# An Improved Result for Positive Measure Reducibility of Quasi-periodic Linear Systems *

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**Abstract**.

*In this paper, by* KAM *method, under weaker small denominator conditions and non-degeneracy conditions, we prove a positive measure reducibility for quasi-periodic linear systems close to constant:* $\dot{X} = (A(\lambda)+F(\varphi,\lambda))X, \quad \dot{\varphi} = \omega$ *where the parameter* $\lambda \in (a,b)$, $\omega$ *is a fixed Diophantine vector, which is a generalization of* Jorba & Simó's *positive measure reducibility result.*

**Key words**: quasi-periodic, reducibility, KAM, non-degeneracy
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## 1 Introduction

Consider the quasi-periodic linear differential equations

$$\dot{X} = A(\omega_1 t, \cdots, \omega_n t)X \tag{1.1}$$

where $t \in \mathbb{R}$, $X \in \mathbb{C}^m$, $A(\omega_1 t, \cdots, \omega_n t)$ is quasi-periodic time-dependent $m \times m$ matrix; $\omega_1, \cdots, \omega_n$, called frequencies, are rational independent.

A quasi-periodic linear system (1.1) is called reducible, if there exists a so called quasi-periodic Lyapunov-Perron(L-P) transformation $X = P(\omega_1 t, \cdots, \omega_n t, \lambda)Y$, such that the transformed system is a linear system with constant coefficients. We say that a linear time-varying change of variables $X = P(\omega_1 t, \cdots, \omega_n t)Y$ is a L-P transformation if $P$ is quasi-periodic in $t$ and non-singular for all $t \in \mathbb{R}$; $P, P^{-1}$, and $\dot{P} = dP/dt$ are bounded in $t \in \mathbb{R}$.

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The reducibility problem of quasi-periodic linear systems has received much attention. For $n = 1$, i.e., the periodic case, the classical Floquet theory shows that there always exists a periodic change of variables $Y = P(\omega_1 t)X$ so that the transformed system has the form $\dot{Y} = \bar{A}Y, \dot{\varphi} = \omega$,where $\bar{A}$ is a constant matrix. For $n > 1$, i.e., quasi-periodic case, the system (1.1) is not always reducible(see [1] for example). In this case, sufficient conditions which guarantee the almostreducibility has been given in earlier works by Coppel [2], Johnson and Sell [3].

The first reducibility result by Kolmogorov-Arnold-Moser (KAM) method was given by Dinaburg and Sinai [4] who proved that the linear equation $\ddot{x} + q(\omega_1 t, \cdots, \omega_n t)x = \lambda x$ is reducible for"most" large enough $\lambda$ in measure sense, where $\omega$ is fixed satisfying the Diophantine condition:

$$|\langle k, \omega\rangle| > \frac{\gamma^{-1}}{|k|^{\sigma}}, \quad 0 \neq k \in \mathbb{Z}^n, \tag{1.2}$$

$\gamma,\ \sigma$ are positive constants. See also Rüssmann [5] for a refined result. Later, Jorba and Simó [6] generalized the result to linear systems of the form

$$\dot{X} = (A + \lambda\bar{Q} + \lambda^2 Q(\omega_1 t, \cdots, \omega_n t))X,\ \ X \in \mathbb{R}^m. \tag{1.3}$$

where $A, \bar{Q}$ are constant diagonal matrices and $Q$ a quasi-periodic analytic matrix with $n$ basic frequencies, $\lambda$ is a small parameter. By KAM method, Jorba and Simó proved that there exists a positive measure Cantor set $\mathcal{E} \subset (0, \lambda_0), \lambda_0 \ll 1$ such that for any $\lambda \in \mathcal{E}$ the system is reducible, provided that the following non-degeneracy conditions hold,

$$|\alpha_i(\lambda) - \alpha_j(\lambda)| > \delta > 0, \quad |\frac{d}{d\lambda}(\alpha_i(\lambda) - \alpha_j(\lambda))| > \chi > 0, \quad \forall 1 \leq i < j \leq m \tag{1.4}$$

where $\alpha_i(\lambda),\ 1 \leq i \leq m$, is the eigenvalue of $\bar{A} = A + \lambda\bar{Q}$. Under , Xu [7] improved the result under weaker non-degeneracy conditions.

Most remarkably, Eliasson [8] gave a full measure reducibility result for the linear Schrödinger equation

$$\frac{d^2x}{dt^2} + (\lambda + Q(\omega t))x = 0.$$

More precisely, he proved that the above equation is reducible for almost all $\lambda \in (a, b)$ in Lebesgue measure sense provided that $Q$ is small. The work was generalized by Krikorian to linear systems with coefficients in $so(3)$ [9]. In a recent paper of Eliasson [10], he proposed the full measure reducibility problem for general parameter-dependent systems

$$\dot{X} = (A(\lambda) + F(\omega_1 t, \cdots, \omega_n t, \lambda))X \tag{1.5}$$

where $t \in \mathbb{R},\ X \in \mathbb{C}^m$, the parameter $\lambda \in \Lambda = (a, b)$, $A$ is a $m \times m$ constant matrix, and $F(\varphi_1, \cdots, \varphi_n, \lambda)$ are analytic mappings from $T^n \times (a, b)$ to $gl(m, \mathbb{C})$, $(\omega_1, \cdots, \omega_n)$ is a Diophantine vector and $|F|$ is sufficiently small.

Our final aim is to prove a full measure reducibility result proposed by Eliasson. Since the existed positive measure results is not enough for this purpose, in this paper we shall give the positive measure reducibility result of (1.5), which generalized the results of Jorba and Simó, and serves as a preparation for dealing with full measure reducibility of (1.5). The non-degeneracy conditions in this paper is closer to Xu's condition.

Equivalently, we consider the following skew-product systems

$$\dot{X} = (A(\lambda) + F(\varphi, \lambda))X, \quad \dot{\varphi} = \omega. \tag{1.6}$$

In order to get positive measure, we have to assume some kind of non-degeneracy conditions (or transverse conditions in Eliasson and Krikorian's terminology ). Without loss of generality, we assume that $A(\lambda) = \mathrm{diag}(A_1(\lambda), \cdots, A_s(\lambda))$ is block-diagonal with

$$\mathrm{dist}(\sigma(A_i), \sigma(A_j)) > \rho > 0, \quad \text{for} \quad i \neq j,$$

where $\sigma(A_i)$ is the set of eigenvalues of $A_i$. Denote by $\alpha_i^\nu, \nu = 1, 2, \cdots, l_i$ all eigenvalues of $A_i$ where $l_i$ is the dimension of $A_i$. Let (See the Appendix for definition)

$$J(k, \lambda) = \mathrm{i}\langle k, \omega\rangle I_{m^2} + (I_m \otimes A(\lambda) - A^T(\lambda) \otimes I_m)$$

$$J_{ij}(k, \lambda) = \mathrm{i}\langle k, \omega\rangle I_{l_i l_j} + (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda) \otimes I_{l_j})$$

$$f_{ij}(k, \lambda) = \det[\mathrm{i}\langle k, \omega\rangle I_{l_i l_j} + (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda) \otimes I_{l_j})],$$

where $A^T,\ A_i^T$ denote transpose matrices of $A,\ A_i$ respectively, $1 \leq i, j \leq s$. By Lemmas in the Appendix,

$$f_{ij}(k, \lambda) = \prod_{\alpha_i \in \sigma(A_i), \alpha_j \in \sigma(A_j)} [\mathrm{i}\langle k, \omega\rangle - (\alpha_i(\lambda) - \alpha_j(\lambda))], \quad i \neq j,$$

and

$$f_{ii}(k, \lambda) = \langle k, \mathrm{i}\omega\rangle^{l_i} \prod_{1 \leq \nu_1 \neq \nu_2 \leq l_i} [\mathrm{i}\langle k, \omega\rangle - (\alpha_i^{\nu_1}(\lambda) - \alpha_i^{\nu_2}(\lambda))].$$

Let

$$g_{ij}(k, \lambda) = \begin{cases} f_{ij} & i \neq j \\ \langle k, \mathrm{i}\omega\rangle^{-l_i} f_{ii}, & i = j, \end{cases}$$

We assume the following non-degeneracy conditions:

**Non-degeneracy Conditions**. For each $i, j$, there exist an integer $d_{ij} > 0$ and $\chi > 0$ such that

$$\left|\frac{\partial^{d_{ij}}}{\partial \lambda^{d_{ij}}} g_{ij}(k, \lambda)\right| > \chi, \tag{1.7}$$

uniformly hold for all $k \in \mathbb{Z}^n$ with $|\langle k, \omega\rangle| \leq \|A_i\| + \|A_j\|$.

*Remark.* The non-degeneracy conditions will guarantee that the small denominator condition holds for "most" parameter $\lambda$.

*Remark.* We only consider those $k$ such that $|\langle k,\omega\rangle| \leq \|A_i\| + \|A_j\|$, because when $|\langle k,\omega\rangle|$ is large enough, the matrix $\mathrm{i}\langle k,\omega\rangle I_{l_il_j} + (I_{l_i}\otimes A_j(\lambda) - A_i^T(\lambda)\otimes I_{l_j})$ is automatically nonsingular and the small denominator condition is obviously satisfied.

*Remark.* The condition (1.7) is not required for $f_{ii}$ with $l_i = 1$. And for system (1.3) considered by Jorba and Simó, $f_{ij} = \langle k,\omega\rangle + \alpha_i(\lambda) - \alpha_j(\lambda)$, so (1.4) implies (1.7).

*Remark.* The non-degeneracy conditions given by Jorba and Simó do not apply to multiple eigenvalue case. However, our non-degeneracy conditions, weaker than that in [JoS], apply to the case when multiple eigenvalues occur at some parameter points. For example, when $s = 1$, *i.e.* only one block, we suppose that each difference $\alpha_{ij}$ of any two eigenvalues $\alpha_i,\ \alpha_j$ of $A(\lambda)$ is a polynomial of parameter $\lambda$ with degree $q_{ij} > 0$, let $d = \sum_{1\leq i\neq j\leq m} q_{ij}$, then for every $k$, the $g(k,\lambda)$ must be a polynomial of $\lambda$ with degree $d$, it is easy to see that the coefficient of the first term is independent of $k$, and our non-degeneracy conditions are satisfied. Moreover, the property that $g(k,\lambda)$ depends analytically(or smoothly) on $\lambda$ can be preserved under small perturbations, so we only need to discuss the preserving of non-degeneracy conditions in the inductive process.

Suppose that $F$ is analytic $g$-valued function defined on

$$W_h(\mathbb{T}^n\times\Lambda) = \{(\theta,\lambda)\in\mathbb{C}^n\times\Lambda|\ \mathrm{dist}(\theta,\mathbb{T}^n) < h\},$$

where $\Lambda = (a,b)$. Let $d = \max\{d_{ij}\}$, for simplicity the norms in our main theorem is defined as

$$\|F\|_h = \max_{0\leq i\leq d}\sup_{(\theta,\lambda)\in W_h(\mathbb{T}^d\times\Lambda)}|\frac{\partial^i F}{\partial\lambda^i}|$$

similarly,

$$\|A\| = \max_{0\leq i\leq d}\sup_{\lambda\in\Lambda}|\frac{\partial^i A(\lambda)}{\partial\lambda^i}|$$

where $|\cdot|$ denotes the matrix norm. Now we are in the position to state our result.

**Theorem 1** *Suppose that $\omega$ is fixed and satisfies the Diophantine condition (1.2), $A(\lambda)$ satisfies the non-degeneracy conditions (1.7), and $\exists\ M > 0$ such that $\|A\| \leq M$. Then there exist $\varepsilon_0 > 0,\ h > 0$, s.t. if $\|F(\cdot,\cdot)\|_h = \varepsilon_1 \leq \varepsilon_0$, the measure of the set of parameter $\lambda$'s for which the system (1.6) is non-reducible is no larger than $CL(10\varepsilon_1)^{\frac{\nu}{5d}}$, where $C,\ \nu,\ b$ are some positive constants, $L$ is the length of the parameter interval $\Lambda$.*

*Remark.* From the remarks of the non-degeneracy conditions we can see that, our theorem implies the result of Jorba & Simó, and is suitable for much more general cases, including the appearance of multiple eigenvalues of $A$ at some parameter points.

## 2 Proof of the Theorem

Let's give an equivalent formulation of the reducibility, which will be useful in the following discussion. Given an analytic quasi-periodic linear system

$$\dot{X} = a(t)X \tag{2.1}$$

or the skew-product system:

$$\dot{X} = A(\varphi)X, \ \dot{\varphi} = \omega \tag{2.2}$$

where $a$, $A$ are in the Lie algebra $g = gl(m, \mathbb{C})$, and the solutions of the equations take values in the Lie group $G = GL(m, \mathbb{C})$. If $A$ is analytic on a complex neighborhood $W_h(\mathbb{T}^n)$ we denote $A \in C_h^\omega(\mathbb{T}^n, g)$. We say that two analytic $g$ -valued functions $A_1$, $A_2 \in C_h^\omega(\mathbb{T}^n, g)$ are conjugated, if there exists a $G$-valued function $P \in C_h^\omega(\mathbb{T}^n, G)$, which is also L-P transformation, such that for two solutions $X_1$, $X_2$ corresponding to $A_1$, $A_2$, we have the relation

$$X_2 = P(\varphi)X_1$$

and denote the conjugate relation by

$$A_1 \equiv A_2 \ (mod \ P)$$

It is easy to see that $A_1 \equiv A_2$ $(mod\ P)$ is equivalent to the following equality:

$$A_2 = L_\omega P \cdot P^{-1} + PA_1P^{-1} \tag{2.3}$$

where $L_\omega = \frac{\partial}{\partial\varphi} \cdot \dot{\varphi}$ is the derivative along frequency vector $\omega$. $A_1$ is said to be reducible if it conjugates to a constant $A_2$. In this paper, we will prove that, for any parameter in a set of positive measure, there exists a L-P transformation $P(\varphi)$ transforming the system $A + F(\varphi)$ into a constant one $A^*$,

We will prove the reducibility by KAM iteration. At each KAM step, we find a L-P transformation close to identity as follows,

$$P(\varphi) = I + Y(\varphi)$$

where $Y(\varphi) \in C_h^\omega(\mathbb{T}^n, g)$, which transforms the quasi-periodic system $\dot{X} = (A+F)X$ into

$$\dot{X} = (L_\omega P \cdot P^{-1} + P(A+F)P^{-1})X.$$

Since $Y$ will be small, $P^{-1}$ can be expanded as

$$P^{-1} = I - Y + C_1Y^2 + O(\|Y\|^3).$$

It follows that

$$\begin{aligned}
&L_\omega P \cdot P^{-1} + P(A+F)P^{-1} \\
&= L_\omega Y \cdot (I - Y + C_1Y^2 + O(\|Y\|^3)) + (I+Y)(A+F)(I - Y + C_1Y^2 + O(\|Y\|^3)) \\
&= A + L_\omega Y + [Y, A] + F - L_\omega Y \cdot Y + [Y, F] + C_1AY^2 - YAY + O(\|Y\|^3)
\end{aligned} \tag{2.4}$$

Roughly speaking, we need to find a small $Y$ so that the transformed system is still of the form $\dot{X} = (A^+ + F^+)X$, where $A^+$ is block diagonal as $A$ and $F^+$ is much smaller than $F$.

For this purpose, we need to find a $Y$ solving the following linearized equation [1]

$$L_\omega Y - [A, Y] = -F \tag{2.5}$$

where $[A, Y] = AY - YA$, and prove

$$F^+ = -L_\omega Y \cdot Y + [Y, F] + C_1 AY^2 - YAY + O(\|Y\|^3),$$

is much smaller.

## 2.1 Solving the Linearized Equation

In the following, we denote by $|N|$ the determinant of a $m \times m$ matrix $N = (n_{ij})$, and $\|N\|$ its operator norm,which is equivalent to $m \times \max |n_{ij}|$. Denote by $\|(v_1, \cdots, v_m)^T\| = \max_{1 \le j \le m} \|v_j\|$ the norm for vectors. For $k \in \mathbb{Z}^n$, denote its module by $|k| = |k_1| + \cdots + |k_n|$. If $f$ is a function, $|f|$ denotes its absolute value. Through the context we use letters $c, C$ to designate positive constants, in order to simplify the computations we allow them to take different values from one line to another even though their actual values do not matter.

Expanding $Y$, $F$ into Fourier series, substituting them into the equation (2.5) and comparing the corresponding Fourier coefficients in both sides of the equation , we have

$$\mathrm{i}\langle k, \omega\rangle Y_k - (AY_k - Y_k A) = -F_k \tag{2.6}$$

the eigenvalues of the linear operator $\mathrm{i}\langle k, \omega\rangle id + [A, \cdot]$ in the left part are

$$\mathrm{i}\langle k, \omega\rangle - (\lambda_i - \lambda_j), \quad 1 \le i, j \le m, \lambda_i, \lambda_j \in \sigma(A).$$

Especially when $k = 0$, the eigenvalues are $\lambda_i - \lambda_j$. Because the matrix $A$ under consideration is diagonal block *i.e.*

$$A = diag(A_1, A_2, \cdots, A_s),$$

and different blocks $A_i$ and $A_j$ have different eigenvalues, *i.e.* $\lambda \ne \mu$ if $\lambda \in A_i$ , $\mu \in A_j$ $(i \ne j)$, from conclusions in Appendix we know that the matrix $I_{l_i} \otimes A_j - A_i^T \otimes I_{l_j}$ is nonsingular if $i \ne j$.

Rewrite $F_k$ in block form $(F_{kij})$, where $F_{kij}$ is a $\dim A_i \times \dim A_j$ matrix. Let

$$F_0^d = diag(F_{011}, F_{022}, \cdots, F_{0ss}),$$

and

$$F_0^* = F_0 - F_0^d.$$

[1] as we will see in the following subsection, the equation can not be solved completely. Some "nice" $x$-independent terms will be absorbed into $A$.

When $k = 0$ the equation (2.6), i.e.,

$$AY_0 - Y_0 A = F_0 \tag{2.7}$$

can not be completely solved due to the multiplicity of eigenvalues. However, the equation

$$AY_0 - Y_0 A = F_0^*$$

has a solution $Y_0 = (Y_{0ij})$, where $Y_{0ii} = 0$ and $Y_{0ij} \quad (i \neq j)$ is the unique solution of

$$A_i Y_{0ij} - Y_{0ij} A_j = F_{0ij}.$$

Moreover, we have the estimate

$$\begin{aligned} \|J_{ij}^{-1}(0,\lambda)\| &\leq \max_{i\neq j} \|[I_{l_j} \otimes A_i(\lambda) - A_j^T(\lambda) \otimes I_{l_i}]^{-1}\| \\ &\leq c\frac{mM^{l_i l_j}}{\rho^{l_i l_j}} \leq C(\rho, m) M^{l_i l_j}, \end{aligned} \tag{2.8}$$

and

$$\max_{0\leq q\leq d} \|\frac{\partial^q J_{ij}^{-1}(0,\lambda)}{\partial \lambda^q}\| = \max_{1\leq q\leq d} \|\frac{\partial^q}{\partial \lambda^q}(\frac{\mathrm{ad}J_{ij}}{\det J_{ij}})\| \leq C(\rho, m, d) M^{(l_i l_j)^2}$$

since $\mathrm{dist}(\sigma(A_i), \sigma(A_j)) > \rho > 0$, for $i \neq j$, where $c$ is a constant. Moreover, we have

$$\begin{aligned} \max_{0\leq q\leq d} \|\frac{\partial^q Y_0(\lambda)}{\partial \lambda^q}\| &\leq C \max_{0\leq q\leq d} \|\frac{\partial^q J_{ij}^{-1}(0,\lambda)}{\partial \lambda^q}\| \cdot \|\frac{\partial^q F_0(\lambda)}{\partial \lambda^q}\| \\ &\leq C(\rho, m, d) M^{m^4} \max_{0\leq q\leq d} \|\frac{\partial^q F_0(\lambda)}{\partial \lambda^q}\|. \end{aligned} \tag{2.9}$$

Now we solve the equation (2.6) for $k \neq 0$. In view of Lemma 3.2 in the Appendix, solving the equation (2.6) is equivalent to solving the following vector equation

$$J(k,\lambda) Y_k'(\lambda) = -F_k'(\lambda) \tag{2.10}$$

which is solvable if and only if $J(k,\lambda)$ is invertible by Corollary 1,2,3 in Appendix. Let $P = I + \sum Y_k$ where $Y_k$ is defined as above. Then the transformed system has the form

$$\dot{X} = (A^+ + F^+) X$$

where

$$\begin{aligned} A^+ &= A + F_0^d \\ F^+ &= -L_\omega Y \cdot Y + [Y, F] + C_1 AY^2 - YAY + O(Y^3) \end{aligned} \tag{2.11}$$

It is obvious that $A^+$ is still block diagonal. In the following, we prove that $F^+$ is much smaller in a smaller domain and $A^+$ still satisfies the non-degeneracy conditions.

*Estimate of $F^+$.*

Firstly, we estimate $Y_k$. In fact, we will prove that the solution $Y_k$ is well controlled if $\exists\ K > 0, \tau > 0$ such that for all $i, j$ , $1 \leq i, j \leq s$,

$$|g_{ij}(k,\lambda)| \geq \frac{K^{-1}}{|k|^{\tau}}, \tag{2.12}$$

In order to give an estimate we now estimate the inverse of the operator $J_{ij}(k,\lambda)$ for $k \neq 0$.

**Lemma 2.1** *For all parameters $\lambda$ satisfying small denominator conditions (2.12) and $k \neq 0$, we have*

$$\|J_{ii}^{-1}(k,\lambda)\| \leq cM^{l_i^2}\gamma^{l_i}K|k|^{l_i^2+l_i\sigma+\tau}, \tag{2.13}$$

$$\|J_{ij}^{-1}(k,\lambda)\| \leq cM^{l_il_j}K|k|^{l_il_j+\tau}, (i \neq j) \tag{2.14}$$

$$\|J^{-1}(k,\lambda)\| \leq cM^{m^2}\gamma^m K^{m^2}|k|^{m^2(\tau+1)+m\sigma}, \tag{2.15}$$

$$\max_{0\leq q\leq d}\|\frac{\partial^q J^{-1}(k,\lambda)}{\partial\lambda^q}\| \leq cM^{m^4}\gamma^{2^d m}K^{2^d m^2}|k|^{2^d(m^2(\tau+1)+m\sigma)} \tag{2.16}$$

*where $c$ is constant.*

Proof. Since for the nonsingular matrix $J_{ij}$, $J_{ij}^{-1} = adJ_{ij}/\det J_{ij}$, where $adJ_{ij}$ is the adjoint matrix of $J_{ij}$, by the definition of $\|J_{ij}\|$ and the small denominator conditions (2.12), it is easy to get the estimations (2.13) and (2.14). Since $\det J = \prod_{1\leq i,j\leq s}\det J_{ij}$, we can similarly obtain the inequalities (2.15) and (2.16). □

Since for $k \neq 0$ solving the equation (2.6) is equivalent to solve the following vector equation

$$J(k,\lambda)Y_k'(\lambda) = -F_k'(\lambda)$$

we have

$$Y_k'(\lambda) = -J^{-1}(k,\lambda)F_k'(\lambda) \tag{2.17}$$

and it is easy to see that $\|Y_k\| = \|Y_k'\|$, $\|F_k\| = \|F_k'\|$ (see Appendix).

In the following, for a $\lambda$-dependent matrix $L(\lambda)$, we denote

$$|L| = \max_{0\leq q\leq d}\|\frac{\partial^q L}{\partial\lambda^q}\|.$$

Note that

$$F_k(\lambda) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} F(x,\lambda)e^{-\mathrm{i}(k,x)}dx. \tag{2.18}$$

since $F \in C_h^\omega(\mathbb{T}^d \times \Lambda, g)$, we have

$$|F_k| \leq |F|_h e^{-|k|h}.$$

consequently, for any $0 < \bar{h} < h$ and $k \neq 0$,

$$\begin{aligned}
|Y_k(\lambda)| &\leq |J^{-1}(k,\lambda)|\cdot|F_k(\lambda)| \\
&\leq CM^{m^4}\gamma^{2^d m}K^{2^d m^2}|k|^{2^d(m^2(\tau+1)+m\sigma)}|F|_h e^{-|k|h} \\
&= CM^{m^4}\gamma^{2^d m}K^{2^d m^2}|F|_h|k|^{2^d(m^2(\tau+1)+m\sigma)}e^{-|k|(h-\bar{h})}e^{-|k|\bar{h}}. \qquad (2.19)
\end{aligned}$$

Since the function $x^{2^d(m^2(\tau+1)+m\sigma)}e^{-x(h-\bar{h})}$ takes its maximal at $x=\frac{2^d(m^2(\tau+1)+m\sigma)}{h-\bar{h}}$, one has

$$\begin{aligned}
|Y_k(\lambda)| &\leq CM^{m^4}\gamma^{2^d m}K^{2^d m^2}|F|_h\Big(\frac{2^d(m^2(\tau+1)+m\sigma)}{(h-\bar{h})e}\Big)^{2^d(m^2(\tau+1)+m\sigma)}e^{-|k|\bar{h}} \\
&= C(m,\sigma,\tau,d)M^{m^4}K^{2^d m^2}|F|_h\frac{e^{-|k|\bar{h}}}{(h-\bar{h})^{2^d(m^2(\tau+1)+m\sigma)}}. \qquad (2.20)
\end{aligned}$$

Let

$$Y(x,\lambda)=\sum_{k\in\mathbb{Z}^n}Y_k(\lambda)e^{\mathrm{i}(k,x)},$$

take $h^+ : 0<h^+<\bar{h}$ such that if $\bar{h}-h^+=h-\bar{h}<1$. Then by the lemma 4 in [6] we have

$$\begin{aligned}
|Y|_{h^+} &\leq \sum_{k\in\mathbb{Z}^d}|Y_k|e^{|k|h^+} \\
&\leq CM^{m^4}|F_0|+CM^{m^4}K^{2^d m^2}\frac{|F|_h}{(h-\bar{h})^{2^d(m^2(\tau+1)+m\sigma)}}\sum_{k\in\mathbb{Z}^d/\{0\}}e^{-(\bar{h}-h^+)|k|} \\
&\leq CM^{m^4}K^{2^d m^2}\frac{|F|_h}{(h-\bar{h})^{2^d(m^2(\tau+1)+m\sigma)}}\Big(\frac{2}{\bar{h}-h^+}\Big)^n\exp\Big(\frac{(\bar{h}-h^+)n}{2}\Big) \\
&\leq C(m,n,\gamma,\sigma,\tau,\rho,d)M^{m^4}K^{2^d m^2}\frac{|F|_h}{(h-h^+)^{2^d(m^2(\tau+1)+m\sigma)+n}}. \qquad (2.21)
\end{aligned}$$

Let $s=2^d(m^2(\tau+1)+m\sigma)+n$, we have

$$|Y|_{h^+}\leq CM^{m^4}K^{2^d m^2}\frac{|F|_h}{(h-h^+)^s} \qquad (2.22)$$

Consequently

$$|L_\omega Y|_{h^+}\leq CM^{m^4}K^{2^d m^2}\frac{|F|_h}{(h-h^+)^{s+1}}$$

$$|L_\omega Y\cdot Y|_{h^+}\leq CM^{2m^4}K^{2^{d+1}m^2}\frac{|F|_h^2}{(h-h^+)^{2s+1}}$$

$$|AY^2|_{h^+} \doteq |YAY|_{h^+} \le CM^{2m^4+1}K^{2^{d+1}m^2}\frac{|F|_h^2}{(h-h^+)^{2s}}$$

$$|[Y,F]|_{h^+} \le 2|Y|_{h^+}\cdot|F|_h \le CM^{m^4}K^{2^dm^2}\frac{|F|_h^2}{(h-h^+)^s}$$

in the end, we have:

$$|F^+|_{h^+} \le CM^{2m^4+1}K^{2^{d+1}m^2}\frac{|F|_h^2}{(h-h^+)^{2s+1}} \tag{2.23}$$

*Verifying the non-degeneracy conditions for $A^+$.*

Note that

$$A^+ = A + F_0^d = \mathrm{diag}(A_1 + F_{011}, \cdots, A_s + F_{0ss})$$

Let

$$f_{ij}^+(k,\lambda) = \det[\mathrm{i}\langle k,\omega\rangle I_{l_il_j} + (I_{l_i}\otimes(A_j(\lambda)+F_{0jj}(\lambda)) - (A_i^T(\lambda)+F_{0ii}^T(\lambda))\otimes I_{l_j})].$$

It is obvious that the new determinant $f_{ij}^+$ is still analytic with respect to $\lambda$.

we can rewrite this determinant as

$$f_{ij}^+(k,\lambda) = f_{ij}(k,\lambda) + Z_{ij}(k,\lambda)$$

where the $Z_{ij}(k,\lambda)$ is a summary of $2^{l_il_j}-1$ determinants $z_t(k,\lambda)$ $(1\le t\le 2^{l_il_j}-1)$. Moreover there exists at least one column in each determinant $z_t$ such that every entry in this column is either zero or of the form $a-b$, where $a$ and $b$ are entries of $F_{0jj}$ and $F_{0ii}$ respectively.

Since $|F_0^d|_h \le |F|_h < \varepsilon$, we have

$$|\frac{\partial^{d_{ij}}}{\partial\lambda^{d_{ij}}}Z_{ij}(k,\lambda)| \le C|A|\varepsilon.$$

Similarly,

$$|\frac{\partial^{d_{ij}}}{\partial\lambda^{d_{ij}}}(g_{ij}^+ - g_{ij})| \le C|A|\varepsilon. \tag{2.24}$$

So we get

$$|\frac{\partial^{d_{ij}}}{\partial\lambda^{d_{ij}}}g_{ij}^+(k,\lambda)| \ge \chi - C|A|\varepsilon \ge \chi - CM\varepsilon. \tag{2.25}$$

The proof is straightforward. Notice here that we only need to consider the case that $|(k,\lambda)|$ are not too large, say $|(k,\lambda)|\le C|A|$, since for large values of $|(k,\lambda)|$, $J^+(k,\lambda)$ are naturally nonsingular and we don't need to preserve the non-degenerate property.

On the other hand, from the perturbation theory of matrix we know that the changing of eigenvalues depends continuously on the entries, and by Ostrowski theorem (see [11]) we can estimate the distance between eigenvalues of any two blocks, *i.e.*

$$\min_{i\ne j}\mathrm{dist}(\sigma(A_i^+),\sigma(A_j^+)) = \rho^+ > \rho - c\varepsilon^{\frac{1}{m}}$$

In summary, we have the following lemma

**Lemma 2.2** *Let $\Lambda \subset (a,b)$ be some parameter segment, $A \in C^{\omega}(\Lambda, g)$ a one parameter family of constant elements, $F \in C_h^{\omega}(\mathbb{T}^n \times \Lambda, g)$ be the perturbation. Suppose that $\exists\ M,\ \varepsilon,\ K > 0$ such that*

*a).* $|A| \leq M, \quad |F|_h < \varepsilon,$
*b).* $\forall \lambda \in \Lambda$, *the non-degeneracy conditions (1.7) and small denominator conditions (2.12) hold.*

*Then there exist $h^+ > 0$ and a map $Y \in C_{h^+}^{\omega}(\mathbb{T}^n \times \Lambda, g)$, and*

$$A^+ \in C^{\omega}(\Lambda, g)$$

$$F^+ \in C_{h^+}^{\omega}(\mathbb{T}^n \times \Lambda, g)$$

*such that,*

*1)* $A^+ + F^+ \equiv A + F, \quad A^+ = A + F_0^d,$
*2) We have the estimation (2.23),*
*3) The non-degeneracy conditions is preserved for $\forall\ i,j$, i.e.*

$$|\frac{\partial^{d_{ij}}}{\partial \lambda^{d_{ij}}} g_{ij}^+(k,\lambda)| \geq \chi - CM\varepsilon,$$

*4)* $\rho^+ > \rho - c\varepsilon^{\frac{1}{m}}, \quad M^+ < M + \varepsilon.$

## 2.2 Iteration

In this subsection, We will show that the perturbation $F$ converges to zero very fast provided that the small divisor conditions hold. In next subsection, we will prove that the set of parameters satisfying the small divisor conditions is of large Lebesgue measure.

Firstly, we give two iterative sequences:

$$h_n = (\frac{1}{2} + \frac{1}{2^n})h_1, \tag{2.26}$$

$$K_n = (\frac{(\frac{6}{5})^n + \frac{1}{\eta}}{h_{n-1} - h_n})^{\nu} = (h_1)^{-\nu} 2^{n\nu} ((\frac{6}{5})^n + \frac{1}{\eta})^{\nu} \tag{2.27}$$

where the constant $\nu \geq d$, and $\eta$ will be taken in the following lemma

**Lemma 2.3** *There exist positive constants $\eta < 1$, $b$ such that, if $\varepsilon_1$ is sufficiently small, then for all $n \geq 1$,*

$$\varepsilon_n \leq \eta^b e^{-(\frac{6}{5})^n}.$$

$$M_n \leq 2^{n-1} M_1$$

Proof. Supposing that we have applied the precedent method until to $n^{th}$ step, and we have got

$$|F_n|_{h_n} \leq \varepsilon_n \leq \eta^b e^{-(\frac{6}{5})^n}$$

and

$$M_n \leq M_{n-1} + \varepsilon_{n-1} \leq 2^{n-1} M_1$$

by induction, we want to show that

$$|F_{n+1}|_{h_{n+1}} \leq \eta^b e^{-(\frac{6}{5})^{n+1}} \tag{2.28}$$

$$M_{n+1} \leq 2^n M_1 \tag{2.29}$$

In fact,(2.29) is satisfied since

$$M_{n+1} \leq M_n + \eta^b e^{-(\frac{6}{5})^n} \leq M_n + 1 \leq 2M_n.$$

To prove (2.28) we need

$$CM_n^{2m^4+1} K_n^{2^{d+1}m^2} \frac{\eta^{2b} e^{-(\frac{6}{5})^{2n}}}{(h-h^+)^{2s+1}} \leq \eta^b e^{-(\frac{6}{5})^{n+1}}$$

then using (2.26) and (2.29), we should have

$$CM_1^{2m^4+1} h_1^{-(2s+1)} 2^{n(2m^4+1)+(n+1)(2s+1)} K_n^{2^{d+1}m^2} \eta^b e^{-(4/5)(\frac{6}{5})^n} \leq 1 \tag{2.30}$$

let $Q_n(\eta) = K_n^{2^{d+1}m^2} \eta^{b-1}$, if we choose

$$b > 2^{d+1} m^2 \nu + 1, \tag{2.31}$$

then by (2.27) we know that when $\eta$ turns smaller, $Q_n$ turns smaller, too. Now, we firstly take an $\eta = \eta_0 < 1$. Since the sequence

$$2^{n(2m^4+1)+(n+1)(2s+1)+nd} Q_n(\eta_0) e^{-(4/5)(\frac{6}{5})^n}$$

is bounded from above, we denote its maximum by $\beta$. In order that the (2.30) be satisfied, it is enough to choose $\eta$ such that

$$Ch_1^{-(2s+1)} M_1^{2m^4+1} \beta\eta \leq 1$$

so, we choose

$$\eta \leq \min\{Ch_1^{2s+1} M_1^{-(2m^4+1)} \beta^{-1}, \eta_0\},$$

and (2.30) is obtained. If we take $\eta = (10\varepsilon_1)^{1/b}$, then it is enough to choose

$$\varepsilon_1 \leq \min\{\frac{Ch_1^{b(2s+1)} M_1^{-b(2m^4+1)}}{10\beta^b}, \eta^b e^{-\frac{6}{5}}\}, \tag{2.32}$$

the lemma is proved.□

From (2.22) we can see that the sequence $|Y_n|_{h_n}$ also converges to zero with "super-exponential" velocity, then $P_n \to I$, and so, the composition of transformations $P_n \circ \cdots \circ P_1$ will converge, too. On the other hand, by lemma 2.2 we know that $\chi_n \geq \chi_{n-1} - CM_n\varepsilon_n$, so

$$\chi_n \geq \chi - C \sum_{1 \leq i \leq n-1} M_i \varepsilon_i \geq \frac{\chi}{2} \tag{2.33}$$

if $\varepsilon_1$ is small enough. Thus, the non-degeneracy conditions are preserved. By the way, we also have the estimate for $\varepsilon_1$ small enough:

$$\rho_n > \rho - c \sum_{1 \leq i \leq n-1} \varepsilon_i^{\frac{1}{m}} \geq \frac{\rho}{2}. \tag{2.34}$$

## 2.3 Measure of the Removed Sets

Finally, we estimate the measure of those removed sets. At the $n^{th}$ step, for $\forall\ i, j,\ 1 \leq i, j \leq s$, we denote the removed set (corresponding to resonance of $\langle k, \omega \rangle$ and $A_i, A_j$) by

$$R^n_{kij} = \{\lambda :\ |g^n_{ij}(k, \lambda)| \leq \frac{K_n^{-1}}{|k|^\tau}\}.$$

and let

$$R^n_k = \bigcup_{1 \leq i,j \leq s} R^n_{kij}$$

$$R^n = \bigcup_{0 \neq k \in \mathbb{Z}^d} R^n_k$$

in order to estimate the measure of $R^n_{kij}$, we introduce the following lemma, which has been proved in[12]

**Lemma 2.4** *Suppose that $g(u)$ is a $C^N$ function on the closure $\bar{I}$, where $I \subset R^1$ is an interval of length $L$. Let $I_h = \{u : |g(u)| \leq h\},\ h > 0$. If for some constant $d > 0$, $|g^{(N)}(u)| \geq d$ for $\forall u \in I$, then $|I_h| \leq cLh^{1/N}$, where $|I_h|$ denotes the Lebesgue measure of $I_h$ and the constant $c = 2(2 + 3 + \cdots + N + d^{-1})$.*

Then, let $L$ be the length of the parameter interval $\Lambda$, we have

$$|R^n_{kij}| \leq cL(\frac{K_n^{-1}}{|k|^\tau})^{1/d_{ij}} \leq cL(\frac{K_n^{-1}}{|k|^\tau})^{1/d}$$

where $c = 2(2 + \cdots + d + \frac{2}{\chi})$, since $g^n_{ij}(k, \lambda) \in C^\omega(\Lambda)$ and using the non-degeneracy conditions and (2.33). Thus,

$$\begin{aligned} |R^n| &\leq Cm^2LK_n^{-\frac{1}{d}} \sum_{0 \neq k \in \mathbb{Z}^r} |k|^{\frac{-\tau}{d}} \\ &\leq Cm^2LK_n^{-\frac{1}{d}} r \sum_{N \geq 1} N^{r-1} N^{\frac{-\tau}{d}} \\ &\leq C(m, r, d, \chi) LK_n^{-\frac{1}{d}} \end{aligned}$$

if $\tau > (r-3)d$, where $r$ denotes the dimension of frequency vector $\omega$ . By (2.27), $K_n > \frac{2^{n\nu}}{\eta^\nu}$, so

$$K_n^{-\frac{1}{d}} \leq \eta^{\frac{\nu}{d}} \cdot \frac{1}{2^{\frac{n\nu}{d}}}.$$

Therefore, notice that $\eta = (10\varepsilon_1)^{1/b}$ and $\nu \geq d$, one has

$$\begin{aligned} |\bigcup_{n=1}^{\infty} R^n| &\leq CL\eta^{\frac{\nu}{d}} \sum_{n=1}^{\infty} 2^{-\frac{n\nu}{d}} \leq CL\eta^{\frac{\nu}{d}} \\ &\leq C(m,r,\gamma,\sigma,\tau,\rho,d,\chi)L(10\varepsilon_1)^{\frac{\nu}{bd}}, \end{aligned}$$

so the proof of the main theorem is finished.

# 3 Appendix

In this section, we introduce some results in matrix theory(see [13], [14] for proof).

**Definition 3.1** *The tensor product of two matrices $A_{m\times n}$, $B_{k\times l}$ is a $mk \times nl$ matrix defined by*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

*Remark.* By the definition of tensor product, it is obvious that if $A$ is block diagonal matrix as $\mathrm{diag}(A_1, A_2, \cdots, A_s)$, the tensor product can be expressed as

$$A \otimes B = \begin{pmatrix} A_1 \otimes B & & & \\ & A_2 \otimes B & & \\ & & \ddots & \\ & & & A_s \otimes B \end{pmatrix}$$

Now, let $A,\ B,\ C$ be $n\times n,\ m\times m,\ n\times m$ matrices respectively, and $X$ be a $n \times m$ unknown matrix. Suppose the eigenvalues of $A,\ B$ are $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_m$. One has the following lemma

**Lemma 3.1** *The eigenvalues of $I_m \otimes A + B \otimes I_n$ are*

$$\alpha_{ij} = \lambda_i + \mu_j, \quad i = 1, \cdots, n;\ j = 1, \cdots, m.$$

*Especially, the eigenvalues of $I_n \otimes A - A \otimes I_n$ are $(\lambda_i - \lambda_j)$ and $0$, $1 \leq i, j \leq n, i \neq j$.*

For any $n \times m$ matrix $C = (C_1, \cdots, C_m)$, where the $C_j$ represent the entries in the $j^{th}$ column, we denote by $C' = (C_1^T, \cdots, C_m^T)^T$ the corresponding $nm$-vector, and we consider the following two equations

$$AX + XB = C \tag{3.1}$$

$$(I_m \otimes A + B^T \otimes I_n)X' = C' \tag{3.2}$$

the following lemma shows that the two equations are equivalent

**Lemma 3.2** *The matrix equation (3.1) is solvable if and only if the vector equation (3.2) is solvable.*

*proof.* Rewrite (3.1) as

$$A(X_1, \cdots, X_m) + XB = (C_1, \cdots, C_m),$$

comparing the corresponding columns in both sides, one has

$$AX_j + \sum_{i=1}^{m} X_i b_{ij} = C_j, \qquad j = 1, \cdots, m.$$

Rewrite tightly these equations into a vector equation, one get (3.2). □

From the two lemmas, one has

**Corollary 1** *If $A(\lambda)$ is a one parameter $m \times m$ matrix defined on $\Lambda$, and $\alpha_j(\lambda)$, $j = 1, \cdots, m$, is the eigenvalue of $A$, let $J(k,\lambda) = \mathrm{i}\langle k, \omega\rangle I_{m^2} - (I_m \otimes A(\lambda) - A(\lambda) \otimes I_m)$, then the eigenvalues of $J(k,\lambda)$ are $\mathrm{i}\langle k,\omega\rangle - (\alpha_i(\lambda) - \alpha_j(\lambda))$, $1 \le i, j \le m$.Therefore, the matrix equation $\mathrm{i}\langle k,\omega\rangle X - (A(\lambda)X - XA(\lambda)) = C$ is solvable for all parameter $\lambda \in \Lambda$, at which $J(k,\lambda)$ is nonsingular.*

**Corollary 2** *If $A$ is a $m \times m$ quasi-diagonal matrix* $\mathrm{diag}(A_1, A_2, \cdots, A_s)$, *where each block $A_i (1 \le i \le s)$ is a $l_i \times l_i$ matrix with $\sum_{i=1}^{s} l_i = m$, then the matrix $(I_m \otimes A - A \otimes I_m)$ is also quasi-diagonal with form as*

$$\begin{pmatrix} I_{l_1} \otimes A - A_1 \otimes I_m & & & \\ & I_{l_2} \otimes A - A_2 \otimes I_m & & \\ & & \ddots & \\ & & & I_{l_s} \otimes A - A_s \otimes I_m \end{pmatrix}$$

*where $I_{l_i} (1 \le i \le s)$ is a $l_i \times l_i$ identity matrix. Thus, $J(k,\lambda)$ is nonsingular if and only if each $l_i \times m$ block* $\mathrm{i}(k,\omega)I_{ml_i} - (I_{l_i} \otimes A - A_i \otimes I_m)$ *is nonsingular.*

From the lemma 3.1 and the last corollary, the following conclusion is obvious.

**Corollary 3** *The eigenvalues of $(I_{l_i} \otimes A - A_i \otimes I_m)$ are just the union of eigenvalues of all the matrices $(I_{l_i} \otimes A_j - A_i \otimes I_{l_j})$, $j = 1, 2, \cdots, s$. So, $J(k,\lambda)$ is nonsingular if and only if for all $i, j (1 \le i, j \le s)$,* $\mathrm{i}(k,\omega)I_{l_i l_j} - (I_{l_i} \otimes A_j(\lambda) - A_i(\lambda) \otimes I_{l_j})$ *is nonsingular.*

*Remark.* We denote $J_{ij}(k,\lambda) = \mathrm{i}(k,\omega)I_{l_i l_j} - (I_{l_i} \otimes A_j(\lambda) - A_i(\lambda) \otimes I_{l_j})$. In fact, if $s = 1$, $J_{ij}(k,\lambda) = J(k,\lambda)$; if $s = m$, $J_{ij}(k,\lambda)$ is just the eigenvalue of $J(k,\lambda)$, *i.e.* $\mathrm{i}(k,\omega) - (\alpha_i(\lambda) - \alpha_j(\lambda))$.

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