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A KAM Theorem for Hamiltonian Partial Differential Equations in Higher Dimensional Spaces*

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Abstract: In this paper, we give a KAM theorem for a class of infinite dimensional nearly integrable Hamiltonian systems. The theorem can be applied to some Hamiltonian partial differential equations in higher dimensional spaces with periodic boundary conditions to construct linearly stable quasi–periodic solutions and its local Birkhoff normal form. The applications to the higher dimensional beam equations and the higher dimensional Schrödinger equations with nonlocal smooth nonlinearity are also given in this paper.

1. Introduction

In late 1980's, motivated by the construction of quasi-periodic solutions for Hamiltonian partial differential equations, the celebrated KAM theory was successfully generalized to infinite dimensional settings by Kuksin [14] and Wayne [20], see also [15–18], which applies to, as typical examples, one-dimensional semi-linear Schrödinger equations

$$\mathrm{i}u_t - u_{xx} + mu = f(u),$$

and wave equations

$$u_{tt} - u_{xx} + mu = f(u),$$

with Dirichlet boundary conditions. When trying to further generalize the KAM theory so as to apply to the one-dimensional wave equations with periodic boundary conditions and higher dimensional Hamiltonian partial differential equations, the multiplicity of the eigenvalues becomes an obstacle. Especially, the multiplicity goes asymptotically to infinity in the higher dimensional case. On one hand, the multiplicity makes the unperturbed part more complicated at succeeding KAM steps, as a consequence solving the

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linearized equations becomes very complicated; on the other hand, it makes the measure estimation very difficult since there are so many non-resonance conditions to be satisfied. For those reasons, there is no KAM theorem for higher dimensional Hamiltonian partial differential equations so far.

To overcome this difficulty, Craig and Wayne retrieved the origination of the KAM method — Newtonian iteration method together with the Liapunov-Schmidt decomposition which involves the Green's function analysis and the control of the inverse of infinite matrices with small eigenvalues. They succeeded in constructing periodic solutions of the one-dimensional semi-linear wave equations with periodic boundary conditions. Bourgain further developed the Craig–Wayne's method and proved the existence of quasi-periodic solutions of partial differential equations in higher dimensional spaces with Dirichlet boundary conditions or periodic boundary conditions. We point out that the Craig-Wayne-Bourgain's method allows one to avoid explicitly using the Hamiltonian structure of the systems. We will not introduce their approaches in detail. The reader is referred to Craig–Wayne [9], Bourgain [3–7].

Comparing with Craig-Wayne-Bourgain's approach, the KAM approach has its own advantages. Besides obtaining the existence results it allows one to construct a local normal form in a neighborhood of the obtained solutions, and this is useful for better understanding of the dynamics. For example, one can obtain the linear stability and zero Liapunov exponents. The question is: Is there a KAM theorem which can be applied to Hamiltonian partial differential equations in higher dimensional spaces? This paper is motivated by this question.

In this paper, we give a KAM theorem which applies to some Hamiltonian partial differential equations in higher dimensional spaces. We use the theorem to construct the quasi-periodic solutions and prove their linear stability. The KAM theorem can be applied to some Hamiltonian partial differential equations *not explicitly containing the space variables and time variable,* including the higher dimensional beam equations

$$u_{tt} + (-\triangle + m)^2 u + f(u) = 0, \quad x \in \mathrm{T}^d$$

and the higher dimensional Schrödinger equations with nonlocal smooth nonlinearities (see Sect. 3 for details)

$$\mathrm{i}u_t + Au + N(u) = 0, \qquad x \in \mathrm{T}^d,$$

as well as one-dimensional wave equations under the periodic boundary conditions.

Different from the finite dimensional case, the KAM theorem may not be true for infinite dimensional nearly integrable Hamiltonian systems. One has to impose further restrictions both on the unperturbed part and on the perturbation besides smallness. In the existent infinite dimensional KAM theorems, e.g., Kuksin [14], Wayne [20] and Pöschel [18], some assumptions on the growth of the normal frequencies and the regularity of the perturbation are required (see $(A1)$–$(A3)$ in the next section). In this paper, we additionally assume that the perturbation has a special form defined in $(A4)$ in the next section. Our proof benefits a lot from such speciality of the perturbation. With the speciality of the form of the perturbation, we can prove that the normal form part of the Hamiltonian remains simple during the iteration. Actually, the normal variables in the normal form part are always uncoupled along the KAM iteration. This makes the measure estimate as easy as the one-dimensional case. Compared with the proof of the existent KAM theorems, *an additional job* done in this paper is to prove that perturbation always has the special form defined in $(A4)$ along the KAM iteration.

We remark that although the assumption $(A4)$ looks artificial, the Hamiltonian systems deriving from the Hamiltonian partial differential equations in T^d *not containing explicitly the space variables and the time variable* do have the special form defined in $(A4)$. And our KAM theorem can be applied to some kind of Hamiltonian partial differential equations, such as the beam equations and the Schödinger equations mentioned above.

Although the applicable equations of the KAM theorem given in this paper are less general, they already have sufficiently strong physical background. And although the existence results are not new, since Bourgain has obtained the results for more general classes of equations [6], the KAM approach may provide more information about the constructed solutions. We are interested in the establishment of a KAM theorem for higher dimensional Hamiltonian partial differential equations. This paper is a step towards this goal. Moreover our proof is simpler compared with Bourgain's proof, we think it is of some interest. With this paper we also hope to call more attention to exploit the inherent properties of the considered equations themselves when studying the dynamics of Hamiltonian partial differential equations. Finally, we remark that a result similar to Bourgain's has been recently announced by Kuksin and Eliasson, but a paper is not yet available.

Since the statement of the main result is a bit long, we postpone it to the next section.

This paper is organized as follows: In Sect. 2 we give an infinite dimensional KAM theorem; in Sect. 3, we give its applications to higher dimensional beam equations and higher dimensional non-local smooth Schrödinger equations. The proof of the KAM theorem is given in Sects. 4, 5, 6. Some technical lemmas are proved in the Appendix.

2. An Infinite Dimensional KAM Theorem for Hamiltonian Partial Differential Equations

In this section, we will formulate an infinite dimensional KAM theorem that can be applied to higher dimensional beam equations, higher dimensional nonlocal smooth Schrödinger equations and one-dimensional wave equations under periodic boundary conditions.

We start by introducing some notations. For given b vectors in Z^d, say $\{i_1, \ldots, i_b\}$, we denote $\mathrm{Z}_1^d = \mathrm{Z}^d \setminus \{i_1, \ldots, i_b\}$. Let $z = (\ldots, z_n, \ldots)_{n\in \mathrm{Z}_1^d}$, and its complex conjugate $\bar{z} = (\ldots, \bar{z}_n, \ldots)_{n\in\mathrm{Z}_1^d}$. We introduce the weighted norm

$$\|z\|_{a,\rho} = \sum_{n\in\mathrm{Z}_1^d} |z_n||n|^a e^{|n|\rho},$$

where $|n| = \sqrt{n_1^2 + \cdots + n_d^2}$, $n = (n_1, \ldots, n_d)$ and $a \geq 0$, $\rho > 0$. Denote a neighborhood of $\mathrm{T}^b \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ by

$$D(r, s) = \{(\theta, I, z, \bar{z}) : |\mathrm{Im}\theta| < r, |I| < s^2, \|z\|_{a,\rho} < s, \|\bar{z}\|_{a,\rho} < s\},$$

where $|\cdot|$ denotes the sup-norm of complex vectors. Moreover, we denote by $\mathcal{O}$ a positive–measure parameter set in R^b.

Let $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n\in\mathrm{Z}_1^d}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n\in\mathrm{Z}_1^d}$, α_n and $\beta_n \in \mathrm{N}$ with finitely many non-zero components of positive integers. The product $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$.

For any given function

$$F(\theta, I, z, \bar{z}) = \sum_{\alpha,\beta} F_{\alpha\beta}(\theta, I) z^\alpha \bar{z}^\beta, \tag{2.1}$$

where $F_{\alpha\beta}$ is a C^1_W function in parameter ξ in the sense of Whitney, we define the weighted norm of F by

$$\|F\|_{D(r,s),\mathcal{O}} \equiv \sup_{\substack{\|z\|_{a,\rho}<s \\ \|\bar{z}\|_{a,\rho}<s}} \sum_{\alpha,\beta} \|F_{\alpha\beta}\| \, |z^\alpha||\bar{z}^\beta|, \tag{2.2}$$

where, if $F_{\alpha\beta} = \sum_{k\in\mathbb{Z}^b, l\in\mathbb{N}^b} F_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k,\theta\rangle}$, ($\langle\cdot,\cdot\rangle$ being the standard inner product in $\mathbb{C}^b$), $\|F_{\alpha\beta}\|$ is short for

$$\|F_{\alpha\beta}\| \equiv \sum_{k,l} |F_{kl\alpha\beta}|_{\mathcal{O}} s^{2|l|} e^{|k|r}, \quad |F_{kl\alpha\beta}|_{\mathcal{O}} \equiv \sup_{\xi\in\mathcal{O}} \left(|F_{kl\alpha\beta}| + |\frac{\partial F_{kl\alpha\beta}}{\partial \xi}| \right) \tag{2.3}$$

(the derivatives with respect to ξ are in the sense of Whitney).

To function F, we associate a Hamiltonian vector field defined by

$$X_F = (F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}_{n\in\mathbb{Z}^d_1}, \{-\mathrm{i}F_{\bar{z}_n}\}_{n\in\mathbb{Z}^d_1}).$$

Its weighted norm is defined by[1]

$$\begin{aligned} \|X_F\|_{D(r,s),\mathcal{O}} \equiv{} & \|F_I\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_\theta\|_{D(r,s),\mathcal{O}} \\ & + \frac{1}{s}\left(\sum_{n\in\mathbb{Z}^d_1} \|F_{z_n}\|_{D(r,s),\mathcal{O}} |n|^{\bar{a}} e^{|n|\rho} + \sum_{n\in\mathbb{Z}^d_1} \|F_{\bar{z}_n}\|_{D(r,s),\mathcal{O}} |n|^{\bar{a}} e^{|n|\rho} \right). \end{aligned} \tag{2.4}$$

Remark. In this paper, we require that $\bar{a} > a$, i.e., the weight of vector fields is a little heavier than that of $z, \bar{z}$. The boundedness of $\|X_F\|_{D_\rho(r,s),\mathcal{O}}$ means X_F sends a decaying z-sequence to a fastly decaying sequence.

The starting point will be a family of integrable Hamiltonians of the form

$$N = \langle \omega(\xi), I \rangle + \sum_{n\in\mathbb{Z}^d_1} \Omega_n(\xi) z_n \bar{z}_n, \tag{2.5}$$

where $\xi \in \mathcal{O}$ is a parameter, the phase space is endowed with the symplectic structure $dI \wedge d\theta + \mathrm{i}\sum_{n\in\mathbb{Z}^d_1} dz_n \wedge d\bar{z}_n$.

For each $\xi \in \mathcal{O}$, the Hamiltonian equations of motion for N, i.e.,

$$\frac{d\theta}{dt} = \omega, \quad \frac{dI}{dt} = 0, \quad \frac{dz_n}{dt} = -\mathrm{i}\Omega_n z_n, \quad \frac{d\bar{z}_n}{dt} = \mathrm{i}\Omega_n \bar{z}_n, \quad n \in \mathbb{Z}^d_1, \tag{2.6}$$

[1] The norm $\|\cdot\|_{D(r,s),\mathcal{O}}$ for scalar functions is defined in (2.2). The vector function $G : D(r,s)\times\mathcal{O} \to \mathbb{C}^m$, $(m < \infty)$ is similarly defined as $\|G\|_{D(r,s),\mathcal{O}} = \sum_{i=1}^m \|G_i\|_{D(r,s),\mathcal{O}}$.

admit special solutions $(\theta, 0, 0, 0) \to (\theta + \omega t, 0, 0, 0)$ that correspond to an invariant torus in the phase space.

Consider now the perturbed Hamiltonian

$$H = N + P = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi). \tag{2.7}$$

Our goal is to prove that, for most values of parameter $\xi \in \mathcal{O}$ (in Lebesgue measure sense), the Hamiltonians $H = N + P$ still admit invariant tori provided that $\|X_P\|_{D(r,s),\mathcal{O}}$ is sufficiently small.

To this end, we need to impose some conditions on $\omega(\xi)$, $\Omega_n(\xi)$ and the perturbation P. As we already remarked, the persistence of the lower dimensional torus may not be true if one only assumes the smallness of the perturbation. This is an essential difference between infinite and finite dimensional cases.

$(A1)$ *Nondegeneracy:* The map $\xi \to \omega(\xi)$ is a C_W^1 diffeomorphism between $\mathcal{O}$ and its image.

$(A2)$ *Asymptotics of normal frequencies:* There exists a $\iota > 0$ such that for all $n = (n_1, \ldots, n_d) \in \mathbb{Z}_1^d$,

$$\Omega_n \neq 0, \quad n \in \mathbb{Z}_1^d, \tag{2.8}$$

$$\Omega_n = \bar{\Omega}_n + \tilde{\Omega}_n, \quad \tilde{\Omega}_n = o(|n|^{-\iota}), \tag{2.9}$$

where $\bar{\Omega}_n$'s are real and independent of ξ while $\tilde{\Omega}_n$'s are C_W^1 functions of ξ with C_W^1-norm bounded by some small positive constant L (depending on $\det(\frac{\partial \omega(\xi)}{\partial \xi})$); furthermore, the asymptotic behavior of $\bar{\Omega}_n$ is assumed to be as follows:

$$\bar{\Omega}_n = |n|^p + o(|n|^p), \quad \bar{\Omega}_n - \bar{\Omega}_m = |n|^p - |m|^p + o(|m|^{-\iota}), \quad |m| \leq |n|, \tag{2.10}$$

where $p \geq 2$ for $d > 1$ or $p \geq 1$ for $d = 1$.

$(A3)$ *Regularity of the perturbation:* The perturbation P is *regular* in the sense that $\|X_P\|_{D(r,s),\mathcal{O}} < \infty$ with $\bar{a} > a$.

$(A4)$ *Special form of the perturbation:* The perturbation is taken from a special class of analytic functions,

$$\mathcal{A} = \left\{ P : P = \sum_{k \in \mathbb{Z}^b, l \in \mathbb{N}^b, \alpha, \beta} P_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k, \theta \rangle} z^\alpha \bar{z}^\beta \right\},$$

where k, α, β has the following relation

$$\sum_{j=1}^{b} k_j i_j + \sum_{n \in \mathbb{Z}_1^d} (\alpha_n - \beta_n) n = 0. \tag{2.11}$$

Remark. Compared with the existent infinite dimensional KAM theorems in literature, we make an additional assumption $(A4)$ on the perturbation. The assumption looks artificial, but it is satisfied by the infinite dimensional Hamiltonian systems derived from Hamiltonian partial differential equations in T^d which *do not explicitly contain the space variables and the time variable,* for example, the Schrödinger equations, wave equations and beam equations in T^d in the introduction.

Now we are ready to state our KAM Theorem.

Theorem 1. *Assume that the unperturbed Hamiltonian N in (2.5) satisfies* $(A1)$ *and* $(A2)$ *and P satisfies* $(A3)$ *and* $(A4)$. *Let $\gamma > 0$ small enough, there is a positive constant $\varepsilon = \varepsilon(b, d, p, \iota, \bar{a} - a, L, \gamma)$ such that if $\|X_P\|_{D(r,s),\mathcal{O}} < \varepsilon$, then the following holds true: There exist a Cantor set $\mathcal{O}_\gamma \subset \mathcal{O}$ with* $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma^\vartheta)$ *(ϑ is specified in Sect. 6) and two maps (analytic in θ and C_W^1 in ξ)*

$$\Psi : \mathrm{T}^b \times \mathcal{O}_\gamma \to D(r, s), \quad \tilde{\omega} : \mathcal{O}_\gamma \to \mathrm{R}^b,$$

where Ψ is $\frac{\varepsilon}{\gamma^2}$-close to the trivial embedding $\Psi_0 : \mathrm{T}^b \times \mathcal{O} \to \mathrm{T}^b \times \{0, 0, 0\}$ and $\tilde{\omega}$ is ε-close to the unperturbed frequency ω, such that for any $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathrm{T}^b$, the curve $t \to \Psi(\theta + \tilde{\omega}(\xi)t, \xi)$ is a quasi-periodic solution of the Hamiltonian equations governed by $H = N + P$. Moreover, the obtained solutions are linearly stable.

Remark 1. In the one dimensional case, the growth of Ω_n can be sub-linear ($0 < p < 1$). But we can not find any interesting application of it.

Remark 2. The regularity $\bar{a} > a$ is used to control the drifting of the normal frequencies which is crucial in the measure estimation of the survived parameters $\mathcal{O} \setminus \mathcal{O}_\gamma$ for our approach. It seems that the restriction is only of technical reasons. One may expect to have a KAM theorem without the regularity assumption $(A3)$. However this problem remains open so far.

Remark 3. The Hamiltonian systems defined by Hamiltonian partial differential equations do have the special form defined in $(A4)$. This fact has been used by many authors when transforming the leading nonlinearity in the perturbation into the partial Birkhoff normal form under Cartesian coordinate systems (see Kuksin–Pöschel [16], Pöschel [17], Craig–Worfolk [10], Bourgain [7, 8], Bambusi [1], Bambusi–Berti [2], and Geng–You [12, 13]). Those papers actually use this property for one or two steps. In this paper, we will use this fact at each step of the KAM iteration. For this purpose, we have to prove that the change of action-angle variables and the KAM iteration preserve the special form of the perturbation defined in $(A4)$.

Remark 4. In the one dimensional case, assumption $(A4)$ is replaced by a kind of decay property in [13]. Since the decay property is weaker than assumption $(A4)$, the KAM theorem assuming only decay property may have more applications, e.g., when the equation depends on the space variable x. However, the proof for the higher dimensional case would be much more complicated and is not available so far.

Remark 5. The parameter γ plays the role of the Diophantine constant for the frequency $\tilde{\omega}$ in the sense that there exists $\tau > 0$ (specified in Sect. 6) such that the frequencies of the obtained KAM tori satisfy the following Diophantine conditions:

$$\langle k, \tilde{\omega} \rangle \geq \frac{\gamma}{2|k|^\tau}, \quad \forall k \in \mathrm{Z}^b \setminus \{0\}.$$

Notice also that $\mathcal{O}_\gamma$ is claimed to be nonempty only for γ small enough.

3. Applications

1. Higher dimensional beam equations. In order to avoid the technical complexities, we apply Theorem 1 to the d-dimensional beam equations with a Fourier multiplier M_ξ. Actually, Theorem 1 can be applied to the higher dimensional beam equations with constant potentials, i.e., $Au \equiv (\triangle^2 + m)^{\frac{1}{2}}u,\ x \in \mathrm{R}^d,\ m > 0$ in (3.1). However, in order to transform the original Hamiltonian into a perturbation of a nonlinear integrable system, the normal form technique is needed. Since the normal form procedure is quite involved and does not fit the main theme of this paper we will handle it in another paper.

Consider

$$\begin{aligned} &u_{tt} + A^2 u + f(u) = 0, \quad Au \equiv (-\triangle + M_\xi)u,\ x \in \mathrm{R}^d,\ t \in \mathrm{R}, \\ &u(t, x_1 + 2\pi, x_2, \dots, x_d) = \cdots = u(t, x_1, x_2, \dots, x_{d-1}, x_d + 2\pi) \\ &\quad = u(t, x_1, x_2, \dots, x_d), \end{aligned} \tag{3.1}$$

where $f(u)$ is a real–analytic function near $u = 0$ with $f(0) = f'(0) = 0$.

Here we assume that the operator $A = -\triangle + M_\xi$ with periodic boundary conditions has eigenvalues $\{\mu_n\}$ satisfying

$$\begin{aligned} \omega_j &= \mu_{i_j} = |i_j|^2 + \xi_j, \quad 1 \le j \le b, \\ \Omega_n &= \mu_n = |n|^2, \quad n \ne i_1, \dots, i_b, \end{aligned} \tag{3.2}$$

and the corresponding eigenfunctions $\phi_n(x) = \sqrt{\frac{1}{(2\pi)^d}} e^{\mathrm{i}\langle n, x\rangle}$ form a basis in the domain of the operator. Assume that $i_1, \dots, i_b \in \mathrm{Z}^d$ are the distinguished sites of Fourier modes (assume $0 \in \{i_1, \dots, i_b\}$ in order to take care of $(\mu_n, k) = (0, 0)$), and $\xi = (\xi_1, \dots, \xi_b)$ is a parameter taking on a closed set $\mathcal{O} \subset \mathrm{R}^b$ of the positive–measure.

Introducing $v = u_t$, (3.1) reads

$$\begin{aligned} u_t &= v, \\ v_t &= -A^2 u - f(u). \end{aligned} \tag{3.3}$$

Letting $q = \frac{1}{\sqrt{2}} A^{\frac{1}{2}} u - \mathrm{i} \frac{1}{\sqrt{2}} A^{-\frac{1}{2}} v$, we obtain

$$\frac{1}{\mathrm{i}} q_t = Aq + \frac{1}{\sqrt{2}} A^{-\frac{1}{2}} f\left(A^{-\frac{1}{2}}\left(\frac{q + \bar{q}}{\sqrt{2}}\right)\right). \tag{3.4}$$

Equation (3.4) can be rewritten as the Hamiltonian equations

$$q_t = \mathrm{i}\frac{\partial H}{\partial \bar{q}} \tag{3.5}$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}(Aq, q) + \int_{\mathrm{T}^d} g\left(A^{-\frac{1}{2}}\left(\frac{q + \bar{q}}{\sqrt{2}}\right)\right) dx, \tag{3.6}$$

where $(\cdot, \cdot)$ denotes the inner product in L^2 and g is a primitive of f.

Let

$$q(x) = \sum_{n \in \mathrm{Z}^d} q_n \phi_n(x).$$

Then system (3.5) is equivalent to the lattice Hamiltonian equations

$$\dot{q}_n = \mathrm{i}\left(\mu_n q_n + \frac{\partial G}{\partial \bar{q}_n}\right), \quad G(q,\bar{q}) \equiv \int_{\mathrm{T}^d} g\left(\sum_{n\in\mathrm{Z}^d} \frac{q_n\phi_n + \bar{q}_n\bar{\phi}_n}{\sqrt{2\mu_n}}\right) dx \tag{3.7}$$

with corresponding Hamiltonian function $H = \sum_{n\in\mathrm{Z}^d} \mu_n q_n \bar{q}_n + G(q,\bar{q})$.

Since $f(u)$ is real analytic in u, $g(q,\bar{q})$ is real analytic in q, $\bar{q}$. Making use of $q(x) = \sum_{n\in\mathrm{Z}^d} q_n\phi_n(x)$ again, we may rewrite g as follows

$$g(q,\bar{q}) = \sum_{\alpha,\beta} g_{\alpha\beta} q^\alpha \bar{q}^\beta \phi^\alpha \bar{\phi}^\beta,$$

hence

$$G(q,\bar{q}) \equiv \int_{\mathrm{T}^d} g\left(\sum_{n\in\mathrm{Z}^d} \frac{q_n\phi_n + \bar{q}_n\bar{\phi}_n}{\sqrt{2\mu_n}}\right) dx = \sum_{\alpha,\beta} G_{\alpha\beta} q^\alpha \bar{q}^\beta,$$

$$G_{\alpha\beta} = 0, \qquad \text{if } \sum_{n\in\mathrm{Z}^d} (\alpha_n - \beta_n) n \neq 0. \tag{3.8}$$

As in [16, 17, 12, 13], the perturbation G in (3.7) has the following regularity property.

Lemma 3.1. *For any fixed $a \geq 0$, $\rho > 0$, the gradient $G_{\bar{q}}$ is real analytic as a map in a neighborhood of the origin with*

$$\|G_{\bar{q}}\|_{a+1,\rho} \leq c\|q\|^2_{a,\rho}. \tag{3.9}$$

Next we introduce standard action-angle variables $(\theta, I) = ((\theta_1,\ldots,\theta_b),(I_1,\ldots,I_b))$ in the $(q_{i_1},\ldots,q_{i_b},\bar{q}_{i_1},\ldots,\bar{q}_{i_b})$-space by letting,

$$I_j = q_{i_j}\bar{q}_{i_j}, \quad j = 1,\ldots,b,$$

and $q_n = z_n$, $\bar{q}_n = \bar{z}_n$, $n \neq i_1,\ldots,i_b$. So system (3.7) becomes

$$\frac{d\theta_j}{dt} = \omega_j + P_{I_j}, \quad \frac{dI_j}{dt} = -P_{\theta_j}, \quad j = 1,\ldots,b, \tag{3.10}$$

$$\frac{dz_n}{dt} = -\mathrm{i}(\Omega_n z_n + P_{\bar{z}_n}), \quad \frac{d\bar{z}_n}{dt} = \mathrm{i}(\Omega_n\bar{z}_n + P_{z_n}),\ n \in \mathrm{Z}^d_1,$$

where P is just G with the $(q_{i_1},\ldots,q_{i_b},\bar{q}_{i_1},\ldots,\bar{q}_{i_b},q_n,\bar{q}_n)$-variables expressed in terms of the $(\theta, I, z_n, \bar{z}_n)$ variables. The Hamiltonian associated to (3.10) (with respect to the symplectic structure $dI \wedge d\theta + \mathrm{i}\sum_{n\in\mathrm{Z}^d_1} dz_n \wedge d\bar{z}_n$) is given by

$$H = \langle\omega(\xi), I\rangle + \sum_{n\in\mathrm{Z}^d_1} \Omega_n(\xi) z_n\bar{z}_n + P(\theta, I, z, \bar{z}, \xi). \tag{3.11}$$

Let's verify that P has the special form defined in $(A4)$ from (3.8), i.e., $P(\theta, I, z, \bar{z}, \xi) \in \mathcal{A}$, which is a key assumption of Theorem 1.

Denote by e_n the infinite dimensional vector with the n^{th} component being 1 and the other components being zero, and $k = (k_1, \ldots, k_b)$, $k_j = \alpha_{i_j} - \beta_{i_j}$, $1 \le j \le b$, then due to (3.8),

$$
\begin{aligned}
G &= \sum_{\sum_{n\in\mathbb{Z}^d}(\alpha_n-\beta_n)n=0} G_{\alpha\beta} q^\alpha \bar{q}^\beta \\
&= \sum_{\sum_{j=1}^b(\alpha_{i_j}-\beta_{i_j})i_j+\sum_{n\in\mathbb{Z}_1^d}(\alpha_n-\beta_n)n=0} G_{\alpha\beta} q_{i_1}^{\alpha_{i_1}} \bar{q}_{i_1}^{\beta_{i_1}} \cdots q_{i_b}^{\alpha_{i_b}} \bar{q}_{i_b}^{\beta_{i_b}} q^{\alpha-\sum_{j=1}^b \alpha_{i_j} e_{i_j}} \bar{q}^{\beta-\sum_{j=1}^b \beta_{i_j} e_{i_j}} \\
&= \sum_{\sum_{j=1}^b(\alpha_{i_j}-\beta_{i_j})i_j+\sum_{n\in\mathbb{Z}_1^d}(\alpha_n-\beta_n)n=0} G_{\alpha\beta} \sqrt{I_1}^{\alpha_{i_1}+\beta_{i_1}} \cdots \sqrt{I_b}^{\alpha_{i_b}+\beta_{i_b}} \\
&\quad \cdot e^{\mathrm{i}\sum_{j=1}^b(\alpha_{i_j}-\beta_{i_j})\theta_j} z^{\alpha-\sum_{j=1}^b \alpha_{i_j} e_{i_j}} \bar{z}^{\beta-\sum_{j=1}^b \beta_{i_j} e_{i_j}} \\
&\triangleq \sum_{\sum_{j=1}^b k_j i_j+\sum_{n\in\mathbb{Z}_1^d}(\alpha_n-\beta_n)n=0} P_{kl\alpha\beta} I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta \equiv P.
\end{aligned}
$$

Thus

$$
P_{kl\alpha\beta} = 0 \quad \text{if} \sum_{j=1}^b k_j i_j + \sum_{n\in\mathbb{Z}_1^d} (\alpha_n - \beta_n) n \ne 0, \tag{3.12}
$$

i.e., $P \in \mathcal{A}$. Moreover the regularity of P holds true:

Lemma 3.2. *For any $\varepsilon > 0$ sufficiently small and $s = \varepsilon^{\frac{1}{2}}$, if $|I| < s^2$ and $\|z\|_{a,\rho} < s$, then*

$$
\|X_P\|_{D(r,s),\mathcal{O}} \le \varepsilon, \quad \bar{a} = a + 1. \tag{3.13}
$$

To this point, we have verified all the assumptions of Theorem 1 for (3.11) with $p = 2$, $\iota = +\infty$, $\bar{a} - a = 1$. Now we are in the position to apply Theorem 1 to get the following result.

Theorem 2. *For any $0 < \gamma \ll 1$, there is a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with $\mathrm{meas}(\mathcal{O}\backslash\mathcal{O}_\gamma) = O(\gamma^\vartheta)$ (ϑ is to be specified in Sect. 6), such that for any $\xi \in \mathcal{O}_\gamma$, Eq. (3.1) parametrized by ξ admits a small-amplitude, quasi-periodic solution of the form*

$$
u(t,x) = \sum_{n\in\mathbb{Z}^d} u_n(\omega_1' t, \ldots, \omega_b' t) e^{\mathrm{i}\langle n,x\rangle},
$$

where $u_n : \mathbb{T}^b \to \mathbb{R}$ and $\omega_1', \ldots, \omega_b'$ are close to the unperturbed frequencies $\omega_1, \ldots, \omega_b$. Moreover, the quasi-periodic solutions we obtained are linearly stable.

Remark 1. Theorem 1 also applies to 1D wave equations with periodic boundary conditions

$$
\begin{aligned}
&u_{tt} - u_{xx} + mu + au^3 + O(u^4) = 0, \quad a \ne 0, \\
&u(t, x + 2\pi) = u(t,x).
\end{aligned} \tag{3.14}
$$

Since the proof follows exactly the same steps as that of the beam equations, we omit it.

2. The higher dimensional nonlocal smooth Schrödinger equations. The dD nonlinear Schrödinger equations to be considered are

$$\mathrm{i}u_t + Au + N(u) = 0, \quad Au = -\triangle u + M_\xi u, \qquad x \in \mathrm{R}^d,\ t \in \mathrm{R} \tag{3.15}$$

with periodic boundary conditions

$$\begin{aligned} u(t, x_1 + 2\pi, x_2, \ldots, x_d) &= \ldots = u(t, x_1, x_2, \ldots, x_{d-1}, x_d + 2\pi) \\ &= u(t, x_1, x_2, \ldots, x_d). \end{aligned}$$

The operator A is the same as that in the beam equations. The nonlinearity we would like to consider is

$$N(u) = f(|u|^2)u \tag{3.16}$$

with the function f real analytic in a neighborhood of $0 \in \mathrm{C}$ and vanishing at zero. However for the sake of regularity imposed on the nonlinearity (see $(A3)$), as those in Pöschel [19] and Bambusi–Berti [2], we have to assume some nonlocal smoothness for the nonlinearity. Thus we actually consider the nonlinearity

$$N(u) = \Psi(f(|\Psi u|^2)\Psi u), \tag{3.17}$$

where $\Psi : u \to \psi * u$ is a convolution operator with a function ψ, which is of smoothness of order $\delta > 0$. More precisely,

$$\|\Psi u\|_{a+\delta,\rho} \le c\|u\|_{a,\rho}. \tag{3.18}$$

Equation (3.15) can be rewritten as a Hamiltonian equation

$$u_t = \mathrm{i}\frac{\partial H}{\partial \bar{u}} \tag{3.19}$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}(Au, u) + \int_{\mathrm{T}^d} g(|\Psi u|^2)\, dx, \tag{3.20}$$

where $(\cdot,\cdot)$ denotes the inner product in L^2 and g is a primitive of f.

Let

$$u(x) = \sum_{n\in\mathrm{Z}^d} q_n\phi_n(x).$$

System (3.19) is then equivalent to the lattice Hamiltonian equations

$$\dot{q}_n = \mathrm{i}(\mu_n q_n + \frac{\partial G}{\partial \bar{q}_n}), \quad G \equiv \int_{\mathrm{T}^d} g(|\Psi u|^2)dx\ , \tag{3.21}$$

with corresponding Hamiltonian function $H = \sum_{n\in\mathrm{Z}^d} \mu_n q_n \bar{q}_n + G$. $\phi_n(x)$, μ_n are defined in the last sub-section.

Since $N(u)$ is real analytic in u, then making use of $u(t,x) = \sum_{n\in\mathrm{Z}^d} q_n(t)\phi_n(x)$, we may rewrite $N(u)$ as follows

$$N(u) = \sum_{\alpha,\beta} N_{\alpha\beta} q^\alpha \bar{q}^\beta \phi^\alpha \bar{\phi}^\beta,$$

hence

$$G \equiv \int_{\mathrm{T}^d} g(|\Psi u|^2)dx = \sum_{\alpha,\beta} G_{\alpha\beta} q^\alpha \bar{q}^\beta,$$

$$G_{\alpha\beta} = 0, \qquad \text{if } \sum_{n\in \mathrm{Z}^d} (\alpha_n - \beta_n)n \neq 0. \tag{3.22}$$

As in [19, 2], the perturbation G in (3.21) has the following regularity property.

Lemma 3.3. *For any fixed $a \geq 0$, $\rho > 0$, the gradient $G_{\bar{q}}$ is real analytic as a map in a neighborhood of the origin with*

$$\|G_{\bar{q}}\|_{a+\delta,\rho} \leq c\|q\|^3_{a,\rho}. \tag{3.23}$$

The same as that of the beam equations, by introducing the standard action-angle variables $(\theta, I) = ((\theta_1, \ldots, \theta_b), (I_1, \ldots, I_b))$ in the $(q_{i_1}, \ldots, q_{i_b}, \bar{q}_{i_1}, \ldots, \bar{q}_{i_b})$-space, we arrive at a Hamiltonian system with the Hamiltonian (with respect to the symplectic structure $dI \wedge d\theta + \mathrm{i}\sum_{n\in \mathrm{Z}_1^d} dz_n \wedge d\bar{z}_n$)

$$H = \langle \omega(\xi), I\rangle + \sum_{n\in \mathrm{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi). \tag{3.24}$$

The same as the beam equations, we can prove that

$$P = \sum_{k\in \mathrm{Z}^b, l\in \mathrm{N}^b, \alpha,\beta} P_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta$$

satisfies the assumption $(A4)$, i.e.,

$$P_{kl\alpha\beta} = 0, \quad \text{if } \sum_{j=1}^{b} k_j i_j + \sum_{n\in \mathrm{Z}_1^d} (\alpha_n - \beta_n)n \neq 0. \tag{3.25}$$

Moreover the regularity of P holds true:

Lemma 3.4. *For any $\varepsilon > 0$ sufficiently small and $s = \varepsilon^{\frac{1}{2}}$, if $|I| < s^2$ and $\|z\|_{a,\rho} < s$, then*

$$\|X_P\|_{D(r,s),\mathcal{O}} \leq \varepsilon, \qquad \bar{a} = a + \delta. \tag{3.26}$$

Remark. When we consider the local smooth nonlinearity $N(u) = f(|u|^2)u$, we can get $\|X_P\|_{D(r,s),\mathcal{O}} \leq \varepsilon$ with $\bar{a} = a$. As a consequence, Theorem 1 can not be applied since $(A3)$ is violated.

So we have verified all the assumptions of Theorem 1 for (3.24) with $p = 2$, $\iota = +\infty$, $\bar{a} - a = \delta > 0$. Then Theorem 1 yields the following result for nonlinear Schrödinger equations.

Theorem 3. *For any $0 < \gamma \ll 1$, there is a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with* $\mathrm{meas}(\mathcal{O}\backslash\mathcal{O}_\gamma) = O(\gamma^\vartheta)$ *(ϑ is to be specified in Sect. 6) such that for any $\xi \in \mathcal{O}_\gamma$, Eq. (3.15) parameterized by $\xi \in \mathcal{O}$ admits a small-amplitude, quasi-periodic solution of the form*

$$u(t, x) = \sum_{n\in \mathrm{Z}^d} u_n(\omega'_1 t, \ldots, \omega'_b t) e^{\mathrm{i}\langle n,x\rangle},$$

where $u_n : \mathrm{T}^b \to \mathrm{R}$ and $\omega'_1, \ldots, \omega'_b$ are close to the unperturbed frequencies $\omega_1, \ldots, \omega_b$. Moreover, the quasi-periodic solutions obtained here are linearly stable.

4. KAM Step

Theorem 1 will be proved by a KAM iteration which involves an infinite sequence of change of variables. Each step of KAM iteration makes the perturbation smaller than in the previous step at the cost of excluding a small set of parameters. We have to prove the convergence of the iteration and estimate the measure of the excluded set after infinite KAM steps.

At the ν–step of the KAM iteration, we consider a Hamiltonian vector field with

$$H_\nu = N_\nu + P_\nu,$$

where N_ν is an "integrable normal form" and $P_\nu \in \mathcal{A}$ is defined in $D(r_\nu, s_\nu) \times \mathcal{O}_{\nu-1}$.

We then construct a map

$$\Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D(r_\nu, s_\nu) \times \mathcal{O}_{\nu-1}$$

so that the vector field $X_{H_\nu \circ \Phi_\nu}$ defined on $D(r_{\nu+1}, s_{\nu+1})$ satisfies

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_\nu} = \|X_{H_\nu\circ\Phi_\nu} - X_{N_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_\nu} \le \varepsilon_\nu^\kappa, \quad \kappa > 1$$

with some new normal form $N_{\nu+1}$. Moreover, the new perturbation $P_{\nu+1}$ still has the special form defined in $(A4)$.

To simplify notations, in what follows, the quantities without subscripts refer to quantities at the ν^{th} step, while the quantities with subscripts $+$ denote the corresponding quantities at the $(\nu+1)^{\text{th}}$ step. Let us then consider the Hamiltonian

$$H = N + P \equiv e + \langle \omega(\xi), I \rangle + \sum_{n\in \mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}, \xi, \varepsilon) \tag{4.1}$$

defined in $D(r, s) \times \mathcal{O}$. We assume that $\xi \in \mathcal{O}$ satisfies (a suitable $\tau > 0$ will be specified later)

$$\begin{aligned}
&|\langle k, \omega(\xi)\rangle| \ge \frac{\gamma}{|k|^\tau}, \quad k \neq 0,\\
&|\langle k, \omega(\xi)\rangle + \Omega_n(\xi)| \ge \frac{\gamma}{|k|^\tau},\\
&|\langle k, \omega(\xi)\rangle + \Omega_n(\xi) + \Omega_m(\xi)| \ge \frac{\gamma}{|k|^\tau},\\
&|\langle k, \omega(\xi)\rangle + \Omega_n(\xi) - \Omega_m(\xi)| \ge \frac{\gamma}{|k|^\tau}, \quad |k| + ||n| - |m|| \neq 0.
\end{aligned} \tag{4.2}$$

Moreover,

$$\|X_P\|_{D(r,s),\mathcal{O}} \le \varepsilon, \tag{4.3}$$

and $P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta$ is in the class $\mathcal{A}$ defined in $(A4)$, i.e.,

$$P_{kl\alpha\beta} = 0 \qquad \text{if } \sum_{j=1}^{b} k_j i_j + \sum_{n\in \mathbb{Z}_1^d} (\alpha_n - \beta_n) n \neq 0. \tag{4.4}$$

Remark 1. According to (4.4), when $k = (k_1, \cdots, k_b) = 0$ and $\alpha = e_n$, $\beta = e_m$, we get

$$P_{0le_ne_m} = 0 \qquad \text{if} \ \sum_{j=1}^{b} k_j i_j + \sum_{n\in Z_1^d} (\alpha_n - \beta_n)n = n - m \neq 0.$$

This means that there are not terms of the form $\sum_{n\neq m} P_{0le_ne_m} I^l z_n \bar{z}_m$ in the perturbation. As a result, the normal variables $z_n, \bar{z}_m$ with $n \neq m$ in the new normal form N_+ will not be coupled.

Remark 2. Compared with the KAM step in existent KAM theorems in the literature, we make an additional assumption $P \in \mathcal{A}$. With this assumption the linearized equations are easy to be solved in Subsect. 4.1, and the new normal form after one step of the iteration still has the form $N_+ \equiv e_+ + \langle \omega_+(\xi), I\rangle + \sum_{n\in Z_1^d} \Omega_n^+(\xi) z_n \bar{z}_n$. This makes the measure estimate available and easier at each KAM step. Subsection 4.5 is an additional work which proves the new perturbation P_+ still has the special form defined in $(A4)$, i.e., $P_+ \in \mathcal{A}$, after one step of the iteration. The proofs in Subsects. 4.2–4.4 are the same as that of the existent KAM theorems.

We now let $0 < r_+ < r$ and define

$$s_+ = \frac{1}{4} s\varepsilon^{\frac{1}{3}}, \quad \varepsilon_+ = c\gamma^{-2}(r - r_+)^{-c}\varepsilon^{\frac{4}{3}}. \tag{4.5}$$

Here and later, the letter c denotes suitable (possibly different) constants that do not depend on the iteration steps.

We now describe how to construct a set $\mathcal{O}_+ \subset \mathcal{O}$ and a change of variables Φ : $D_+ \times \mathcal{O}_+ = D(r_+, s_+) \times \mathcal{O}_+ \to D(r, s) \times \mathcal{O}$ such that the transformed Hamiltonian $H_+ = N_+ + P_+ \equiv H \circ \Phi$ satisfies all the above iterative assumptions with new parameters s_+, ε_+, r_+ and with $\xi \in \mathcal{O}_+$.

4.1. Solving the linearized equations. Expand P into the Fourier-Taylor series

$$P = \sum_{k,l,\alpha,\beta} P_{kl\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle} I^l z^\alpha \bar{z}^\beta,$$

where $k \in Z^b, l \in N^b$ and the multi–indices α and β run over the set of all infinite dimensional vectors $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n\in Z_1^d}$ with finitely many nonzero components of positive integers.

Let R be the truncation of P given by

$$\begin{aligned} R(\theta, I, z, \bar{z}) = R_0 + R_1 + R_2 = &\sum_{k,|l|\leq 1} P_{kl00} e^{\mathrm{i}\langle k,\theta\rangle} I^l \\ &+ \sum_{k,n} (P_n^{k10} z_n + P_n^{k01} \bar{z}_n) e^{\mathrm{i}\langle k,\theta\rangle} \\ &+ \sum_{k,n,m} (P_{nm}^{k20} z_n z_m + P_{nm}^{k11} z_n \bar{z}_m + P_{nm}^{k02} \bar{z}_n \bar{z}_m) e^{\mathrm{i}\langle k,\theta\rangle}, \end{aligned} \tag{4.6}$$

where $P_n^{k10} = P_{kl\alpha\beta}$ with $\alpha = e_n$, $\beta = 0$, here e_n denotes the vector with the n^{th} component being 1 and the other components being zero; $P_n^{k01} = P_{kl\alpha\beta}$ with $\alpha = 0$, $\beta = e_n$;

$P^{k20}_{nm} = P_{kl\alpha\beta}$ with $\alpha = e_n + e_m, \beta = 0$; $P^{k11}_{nm} = P_{kl\alpha\beta}$ with $\alpha = e_n, \beta = e_m$; $P^{k02}_{nm} = P_{kl\alpha\beta}$ with $\alpha = 0, \beta = e_n + e_m$. Due to assumption $(A4)$, $P \in \mathcal{A}$ implies that

$$\begin{aligned}
P_{kl00} &= 0, \qquad \text{if} \sum_{j=1}^{b} k_j i_j \neq 0, \\
P^{k10}_{n} &= 0, \qquad \text{if} \sum_{j=1}^{b} k_j i_j + n \neq 0, \\
P^{k01}_{n} &= 0, \qquad \text{if} \sum_{j=1}^{b} k_j i_j - n \neq 0, \\
P^{k20}_{nm} &= 0, \qquad \text{if} \sum_{j=1}^{b} k_j i_j + n + m \neq 0, \\
P^{k11}_{nm} &= 0, \qquad \text{if} \sum_{j=1}^{b} k_j i_j + n - m \neq 0, \\
P^{k02}_{nm} &= 0, \qquad \text{if} \sum_{j=1}^{b} k_j i_j - n - m \neq 0.
\end{aligned} \tag{4.7}$$

Remark. The special form of P defined in $(A4)$, i.e., $P \in \mathcal{A}$, is crucial in this paper. With P of such special form, one knows that $P^{k11}_{nm} = 0$ if $k = 0$ and $n \neq m$, then the terms $P^{011}_{nm} z_n \bar{z}_m$ with $n \neq m$ are absent, i.e., z_n, z_m with $n \neq m$ are uncoupled in the new normal form.

Rewrite H as $H = N + R + (P - R)$. By the choice of s_+ in (4.5) and the definition of the norms, it follows immediately that

$$\|X_R\|_{D(r,s),\mathcal{O}} \leq \|X_P\|_{D(r,s),\mathcal{O}} \leq \varepsilon. \tag{4.8}$$

Moreover, we take $s_+ \ll s$ such that in a domain $D(r, s_+)$,

$$\|X_{(P-R)}\|_{D(r,s_+)} \; < \; \mathrm{c}\, \varepsilon_+. \tag{4.9}$$

In the following, we will look for an F in the class $\mathcal{A}$, defined in a domain $D_+ = D(r_+, s_+)$, such that the time one map ϕ^1_F of the Hamiltonian vector field X_F defines a map from $D_+ \to D$ and transforms H into H_+. More precisely, by second order Taylor formula, we have

$$\begin{aligned}
H \circ \phi^1_F &= (N + R) \circ \phi^1_F + (P - R) \circ \phi^1_F \\
&= N + \{N, F\} + R \\
&\quad + \int_0^1 (1 - t)\{\{N, F\}, F\} \circ \phi^t_F dt + \int_0^1 \{R, F\} \circ \phi^t_F dt + (P - R) \circ \phi^1_F \\
&= N_+ + P_+ + \{N, F\} + R - P_{0000} - \langle \omega', I \rangle - \sum_n P^{011}_{nn} z_n \bar{z}_n,
\end{aligned} \tag{4.10}$$

where

$$\omega' = \int \frac{\partial P}{\partial I} d\theta|_{z=\bar{z}=0, I=0},$$

$$N_+ = N + P_{0000} + \langle \omega', I \rangle + \sum_n P_{nn}^{011} z_n \bar{z}_n, \tag{4.11}$$

$$P_+ = \int_0^1 (1-t)\{\{N, F\}, F\} \circ \phi_F^t dt + \int_0^1 \{R, F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^1. \tag{4.12}$$

Remark. Generally speaking, $\sum_n P_{nn}^{011} z_n \bar{z}_n$ should be replaced in (4.11) by $\sum_{|n|=|m|} P_{nm}^{011} z_n \bar{z}_m$, but in terms of (4.7), $P_{nm}^{011} = 0$ if $n \neq m$. Hence the terms $\sum_{n \neq m} P_{nm}^{011} z_n \bar{z}_m$ are absent. Thus N_+ has the same form as that in the first step.

We shall find a function F of the form

$$\begin{aligned} F(\theta, I, z, \bar{z}) &= F_0 + F_1 + F_2 \\ &= \sum_{k \neq 0, |l| \leq 1} F_{kl00} e^{\mathrm{i}\langle k, \theta \rangle} I^l + \sum_{k,n} (F_n^{k10} z_n + F_n^{k01} \bar{z}_n) e^{\mathrm{i}\langle k, \theta \rangle} \\ &\quad + \sum_{k,n,m} (F_{nm}^{k20} z_n z_m + F_{nm}^{k02} \bar{z}_n \bar{z}_m) e^{\mathrm{i}\langle k, \theta \rangle} + \sum_{|k|+||n|-|m|| \neq 0} F_{nm}^{k11} z_n \bar{z}_m e^{\mathrm{i}\langle k, \theta \rangle} \end{aligned} \tag{4.13}$$

satisfying the equation

$$\{N, F\} + R - P_{0000} - \langle \omega', I \rangle - \sum_n P_{nn}^{011} z_n \bar{z}_n = 0. \tag{4.14}$$

Lemma 4.1. *F satisfies (4.14) and is in $\mathcal{A}$ if the Fourier coefficients of F are defined by the following equations:*

$$\begin{aligned} (\langle k, \omega \rangle) F_{kl00} &= \mathrm{i} P_{kl00}, \quad k \neq 0, |l| \leq 1, \\ (\langle k, \omega \rangle - \Omega_n) F_n^{k10} &= \mathrm{i} P_n^{k10}, \\ (\langle k, \omega \rangle + \Omega_n) F_n^{k01} &= \mathrm{i} P_n^{k01}, \\ (\langle k, \omega \rangle - \Omega_n - \Omega_m) F_{nm}^{k20} &= \mathrm{i} P_{nm}^{k20}, \\ (\langle k, \omega \rangle - \Omega_n + \Omega_m) F_{nm}^{k11} &= \mathrm{i} P_{nm}^{k11}, \quad |k| + ||n| - |m|| \neq 0, \\ (\langle k, \omega \rangle + \Omega_n + \Omega_m) F_{nm}^{k02} &= \mathrm{i} P_{nm}^{k02}. \end{aligned} \tag{4.15}$$

Proof. Inserting F, defined in (4.13), into (4.14) one sees that (4.14) is equivalent to the following equations

$$\begin{aligned} \{N, F_0\} + R_0 &= P_{0000} + \langle \omega', I \rangle, \\ \{N, F_1\} + R_1 &= 0, \\ \{N, F_2\} + R_2 &= \sum_n P_{nn}^{011} z_n \bar{z}_n. \end{aligned} \tag{4.16}$$

By comparing the coefficients, the first equation in (4.16) is obviously equivalent to the first equation in (4.15). To solve $\{N, F_1\} + R_1 = 0$, we note that

$$\{N, F_1\} = \mathrm{i} \sum_{k,n} (\langle k, \omega \rangle F_n^{k10} z_n - \Omega_n F_n^{k10} z_n) e^{\mathrm{i}\langle k, \theta \rangle}$$

$$
\begin{aligned}
&+\mathrm{i}\sum_{k,n}(\langle k,\omega\rangle F_n^{k01}\bar{z}_n+\Omega_n F_n^{k01}\bar{z}_n)e^{\mathrm{i}\langle k,\theta\rangle}\\
=\ &\mathrm{i}\sum_{k,n}(\langle k,\omega\rangle-\Omega_n)F_n^{k10}z_n e^{\mathrm{i}\langle k,\theta\rangle}\\
&+\mathrm{i}\sum_{k,n}(\langle k,\omega\rangle+\Omega_n)F_n^{k01}\bar{z}_n e^{\mathrm{i}\langle k,\theta\rangle}\ . \qquad (4.17)
\end{aligned}
$$

It follows that F_n^{k10}, F_n^{k01} are determined by the linear algebraic systems

$$
\begin{aligned}
&\mathrm{i}(\langle k,\omega\rangle-\Omega_n)F_n^{k10}+R_n^{k10}=0, \quad n\in \mathrm{Z}_1^d, k\in \mathrm{Z}^b,\\
&\mathrm{i}(\langle k,\omega\rangle+\Omega_n)F_n^{k01}+R_n^{k01}=0, \quad n\in \mathrm{Z}_1^d, k\in \mathrm{Z}^b.
\end{aligned}
$$

Similarly, from

$$
\begin{aligned}
\{N,F_2\}=\ &\mathrm{i}\sum_{k,n,m}(\langle k,\omega\rangle F_{nm}^{k20}z_nz_m-\Omega_nF_{nm}^{k20}z_nz_m-\Omega_mF_{nm}^{k20}z_nz_m)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+\mathrm{i}\sum_{|k|+||n|-|m||\neq 0}(\langle k,\omega\rangle F_{nm}^{k11}z_n\bar{z}_m-\Omega_nF_{nm}^{k11}z_n\bar{z}_m+\Omega_mF_{nm}^{k11}z_n\bar{z}_m)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+\mathrm{i}\sum_{k,n,m}(\langle k,\omega\rangle F_{nm}^{k02}\bar{z}_n\bar{z}_m+\Omega_nF_{nm}^{k02}\bar{z}_n\bar{z}_m+\Omega_mF_{nm}^{k02}\bar{z}_n\bar{z}_m)e^{\mathrm{i}\langle k,\theta\rangle}\\
=\ &\mathrm{i}\sum_{k,n,m}(\langle k,\omega\rangle-\Omega_n-\Omega_m)F_{nm}^{k20}z_nz_me^{\mathrm{i}\langle k,\theta\rangle}\\
&+\mathrm{i}\sum_{|k|+||n|-|m||\neq 0}(\langle k,\omega\rangle-\Omega_n+\Omega_m)F_{nm}^{k11}z_n\bar{z}_me^{\mathrm{i}\langle k,\theta\rangle}\\
&+\mathrm{i}\sum_{k,n,m}(\langle k,\omega\rangle+\Omega_n+\Omega_m)F_{nm}^{k02}\bar{z}_n\bar{z}_me^{\mathrm{i}\langle k,\theta\rangle}\ , \qquad (4.18)
\end{aligned}
$$

it follows that F_{nm}^{k20}, F_{nm}^{k11} and F_{nm}^{k02} are determined by the following linear algebraic systems

$$
\begin{aligned}
&(\langle k,\omega\rangle-\Omega_n-\Omega_m)F_{nm}^{k20}=\mathrm{i}R_{nm}^{k20}, \quad n,m\in \mathrm{Z}_1^d,\ k\in \mathrm{Z}^b,\\
&(\langle k,\omega\rangle-\Omega_n+\Omega_m)F_{nm}^{k11}=\mathrm{i}R_{nm}^{k11}, \quad |k|+||n|-|m||\neq 0, \qquad (4.19)\\
&(\langle k,\omega\rangle+\Omega_n+\Omega_m)F_{nm}^{k02}=\mathrm{i}R_{nm}^{k02}, \quad n,m\in \mathrm{Z}_1^d,\ k\in \mathrm{Z}^b.
\end{aligned}
$$

Moreover, $P\in\mathcal{A}$ implies $F\in\mathcal{A}$. □

4.2. Estimation on the coordinate transformation. We proceed to estimate X_F and ϕ_F^1. We start with the following

Lemma 4.2. *Let* $D_i=D(r_++\frac{i}{4}(r-r_+),\frac{i}{4}s)$, $0<i\leq 4$. *Then*

$$
\|X_F\|_{D_3,\mathcal{O}}\leq c\gamma^{-2}(r-r_+)^{-c}\varepsilon. \qquad (4.20)
$$

Proof. By (4.2), Lemma 4.1, Assumptions (A1) and (A2), we have

$$\begin{aligned}
&|F_{kl00}|_{\mathcal{O}} \le (|\langle k,\omega\rangle|^{-1}|P_{kl00}|)_{\mathcal{O}} < c\gamma^{-2}|k|^{2\tau+1}|P_{kl00}|_{\mathcal{O}}, \quad k\neq 0;\\
&|F_n^{k10}|_{\mathcal{O}} \le c\gamma^{-2}|k|^{2\tau+1}|P_n^{k10}|;\\
&|F_n^{k01}|_{\mathcal{O}} \le c\gamma^{-2}|k|^{2\tau+1}|P_n^{k01}|;\\
&|F_{nm}^{k20}|_{\mathcal{O}} \le c\gamma^{-2}|k|^{2\tau+1}|P_{nm}^{k20}|;\\
&|F_{nm}^{k11}|_{\mathcal{O}} \le c\gamma^{-2}|k|^{2\tau+1}|P_{nm}^{k11}|, \quad |k|+||n|-|m||\neq 0;\\
&|F_{nm}^{k02}|_{\mathcal{O}} \le c\gamma^{-2}|k|^{2\tau+1}|P_{nm}^{k02}|.
\end{aligned} \tag{4.21}$$

It follows that

$$\begin{aligned}
\frac{1}{s^2}\|F_\theta\|_{D_3,\mathcal{O}} \le \frac{1}{s^2}\Bigg(&\sum_{k,|l|\le 1}|F_{kl00}|\cdot s^{2|l|}\cdot|k|\cdot e^{|k|(r-\frac14(r-r_+))}\\
&+\sum_{k,n}|F_n^{k10}|\cdot|z_n|\cdot|k|\cdot e^{|k|(r-\frac14(r-r_+))}\\
&+\sum_{k,n}|F_n^{k01}|\cdot|\bar z_n|\cdot|k|\cdot e^{|k|(r-\frac14(r-r_+))}\\
&+\sum_{k,n,m}|F_{nm}^{k20}|\cdot|z_n|\cdot|z_m|\cdot|k|\cdot e^{|k|(r-\frac14(r-r_+))}\\
&+\sum_{|k|+||n|-|m||\neq 0}|F_{nm}^{k11}|\cdot|z_n|\cdot|\bar z_m|\cdot|k|\cdot e^{|k|(r-\frac14(r-r_+)}\Bigg)\\
&+\sum_{k,n,m}|F_{nm}^{k02}|\cdot|\bar z_n|\cdot|\bar z_m|\cdot|k|\cdot e^{|k|(r-\frac14(r-r_+))})\\
\le\ & c\gamma^{-2}(r-r_+)^{-c}\|X_R\|\\
\le\ & c\gamma^{-2}(r-r_+)^{-c}\varepsilon.
\end{aligned} \tag{4.22}$$

Similarly,

$$\|F_I\|_{D_3,\mathcal{O}} = \sum_{|l|=1}|F_{kl00}|e^{|k|(r-\frac14(r-r_+))} \le c\gamma^{-2}(r-r_+)^{-c}\varepsilon.$$

Now we estimate $\|X_{F_1}\|_{D_3,\mathcal{O}}$. Since

$$\begin{aligned}
\|F_{1_{z_n}}\|_{D_3,\mathcal{O}} &= \|\sum_k F_n^{k10}e^{\mathrm{i}\langle k,\theta\rangle}\|_{D_3,\mathcal{O}}\\
&\le \sum_k |F_n^{k10}|e^{|k|(r-\frac14(r-r_+))}\\
&\le c\gamma^{-2}\sum_k |P_n^{k10}||k|^{2\tau+1}e^{|k|(r-\frac14(r-r_+))}
\end{aligned}$$

and similarly

$$\|F_{1_{\bar z_n}}\|_{D_3,\mathcal{O}} \le c\gamma^{-2}\sum_k |P_n^{k01}||k|^{2\tau+1}e^{|k|(r-\frac14(r-r_+))},$$

it follows from the definition of the weighted norm[2] that

$$\|X_{F_1}\|_{D_3,\mathcal{O}} \le \frac{c}{s}\left(\sum_n \|F_{1_{z_n}}\|_{D_3,\mathcal{O}}|n|^{\bar{a}}e^{|n|\rho} + \sum_n \|F_{1_{\bar{z}_n}}\|_{D_3,\mathcal{O}}|n|^{\bar{a}}e^{|n|\rho}\right)$$
$$\le c\gamma^{-2}(r-r_+)^{-c}\|X_R\| \le c\gamma^{-2}(r-r_+)^{-c}\varepsilon. \tag{4.23}$$

Moreover,

$$\|F_{2_{z_n}}\|_{D_3,\mathcal{O}} = \|\sum_{k,m} F_{nm}^{k20} z_m e^{\mathrm{i}\langle k,\theta\rangle}\|_{D_3,\mathcal{O}} + \|\sum_{k,m} F_{nm}^{k11}\bar{z}_m e^{\mathrm{i}\langle k,\theta\rangle}\|_{D_3,\mathcal{O}}$$
$$\le c\gamma^{-2}\left(\sum_{k,m}|P_{nm}^{k20}||z_m||k|^{2\tau+1}e^{|k|(r-\frac{1}{4}(r-r_+)})\right.$$
$$\left.+\sum_{k,m}|P_{nm}^{k11}||\bar{z}_m||k|^{2\tau+1}e^{|k|(r-\frac{1}{4}(r-r_+))}\right); \tag{4.24}$$

and similarly

$$\|F_{2_{\bar{z}_n}}\|_{D_3,\mathcal{O}} \le c\gamma^{-2}\left(\sum_{k,m}|P_{mn}^{k11}||z_m||k|^{2\tau+1}e^{|k|(r-\frac{1}{4}(r-r_+))}\right.$$
$$\left.+\sum_{k,m}|P_{nm}^{k02}||\bar{z}_m||k|^{2\tau+1}e^{|k|(r-\frac{1}{4}(r-r_+))}\right).$$

Hence we have

$$\|X_{F_2}\|_{D_3,\mathcal{O}} \le \frac{c}{s}\left(\sum_n \|F_{2_{z_n}}\|_{D_3,\mathcal{O}}|n|^{\bar{a}}e^{|n|\rho} + \sum_n \|F_{2_{\bar{z}_n}}\|_{D_3,\mathcal{O}}|n|^{\bar{a}}e^{|n|\rho}\right)$$
$$\le c\gamma^{-2}(r-r_+)^{-c}\|X_R\|$$
$$\le c\gamma^{-2}(r-r_+)^{-c}\varepsilon. \tag{4.25}$$

The conclusion of the lemma follows from the estimates above. □

In the next lemma, we give some estimates for ϕ_F^t. The formula (4.26) will be used to prove our coordinate transformation is well defined. Inequality (4.27) will be used to check the convergence of the iteration.

Lemma 4.3. *Let* $\eta = \varepsilon^{\frac{1}{3}}$, $D_{i\eta} = D(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}\eta s)$, $0 < i \le 4$. *If* $\varepsilon \ll (\frac{1}{2}\gamma^2(r-r_+)^c)^{\frac{3}{2}}$, *we then have*

$$\phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \le t \le 1. \tag{4.26}$$

Moreover,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} < c\gamma^{-2}(r-r_+)^{-c}\varepsilon. \tag{4.27}$$

[2] Recall (2.4), the definition of the norm.

Proof. Let

$$\|D^m F\|_{D,\mathcal{O}} = \max\left\{ \left\| \frac{\partial^{|i|+|l|+|\alpha|+|\beta|}}{\partial\theta^i \partial I^l \partial z_{\vec{n}}^{\alpha} \partial \bar{z}_{\vec{n}}^{\beta}} F \right\|_{D,\mathcal{O}}, |i| + |l| + |\alpha| + |\beta| = m \geq 2 \right\}.$$

Notice that F is a polynomial of degree 1 in I and degree 2 in $z, \bar{z}$. From (2.4), (4.25) and the Cauchy inequality, it follows that

$$\|D^m F\|_{D_2,\mathcal{O}} < c\gamma^{-2}(r - r_+)^{-c}\varepsilon, \tag{4.28}$$

for any $m \geq 2$.

To get the estimates for ϕ_F^t, we start from the integral equation,

$$\phi_F^t = id + \int_0^t X_F \circ \phi_F^s \, ds$$

so that $\phi_F^t : D_{2\eta} \to D_{3\eta}, \quad -1 \leq t \leq 1$, which follows directly from (4.28). Since

$$D\phi_F^t = Id + \int_0^t (DX_F) D\phi_F^s \, ds = Id + \int_0^t J(D^2 F) D\phi_F^s \, ds,$$

where J denotes the standard symplectic matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, it follows that

$$\|D\phi_F^t - Id\| \leq 2\|D^2 F\| < c\gamma^{-2}(r - r_+)^{-c}\varepsilon. \tag{4.29}$$

Consequently Lemma 4.3 follows. □

4.3. Estimation for the new normal form. The map ϕ_F^1 defined above transforms H into $H_+ = N_+ + P_+$(see (4.10) and (4.14)). Due to the special form of P defined in $(A4)$, the terms in $\sum_{n,m} P_{nm}^{011} z_n \bar{z}_m$ with $n \neq m$ are absent, i.e., z_n, z_m with $n \neq m$ are uncoupled. Hence compared with the normal form in [13], here the normal form N_+ is simpler

$$\begin{aligned} N_+ &= N + P_{0000} + \langle \omega', I \rangle + \sum_n P_{nn}^{011} z_n \bar{z}_n \\ &= e_+ + \langle \omega_+, I \rangle + \sum_n \Omega_n^+ z_n \bar{z}_n, \end{aligned} \tag{4.30}$$

where

$$e_+ = e + P_{0000}, \; \omega_+ = \omega + P_{0l00}(|l| = 1), \; \Omega_n^+ = \Omega_n + P_{nn}^{011}. \tag{4.31}$$

Now we prove that N_+ has the same properties as N. By regularity of P, set[3] $\delta = \min\{\iota, \bar{a} - a\}$, then we have

$$|\omega_+ - \omega|_{\mathcal{O}} < \varepsilon, \quad |P_{nn}^{011}|_{\mathcal{O}} < \varepsilon |n|^{-\delta}. \tag{4.32}$$

[3] Recall (2.9) and (2.4).

It follows that

$$|\langle k, \omega + P_{0l00}\rangle| \geq |\langle k, \omega\rangle| - |\langle k, P_{0l00}\rangle| \geq \frac{\gamma}{|k|^\tau} - \varepsilon|k| \geq \frac{\gamma_+}{|k|^\tau}, \quad k \neq 0, \tag{4.33}$$

$$|\langle k, \omega + P_{0l00}\rangle + \Omega_n^+| \geq |\langle k, \omega\rangle + \Omega_n| - |\langle k, P_{0l00}\rangle + P_{nn}^{011}| \geq \frac{\gamma_+}{|k|^\tau}, \tag{4.34}$$

and

$$|\langle k, \omega + P_{0l00}\rangle + \Omega_n^+ + \Omega_m^+| \geq \frac{\gamma_+}{|k|^\tau}, \tag{4.35}$$

$$|\langle k, \omega + P_{0l00}\rangle + \Omega_n^+ - \Omega_m^+| \geq \frac{\gamma_+}{|k|^\tau}, \quad |k| + ||n| - |m|| \neq 0, \tag{4.36}$$

provided that $\varepsilon|k|^{\tau+1} \leq c(\gamma - \gamma_+)$. This means that in the succeeding KAM step, small divisor conditions are automatically satisfied for $|k| \leq K$, where $\varepsilon K^{\tau+1} \leq c(\gamma - \gamma_+)$. The following bounds will be used for the measure estimates:

$$|\omega_+ - \omega|_{\mathcal{O}} < \varepsilon, \quad |\Omega_n^+ - \Omega_n|_{\mathcal{O}} < \varepsilon|n|^{-\delta}. \tag{4.37}$$

4.4. Estimation for the new perturbation. Since

$$\begin{aligned} P_+ &= \int_0^1 (1-t)\{\{N, F\}, F\} \circ \phi_F^t dt + \int_0^1 \{R, F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^1 \\ &= \int_0^1 \{R(t), F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^1, \end{aligned}$$

where $R(t) = (1-t)(N_+ - N) + tR$. Hence

$$X_{P_+} = \int_0^1 (\phi_F^t)^* X_{\{R(t), F\}} dt + (\phi_F^1)^* X_{(P-R)}.$$

According to Lemma 4.3,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} < c\gamma^{-2}(r - r_+)^{-c}\varepsilon, \quad -1 \leq t \leq 1,$$

thus

$$\|D\phi_F^t\|_{D_{1\eta}} \leq 1 + \|D\phi_F^t - Id\|_{D_{1\eta}} \leq 2, \quad -1 \leq t \leq 1.$$

Due to Lemma 7.3,

$$\|X_{\{R(t), F\}}\|_{D_{2\eta}} \leq c\gamma^{-2}(r - r_+)^{-c}\eta^{-2}\varepsilon^2,$$

and

$$\|X_{(P-R)}\|_{D_{2\eta}} \leq c\eta\varepsilon,$$

we have

$$\|X_{P_+}\|_{D(r_+, s_+)} \leq c\eta\varepsilon + c\gamma^{-2}(r - r_+)^{-c}\eta^{-2}\varepsilon^2 \leq c\varepsilon_+.$$

4.5. Verification of $(A4)$ *after one KAM step.* The assumption $P \in \mathcal{A}$ defined in $(A4)$ is used to guarantee that the normal form at each KAM step has the same form as in the first step. To complete one KAM step, we need to prove the new perturbation P_+ still has the special form defined in $(A4)$, i.e., $P_+ \in \mathcal{A}$.

Note that

$$P_+ = P - R + \{P, F\} + \frac{1}{2!}\{\{N, F\}, F\} + \frac{1}{2!}\{\{P, F\}, F\} + \cdots + \frac{1}{n!}\{\cdots\{N, \underbrace{F\}\cdots, F}_{n}\} + \frac{1}{n!}\{\cdots\{P, \underbrace{F\}\cdots, F}_{n}\} + \cdots.$$

Since $P \in \mathcal{A}$ we have $P - R \in \mathcal{A}$. Moreover, $\{N, F\} = P_{0000} + \langle \omega', I\rangle + \sum_n P^{011}_{nn} z_n \bar{z}_n - R \in \mathcal{A}$. Hence the key point is to prove that $\mathcal{A}$ is closed under the Poisson bracket. To this end, we prove the following lemma.

Lemma 4.4. *If* $G(\theta, I, z, \bar{z}), F(\theta, I, z, \bar{z}) \in \mathcal{A}$*, then* $B(\theta, I, z, \bar{z}) = \{G, F\} \in \mathcal{A}$.

Proof. Let

$$G = \sum_{k_1,\alpha_1,\beta_1} G_{k_1\alpha_1\beta_1}(I)e^{\mathrm{i}\langle k_1,\theta\rangle} z^{\alpha_1}\bar{z}^{\beta_1},$$
$$F = \sum_{k_2,\alpha_2,\beta_2} F_{k_2\alpha_2\beta_2}(I)e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_2}\bar{z}^{\beta_2},$$

where the summations are taken over

$$\{(k_1, \alpha_1, \beta_1), \quad \sum_{j=1}^{b} k_{1j} i_j + \sum_{n\in \mathbb{Z}_1^d} (\alpha_{1n} - \beta_{1n})n = 0\}, \tag{4.38}$$

and

$$\{(k_2, \alpha_2, \beta_2), \quad \sum_{j=1}^{b} k_{2j} i_j + \sum_{n\in \mathbb{Z}_1^d} (\alpha_{2n} - \beta_{2n})n = 0\} \tag{4.39}$$

respectively. Since

$$\begin{aligned}\{G, F\} = &\sum_{A_1}\sum_{A_2}\langle \frac{\partial G_{k_1\alpha_1\beta_1}(I)}{\partial I}, \mathrm{i}k_2\rangle F_{k_2\alpha_2\beta_2}(I)e^{\mathrm{i}\langle k_1,\theta\rangle} z^{\alpha_1}\bar{z}^{\beta_1} e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_2}\bar{z}^{\beta_2}\\
&-\sum_{A_1}\sum_{A_2}\langle \mathrm{i}k_1, \frac{\partial F_{k_2\alpha_2\beta_2}(I)}{\partial I}\rangle G_{k_1\alpha_1\beta_1}(I)e^{\mathrm{i}\langle k_1,\theta\rangle} z^{\alpha_1}\bar{z}^{\beta_1} e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_2}\bar{z}^{\beta_2}\\
&+\mathrm{i}\sum_{m}\sum_{A_3} G_{k_1\alpha_1\beta_1}(I)F_{k_2\alpha_2\beta_2}(I)e^{\mathrm{i}\langle k_1,\theta\rangle}e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_1-e_m}\bar{z}^{\beta_1} z^{\alpha_2}\bar{z}^{\beta_2-e_m}\\
&-\mathrm{i}\sum_{m}\sum_{A_4} G_{k_1\alpha_1\beta_1}(I)F_{k_2\alpha_2\beta_2}(I)e^{\mathrm{i}\langle k_1,\theta\rangle}e^{\mathrm{i}\langle k_2,\theta\rangle} z^{\alpha_1}\bar{z}^{\beta_1-e_m} z^{\alpha_2-e_m}\bar{z}^{\beta_2}\\
= &\sum_{A_5} B_{(k_1+k_2)(\alpha_1+\alpha_2)(\beta_1+\beta_2)}(I)e^{\mathrm{i}\langle k_1+k_2,\theta\rangle} z^{\alpha_1+\alpha_2}\bar{z}^{\beta_1+\beta_2}\\
&+\sum_{A_6} B_{(k_1+k_2)(\alpha_1+\alpha_2-e_m)(\beta_1+\beta_2-e_m)}(I)e^{\mathrm{i}\langle k_1+k_2,\theta\rangle} z^{\alpha_1+\alpha_2-e_m}\bar{z}^{\beta_1+\beta_2-e_m},\end{aligned} \tag{4.40}$$

where A_1 denotes

$$\sum_{j=1}^{b} k_{1j} i_j + \sum_{n \in \mathbb{Z}_1^d} (\alpha_{1n} - \beta_{1n}) n = 0,$$

A_2 denotes

$$\sum_{j=1}^{b} k_{2j} i_j + \sum_{n \in \mathbb{Z}_1^d} (\alpha_{2n} - \beta_{2n}) n = 0,$$

A_3 denotes

$$\sum_{j=1}^{b} k_{1j} i_j + (\alpha_{1m} - 1 - \beta_{1m}) m + \sum_{n \in \mathbb{Z}_1^d \setminus \{m\}} (\alpha_{1n} - \beta_{1n}) n = -m,$$
$$\sum_{j=1}^{b} k_{2j} i_j + (\alpha_{2m} - (\beta_{2m} - 1)) m + \sum_{n \in \mathbb{Z}_1^d \setminus \{m\}} (\alpha_{2n} - \beta_{2n}) n = m,$$

A_4 denotes

$$\sum_{j=1}^{b} k_{1j} i_j + (\alpha_{1m} - (\beta_{1m} - 1)) m + \sum_{n \in \mathbb{Z}_1^d \setminus \{m\}} (\alpha_{1n} - \beta_{1n}) n = m,$$
$$\sum_{j=1}^{b} k_{2j} i_j + (\alpha_{2m} - 1 - \beta_{2m}) m + \sum_{n \in \mathbb{Z}_1^d \setminus \{m\}} (\alpha_{2n} - \beta_{2n}) n = -m,$$

A_5 denotes

$$\sum_{j=1}^{b} (k_{1j} + k_{2j}) i_j + \sum_{n \in \mathbb{Z}_1^d} (\alpha_{1n} + \alpha_{2n} - \beta_{1n} - \beta_{2n}) n = 0,$$

A_6 denotes

$$\sum_{j=1}^{b} (k_{1j} + k_{2j}) i_j + ((\alpha_{1m} + \alpha_{2m} - 1) - (\beta_{1m} + \beta_{2m} - 1)) m$$
$$+ \sum_{n \in \mathbb{Z}_1^d \setminus \{m\}} ((\alpha_{1n} + \alpha_{2n}) - (\beta_{1n} + \beta_{2n})) n = 0.$$

Thus Lemma 4.4 is obtained. □

Corollary 1. *The new perturbation* $P_+ \in \mathcal{A}$.

5. Iteration Lemma and Convergence

For any given $s, \varepsilon, r, \gamma$ and for all $\nu \geq 1$, we define the following sequences

$$\begin{aligned}
r_\nu &= r\left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right), \\
\varepsilon_\nu &= c\gamma^{-2}(r_{\nu-1} - r_\nu)^{-c}\varepsilon_{\nu-1}^{\frac{4}{3}}, \\
\gamma_\nu &= \gamma\left(1 - \sum_{i=2}^{\nu+1} 2^{-i}\right), \\
\eta_\nu &= \varepsilon_\nu^{\frac{1}{3}}, \quad L_\nu = L_{\nu-1} + \varepsilon_{\nu-1}, \\
s_\nu &= \frac{1}{4}\eta_{\nu-1}s_{\nu-1} = 2^{-2\nu}\left(\prod_{i=0}^{\nu-1}\varepsilon_i\right)^{\frac{1}{3}} s_0, \\
K_\nu &= c(\varepsilon_\nu^{-1}(\gamma_\nu - \gamma_{\nu+1}))^{\frac{1}{\tau+1}},
\end{aligned} \tag{5.1}$$

where c is a constant, and the parameters $r_0, \varepsilon_0, \gamma_0, L_0, s_0$ and K_0 are defined to be $r, \varepsilon, \gamma, L, s$ and 1 respectively. Note that

$$\Psi(r) = \prod_{i=1}^{\infty}[(r_{i-1} - r_i)^{-c}]^{(\frac{3}{4})^i}$$

is a well–defined function of r.

5.1. Iteration lemma. The preceding analysis can be summarized as follows.

Lemma 5.1. *Let ε is small enough and $\nu \geq 0$. Suppose that*

(1) $N_\nu = e_\nu + \langle \omega_\nu(\xi), I\rangle + \sum_n \Omega_n^\nu(\xi) z_n \bar{z}_n$ is a normal form with parameters ξ satisfying

$$\begin{aligned}
|\langle k, \omega_\nu\rangle| &\geq \frac{\gamma_\nu}{|k|^\tau}, \quad k \neq 0, \\
|\langle k, \omega_\nu\rangle + \Omega_n^\nu| &\geq \frac{\gamma_\nu}{|k|^\tau}, \\
|\langle k, \omega_\nu\rangle + \Omega_n^\nu + \Omega_m^\nu| &\geq \frac{\gamma_\nu}{|k|^\tau}, \\
|\langle k, \omega_\nu\rangle + \Omega_n^\nu - \Omega_m^\nu| &\geq \frac{\gamma_\nu}{|k|^\tau}, \quad |k| + ||n| - |m|| \neq 0
\end{aligned}$$

on a closed set $\mathcal{O}_\nu$ of R^b;

(2) $\omega_\nu(\xi)$, $\Omega_n^\nu(\xi)$ are C_W^1 smooth and satisfy

$$|\omega_\nu - \omega_{\nu-1}|_{\mathcal{O}_\nu} \leq \varepsilon_{\nu-1}, \quad |\Omega_n^\nu - \Omega_n^{\nu-1}|_{\mathcal{O}_\nu} \leq \varepsilon_{\nu-1}|n|^{-\delta};$$

(3) P_ν *has the special form defined in* (A4) *(i.e.,* $P_\nu \in \mathcal{A}$*) and*

$$\|X_{P_\nu}\|_{D(r_\nu,s_\nu),\mathcal{O}_\nu} \le \varepsilon_\nu.$$

Then there is a subset $\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu$,

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \left(\bigcup_{|k|>K_\nu} \mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) \right),$$

where

$$\mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) = \left\{ \xi \in \mathcal{O}_\nu : \begin{array}{l} |\langle k, \omega_{\nu+1}\rangle| < \frac{\gamma_{\nu+1}}{|k|^\tau} \; |\langle k, \omega_{\nu+1}\rangle + \Omega_n^{\nu+1}| < \frac{\gamma_{\nu+1}}{|k|^\tau}, \text{ or} \\ |\langle k, \omega_{\nu+1}\rangle \pm \Omega_n^{\nu+1} \pm \Omega_m^{\nu+1}| < \frac{\gamma_{\nu+1}}{|k|^\tau}, \end{array} \right\}$$

with $\omega_{\nu+1} = \omega_\nu + P_{0l00}^\nu$*, and a symplectic transformation of variables*

$$\Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D(r_\nu, s_\nu), \tag{5.2}$$

such that on $D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$, $H_{\nu+1} = H_\nu \circ \Phi_\nu$ *has the form*

$$H_{\nu+1} = e_{\nu+1} + \langle \omega_{\nu+1}, I\rangle + \sum_n \Omega_n^{\nu+1} z_n \bar{z}_n + P_{\nu+1}, \tag{5.3}$$

with

$$|\omega_{\nu+1} - \omega_\nu|_{\mathcal{O}_{\nu+1}} \le \varepsilon_\nu, \quad |\Omega_n^{\nu+1} - \Omega_n^\nu|_{\mathcal{O}_{\nu+1}} \le \varepsilon_\nu |n|^{-\delta}. \tag{5.4}$$

And also $P_{\nu+1}$ *has the special form defined in* (A4) *(i.e.,* $P_{\nu+1} \in \mathcal{A}$*) with*

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}} \le \varepsilon_{\nu+1}. \tag{5.5}$$

5.2. Convergence. Suppose that the assumptions of Theorem 1 are satisfied. Recall that

$$\varepsilon_0 = \varepsilon, r_0 = r, \gamma_0 = \gamma, s_0 = s, L_0 = L, N_0 = N, P_0 = P,$$

$$\mathcal{O}_0 = \left\{ \xi \in \mathcal{O} : \begin{array}{ll} |\langle k, \omega(\xi)\rangle| \ge \frac{\gamma_0}{|k|^\tau}, & k \ne 0, \\ |\langle k, \omega(\xi)\rangle + \Omega_n| \ge \frac{\gamma_0}{|k|^\tau}, & \\ |\langle k, \omega(\xi)\rangle + \Omega_n + \Omega_m| \ge \frac{\gamma_0}{|k|^\tau}, & \\ |\langle k, \omega(\xi)\rangle + \Omega_n - \Omega_m| \ge \frac{\gamma_0}{|k|^\tau}, & |k| + ||n| - |m|| \ne 0 \end{array} \right\},$$

the assumptions of the iteration lemma are satisfied when $\nu = 0$ if ε_0 and γ_0 are sufficiently small. Inductively, we obtain the following sequences:

$$\begin{aligned} &\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu, \\ &\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D(r_0, s_0), \nu \ge 0, \\ &H \circ \Psi^\nu = H_{\nu+1} = N_{\nu+1} + P_{\nu+1}. \end{aligned}$$

Let $\tilde{\mathcal{O}} = \cap_{\nu=0}^\infty \mathcal{O}_\nu$. As in [17, 18], thanks to Lemma 4.3, it concludes that N_ν, Ψ^ν, $D\Psi^\nu$, ω_ν converge uniformly on $D(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ with

$$N_\infty = e_\infty + \langle \omega_\infty, I\rangle + \sum_n \Omega_n^\infty z_n \bar{z}_n.$$

Since

$$\varepsilon_{\nu+1} = c\gamma_\nu^{-2}(r_\nu - r_{\nu+1})^{-c}\varepsilon_\nu^{\frac{4}{3}} \leq (c\gamma^{-2}\Psi(r)\varepsilon)^{(\frac{4}{3})^\nu},$$

it follows that $\varepsilon_{\nu+1} \to 0$ provided that ε is sufficiently small.

Let ϕ_H^t be the flow of X_H. Since $H \circ \Psi^\nu = H_{\nu+1}$, we have

$$\phi_H^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{H_{\nu+1}}^t. \tag{5.6}$$

The uniform convergence of Ψ^ν, $D\Psi^\nu$, ω_ν and X_{H_ν} implies that the limits can be taken on both sides of (5.6). Hence, on $D(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ we get

$$\phi_H^t \circ \Psi^\infty = \Psi^\infty \circ \phi_{H_\infty}^t \tag{5.7}$$

and

$$\Psi^\infty : D(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}} \to D(r, s) \times \mathcal{O}.$$

It follows from (5.7) that

$$\phi_H^t(\Psi^\infty(\mathrm{T}^b \times \{\xi\})) = \Psi^\infty\phi_{N_\infty}^t(\mathrm{T}^b \times \{\xi\}) = \Psi^\infty(\mathrm{T}^b \times \{\xi\})$$

for $\xi \in \tilde{\mathcal{O}}$. This means that $\Psi^\infty(\mathrm{T}^b \times \{\xi\})$ is an embedded torus which is invariant for the original perturbed Hamiltonian system at $\xi \in \tilde{\mathcal{O}}$. We remark here that the frequencies $\omega_\infty(\xi)$ associated to $\Psi^\infty(\mathrm{T}^b \times \{\xi\})$ are slightly different from $\omega(\xi)$. The normal behavior of the invariant torus is governed by normal frequencies Ω_n^∞. □

6. Measure Estimates

For notational convenience, let $\mathcal{O}_{-1} = \mathcal{O}$, $K_{-1} = 0$. Then at ν^{th} step of KAM iteration, we have to exclude the following resonant set

$$\mathcal{R}^\nu = \bigcup_{|k|>K_{\nu-1},n,m} (\mathcal{R}_k^\nu \bigcup \mathcal{R}_{kn}^\nu \bigcup \mathcal{R}_{knm}^\nu),$$

where

$$\mathcal{R}_k^\nu = \{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\xi)\rangle| < \frac{\gamma_\nu}{|k|^\tau}\}, \tag{6.1}$$

$$\mathcal{R}_{kn}^\nu = \{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\xi)\rangle + \Omega_n^\nu| < \frac{\gamma_\nu}{|k|^\tau}\}, \tag{6.2}$$

$$\mathcal{R}_{knm}^\nu = \{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\xi)\rangle \pm \Omega_n^\nu \pm \Omega_m^\nu| < \frac{\gamma_\nu}{|k|^\tau}\}. \tag{6.3}$$

Remark. From Sect. 4.3, one has that at the ν^{th} step, small divisor conditions are automatically satisfied for $|k| \leq K_{\nu-1}$. Hence, we only need to excise the above resonant set $\mathcal{R}^\nu$. Note that due to the *Special form of the perturbation* (see ($A4$) and (4.7)), there are not the terms of the form $\sum_{n\neq m} P_{\nu nm}^{011} z_n \bar{z}_m$ in the perturbation P_ν, thus we need not consider small divisors $\langle k, \omega\rangle + \Omega_n^\nu - \Omega_m^\nu$ with $k = 0, n \neq m$, which is crucial for this paper.

Lemma 6.1. *For any fixed* $|k| > K_{\nu-1}$, *n and m,*

$$\operatorname{meas}\left(\mathcal{R}_k^\nu \bigcup \mathcal{R}_{kn}^\nu \bigcup \mathcal{R}_{knm}^\nu\right) < c\frac{\gamma_\nu}{|k|^{\tau+1}}.$$

Proof. Recall that $\omega_\nu(\xi) = \omega(\xi) + \sum_{j=0}^{\nu-1} P_{0l00}^j(\xi)$ with

$$|\sum_{j=0}^{\nu-1} P_{0l00}^j(\xi)|_{\mathcal{O}_{\nu-1}} \leq \varepsilon, \tag{6.4}$$

and $\Omega_n^\nu(\xi) = \Omega_n(\xi) + \sum_{j=0}^{\nu-1} P_{nn}^{011,j}$ with

$$\left|\sum_{j=0}^{\nu-1} P_{nn}^{011,j}\right|_{\mathcal{O}_{\nu-1}} \leq \varepsilon|n|^{-\delta}. \tag{6.5}$$

It follows that [4]

$$\left|\frac{\partial(\langle k, \omega_\nu(\xi)\rangle \pm \Omega_n^\nu \pm \Omega_m^\nu)}{\partial\xi}\right| \geq c|k|,$$

then the proof of this lemma is evident; we omit it. □

Lemma 6.2. *The total measure we need to exclude along the KAM iteration is*

$$\operatorname{meas}\left(\bigcup_{\nu\geq 0}\mathcal{R}^\nu\right) = \operatorname{meas}\left[\bigcup_{\nu\geq 0}\bigcup_{|k|>K_{\nu-1},n,m}\left(\mathcal{R}_k^\nu \bigcup \mathcal{R}_{kn}^\nu \bigcup \mathcal{R}_{knm}^\nu\right)\right] < c\gamma^\vartheta, \quad \vartheta > 0.$$

Proof. We estimate

$$\operatorname{meas}\left(\bigcup_{|k|>K_{\nu-1}}\bigcup_{n,m}\{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\xi)\rangle + \Omega_n^\nu - \Omega_m^\nu| < \frac{\gamma_\nu}{|k|^\tau}\}\right),$$

which is the most complicated case. We divide the proof into several cases according to p, d.

Case 1. $p = 1, d = 1$ and $p = 2, d > 1$. The case $p = 1, d = 1$ has been proved by Pöschel in [17]. The case $p = 2, d > 1$ can be proved similarly. In fact, suppose that $|n|^2 - |m|^2 = l \geq 0$. If $l > c|k|$, $\mathcal{R}_{knm}^{\nu+1} = \emptyset$; if $l \leq c|k|$, then according to assumption $(A2)$ and (6.5), we have

$$|\Omega_n^\nu - \Omega_m^\nu - l| \leq O(|m|^{-\delta}).$$

It follows that

$$\mathcal{R}_{knm}^\nu \subseteq Q_{klm}^\nu \overset{\text{def}}{=} \{\xi : |\langle k, \omega_\nu\rangle + l| < \frac{\gamma_\nu}{|k|^\tau} + O(|m|^{-\delta})\}. \tag{6.6}$$

[4] Here $|\cdot|$ denotes ℓ^1–norm.

Moreover, $Q^{\nu}_{klm} \subseteq Q^{\nu}_{klm_0}$ for $|m| \geq |m_0|$. Due to Lemma 6.1, one has

$$\begin{aligned}\text{meas}\left(\bigcup_{l\leq c|k|}\ \bigcup_{|n|^2-|m|^2=l} \mathcal{R}^{\nu}_{knm}\right) &\leq \sum_{l\leq c|k|}\ \sum_{|m|<|m_0|} \text{meas}(\mathcal{R}^{\nu}_{knm}) + \sum_{l\leq c|k|} \text{meas}(Q_{klm_0}) \\ &< c\left(\frac{\gamma|m_0|^{C(d)}}{|k|^{\tau}} + O(|m_0|^{-\delta})\right), \qquad (6.7)\end{aligned}$$

where $C(d)$ is a constant depending only on space dimension d. By choosing $\frac{\gamma|m_0|^{C(d)}}{|k|^{\tau}} = |m_0|^{-\delta}$, i.e.

$$|m_0| = \left(\frac{|k|^{\tau}}{\gamma}\right)^{\frac{1}{\delta+C(d)}},$$

we arrive at

$$\text{meas}\left(\bigcup_{l\leq c|k|}\ \bigcup_{|n|^2-|m|^2=l} \mathcal{R}^{\nu}_{knm}\right) < c\frac{\gamma^{\frac{\delta}{\delta+C(d)}}}{|k|^{\frac{\delta\tau}{\delta+C(d)}}}. \qquad (6.8)$$

Case 2. $p > 2$ for $d > 1$ and $p > 1$ for $d = 1$. Without loss of generality, we assume that $|n| \geq |m|$. If $|n| = |m|$, the proof proceeds in the same way as in Case 1. If $|n| > |m|$, we have

$$\begin{aligned}|n|^p - |m|^p &\geq \frac{1}{2}|n|^{p-2}(|n|^2 - |m|^2), \quad \text{for } d > 1,\\ |n|^p - |m|^p &\geq |n|^{p-1}(|n| - |m|), \quad \text{for } d = 1.\end{aligned}$$

If $|n| > c|k|^{\frac{1}{p-2}}$ for $d > 1$ or $|n| > c|k|^{\frac{1}{p-1}}$ for $d = 1$, we get $\mathcal{R}^{\nu}_{knm} = \emptyset$; if $|n| \leq c|k|^{\frac{1}{p-2}}$ for $d > 1$, it follows from Lemma 6.1 that

$$\text{meas}\Big(\bigcup_{|n|\neq|m|} \mathcal{R}^{\nu}_{knm}\Big) = \text{meas}\Big(\bigcup_{|n|\neq|m|;\,|n|,|m|\leq c|k|^{\frac{1}{p-2}}} \mathcal{R}^{\nu}_{knm}\Big) < c\frac{\gamma}{|k|^{\tau+1-\frac{C(d)}{p-2}}}. \qquad (6.9)$$

The case of $d = 1$ can be proved analogously.

Let $\vartheta = \frac{\delta}{\delta+C(d)}$, $\tau > \frac{(b+1)(\delta+C(d))}{\delta} + \frac{C(d)}{p-2}$ (here $p > 2$),

$$\begin{aligned}\text{meas}(\mathcal{R}^{\nu}) &\leq c\ \text{meas}\left(\bigcup_{|k|>K_{\nu-1}}\ \bigcup_{n,m} \mathcal{R}^{\nu}_{knm}\right)\\ &\leq c \sum_{|k|>K_{\nu-1}} \text{meas}\left(\bigcup_{n,m} \mathcal{R}^{\nu}_{knm}\right)\\ &< c\gamma^{\vartheta}(1+K_{\nu-1})^{-1}. \qquad (6.10)\end{aligned}$$

The above upper bound (6.10) gives out the measure estimate of excluded parameter set at the ν^{th} step of KAM iteration, then along the KAM iteration, the total measure we need to excise has the upper bound

$$\begin{aligned}\text{meas}\left(\bigcup_{\nu\geq 0}\mathcal{R}^{\nu}\right) &\leq \sum_{\nu\geq 0}\text{meas}\left(\mathcal{R}^{\nu}\right)\\ &< c\gamma^{\vartheta}\sum_{\nu\geq 0}(1+K_{\nu-1})^{-1} < c\gamma^{\vartheta}. \end{aligned} \tag{6.11}$$

So Lemma 6.2 follows.

Remark 1. Note that the regularity is crucial in (6.6)–(6.8).

Remark 2. The case $p=1, d\geq 2$ is not considered in this paper since we can not prove Lemma 6.2 in this case. This is the reason why Theorem 1 can not be applied to higher dimensional wave equations.

7. Appendix

Lemma 7.1.

$$\|FG\|_{D(r,s)} \leq \|F\|_{D(r,s)}\|G\|_{D(r,s)}.$$

Proof. Since $(FG)_{kl\alpha\beta} = \sum_{k',l',\alpha',\beta'} F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'}$, we have

$$\begin{aligned}\|FG\|_{D(r,s)} &= \sup_{\substack{\|z\|_\rho<s\\ \|\bar z\|_\rho<s}}\sum_{k,l,\alpha,\beta}|(FG)_{kl\alpha\beta}|s^{2l}|z^{\alpha}||\bar z^{\beta}|e^{|k|r}\\ &\leq \sup_{\substack{\|z\|_\rho<s\\ \|\bar z\|_\rho<s}}\sum_{k,l,\alpha,\beta}\sum_{k',l',\alpha',\beta'}|F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'}|s^{2l}|z^{\alpha}||\bar z^{\beta}|e^{|k|r}\\ &\leq \|F\|_{D(r,s)}\|G\|_{D(r,s)},\end{aligned}$$

and the proof is finished. □

Lemma 7.2 (Cauchy inequalities).

$$\begin{aligned}\|F_{\theta}\|_{D(r-\sigma,s)} &\leq \frac{c}{\sigma}\|F\|_{D(r,s)},\\ \|F_I\|_{D(r,\frac{1}{2}s)} &\leq \frac{c}{s^2}\|F\|_{D(r,s)},\end{aligned}$$

and

$$\begin{aligned}\|F_{z_n}\|_{D(r,\frac{1}{2}s)} &\leq \frac{c}{s}\|F\|_{D(r,s)}|n|^{a}e^{|n|\rho},\\ \|F_{\bar z_n}\|_{D(r,\frac{1}{2}s)} &\leq \frac{c}{s}\|F\|_{D(r,s)}|n|^{a}e^{|n|\rho}.\end{aligned}$$

Let $\{\cdot, \cdot\}$ denote the Poisson bracket of smooth functions, i.e.,

$$\{F, G\} = \left\langle \frac{\partial F}{\partial I}, \frac{\partial G}{\partial \theta} \right\rangle - \left\langle \frac{\partial F}{\partial \theta}, \frac{\partial G}{\partial I} \right\rangle + \mathrm{i} \sum_n \left(\frac{\partial F}{\partial z_n} \frac{\partial G}{\partial \bar{z}_n} - \frac{\partial F}{\partial \bar{z}_n} \frac{\partial G}{\partial z_n} \right),$$

then we have the following lemma:

Lemma 7.3 *If*

$$\|X_F\|_{D(r,s)} < \varepsilon', \quad \|X_G\|_{D(r,s)} < \varepsilon'',$$

then

$$\|X_{\{F,G\}}\|_{D(r-\sigma,\eta s)} < c\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon'', \quad \eta \ll 1.$$

In particular, if $\eta \sim \varepsilon^{\frac{1}{3}}$, $\varepsilon', \varepsilon'' \sim \varepsilon$, *we have* $\|X_{\{F,G\}}\|_{D(r-\sigma,\eta s)} \sim \varepsilon^{\frac{4}{3}}$.

For the proof, see [13]. □

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