \tag{1.3}$$

Eqs. (1.1), (1.2) may serve as the perturbation of integrable Hamiltonian equation

$$u_{tt} + u_{xxxx} + mu = 0, \tag{1.4}$$

with hinged boundary conditions (1.2). There are two different approaches to deal with the quasiperiodic solutions of Hamiltonian PDEs: one is the infinite-dimensional KAM theory which is the extension of the classical KAM theory, see Wayne [15], Kuksin [11], Pöschel [13], Chierchia and You [6]; the other one is based on Lyapunov–Schmidt procedure and techniques by Fröhlich and Spencer [10], which is established by Craig and Wayne [5] and improved by Bourgain [2–4]. Recently, Bambusi [1] found a simple way (also based on the Lyapunov–Schmidt reduction) to construct periodic solutions of 1D wave equation as well as beam equations. In his approach, the small divisor problem is cleverly avoided. However, his method fails to construct quasi-periodic solutions. In this paper, we will use the KAM approach originating from Kuksin, Wayne and Pöschel. Actually, our proof follows the line of Pöschel [13] for wave equations with some minor modifications.

A rough description of our results is as follows. Consider Eq. (1.1) with (1.2), then for all $m > 0$ but a set of small Lebesgue measure, there exist small-amplitude quasi-periodic solutions for (1.1) corresponding to a n-dimensional KAM tori of an associated infinite-dimensional Hamiltonian system. Moreover (as usual in the KAM approach) one obtains, for the constructed solutions, a local normal form which provides linear stability. In this sense, the result obtained by KAM theorem is a little stronger than that obtained by the methods of Craig, Wayne and Bourgain.

The operator $A = d^4/dx^4 + m$ has an orthonormal basis of eigenfunctions $\{\phi_j = \sqrt{2/\pi}\sin(jx)\}_{j=1}^{\infty}$ and corresponding frequencies $\{\omega_j = \sqrt{j^4 + m}\}_{j=1}^{\infty}$. Consider linearized equation (1.4), whose solutions are given by

$$u(x,t) = \sum_{j \in J} I_j \cos(\omega_j t + \xi_j)\phi_j(x), \quad J = \{j_1, \ldots, j_n\}, \tag{1.5}$$

with amplitudes $I_j \geqslant 0$ and initial phase ξ_j. The motions are periodic, quasi-periodic, respectively, depending on whether one or finitely many eigenfunctions are excited. In particular, for every choice

$$J = \{j_1 < j_2 < \cdots < j_n\} \subset \mathbb{N},$$

the corresponding quasi-periodic motions form an invariant n-dimensional torus with frequencies $\omega_{j_1}, \ldots, \omega_{j_n}$ in a suitable phase space. In addition, such torus is linearly stable, and all solutions have zero Lyapunov exponents.

Upon restoring the nonlinearity f, the invariant tori with those quasi-periodic solutions will not entirely persist due to resonances among the modes and the strong perturbing effect of f for large amplitudes. In a sufficiently small neighborhood of the origin, however, there does exist a large Cantor family of rotational n-tori which are only slightly deformed.

The following is the main result of this paper.

Theorem 1 (Main Theorem). *Consider* 1D *nonlinear beam equation* (1.1) *with the boundary condition* (1.2)*, with f a real analytic and odd function of the form* (1.3)*. Then for each index set $J = \{j_1 < \cdots < j_n\}$ with $n \geqslant 1$, there exists, for all $m > 0$ but a set of small Lebesgue measure, a Cantor manifold $\mathcal{E}_J$ of real analytic, linearly stable and Diophantine n-tori in an associated phase space carrying quasi-periodic solutions of the nonlinear beam equation.*

Remark 1. The assumption that f is odd in u is necessary. The solutions constructed below are real analytic sine-series, hence in a neighborhood of $x = 0$ they are defined, odd, and satisfy the equation. Adding the equations for $u(x,t)$ and $u(-x,t)$ one obtains $f(u) + f(-u) = 0$.

Remark 2. The result remains true for odd nonlinearities f of the form

$$f(x,u) = au^3 + \sum_{k\geqslant 5} f_k(x)u^k, \quad a \neq 0,$$

where the coefficients f_k are real analytic in x.

Remark 3. In the case $n = 1$, We will obtain the existence of periodic solutions. But with respect to periodic solutions, there have been better results even to the nonautonomous equations (see [7,9,12]).

Remark 4. The frequencies of the Diophantine tori are also under control. They are

$$\omega_*(\xi) = \omega_J + A_J\xi + \mathrm{O}\big(\|\xi\|^2\big),$$

where $\omega_J = (\omega_{j_1}, \ldots, \omega_{j_n})$, A_J is the $(n \times n)$-matrix with coefficients $A_{kl} = (6/\pi)((4 - \delta_{kl})/(\omega_{j_k}\omega_{j_l}))$ and ξ is a real parameter in $\mathbb{R}^n$. Furthermore, we may choose any finite number n of orthonormal eigenfunctions $\phi_{j_1}, \ldots, \phi_{j_n}$ and renumber them in such a way that they become the first n eigenfunctions.

Remark 5. One can use the methods of Craig, Wayne and Bourgain, showing the existence of quasi-periodic solutions, but it is much more involved.

The rest of the paper is organized as follows. In Section 2 we formulate an infinite-dimensional KAM theorem for nonlinear partial differential equations, which is given by Pöschel [14]. In Section 3 the Hamiltonian function is written in infinitely many coordinates, which is then put into partial normal form in Section 4. In Section 5 we complete the proof of Main Theorem by applying an infinite-dimensional KAM theorem given by Pöschel [14]. Some technical lemmata are proved in the Appendix.

2. An infinite-dimensional KAM theorem for partial differential equations

We consider small perturbations of an infinite-dimensional Hamiltonian in the parameter dependent normal form

$$N = \sum_{1\leqslant j\leqslant n} \omega_j(\xi) y_j + \frac{1}{2}\sum_{j\geqslant 1} \Omega_j(\xi)\big(u_j^2 + v_j^2\big)$$

on a phase space

$$\mathcal{P}^{a,\rho} = \mathbb{T}^n \times \mathbb{R}^n \times \mathcal{H}^{a,\rho} \times \mathcal{H}^{a,\rho} \ni (x, y, u, v),$$

where $\mathbb{T}^n$ is the usual n-torus with $1 \leqslant n < \infty$, and $\mathcal{H}^{a,\rho}$ is the Hilbert space of all real (later complex) sequences $w = (w_1, w_2, \ldots)$ with

$$\|w\|_{a,\rho}^2 = \sum_{j\geqslant 1} |w_j|^2 j^{2a} e^{2\rho j} < \infty,$$

where $a \geqslant 0$ and $\rho > 0$. The frequencies $\omega = (\omega_1, \ldots, \omega_n)$ and $\Omega = (\Omega_1, \Omega_2, \ldots)$ depend on n parameters $\xi \in \Pi \subset \mathbb{R}^n$, a closed bounded set of positive Lebesgue measure, in a way described below.

The Hamiltonian equations of motion of N are

$$\dot{x} = \omega(\xi), \qquad \dot{y} = 0, \qquad \dot{u} = \Omega(\xi)v, \qquad \dot{v} = -\Omega(\xi)u,$$

where $(\Omega u)_j = \Omega_j u_j$. Hence, for each $\xi \in \Pi$, there is an invariant n-dimensional torus $\mathcal{T}_0^n = \mathbb{T}^n \times \{0, 0, 0\}$ with frequencies $\omega(\xi)$, which corresponds to an elliptic fixed point in its attached uv-space with frequencies $\Omega(\xi)$. Hence $\mathcal{T}_0^n$ is linearly stable. The aim is to prove the persistence of a large portion of this family of linearly stable rotational tori under small perturbations $H = N + P$ of the Hamiltonian N. To this end the following assumptions are made.

Assumption A (*Nondegeneracy*). The map $\xi \to \omega(\xi)$ is a Lipeomorphism. Moreover, for all integer vectors $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leqslant |l| \leqslant 2$,

$$\big|\big\{\xi\colon \big(k, \omega(\xi)\big) + \big(l, \Omega(\xi)\big) = 0\big\}\big| = 0$$

and

$$\big(l, \Omega(\xi)\big) \neq 0 \quad \text{on } \Pi,$$

where $|\cdot|$ denotes Lebesgue measure for sets, $|l| = \sum_j |l_j|$ for integer vectors, and $(\cdot, \cdot)$ is the usual scalar product.

Assumption B (*Spectral Asymptotics*). There exist $d \geqslant 1$ and $\delta < d - 1$ such that

$$\Omega_j(\xi) = j^d + \cdots + \mathrm{O}(j^\delta),$$

where the dots stand for fixed lower order terms in j, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\overline{\Omega}$ with $\overline{\Omega}_j = j^d + \cdots$ such that the tails $\widetilde{\Omega}_j = \Omega_j - \overline{\Omega}_j$ give rise to a Lipschitz map

$$\widetilde{\Omega} : \Pi \to \mathcal{H}_\infty^{-\delta},$$

where $\mathcal{H}_\infty^p$ is the space of all real sequences with finite norm $\|w\|_p = \sup_j |w_j| j^p$. Note that the coefficient of j^d can always be normalized to one by rescaling the time. So there is no loss of generality by this assumption. Also, there is no restriction on finite numbers of frequencies.

Assumption C (*Regularity*). The perturbation P is real analytic in the space coordinates and Lipschitz in the parameters, and for each $\xi \in \Pi$ its Hamiltonian vector field $X_P = (P_y, -P_x, P_v, -P_u)^T$ defines near $\mathcal{T}_0^n$ a real analytic map

$$X_P : \mathcal{P}^{a,\rho} \to \mathcal{P}^{\bar{a},\rho}, \quad \begin{cases} \bar{a} \geqslant a & \text{for } d > 1, \\ \bar{a} > a & \text{for } d = 1. \end{cases}$$

We may also assume that $a - \bar{a} \leqslant \delta < d - 1$ by increasing δ, if necessary.

To make this quantitative we introduce complex $\mathcal{T}_0^n$-neighbourhoods

$$D(s,r):\ |\operatorname{Im} x| < s,\ |y| < r^2,\ \|u\|_{a,\rho} + \|v\|_{a,\rho} < r,$$

where $|\cdot|$ denotes the sup-norm for complex vectors, and weighted phase space norms

$$\|W\|_r = \|W\|_{\bar{a},r} = |X| + \frac{1}{r^2}|Y| + \frac{1}{r}\|U\|_{\bar{a},\rho} + \frac{1}{r}\|V\|_{\bar{a},\rho}$$

for $W = (X, Y, U, V)$.Then we assume that X_P is real analytic in $D(s,r)$ for some positive s, r uniformly in ξ with finite norm $\|X_P\|_{r,D(s,r)} = \sup_{D(s,r)} \|X_P\|_r$, and that the same holds for its Lipschitz semi-norm

$$\|X_P\|_r^L = \sup_{\xi \neq \zeta} \frac{\|\Delta_{\xi\zeta} X_P\|_r}{|\xi - \zeta|},$$

where $\Delta_{\xi\zeta} X_P = X_P(\cdot, \xi) - X_P(\cdot, \zeta)$, and where the supremum is taken over Π.

To state the main results we assume that

$$|\omega|_\Pi^L + \|\Omega\|_{-\delta,\Pi}^L \leqslant M < \infty, \qquad |\omega^{-1}|_{\omega(\Pi)}^L \leqslant L < \infty,$$

where the Lipschitz semi-norms are defined analogously to $\|X_P\|_r^L$. Moreover, we introduce the notations

$$\langle l \rangle_d = \max\Bigl(1, \Bigl|\sum j^d l_j\Bigr|\Bigr), \qquad A_k = 1 + |k|^\tau,$$

where $\tau \geqslant n + 1$. Finally, let $\mathcal{Z} = \{(k,l) \neq 0, |l| \leqslant 2\} \subset \mathbb{Z}^n \times \mathbb{Z}^\infty$.

Theorem 2 (Infinite-Dimensional KAM Theorem, Pöschel [14]). *Suppose* $H = N + P$ *satisfies Assumptions* A, B, C, *and*

$$\varepsilon = \|X_P\|_{r,D(s,r)} + \frac{\alpha}{M}\|X_P\|^L_{r,D(s,r)} \leqslant \gamma\alpha, \tag{2.1}$$

where $0 < \alpha \leqslant 1$ *is another parameter, and* γ *depends on* n, τ *and* s. *Then there exists a Cantor set* $\Pi_\alpha \subset \Pi$, *a Lipschitz continuous family of torus embeddings* $\Phi : \mathbb{T}^n \times \Pi_\alpha \to \mathcal{P}^{\bar{a},\rho}$, *and a Lipschitz continuous map* $\omega_* : \Pi_\alpha \to \mathbb{R}^n$, *such that for each* $\xi \in \Pi_\alpha$, *the map* Φ *restricted to* $\mathbb{T}^n \times \{\xi\}$ *is a real analytic embedding of a rotational torus with frequencies* $\omega_*(\xi)$ *for the Hamiltonian* H *at* ξ.

Each embedding is real analytic on $|\operatorname{Im} x| < \frac{s}{2}$, *and*

$$\|\Phi - \Phi_0\|_r + \frac{\alpha}{M}\|\Phi - \Phi_0\|^L_r \leqslant c\varepsilon/\alpha,$$
$$|\omega_* - \omega| + \frac{\alpha}{M}|\omega_* - \omega|^L \leqslant c\varepsilon,$$

uniformly on that domain and Π_α, *where* Φ_0 *is the trivial embedding* $\mathbb{T}^n \times \Pi \to \mathcal{T}^n_0$, *and* $c \leqslant \gamma^{-1}$ *depends on the same parameters as* γ.

Moreover, there exist Lipschitz maps ω_ν *and* Ω_ν *on* Π *for* $\nu \geqslant 0$ *satisfying* $\omega_0 = \omega$, $\Omega_0 = \Omega$ *and*

$$|\omega_\nu - \omega| + \frac{\alpha}{M}|\omega_\nu - \omega|^L \leqslant c\varepsilon,$$
$$\|\Omega_\nu - \Omega\|_{-\delta} + \frac{\alpha}{M}\|\Omega_\nu - \Omega\|^L_{-\delta} \leqslant c\varepsilon,$$

such that $\Pi \setminus \Pi_\alpha \subset \bigcup \mathcal{R}^\nu_{kl}(\alpha)$, *where*

$$\mathcal{R}^\nu_{kl}(\alpha) = \left\{\xi \in \Pi:\ \left|\big(k, \omega_\nu(\xi)\big) + \big(l, \Omega_\nu(\xi)\big)\right| < \alpha\frac{\langle l\rangle_d}{A_k}\right\},$$

and the union is taken over all $\nu \geqslant 0$ *and* $(k, l) \in \mathcal{Z}$ *such that* $|k| > K_0 2^{\nu-1}$ *for* $\nu \geqslant 1$ *with a constant* $K_0 \geqslant 1$ *depending only on* n *and* τ.

Remark 1. Around each torus there exists another normal form of the Hamiltonian having an elliptic fixed point in the uv-space. Thus all the tori are linearly stable. Moreover, their frequencies are Diophantine.

Remark 2. The role of the parameter α is the following. In applications the size of the perturbation usually depends on a small parameter, for example the size of the neighbourhood around an elliptic fixed point. One then wants to choose α as another function of this parameter in order to obtain useful estimates for $|\Pi \setminus \Pi_\alpha|$.

Remark 3. Theorem 2 only requires the frequency map $\xi \to \omega(\xi)$ to be Lipschitz continuous, but not to be a homeomorphism or Lipeomorphism.

3. The Hamiltonian setting of beam equations

Let us rewrite the beam equation (1.1) as follows

$$u_{tt} + Au = f(u), \quad Au \equiv u_{xxxx} + mu, \ x, t \in \mathbb{R}, \tag{3.1}$$

$$u(0,t) = u_{xx}(0,t) = u(\pi,t) = u_{xx}(\pi,t) = 0, \tag{3.2}$$

Eq. (3.1) may be rewritten as

$$\dot{u} = v, \qquad \dot{v} + Au = f(u), \tag{3.3}$$

which, as is well known, may be viewed as the (infinite-dimensional) Hamiltonian equations $\dot{u} = H_v$, $\dot{v} = -H_u$ associated to the Hamiltonian

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)\,dx, \tag{3.4}$$

where g is a primitive of $(-f)$ (with respect to the u variable) and $\langle\cdot,\cdot\rangle$ denotes the scalar product in L^2.

As in [13], we introduce coordinates $q = (q_1, q_2, \ldots)$, $p = (p_1, p_2, \ldots)$ through the relations

$$u(x) = \sum_{j\geqslant 1} \frac{q_j}{\sqrt{\omega_j}}\phi_j(x), \qquad v = \sum_{j\geqslant 1} \sqrt{\omega_j}\, p_j \phi_j(x),$$

where $\phi_j = \sqrt{2/\pi}\sin jx$ for $j = 1, 2, \ldots$ are the orthonormal eigenfunctions of the operator A with eigenvalues $\omega_j^2 = j^4 + m$. The coordinates are taken from some Hilbert space $\mathcal{H}^{a,\rho}$ of all real valued sequences $w = (w_1, w_2, \ldots)$ with finite norm

$$\|w\|_{a,\rho}^2 = \sum_{j\geqslant 1} |w_j|^2 j^{2a} e^{2j\rho}.$$

Below we will assume that $a \geqslant 0$ and $\rho > 0$. We formally obtain the Hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geqslant 1}\omega_j\bigl(p_j^2 + q_j^2\bigr) + \int_0^\pi g\Biggl(\sum_{j\geqslant 1}\frac{q_j}{\sqrt{\omega_j}}\phi_j\Biggr)dx \tag{3.5}$$

with the lattice Hamiltonian equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \omega_j p_j, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} = -\omega_j q_j - \frac{\partial G}{\partial q_j}. \tag{3.6}$$

Rather than discussing the above formal validity, we shall, following [13] or [6], use the following elementary observation (proved in the Appendix):

Lemma 3.1. *Let I be an interval and let*

$$t \in I \to \bigl(q(t), p(t)\bigr) \equiv \bigl(\{q_j(t)\}_{j\geqslant 1}, \{p_j(t)\}_{j\geqslant 1}\bigr)$$

be an analytic solution of (3.6) *such that*

$$\sup_{t\in I}\sum_{j\geqslant 1}\big(\left|q_j(t)\right|^2+\left|p_j(t)\right|^2\big)j^{2a}e^{2j\rho}<\infty \tag{3.7}$$

for some $\rho>0$ *and* $a\geqslant 0$. *Then*

$$u(t,x)\equiv\sum_{j\geqslant 1}\frac{q_j(t)}{\sqrt{\omega_j}}\phi_j(x),$$

is an analytic solution of (3.1).

Next we consider the regularity of the gradient of G. To this end, let $\mathcal{H}_b^2$ and L^2, respectively, be the Hilbert spaces of all bi-infinite, square summable sequences with complex coefficients and all square-integrable complex valued functions on $[-\pi,\pi]$. Let

$$\mathcal{F}:\mathcal{H}_b^2\to L^2,\quad q\mapsto\mathcal{F}q=\frac{1}{\sqrt{2\pi}}\sum_j q_je^{\mathrm{i}jx},$$

be the inverse discrete Fourier transform, which defines an isometry between the two spaces. The subspaces $\mathcal{H}_b^{a,\rho}\subset\mathcal{H}_b^2$ consist, by definition, of all bi-infinite sequences with finite norm

$$\|q\|_{a,\rho}^2=|q_0|^2+\sum_j|q_j|^2|j|^{2a}e^{2|j|\rho}.$$

Through $\mathcal{F}$ they define subspaces $W^{a,\rho}\subset L^2$ that are normed by setting $\|\mathcal{F}q\|_{a,\rho}=\|q\|_{a,\rho}$.

Lemma 3.2. *For* $a>\frac{1}{2}$, *the space* $\mathcal{H}_b^{a,\rho}$ *is a Hilbert algebra with respect to convolution of sequences, and*

$$\|q*p\|_{a,\rho}\leqslant c\|q\|_{a,\rho}\|p\|_{a,\rho}$$

with a constant c *depending only on* a. *Consequently,* $W^{a,\rho}$ *is a Hilbert algebra with respect to multiplication of functions.*

Lemma 3.3. *For* $a\geqslant 0$ *and* $\rho>0$, *the gradient* G_q *is real analytic as a map from some neighbourhood of the origin in* $\mathcal{H}^{a,\rho}$ *into* $\mathcal{H}^{a+2,\rho}$, *with*

$$\|G_q\|_{a+2,\rho}=\mathrm{O}\big(\|q\|_{a,\rho}^3\big).$$

The proof of Lemma 3.2 and Lemma 3.3 is given in the Appendix.

For the nonlinearity $(-u^3)$ we find

$$G=\frac{1}{4}\int_0^{\pi}\left|u(x)\right|^4dx=\frac{1}{4}\sum_{i,j,k,l}G_{ijkl}q_iq_jq_kq_l \tag{3.8}$$

with

$$G_{ijkl} = \frac{1}{\sqrt{\omega_i\omega_j\omega_k\omega_l}} \int_0^\pi \phi_i\phi_j\phi_k\phi_l \, dx. \tag{3.9}$$

It is not difficult to verify that $G_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$, for some combination of plus and minus signs. Particularly, we have

$$G_{iijj} = \frac{1}{2\pi} \frac{2 + \delta_{ij}}{\omega_i\omega_j} \tag{3.10}$$

by elementary calculation.

In the following, we focus on the nonlinearity $(-u^3)$, since a non-zero coefficient in front of u^3 makes no difference.

4. Partial Birkhoff normal form

Next we transform the Hamiltonian (3.5) into some partial Birkhoff form of order four so that it may serve as a small perturbation of some nonlinear integrable system in a sufficiently small neighborhood of the origin.

We first switch to complex coordinates

$$z_j = \frac{1}{\sqrt{2}}(q_j + i p_j), \qquad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - i p_j),$$

then Hamiltonian (with respect to the symplectic structure $\mathrm{i}\sum_j dz_j \wedge d\bar{z}_j$) is given by

$$H = \Lambda + G = \sum_j \omega_j |z_j|^2 + \int_0^\pi g\left(\sum_j \frac{z_j + \bar{z}_j}{\sqrt{2\omega_j}} \phi_j\right) dx. \tag{4.1}$$

Lemma 4.1. *If i, j, k, l are non-zero integers, such that $i \pm j \pm k \pm l = 0$, but $(i, j, k, l) \neq (p, -p, q, -q)$, then for $m \notin S$ (S is a set of small measure in $(0, \infty)$),*

$$|\omega_i \pm \omega_j \pm \omega_k \pm \omega_l| \geqslant \frac{1}{\sqrt{h^4 + m}^3}, \quad h = \min\big(|i|, |j|, |k|, |l|\big). \tag{4.2}$$

The long but elementary proof is given in the Appendix. We remark that the result in the above lemma for beam equations is weaker than the corresponding result for wave equations (see [13]) where (4.2) holds for all m. This is due to the difference of eigenvalues in two cases.

Proposition 4.1. *For each finite $n \geqslant 1$ and all $m \in (0, \infty) \setminus S$ (S is a set of small measure in $(0, \infty)$), there exists a real analytic, symplectic change of coordinates X_F^1 in some neighborhood of the origin that takes the Hamiltonian $H = \Lambda + G$ with nonlinearity (3.8) into*

$$H \circ X_F^1 = \Lambda + \overline{G} + \widehat{G} + K,$$

where $X_{\overline{G}}, X_{\widehat{G}}, X_K : \mathcal{H}^{a,\rho} \to \mathcal{H}^{a+2,\rho}$,

$$\overline{G} = \frac{1}{2} \sum_{\min(i,j) \leqslant n} \overline{G}_{ij} |z_i|^2 |z_j|^2$$

with $\overline{G}_{ij} = (6/\pi)((4 - \delta_{ij})/\omega_i \omega_j)$, *and*

$$|\widehat{G}| = \mathrm{O}\big(\|\hat{z}\|_{a,\rho}^4\big), \qquad |K| = \mathrm{O}\big(\|z\|_{a,\rho}^6\big),$$

$\hat{z} = (z_{n+1}, z_{n+2}, \ldots)$. *Moreover, the neighbourhood can be chosen uniformly for every compact* m*-interval in* $(0, \infty)$, *and the dependence of* F *on* m *is real analytic.*

Thus, Hamiltonian $\Lambda + \overline{G}$ is integrable with integrals $|z_j|^2$, $j = 1, 2, \ldots,$ while the non-normalized fourth order term $\widehat{G}$ is not integrable, but independent of the first n modes.

Proof. For convenience we introduce another coordinates $(\ldots, w_{-2}, w_{-1}, w_1, w_2, \ldots)$ in $\mathcal{H}_b^{a,\rho}$ by setting $z_j = w_j$, $\bar{z}_j = w_{-j}$. The Hamiltonian then reads

$$\begin{aligned} H &= \Lambda + G \\ &= \sum_{j \geqslant 1} \omega_j z_j \bar{z}_j + \frac{1}{4} \sum_{i,j,k,l} G_{ijkl} (z_i + \bar{z}_i) \cdots (z_l + \bar{z}_l) \\ &= \sum_{j \geqslant 1} \omega_j w_j w_{-j} + \sum_{i,j,k,l} G_{ijkl} w_i w_j w_k w_l. \end{aligned}$$

by second order Taylor formula, we formally obtain

$$\begin{aligned} H \circ X_F^1 &= H + \{H, F\} + \int_0^1 (1-t) \big\{\{H, F\}, F\big\} \circ X_F^t \, dt \\ &= \Lambda + G + \{\Lambda, F\} + \{G, F\} + \int_0^1 (1-t) \big\{\{H, F\}, F\big\} \circ X_F^t \, dt, \end{aligned}$$

where $\{H, F\}$ denotes the Poisson bracket of H and F. We will find a function F of the form

$$F = \sum_{i,j,k,l} F_{ijkl} w_i w_j w_k w_l$$

satisfying the equation

$$G + \{\Lambda, F\} = \Bigg(\sum_{(i,j,k,l) \in \mathcal{N}_n} + \sum_{(i,j,k,l) \notin \mathcal{L}_n} \Bigg) G_{ijkl} w_i w_j w_k w_l = \overline{G} + \widehat{G},$$

where

$$\mathcal{L}_n = \big\{(i, j, k, l) \in \mathbb{Z}^4 \colon 0 \neq \min\big(|i|, |j|, |k|, |l|\big) \leqslant n\big\},$$

and $\mathcal{N}_n \subset \mathcal{L}_n$ is the subset of all $(i,j,k,l) \equiv (p,-p,q,-q)$. That is, they are of the form $(p,-p,q,-q)$ or some permutation of it. While $\{\Lambda, F\} = -\mathrm{i}\sum_{i,j,k,l}(\pm\omega_i \pm \omega_j \pm \omega_k \pm \omega_l)F_{ijkl}w_iw_jw_kw_l$, according to Lemma 4.1, we obtain

$$\mathrm{i}F_{ijkl} = \begin{cases} \frac{G_{ijkl}}{\pm\omega_i \pm\omega_j \pm\omega_k \pm\omega_l} & \text{for } (i,j,k,l) \in \mathcal{L}_n \setminus \mathcal{N}_n, \\ 0 & \text{otherwise.} \end{cases}$$

Returning into the notations $z_j, \bar{z}_j$, we have

$$\overline{G} = \frac{1}{2}\sum_{\min(i,j)\leqslant n} \overline{G}_{ij}|z_i|^2|z_j|^2$$

with

$$\overline{G}_{ij} = \begin{cases} 24G_{iijj} = \frac{24}{\pi}\frac{1}{\omega_i\omega_j} & \text{for } i \neq j, \\ 12G_{iiii} = \frac{18}{\pi}\frac{1}{\omega_i\omega_j} & \text{for } i = j, \end{cases}$$

by (3.10), while $\widehat{G}$ is independent of the first n coordinates. Hence we formally have $H \circ X_F^1 = \Lambda + \overline{G} + \widehat{G} + K$ as claimed.

To prove analyticity and regularity of the preceding transformation we first show that

$$X_F : \mathcal{H}_b^{a,\rho} \to \mathcal{H}_b^{a+2,\rho}.$$

Indeed, by Lemma 4.1 and Eq. (3.9), and with $\widetilde{w}_j = \frac{|w_j|+|w_{-j}|}{|j|}$, then for majority of $m > 0$, we have

$$\left|\frac{\partial F}{\partial w_l}\right| \leqslant \sum_{\pm i\pm j\pm k=l} |F_{ijkl}||w_iw_jw_k| \leqslant \frac{c}{|l|}\sum_{\pm i\pm j\pm k=l}\frac{|w_iw_jw_k|}{|ijk|}$$
$$\leqslant \frac{c}{|l|}\sum_{i+j+k=l}\widetilde{w}_i\widetilde{w}_j\widetilde{w}_k = \frac{c}{|l|}(\widetilde{w} * \widetilde{w} * \widetilde{w})_l.$$

Hence by Lemma 3.2,

$$\|F_w\|_{a+2,\rho} \leqslant c\|\widetilde{w} * \widetilde{w} * \widetilde{w}\|_{a+1,\rho} \leqslant c\|\widetilde{w}\|_{a+1,\rho}^3 \leqslant c\|w\|_{a,\rho}^3. \tag{4.3}$$

The analyticity of F_w follows from the analyticity of each component function and its local boundedness. Hence in a sufficiently small neighborhood of the origin in $\mathcal{H}^{a,\rho}$ the time-1-map X_F^1 is well defined with the estimates

$$\left\|X_F^1 - id\right\|_{a+2,\rho} = \mathrm{O}\left(\|w\|_{a,\rho}^3\right), \qquad \left\|DX_F^1 - I\right\|_{a+2,a,\rho}^{\mathrm{op}} = \mathrm{O}\left(\|w\|_{a,\rho}^2\right),$$

where the operator norm $\|\cdot\|_{r,a,\rho}^{\mathrm{op}}$ is defined by

$$\|A\|_{r,a,\rho}^{\mathrm{op}} = \sup_{w\neq 0}\frac{\|Aw\|_{r,\rho}}{\|w\|_{a,\rho}}.$$

Obviously, $\|DX_F^1 - I\|_{a+2,a+2,\rho}^{\mathrm{op}} \leqslant \|DX_F^1 - I\|_{a+2,a}^{\mathrm{op}}$, while in a sufficiently small neighbourhood of the origin, DX_F^1 defines an isomorphism of $\mathcal{H}^{a+2,\rho}$. Since

$$X_H : \mathcal{H}_{a,\rho} \to \mathcal{H}^{a+2,\rho},$$

then

$$X_{H\circ X_F^1}:\mathcal{H}^{a,\rho}\to\mathcal{H}^{a+2,\rho}.$$

The same holds for the Lie bracket: The boundedness of $\|DX_F\|^{\mathrm{op}}_{a+2,a,\rho}$ implies that

$$[X_F,X_H]=X_{\{H,F\}}:\mathcal{H}^{a,\rho}\to\mathcal{H}^{a+2,\rho}.$$

These two facts show that $X_K:\mathcal{H}^{a,\rho}\to\mathcal{H}^{a+2,\rho}$. The analogous claims for $X_{\overline{G}}$ and $X_{\widehat{G}}$ are obvious. □

5. Proof of main theorem

We prove Theorem 1 by applying Theorem 2, in fact, we may further prove the following theorem.

Theorem 3. *Consider the Hamiltonian*

$$H=\Lambda+\overline{G}+\widehat{G}+K,$$

where

$$\Lambda=(\alpha,I)+(\beta,Z),$$
$$\overline{G}=\frac{1}{2}(AI,I)+(BI,Z),$$
$$|\widehat{G}|=\mathrm{O}\big(\|\hat{z}\|^4_{a,\rho}\big),\qquad |K|=\mathrm{O}\big(\|z\|^6_{a,\rho}\big),$$

with $\alpha=(\omega_1,\ldots,\omega_n)$, $\beta=(\omega_{n+1},\omega_{n+2},\ldots)$, $A=(\overline{G}_{ij})_{1\leqslant i,j\leqslant n}$, $B=(\overline{G}_{ij})_{1\leqslant j\leqslant n<i}$ *and* $I=(|z_1|^2,\ldots,|z_n|^2)$, $Z=(|z_{n+1}|^2,|z_{n+2}|^2,\ldots)$. *Furthermore,* $X_{\overline{G}},X_{\widehat{G}},X_K:\mathcal{H}^{a,\rho}\to\mathcal{H}^{a+2,\rho}$, *then for* $0<\alpha\leqslant 1$ *another parameter depending on* n, τ *and* s, *there exists a Cantor set* $\Pi_\alpha\subset\Pi$, *a Lipschitz continuous family of torus embeddings* $\Phi:\mathbb{T}^n\times\Pi_\alpha\to\mathcal{P}^{a+2,\rho}$ *which is a higher order perturbation of the inclusion map* $\Phi_0:\mathbb{T}^n\times\Pi\to\mathcal{T}_0^n$, *and a Lipschitz continuous map* $\omega_*:\Pi_\alpha\to\mathbb{R}^n$, *such that for each* $\xi\in\Pi_\alpha$, *the map* Φ *restricted to* $\mathbb{T}^n\times\{\xi\}$ *is a real analytic embedding of a rotational torus with frequencies* $\omega_*(\xi)$ *for the Hamiltonian* H *at* ξ.

Remark. Theorem 1 is the direct consequence of Theorem 3.

Proof. We introduce symplectic polar and real coordinates by setting

$$z_j=\begin{cases}\sqrt{(\xi_j+y_j)}e^{-\mathrm{i}x_j}, & 1\leqslant j\leqslant n,\\ \frac{1}{\sqrt{2}}(u_j+\mathrm{i}v_j), & j\geqslant n+1,\end{cases}$$

depending on parameters $\xi\in\Pi=[0,1]^n$. Then we have

$$\mathrm{i}\sum_{j\geqslant 1}dz_j\wedge d\bar{z}_j=\sum_{1\leqslant j\leqslant n}dx_j\wedge dy_j+\sum_{j\geqslant n+1}du_j\wedge dv_j,$$

$I = \xi + y$ and $Z = \frac{1}{2}(u^2 + v^2)$, with the obvious component-by-component interpretation. The normal form becomes

$$\Lambda + \overline{G} = \big(\omega(\xi), y\big) + \frac{1}{2}\big(\Omega(\xi), u^2 + v^2\big) + \widetilde{G}$$

with frequencies $\omega(\xi) = \alpha + A\xi$, $\Omega(\xi) = \beta + B\xi$ and remainder $\widetilde{G} = \mathrm{O}(|y|^2) + \mathrm{O}(|y|\|u^2 + v^2\|)$. The total Hamiltonian is $H = N + P$ with $P = \widetilde{G} + \widehat{G} + K$.

In the following, we will verify the assumptions A, B and C for the above Hamiltonian. Since

$$A = (\overline{G}_{ij})_{1\leqslant i,j\leqslant n} = \frac{6}{\pi}\begin{pmatrix} \frac{3}{\omega_1^2} & \frac{4}{\omega_1\omega_2} & \cdots & \frac{4}{\omega_1\omega_n} \\ \frac{4}{\omega_2\omega_1} & \frac{3}{\omega_2^2} & \cdots & \frac{4}{\omega_2\omega_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{4}{\omega_n\omega_1} & \frac{4}{\omega_n\omega_2} & \cdots & \frac{3}{\omega_n^2} \end{pmatrix}$$

then $\det(A) = (-1)^{n-1}G^n/(\omega_1^2\cdots\omega_n^2)(4n-1) \neq 0$. Thus A is invertible and the map $\xi \to \omega(\xi)$ is a lipeomorphism of $\mathbb{R}^n$ onto itself. The measure condition is satisfied, since $(k, \omega(\xi)) + (l, \Omega(\xi))$ is a non-trivial affine function of ξ which vanishes on a codimension 1 subspace. Finally, clearly $(l, \beta) \neq 0$, for $1 \leqslant |l| \leqslant 2$, and $B\xi$ is small because of $|\xi|$ small with

$$B = (\overline{G}_{ij})_{1\leqslant j\leqslant n<i} = \frac{6}{\pi}\begin{pmatrix} \frac{4}{\omega_{n+1}\omega_1} & \cdots & \frac{4}{\omega_{n+1}\omega_n} \\ \frac{4}{\omega_{n+2}\omega_1} & \cdots & \frac{4}{\omega_{n+2}\omega_n} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

hence $(l, \Omega(\xi)) \neq 0$ on Π. Consequently the Assumption A is satisfied.

To verify the Assumption B, we note that

$$\omega_j = \sqrt{j^4 + m} = j^2 + \frac{m}{2j^2} + \mathrm{O}\big(j^{-6}\big).$$

Then

$$\Omega_{j-n} = (\beta + B\xi)_{j-n} = \omega_j + \frac{(v, \xi)}{\omega_j},$$

with $v = 24/\pi(\omega_1^{-1}, \ldots, \omega_n^{-1})$, by the notation. This gives the asymptotic expansion

$$\Omega_{j-n} = j^2 + \frac{m_\xi}{j^2} + \mathrm{O}\big(j^{-6}\big),$$

$m_\xi = \frac{1}{2}m + (v, \xi)$. The Assumption B is satisfied if one takes $d = 2$, $\delta = -2$ and $\overline{\Omega} = \beta$ since $B_{ij} = \mathrm{O}(i^{-2})$ uniformly for $1 \leqslant j \leqslant n$, $\widetilde{\Omega}_j = \Omega_j - \overline{\Omega}_j$ is a Lipschitz map from Π to $\mathcal{H}_\infty^2$.

The Assumption C is verified as follows. Since $X_{\overline{G}} : \mathcal{H}^{a,\rho} \to \mathcal{H}^{a+2,\rho}$, then $X_{\widetilde{G}} : \mathcal{H}^{a,\rho} \to \mathcal{H}^{a+2,\rho}$. Therefore in terms of regularity of $X_{\widetilde{G}}, X_{\widehat{G}}, X_K$, we obtain

$$X_P : \mathcal{H}^{a,\rho} \to \mathcal{H}^{a+2,\rho}.$$

Thus the Assumption C holds true.

Furthermore, we choose γ, α such that $c_1 r^2 \leqslant \gamma\alpha \leqslant c_2 r^{\frac{4}{3}}$, where c_1, c_2 are constants. Consider the phase space domain

$$D(1,r)\colon\ |\operatorname{Im} x| < 1,\ |y| < r^2,\ \|u\|_{a,\rho} + \|v\|_{a,\rho} < r,$$

and the parameter domain

$$E_{\alpha,r} = U_\alpha E_r, \quad E_r = \left\{\xi\colon\ 0 < \xi < r^{4/3}\right\},$$

where $U_\alpha E_r$ is the subset of all points in E_r with boundary distance greater than α.

The total Hamiltonian H is well defined on these domains. And $|\widetilde{G}| = \mathrm{O}(r^4)$, $|\widehat{G}| = \mathrm{O}(r^4)$, $|K| = \mathrm{O}(r^6)$ on $D(1,r)$. Using Cauchy estimates, we obtain

$$\|X_{\widetilde{G}}\|_{r/2,D(1/2,r/2)} + \|X_{\overline{G}}\|_{r/2,D(1/2,r/2)} + \|X_K\|_{r/2,D(1/2,r/2)} = \mathrm{O}\big(r^2\big).$$

Using again Cauchy estimates with respect to ξ on $E_{\alpha,r}$, we have

$$\|X_{\widetilde{G}}\|^L_{r/2} + \|X_{\overline{G}}\|^L_{r/2} + \|X_K\|^L_{r/2} = \mathrm{O}\big(r^2/\alpha\big).$$

Altogether we obtain

$$\|X_P\|_{r/2,D(1/2,r/2)} + \alpha\|X_P\|^L_{r/2,D(1/2,r/2)} = \mathrm{O}\big(r^2\big).$$

Thus Eq. (2.1) holds true.

In conclusion, all the conditions of Theorem 2 are satisfied, thus Theorem 3 is obtained by applying Theorem 2. □

Appendix A

A.1. The proof of Lemma 3.1

From the hypotheses it follows that the eigenfunctions ϕ_n are analytic and bounded with, in particular,

$$\sup_{\mathbb{R}}\big(|\phi_n'| + |\phi_n''|\big) \leqslant c\omega_n.$$

Thus the sum defining $u(x,t)$ is uniformly convergent in some complex neighbourhood of the x-interval $[0,\pi]$ and some complex disc around a given t in I. Therefore u is real analytic in x and t, and we may differentiate under the summation sign. Thus

$$\frac{\partial G}{\partial q_j} = -\frac{1}{\sqrt{\omega_j}}\big\langle f(u), \phi_j\big\rangle,$$

hence

$$\begin{aligned} u_{tt} + Au &= \sum_{j\geqslant 1}\frac{\ddot{q}_j}{\sqrt{\omega_j}}\phi_j + \sum_{j\geqslant 1}\frac{q_j}{\sqrt{\omega_j}}A\phi_j \\ &= \sum_{j\geqslant 1}\frac{1}{\sqrt{\omega_j}}\left(-\omega_j^2 q_j - \frac{\partial G}{\partial q_j}\omega_j\right)\phi_j + \sum_{j\geqslant 1}\frac{q_j}{\sqrt{\omega_j}}\omega_j^2\phi_j \\ &= \sum_{j\geqslant 1}\big\langle f(u), \phi_j\big\rangle\phi_j = f(u). \end{aligned}$$

We conclude that

$$u_{tt} + Au = f(u)$$

as we wanted to show.

A.2. The Banach algebra property

Consider the Hilbert space $\mathcal{H}^{a,\rho}$ of all doubly infinite complex sequences $q = (\dots, q_{-1}, q_0, q_1, \dots)$ with

$$\|q\|_{a,\rho}^2 = \sum_j |q_j|^2 [j]^{2a} e^{2\rho|j|} < \infty, \quad [j] = \max\bigl(|j|, 1\bigr).$$

The convolution $q * p$ of two such sequences is defined by $(q * p)_j = \sum_k q_{j-k} p_k$.

A.3. The proof of Lemma 3.2

Let $\gamma_{jk} = ([j-k][k]/[j])$. By the Schwarz inequality,

$$\left|\sum_k x_k\right|^2 = \left|\sum_k \frac{\gamma_{jk}^a x_k}{\gamma_{jk}^a}\right|^2 \leqslant c_j^2 \sum_k \gamma_{jk}^{2a} |x_k|^2, \quad c_j^2 = \sum_k \frac{1}{\gamma_{jk}^{2a}},$$

for all j. We have

$$\frac{1}{\gamma_{jk}} \leqslant \frac{[j-k] + [k]}{[j-k][k]} = \frac{1}{[j-k]} + \frac{1}{[k]},$$

so that

$$c_j^2 \leqslant \sum_k \left(\frac{1}{[j-k]} + \frac{1}{[k]}\right)^{2a} \leqslant 4^a \sum_k \frac{1}{[k]^{2a}} \stackrel{\text{def}}{=} c^2 < \infty$$

for all j. It follows that for $\rho > 0$

$$\begin{aligned}
\|q * p\|_{a,\rho}^2 &= \sum_j [j]^{2a} \left|\sum_k q_{j-k} p_k\right|^2 e^{2\rho|j|} \\
&\leqslant c^2 \sum_j [j]^{2a} \sum_k \gamma_{jk}^{2a} |q_{j-k} p_k|^2 e^{2\rho|j-k|} e^{2\rho|k|} \\
&= c^2 \sum_{j,k} [j-k]^{2a} |q_{j-k}|^2 e^{2\rho|j-k|} [k]^{2a} |p_k|^2 e^{2\rho|k|} = c^2 \|q\|_{a,\rho}^2 \|p\|_{a,\rho}^2.
\end{aligned}$$

A.4. The proof of Lemma 3.3

Due to

$$G = \frac{1}{4} \int_0^\pi \bigl|u(x)\bigr|^4 dx = \frac{1}{4} \sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l$$

then

$$\frac{\partial G}{\partial q_l}=\sum_{i,j,k}G_{ijkl}q_iq_jq_k.$$

Hence

$$\begin{aligned}
\|G_q\|_{a+2,\rho}^2&=\sum_{l\geqslant 1}|G_{q_l}|^2l^{2(a+2)}e^{2\rho l}\\
&\leqslant c\sum_{l\geqslant 1}\sum_{\pm i\pm j\pm k=l}\left(\frac{1}{\sqrt{\omega_i\omega_j\omega_k\omega_l}}|q_iq_jq_k|\right)^2l^{2(a+2)}e^{2\rho l}\\
&\leqslant c\sum_{l\geqslant 1}\left(\frac{1}{l}\right)^2\sum_{\pm i\pm j\pm k=l}\left(\frac{q_iq_jq_k}{|i||j||k|}\right)^2l^{2(a+2)}e^{2\rho l}\\
&\leqslant c\sum_{l\geqslant 1}\frac{1}{l^2}(\tilde q*\tilde q*\tilde q)_l^2l^{2(a+2)}e^{2\rho l}\leqslant c\sum_{l\geqslant 1}(\tilde q*\tilde q*\tilde q)_l^2l^{2(a+1)}e^{2\rho l}\\
&\leqslant c\|\tilde q*\tilde q*\tilde q\|_{a+1,\rho}^2\leqslant c\big(\|\tilde q\|_{a+1,\rho}^2\big)^3\leqslant c\big(\|q\|_{a,\rho}^2\big)^3.
\end{aligned}$$

So

$$\|G_q\|_{a+2,\rho}\leqslant c\big(\|q\|_{a,\rho}\big)^3.$$

The regularity of X_G follows from the regularity of its components.

A.5. The proof of Lemma 4.1

Without loss of generality, we may assume that $i\leqslant j\leqslant k\leqslant l$. The condition $i\pm j\pm k\pm l=0$ then reduces to two possibilities, either $i-j-k+l=0$ or $i+j+k-l=0$.

We have to study divisors of the form $\delta=\pm\omega_i\pm\omega_j\pm\omega_k\pm\omega_l$ for all possible combinations of plus and minus signs. To this end, we distinguish them according to their number of minus signs. To shorten notation we let for example $\delta_{++-+}=\omega_i+\omega_j-\omega_k+\omega_l$. Similarly for all other combinations of plus and minus signs.

Case 0: No minus sign. This case is trivial.

Case 1: One minus sign. Obviously, $\delta_{++-+},\delta_{+-++},\delta_{-+++}\geqslant 1$. The remaining is $\delta=\delta_{+++-}$, we consider δ as a function of m and notice that

$$\delta'(m)=\frac{1}{2}\left(\frac{1}{\omega_i}+\frac{1}{\omega_j}+\frac{1}{\omega_k}-\frac{1}{\omega_l}\right)\geqslant\frac{1}{2\omega_i}>0$$

then δ is strictly monotone increasing in m. Thus

$$\left|\left\{m\in(0,\infty):\ \delta(m)<\frac{1}{\sqrt{h^4+m}^3}\right\}\right|<\frac{2}{h^4+m}<\frac{2}{h^4}.$$

So after excising a set of small measure, we obtain that

$$\delta(m)\geqslant\frac{1}{\sqrt{h^4+m}^3}.$$

Case 2: Two minus signs. Here we have $\delta_{-+-+}, \delta_{--++} \geqslant \delta_{+--+}$, and all other cases reduce to these ones by inverting the signs. So it suffices to study $\delta = \delta_{+--+}$. The function $f(t) = \sqrt{t^4 + m}$ is monotone increasing and convex for $t \geqslant 0$. Hence we have the estimate $\omega_l - \omega_k \geqslant \omega_{l-p} - \omega_{k-p}$ for every $0 \leqslant p \leqslant k$.

In the case $i + j + k = l$ we thus obtain

$$\omega_l - \omega_k \geqslant \omega_{l-k+i} - \omega_i = \omega_{j+2i} - \omega_i,$$

hence

$$\delta = \omega_i - \omega_j - \omega_k + \omega_l \geqslant \omega_i - \omega_j + \omega_{j+2i} - \omega_i = \omega_{j+2i} - \omega_j$$
$$\geqslant 2(\omega_{j+i} - \omega_j) \geqslant 2if'(j) \geqslant \frac{i}{\sqrt{1+m}},$$

using the mean value theorem and the monotonicity of f'.

With the other alternative $i - j - k + l = 0$, we have $j - i = l - k \neq 0$, hence $\omega_l - \omega_k \geqslant \omega_{j+1} - \omega_{i+1}$ and $\omega_{j+1} - \omega_j \geqslant \omega_{i+2} - \omega_{i+1}$. So we obtain

$$\delta \geqslant \omega_{j+1} - \omega_{i+1} - \omega_j + \omega_i \geqslant \omega_{i+2} - 2\omega_{i+1} + \omega_i$$
$$\geqslant cf''(i) \geqslant \frac{1}{\sqrt{i^4+m}^3}.$$

Case 3 *and* 4: Three and four minus signs. These ones reduces to Cases 1 and 0, respectively.

Thus these bounds give the claimed estimate and Lemma 4.1 follows.

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