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The rigidity of reducibility of cocycles on $SO(N, \mathbb{R})$

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Abstract
In this paper, we prove that for almost all $\alpha \in \mathbb{R}/\mathbb{Z}$ and constant $C \in SO(N, \mathbb{R})$, if an analytic one-dimensional quasi-periodic cocycle (α, A) on $SO(N, \mathbb{R})$ is conjugated to the constant cocycle (α, C) by a measurable conjugacy $(0, B)$, then it can be analytically reduced to (α, C). The proof is based on Krikorian's renormalization scheme and a local result (i.e. A is close to a constant) in Hou and You (*Discrete Contin. Dyn. Syst.* at press).

Mathematics Subject Classification: 37C05, 37C15, 37A20, 34A30, 34C27, 34F05

1. Introduction

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus, and we will use $\mathbb{T}$ instead of $\mathbb{T}^1$ to represent the 1-dimensional torus. We denote by G some Lie subgroup of $GL(N, \mathbb{R})$ or $GL(N, \mathbb{C})$ (N is a positive integer). We also introduce the set $SW^r(\mathbb{T}^d, G)$ ($r = 0, 1, \ldots, \infty, \omega$) of all $(\alpha, A) \in \mathbb{T}^d \times C^r(\mathbb{T}^d, G)$, which represent a C^r (discrete-time) d-dimensional quasi-periodic cocycle on G, i.e. a diffeomorphism of $\mathbb{T}^d \times G$ of the form

$$\begin{aligned}(\alpha, A) : \mathbb{T}^d \times G &\to \mathbb{T}^d \times G \\ (x, y) &\mapsto (x + \alpha, A(x)y).\end{aligned}$$

For any $(\alpha, A) \in SW^r(\mathbb{T}^d, G)$, we will call $\mathbb{T}^d$ the base space of it. When $\alpha \in \mathbb{T}^d$ is fixed the set of all such $(\alpha, A) \in SW^r(\mathbb{T}^d, G)$ will be denoted by $SW^r_\alpha(\mathbb{T}^d, G)$.

We say that a cocycle $(\alpha, A) \in SW^r(\mathbb{T}^d, G)$ is C^s $(0 \leqslant s \leqslant r)$ or measurably conjugated to $(\alpha, \widetilde{A})$ if there exists a C^s differentiable or measurable $B : \mathbb{T}^d \to G$ such that

$$\begin{aligned}Ad(B).(\alpha, A) :&= (0, B) \circ (\alpha, A) \circ (0, B)^{-1} \\ &= (\alpha, B(\cdot + \alpha)AB^{-1}) \\ &= (\alpha, \widetilde{A}).\end{aligned}$$

When $\widetilde{A}$ is a constant, we say that the cocycle $(\alpha, \widetilde{A})$ is constant and that (α, A) is C^s or measurably reducible. Note that $A \in C^0(\mathbb{T}^d, G)$ is homotopic to a constant if (α, A) is continuously reducible.

Denote the iteration of $(\alpha, A) \in SW^r(\mathbb{T}^d, G)$ by $(\alpha, A)^n = (n\alpha, A_n)$, where $A_0 \equiv I$ (I denotes the identity element) and for $n \geqslant 1$

$$\begin{cases} A_n(\cdot) = A(\cdot + (n-1)\alpha) \cdots A(\cdot), \\ A_{-n}(\cdot) = A(\cdot - n\alpha)^{-1} \cdots A(\cdot - \alpha)^{-1}. \end{cases}$$

If (α, A) is C^s $(0 \leqslant s \leqslant r)$ reducible to some constant (α, C), i.e. there exists $B \in C^s(\mathbb{T}^d, G)$ such that $B(\cdot + \alpha)AB^{-1} = C$, its dynamic is well understood since

$$(\alpha, A)^n = (n\alpha, A_n) = (n\alpha, B(\cdot + \alpha)C^n B^{-1}).$$

Usually, a higher differentiable conjugacy carries more information than a lower one. For example, if B is C^r differentiable all kth $(1 \leqslant k \leqslant r$ as $r < \infty$, or $1 \leqslant k < \infty$ as $r = \infty, \omega)$ derivatives of A_n will be uniformly bounded. Moreover, if $\{C^n : n \in \mathbb{Z}\}$ are precompact in G, $\{A_n : n \in \mathbb{Z}\}$ are then precompact in $C^s(\mathbb{T}^d, G)$ $(0 \leqslant s < r)$ w.r.t. C^s topology.

In general, lower differentiable reducibility of a cocycle does not always imply higher differentiable reducibility even if the cocycle we consider is smooth enough. A natural question is: when is a measurably or continuously reducible C^r cocycle C^r reducible? Some arithmetic conditions will be necessary when considering such a rigidity problem. To see this, let us check the differential reducibility of cocycle $(\alpha, R_{\varphi(x)}) \in SW^\omega(\mathbb{T}^d, SO(2, \mathbb{R}))$, where R_θ indicates the rotation of angle θ and $\varphi \in C^\omega(\mathbb{T}^d, \mathbb{R})$. In this special case, analytic reducibility amounts to finding a solution $f \in C^\omega(\mathbb{T}^d, \mathbb{R})$ of the equation

$$f(x + \alpha) - f(x) = \varphi(x) - \int_{\mathbb{T}^d} \varphi(\theta)\,\mathrm{d}\theta.$$

Sometimes, the above equation may not have an analytic solution. However, as $\alpha \in DC(\gamma, \tau)$ for some $\gamma, \tau > 0$, one can find a solution $f \in C^\omega(\mathbb{T}^d, \mathbb{R})$. Here we say $\alpha \in DC(\gamma, \tau)$ if

$$|\langle k, \alpha\rangle|_{\mathbb{Z}} \geqslant \frac{\gamma^{-1}}{|k|^\tau}, \qquad 0 \neq k \in \mathbb{Z}^d,$$

where $|x|_{\mathbb{Z}} = \min\{|x - k| : k \in \mathbb{Z}\}$. Let $DC = \bigcup_{\gamma,\tau>0} DC(\gamma, \tau)$ be the set of all Diophantine vectors. Note that $\bigcup_{\gamma>0} DC(\gamma, \tau)$ has full Haar measure provided $\tau > d - 1$.

It is well known that one can define a fibred rotation number for any cocycle (α, A) in $SW^0(\mathbb{T}^d, SL(2, \mathbb{R}))$ with A being homotopic to a constant [JM]. Eliasson has proved in [E1] a local result[1] that for $A \in C^\omega(\mathbb{T}^d, SL(2, \mathbb{R}))$ and Diophantine α, (α, A) will be C^ω reducible when A is sufficiently close to a constant and its fibred rotation number ρ equals $\frac{1}{2}\langle k, \alpha\rangle$ $(k \in \mathbb{Z}^d)$ or is Diophantine w.r.t. α, that is

$$|\langle k, \alpha\rangle - 2\rho|_{\mathbb{Z}} \geqslant \frac{\gamma}{(1 + |k|)^\tau}, \qquad k \in \mathbb{Z}^d, \quad \gamma, \tau > 0.$$

For reference, see also the earlier works by Dinaburg and Sinai [DiS], Rüssmann [R], Moser and Pöschel [MP] and works in [E2, E3, Kr1, Kr2, Kr3, HeY]. A non-perturbative version of Eliasson's local result was given recently by Puig [P] and Avila and Jitomirskaya [AJ].

Global reducibility (reducibility of a cocycle which is not necessarily close to a constant) is a more difficult problem. When the base space is $\mathbb{T}$, a so-called renormalization method can be applied. In [Ry], Rychlik developed a renormalization scheme via return maps on base space in an attempt to prove C^0-density of reducible cocycles in $SW^0_{\alpha_0}(\mathbb{T}, SU(2))$ (where α_0

[1] This result was originally stated for quasi-periodic linear differential equations on $\mathbb{T}^d \times SL(2, \mathbb{R})$ in [E1], but the proof also works for discrete-time cocycles in $SW^\omega(\mathbb{T}^d, SL(2, \mathbb{R}))$.

is the golden mean). Later, Krikorian introduced a different approach via renormalization of $\mathbb{Z}^2$-actions and proved a series of remarkable global reducibility results in [Kr4, Kr5]. Here we also mention the work by Avila and Krikorian in [AK] on the reducibility of cocycles in $SW^{\omega}(\mathbb{T}, SL(2, \mathbb{R}))$. In [AK], it was proved that for $\alpha \in RDC$ (it will be defined later) and $A \in C^{\omega}(\mathbb{T}, SL(2, \mathbb{R}))$ which is homotopic to a constant, (α, A) is analytically reducible if it can be conjugated to some cocycle on $SO(2, \mathbb{R})$ by a $B \in L^2(\mathbb{T}, SL(2, \mathbb{R}))$ and the fibred rotation number of it is Diophantine w.r.t. α. Their result implies that for almost all α and $C \in SO(2, \mathbb{R})$, if $(\alpha, A) \in SW^{\omega}(\mathbb{T}, SL(2, \mathbb{R}))$ is L^2 reduced to (α, C), then it is analytically reducible.

In this paper, we try to get an analogous global result for cocycles in $SW^{\omega}(\mathbb{T}, SO(N, \mathbb{R}))$ via Krikorian's renormalization scheme. Note that the fibred rotation number plays an important role in the analytic reducibility of a cocycle in $SW^{\omega}(\mathbb{T}, SL(2, \mathbb{R}))$, while the fibred rotation vector (higher dimensional analogue of fibred rotation number) is not always well defined for a cocycle in $SW^{\omega}(\mathbb{T}, SO(N, \mathbb{R}))$ as $N \geqslant 3$. However, when a cocycle in $SW^{\omega}_{\alpha}(\mathbb{T}, SO(N, \mathbb{R}))$ can be measurably conjugated to a constant (α, C), one can take the eigenvalues of C into account. We will prove that for almost all α the measurable reducibility does imply the analytic reducibility when the eigenvalues of C satisfy some arithmetic conditions. In our proof the eigenvalues of C play the role of fibred rotation number. We remark that we do not touch the problem when a cocycle is measurably reducible. Finding criteria of measurable reducibility of a cocycle remains an interesting open problem.

Before introducing our result precisely, we introduce some notation. Denote by $RDC(\gamma, \tau)$ the set of irrational $\alpha \in [0, 1)$ satisfying $G^n(\alpha) \in DC(\gamma, \tau)$ for infinitely many n, where $G : [0, 1) \to [0, 1)$ is the Gauss map $G(x) = \{x^{-1}\}$ ($\{\cdot\}$ denotes the fractional part). $RDC(\gamma, \tau)$ has full Lebesgue measure as long as $DC(\gamma, \tau)$ has positive Lebesgue measure (since G is ergodic w.r.t. the probability measure $\frac{\mathrm{d}x}{(1+x)\ln 2}$). Let $RDC = \bigcup_{\gamma,\tau>0} RDC(\gamma, \tau)$.

For any given $\alpha \in \mathbb{R}^d$, denote by $\Upsilon(\alpha)$ the set of all $\vartheta = (\vartheta_1, \ldots, \vartheta_N) \in \mathbb{R}^N$, such that for some $\sigma, \nu > 0$

$$|\langle k, \alpha\rangle - (\vartheta_i - \vartheta_j)|_{\mathbb{Z}} \geqslant \frac{\sigma}{(1 + |k|)^{\nu}}, \qquad k \in \mathbb{Z}^d,\ i \neq j.$$

It is obvious that $\Upsilon(\alpha)$ has full Lebesgue measure. Denote by $\Sigma(\alpha)$ the set of all $A \in U(N)$ which has eigenvalues $\{\mathrm{e}^{2\pi \mathrm{i}\vartheta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\vartheta_N}\}$ with $(\vartheta_1, \ldots, \vartheta_N) \in \Upsilon(\alpha)$.

We denote by $C^{\omega}_h(\mathbb{T}^d, M)$ ($M \subseteq gl(N, \mathbb{C})$) the set of all $F : \mathbb{T}^d \to M$ which has holomorphic extension in the complex neighbourhood

$$\mathbb{T}^d_h := \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |\mathrm{Im}\, z_i| < h, i = 1, \ldots, d\}/\mathbb{Z}^d.$$

Let $\|\cdot\|$ be the usual matrix norm, and for any $F \in C^{\omega}_h(\mathbb{T}^d, M)$, we define

$$\|F\|_h := \sup_{x \in \mathbb{T}^d_h} \|F(x)\|.$$

For $E \in L^1(\mathbb{T}^d, gl(N, \mathbb{C}))$, define

$$[E] := \inf_{1 \leqslant s \leqslant N} \sup\{|\widehat{E}_{s,t}(r)| : 1 \leqslant t \leqslant N, r \in \mathbb{Z}^d\},$$

where $\widehat{E}(r)$ is the rth Fourier coefficient of E, that is

$$\widehat{E}(r) = \int_{\mathbb{T}^d} E(x)\mathrm{e}^{-2\pi \mathrm{i}\langle r, x\rangle}\, \mathrm{d}x,$$

and $\widehat{E}_{s,t}(r)$ is the (s, t) component of the matrix $\widehat{E}(r)$. Note that for any $E : \mathbb{T}^d \to U(N)$ measurable, $E \in L^1(\mathbb{T}^d, U(N))$ and $[E] > 0$.

The following proportion is a local rigidity result of reducibility of quasi-periodic cocycles on $U(N)$, proved in [HoY] by the KAM method.

Proposition 1.1 ([HoY]). *Let $\alpha \in DC(\gamma,\tau)$ $(\gamma,\tau > 0)$, $F \in C_h^\omega(\mathbb{T}^d, u(N))$ $(h > 0)$ and $A \in U(N)$. Assume that (α, Ae^F) is conjugated to a constant (α, C) with $C \in \Sigma(\alpha)$ by a measurable $B : \mathbb{T}^d \to U(N)$, and there exists $\delta_0 > 0$ such that $[B] \geqslant \delta_0$. Then there exists $\delta = \delta(d, h, \gamma, \tau, \delta_0) > 0$, if furthermore $\|F\|_h < \delta$, there is a $\widetilde{B} \in C^\omega(\mathbb{T}^d, U(N))$ satisfying $\widetilde{B}(x) = B(x)$ for a.e. $x \in \mathbb{T}^d$, which also reduces (α, Ae^F) to (α, C).*

In this paper, we will use proposition 1.1 to prove a global rigidity result as in the following theorem.

Theorem 1.1. *Let $\alpha \in RDC$. Assume that $(\alpha, A) \in SW^\omega(\mathbb{T}, SO(N,\mathbb{R}))$ is reduced to a constant (α, C) with $C \in \Sigma(\alpha) \cap SO(N,\mathbb{R})$ by a measurable $B : \mathbb{T} \to SO(N,\mathbb{R})$. Then there is a $\widetilde{B} \in C^\omega(\mathbb{T}, SO(N,\mathbb{R}))$ satisfying $\widetilde{B}(x) = B(x)$ for a.e. $x \in \mathbb{T}$, which also reduced (α, A) to (α, C) .*

Similarly, one can consider the flow $(\varpi t, Z^t(\cdot))$ on $\mathbb{T}^2 \times SO(N,\mathbb{R})$ of differential equation

$$\begin{cases} \dot{X} = A(\theta)X, \\ \dot{\theta} = \varpi, \end{cases}$$

where $\varpi \in \mathbb{R}^2$ and $A \in C^\omega(\mathbb{T}^2, SO(N,\mathbb{R}))$ satisfies the continuous-time cocycle property

$$Z^{t+s}(\cdot) = Z^t(\cdot + \varpi s)Z^s(\cdot).$$

Similarly to corollary 2.4 in [Kr5], we have the following corollary.

Corollary 1.1. *Let $\varpi = (\varpi_1, \varpi_2) \in \mathbb{R}^2$ with $\varpi_1 \neq 0$ and $\{\varpi_2/\varpi_1\} \in RDC$. If the flow $(\varpi t, Z^t)$ on $\mathbb{T}^2 \times SO(N,\mathbb{R})$ is conjugated to $(\varpi t, e^{tF_0})$ with $e^{F_0} \in \Sigma(\varpi_2/\varpi_1) \cap SO(N,\mathbb{R})$ by a measurable $B : \mathbb{T}^2 \to SO(N,\mathbb{R})$, then there exists $\widetilde{B} \in C^\omega(\mathbb{T}^2, SO(N,\mathbb{R}))$ conjugating $(\varpi t, Z^t)$ to $(\varpi t, e^{tF_0})$ with $B(\phi,\theta) = \widetilde{B}(\phi,\theta)$ for a.e. $(\phi,\theta) \in \mathbb{T}^2$.*

The proof of theorem 1.1 is essentially an application of Krikorian's renormalization scheme. The idea of renormalization is to define a sequence of cocycles $(\alpha_n, A^{(n)})$ by some renormalization transformations so that a subsequence $(\alpha_{n_k}, A^{(n_k)})$ analytically converges to a constant. Meanwhile, the renormalization does not change the reducibility. By the local result of proposition 1.1, we obtain the analytic reducibility of $(\alpha_{n_k}, A^{(n_k)})$ for sufficiently large k, which implies the analytic reducibility of (α, A).

Outline of the paper. We introduce some estimates in section 2 which are needed in section 4. In section 3 we give the definition of the $\mathbb{Z}^2$-action and renormalization of it, which is due to Krikorian. In section 4 we use the renormalization of the $\mathbb{Z}^2$-action to prove theorem 1.1.

2. Some estimates

In this section, we give some estimates needed for the proof of theorem 1.1.

2.1. Estimates of derivative for cocycle

The following inequality was given in [AK] by Avila and Krikorian on estimation of derivatives of A_n for any $(\alpha, A) \in SW^K(\mathbb{T}, SO(N,\mathbb{R}))$ $(K \geqslant 1)$. In our case the proof is simpler.

Lemma 2.1. *Assume $A \in C^K(\mathbb{R}/\mathbb{Z}, SO(N,\mathbb{R}))$ $(1 \leqslant K < \infty)$. Then for every $0 \leqslant r \leqslant K$ and any $x \in \mathbb{R}/\mathbb{Z}$, we have*

$$||\partial^r A_n(x)|| \leqslant n^r C^r ||\partial^r A||_0,$$

where $\|\cdot\|_0$ denotes the C^0 norm and C is an absolute constant.

Proof. Note first that $||A_n(x)|| \leqslant 1$ for all $n \in \mathbb{Z}$. We compute

$$\partial^r A_n(x) = \partial^r \left(\prod_{k=n-1}^{0} A(x + k\alpha) \right)(x),$$

which by Leibniz formula is a sum of n^r terms of the form ($s \leqslant r$)

$$I_{(i^*)}(x) = \left(\prod_{l=n-1}^{i_1+1} A(x + l\alpha) \right) \partial^{m_1} A(x + i_1\alpha) \left(\prod_{l=i_1-1}^{i_2+1} A(x + l\alpha) \right) \cdots \partial^{m_s} A(x + i_s\alpha) \left(\prod_{l=i_s-1}^{0} A(x + l\alpha) \right),$$

where i^* runs through $\mathcal{I} = \{0, \ldots, n-1\}^{1,\ldots,r}$ and $\{i_1, \ldots, i_s\} = i^*(\{1, \ldots, r\})$ satisfy $n - 1 \geqslant i_1 > \cdots > i_s \geqslant 0$ and $m_l = \sharp(i^*)^{-1}(i_l)$ (notice $m_1 + \cdots + m_s = r$). Each term $I_{(i^*)}$ can be written as

$$I_{(i^*)}(x) = \left(\prod_{l=n-1}^{0} A(x + l\alpha) \right) Ad \left(\left(\prod_{l=i_1-1}^{0} A(x + l\alpha) \right)^{-1} \right) (A(x + i_1\alpha)^{-1} \partial^{m_1} A(x + i_1\alpha)) \cdots Ad \left(\left(\prod_{l=i_s-1}^{0} A(x + l\alpha) \right)^{-1} \right) (A(x + i_s\alpha)^{-1} \partial^{m_s} A(x + i_s\alpha)).$$

It follows that

$$||I_{(i^*)}(x)|| \leqslant \prod_{p=1}^{s} ||\partial^{m_p} A||_0.$$

From this and the convexity (Hadamard–Kolmogorov) inequalities [K]

$$||\partial^m A||_0 \leqslant C||A||_0^{1-\frac{m}{r}} ||\partial^r A||_0^{\frac{m}{r}} \leqslant C||\partial^r A||_0^{\frac{m}{r}}, \qquad 0 \leqslant m \leqslant r,$$

we have (using $\sum_{p=1}^{s} m_p = r$)

$$||I_{(i^*)}||_0 \leqslant \prod_{p=1}^{s} (C||\partial^r A||_0^{\frac{m}{r}}) \leqslant C^r ||\partial^r A||_0.$$

It can be concluded that

$$||\partial^r A_n||_0 \leqslant \sum_{i^* \in \mathcal{I}} ||I_{(i^*)}||_0 \leqslant n^r C^r ||\partial^r A||_0.$$

□

2.2. L^1-estimates

Lemma 2.2. *Let $B : \mathbb{R}/\mathbb{Z} \to SO(N, \mathbb{R})$ measurable. For x_* being a measurable continuity point of B, we have*

$$\lim_{t \to 0+} \int_{[0,1]} ||B(x_* + tx)B(x_*)^T - I|| \, \mathrm{d}x = 0,$$

where I denotes the identity matrix.

Proof. Denote by X the set of x_* which is a measurable continuity point of B. By the well-known Lebesgue density theorem, X has full Lebesgue measure.

Fix $x_* \in X$. For any $\epsilon > 0$, let $I_t(\epsilon) = \{x \in [x_*, x_* + t] : ||B(x)B(x_*)^T - I|| < \epsilon\}$ and $J_t(\epsilon) = [0, t] - I_t(\epsilon)$; we then have

$$\lim_{t\to 0+} \frac{m(J_t(\epsilon))}{t} = 0,$$

where m denotes Lebesgue measure. Now we have

$$\begin{aligned}\int_{[0,1]} &||B(x_* + tx)B(x_*)^T - I||\,\mathrm{d}x \\ &= \frac{1}{t}\int_{[x_*, x_*+t]} ||B(x)B(x_*)^T - I||\,\mathrm{d}x \\ &= \frac{1}{t}\left(\int_{I_t(\epsilon)} + \int_{J_t(\epsilon)}\right) ||B(x)B(x_*)^T - I||\,\mathrm{d}x \\ &\leqslant \epsilon + \frac{m(J_t(\epsilon))}{t} \to \epsilon\end{aligned}$$

as $t \to 0+$. Since the inequality holds for any $\epsilon > 0$, the lemma is proved. □

2.3. *Estimates of Fourier coefficients*

Lemma 2.3. *Given* $B, \Delta : \mathbb{R}/\mathbb{Z} \to gl(N, \mathbb{C})$ *measurable with*

$$\int_{\mathbb{T}} ||\Delta(x)||\,\mathrm{d}x < \epsilon$$

for some $\epsilon > 0$. *Then* $[B(I + \Delta)] > [B] - \epsilon$.

Proof. Note that for any $r \in \mathbb{Z}$

$$\int_{\mathbb{T}} ||B(x)\Delta(x)\mathrm{e}^{-2\pi i r x}||\,\mathrm{d}x \leqslant \int_{\mathbb{T}} ||\Delta(x)||\,\mathrm{d}x < \epsilon.$$

Thus $[B(I + \Delta)] > [B] - \epsilon$. □

The following corollary is a consequence of the above lemma.

Corollary 2.1. *Let* B_n *and* C_n *be sequences of a measurable map from* $\mathbb{R}/\mathbb{Z}$ *to* $SO(N, \mathbb{R})$ *such that* $[C_n] \geqslant \varrho$ *for some* $\varrho > 0$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{T}} ||B_n(x) - C_n(x)||\,\mathrm{d}x = 0.$$

Then $\liminf_{n\to\infty}[B_n] \geqslant \varrho$.

3. $\mathbb{Z}^2$-action and renormalization

A C^r fibred $\mathbb{Z}^2$-action Φ is a homomorphism from the additive group $\mathbb{Z}^2$ to the composition group $\Omega^r = \mathbb{R} \times C^r(\mathbb{R}, SO(N, \mathbb{R}))$, where Ω^r is viewed as a subset of $\mathrm{Diff}^r(\mathbb{R} \times SO(N, \mathbb{R}))$:

$$\begin{aligned}(\alpha, A) : \mathbb{R} \times SO(N, \mathbb{R}) &\to \mathbb{R} \times SO(N, \mathbb{R}) \\ (x, y) &\mapsto (x + \alpha, A(x)y).\end{aligned}$$

That is: $\Phi(n,m)\circ\Phi(n',m')=\Phi(n+n',m+m')$, for any $(n,m),(n',m')\in\mathbb{Z}^2$. Denote by Λ^r all such C^r fibred $\mathbb{Z}^2$-action, endowed with the topology induced by the embedding

$$\Lambda^r\to SW^r(\mathbb{R},SO(N,\mathbb{R}))\times SW^r(\mathbb{R},SO(N,\mathbb{R})),\ \Phi\to(\Phi(1,0),\Phi(0,1)).$$

Let $\{(\alpha,A),(\widetilde{\alpha},\widetilde{A})\}$ denote the $\mathbb{Z}^2$-action Φ with $\Phi(1,0)=(\alpha,A)$ and $\Phi(0,1)=(\widetilde{\alpha},\widetilde{A})$.

Let $\gamma^{\Phi}_{n,m}=\Pi_1\circ\Phi(n,m)$ and $A^{\Phi}_{n,m}=\Pi_2\circ\Phi(n,m)$, where

$$\Pi_1:\mathbb{R}\times C^r(\mathbb{R},SO(N,\mathbb{R}))\to\mathbb{R},\quad \Pi_2:\mathbb{R}\times C^r(\mathbb{R},SO(N,\mathbb{R}))\to C^r(\mathbb{R},SO(N,\mathbb{R}))$$

be the coordinate projections. Denote by Λ^r_0 the set of $\Phi\in\Lambda^r$ with $\gamma^{\Phi}_{1,0}=1$ and Γ^r the set of $\Phi\in\Lambda^r$ with $\Pi_1\circ\Phi:\mathbb{Z}^2\to\mathbb{R}$ being injective. Let $\alpha^{\Phi}=\gamma^{\Phi}_{0,1}$ when $\Phi\in\Lambda^r_0$ and $\Gamma^r_0=\Gamma^r\bigcap\Lambda^r_0=\{\Phi\in\Lambda^r_0:\alpha^{\Phi}$ is irrational$\}$.

3.1. Some operations

Let $x_*\in\mathbb{R}$, define $T_{x_*}:\Lambda^r\to\Lambda^r$ as

$$T_{x_*}(\Phi)(n,m)=(\gamma^{\Phi}_{n,m},A^{\Phi}_{n,m}(\cdot+x_*)).$$

Let $\lambda\neq 0$, define $M_\lambda:\Lambda^r\to\Lambda^r$ as

$$M_\lambda(\Phi)(n,m)=(\lambda^{-1}\gamma^{\Phi}_{n,m},A^{\Phi}_{n,m}(\lambda\cdot)).$$

Let $U\in GL(N,\mathbb{Z})$, define $N_U:\Lambda^r\to\Lambda^r$ as

$$N_U(\Phi)(n,m)=\Phi((n,m)U^{-T}).$$

The operations T_{x_*}, M_λ, N_U will be called translation, rescaling and base change, respectively. Note that $T_{x_*}T_{y_*}=T_{x_*+y_*}$, $M_\lambda M_{\lambda'}=M_{\lambda\lambda'}$ and $N_UN_{U'}=N_{UU'}$. Moreover, base changes commute with translations and rescalings. And it is easy to see that $T_{x_*}M_\lambda=M_\lambda T_{\lambda x_*}$.

Let $B\in C^r(\mathbb{R},SO(N,\mathbb{R}))$ (or measurable), define

$$Ad(B).\Phi(n,m)=(0,B)\circ\Phi(n,m)\circ(0,B)^{-1}.$$

We will say that Φ is C^r conjugated (or measurably conjugated) to $Ad(B).\Phi$ by B.

3.2. Continued fraction expansion

Let $\alpha\in(0,1)$ be irrational. We will fix some notation regarding the continued fraction expansion

$$\alpha=\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}}.$$

We set $\alpha_0=\alpha$, and

$$\alpha_n=\cfrac{1}{a_{n+1}+\cfrac{1}{a_{n+2}+\cdots}}.$$

In fact, $\alpha_n=G^n(\alpha)$, recall that G is the Gauss map. The coefficients a_n are given by $a_n=[\alpha^{-1}_{n-1}]$ ($[\cdot]$ denotes the integer part). We also set $a_0=0$ for convenience.

Let $\beta_n=\prod_{j=0}^n\alpha_j$. Define

$$Q_0=\begin{bmatrix}q_0 & p_0\\ q_{-1} & p_{-1}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$Q_n=\begin{bmatrix}q_n & p_n\\ q_{n-1} & p_{n-1}\end{bmatrix}=\begin{bmatrix}a_n & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}q_{n-1} & p_{n-1}\\ q_{n-2} & p_{n-2}\end{bmatrix}.$$

It is easy to see that $Q_n = U(\alpha_n)\cdots U(\alpha_1)$ with

$$U(x) = \begin{bmatrix} [x^{-1}] & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus we have $\det(Q_n) = q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

Note that

$$\beta_n = (-1)^n (q_n\alpha - p_n) = \frac{1}{q_{n+1} + \alpha_{n+1} q_n};$$

$$\frac{1}{q_n + q_{n+1}} < \beta_n < \frac{1}{q_{n+1}}.$$

3.3. Renormalization

Renormalization operator $R : \Gamma_0^r \to \Gamma_0^r$ is defined as $R(\Phi) = M_{\alpha^\Phi} N_{U(\alpha)}(\Phi)$. Note that if $\Phi \in \Gamma_0^r$ then $\alpha^{R(\Phi)} = G(\alpha^\Phi)$ and so

$$R^n(\Phi) = M_{\alpha_{n-1}} N_{U(\alpha_{n-1})} \cdots M_{\alpha_0} N_{U(\alpha_0)}(\Phi) = M_{\beta_{n-1}} N_{Q_n}(\Phi).$$

3.4. Normalized actions

A fibred $\mathbb{Z}^2$-action $\Phi \in \Lambda_0^r$ is called normalized if $\Phi(1, 0) = (1, I)$. If Φ is normalized then $\Phi(0, 1) = (\alpha, A)$ can be viewed as a cocycle in $SW^r(\mathbb{T}, SO(N, \mathbb{R}))$, since A is automatically $\mathbb{Z}$-periodic. Conversely, we can associate a normalized fibred $\mathbb{Z}^2$-action $\Phi = \{(1, I), (\alpha, A)\}$ with any given $(\alpha, A) \in SW^r(\mathbb{T}, SO(N, \mathbb{R}))$.

The following lemma is about the conjugation of a general fibred $\mathbb{Z}^2$-action to a normalized one. We omit the proof because it is similar to that of lemma 4.1 in [AK].

Lemma 3.1. *Any $\Phi \in \Lambda_0^\omega$ can be C^ω conjugated to a normalized one. Moreover, for a sequence of $\Phi_n = \{(1, C^{(n)}), (\gamma_n, D^{(n)})\} \in \Lambda_0^\omega$, if $C^{(n)}$ admit a sequence of analytic extension in a uniform domain $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 2, |\mathrm{Im}\, z| < h\}$ $(h > 0)$ converging uniformly to I, then there exists $\delta > 0$ and a sequence of $B_n \in C^\omega(\mathbb{R}, SO(N, \mathbb{R}))$, which conjugate Φ_n to a sequence of normalized fibred $\mathbb{Z}^2$-actions and admit a sequence of analytic extension in the domain $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1, |\mathrm{Im}\, z| < \delta\}$ converging uniformly to I.*

3.5. Reducibility

We say that a fibred $\mathbb{Z}^2$-action Φ with $\Phi(n, m) = (\gamma_{n,m}^\Phi, A_{n,m}^\Phi)$ is a constant if $A_{n,m}^\Phi$ $((n, m) \in \mathbb{Z}^2)$ are all constants. A C^r fibred $\mathbb{Z}^2$-action Φ is said to be C^r reducible (measurably reducible) if it can be C^r conjugated (measurably conjugated) to a constant.

One sees that the C^r reducibility (measurable reducibility) is invariant under the translation, the rescaling, the base change and the C^r conjugation (measurable conjugation) defined above and thus invariant under renormalization.

Note that $(\alpha, A) \in SW^r(\mathbb{T}, SO(N, \mathbb{R}))$ is C^r reducible (measurably reducible) if and only if $\{(1, I), (\alpha, A)\}$ is C^r reducible (measurably reducible), and the proof is similar to that of lemma 4.3 in [AK].

4. Analytic reducibility

In this section we prove theorem 1.1. The strategy of renormalization is to reduce the reducibility of (α, A) to the reducibility of a cocycle close to a constant. For any given fibred $\mathbb{Z}^2$-action $\Phi = \{(1, I), (\alpha, A)\}$ with irrational α, note that

$$R^n(\Phi) = \{(1, A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}\cdot)), (\alpha_n, A_{(-1)^n q_n}(\beta_{n-1}\cdot))\}, \qquad n = 1, 2, \ldots.$$

Lemma 4.1. *Assume that* $\alpha \in [0, 1)$ *is irrational and* $A \in C^\omega(\mathbb{T}, SO(N, \mathbb{R}))$. *Let*

$$U_n(\cdot) = A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}\cdot), \qquad V_n(\cdot) = A_{(-1)^n q_n}(\beta_{n-1}\cdot).$$

Then for any $r \geqslant 1$, U_n *and* V_n *are* C^r *bounded. Moreover, there exists* $\delta > 0$, *such that* U_n *and* V_n *admit uniformly bounded analytic extensions in the domain* $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta\}$.

Proof. Applying lemma 2.1 to both (α, A) and $(\alpha, A)^{-1}$, we obtain

$$||\partial^r A_k||_0 \leqslant |k|^r C^r ||\partial^r A||_0, \quad k \in \mathbb{Z},$$

where C is an absolute constant. It follows that for all $x \in \mathbb{R}$

$$||\partial^r U_n(x)|| \leqslant |\beta_{n-1} q_{n-1}|^r C^r ||\partial^r A||_0 \leqslant C^r ||\partial^r A||_0.$$

We assume A admits an analytic extension in $\{z \in \mathbb{C} : |\mathrm{Im}\, z| \leqslant h\}$ for some $h > 0$; then by the Cauchy formula

$$||\partial^r A||_0 \leqslant C_1 \frac{r!}{h^r},$$

where C_1 is another absolute constant. Now for any $x_0 \in \mathbb{R}$

$$\sum_{r=0}^{\infty} \frac{|z - x_0|^r}{r!} ||\partial^r U_n(x_0)|| \leqslant C_1 \sum_{r=0}^{\infty} \left(C \frac{|z - x_0|}{h} \right)^r.$$

Thus

$$\sum_{r=0}^{\infty} \frac{(z - x_0)^r}{r!} \partial^r U_n(x_0)$$

is convergent as $|z - x_0| < \delta$ for any $\delta \in (0, h)$ satisfying $\delta C < h$. Then for such δ, U_n ($n = 1, 2, \ldots$) admit uniformly bounded analytic extensions in $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta\}$.

And one can prove the conclusion for V_n in the same way. □

We also need the following fact.

Lemma 4.2. *If* $\vartheta = (\vartheta_1, \ldots, \vartheta_N) \in \Upsilon(\alpha)$, *then for all* $n \in \mathbb{N}$, $(-1)^n \beta_{n-1}^{-1} \vartheta \in \Upsilon(\alpha_n)$.

Proof. Without loss of generality, we assume

$$|\vartheta_i - \vartheta_j| \leqslant 2, \qquad i, j \in \{1, \ldots, N\}.$$

There exist $\sigma, \nu > 0$, such that for any $k, l \in \mathbb{Z}$ and $i \neq j$

$$|k\alpha - (\vartheta_i - \vartheta_j) - l| \geqslant \frac{\sigma}{(1 + |k|)^\nu}.$$

When $|l| > 2|k| + 3$, for $i \neq j$

$$|k\alpha - (\vartheta_i - \vartheta_j) - l| \geqslant |l| - |k| - 2 \geqslant 2|k| + 3 - |k| - 2 = |k| + 1.$$

Note that

$$\begin{aligned}
&|k\alpha_n - (-1)^n \beta_{n-1}^{-1}(\vartheta_i - \vartheta_j) - l| \\
&\quad = \beta_{n-1}^{-1} |(-1)^n k\beta_n - (\vartheta_i - \vartheta_j) - (-1)^n l \beta_{n-1}| \\
&\quad = \beta_{n-1}^{-1} |(q_n\alpha - p_n)k - (\vartheta_i - \vartheta_j) + (q_{n-1}\alpha - p_{n-1})l| \\
&\quad = \beta_{n-1}^{-1} |(q_n k + q_{n-1} l)\alpha - (\vartheta_i - \vartheta_j) - (p_n + p_{n-1})| \\
&\quad \geqslant \frac{\beta_{n-1}^{-1}\sigma}{(1 + q_n|k| + q_{n-1}|l|)^\nu}.
\end{aligned}$$

When $|l| \leqslant 2|k|+3$, for $i \neq j$

$$
\begin{aligned}
&|k\alpha_n - (-1)^n \beta_{n-1}^{-1}(\vartheta_i - \vartheta_j) - l| \\
&\qquad \geqslant \frac{\beta_{n-1}^{-1}\sigma}{(1+q_n|k|+q_{n-1}|l|)^\nu} \\
&\qquad \geqslant \frac{\beta_{n-1}^{-1}\sigma}{(1+q_n|k|+2q_n|k|+3q_n)^\nu} \\
&\qquad \geqslant \frac{\beta_{n-1}^{-1}\sigma}{(4q_n)^\nu(|k|+1)^\nu}.
\end{aligned}
$$

Then there exists $\sigma_n > 0$, such that for any $k, l \in \mathbb{Z}$ and $i \neq j$

$$|k\alpha - (-1)^n \beta_{n-1}^{-1}(\vartheta_i - \vartheta_j) - l| \geqslant \frac{\sigma_n}{(1+|k|)^\nu}.$$

□

Now we are in a position to prove theorem 1.1. Let $\Phi = \{(1, I), (\alpha, A)\}$ be a normalized fibred $\mathbb{Z}^2$-action such that $\alpha \in RDC(\gamma, \tau)$ for some $\gamma, \tau > 0$, and there exists a measurable $B : \mathbb{R}/\mathbb{Z} \to SO(N, \mathbb{R})$ satisfying

$$Ad(B).(\alpha, A) = (\alpha, C)$$

with C being a constant in $\Sigma(\alpha) \cap SO(N, \mathbb{R})$. Without loss of generality, we assume 0 is a measurable continuity point of B and B^T (if it is not the case we can make some translation). We also assume $N = 2m$, and the proof for odd N is similar.

Let $H(\cdot) = \mathrm{e}^{h(\cdot)}$, where h is a map from $\mathbb{R}^m$ to $u(2m)$ defined as

$$h(x_1, \ldots, x_m) = 2\pi \mathrm{i}\,\mathrm{diag}(x_1, -x_1, \ldots, x_m, -x_m).$$

It is obvious that $H(x+y) = H(x)H(y) = H(y)H(x)$ for any $x, y \in \mathbb{R}^m$ and $H(k) = I$ for any $k \in \mathbb{Z}^m$. Thus H can be viewed as a homomorphism from $\mathbb{R}^m/\mathbb{Z}^m$ to $U(2m)$.

Lemma 4.3. *There exist $S \in U(2m)$ and $\rho \in \mathbb{R}^m$ such that $C = S^* H(\rho) S$. Moreover there exist a linear subspace $V_\rho \supseteq \{t\rho : t \in \mathbb{R}\}$ of $\mathbb{R}^m$ and a positive integer K_ρ, such that for any $\phi \in V_\rho$ we have $S^* H(\phi) S \in SO(2m, \mathbb{R})$ and there exists $\phi^{(0)} \in V_\rho \cap [-K_\rho, K_\rho]^m$ satisfying $H(\phi^{(0)}) = H(\phi)$.*

Proof. There exists $X \in SO(2m, \mathbb{R})$ such that $C = \mathrm{e}^X$. It is well known that any $X \in gl(2m, \mathbb{C})$ satisfying $X^*X = XX^*$ can be diagonalized by a unitary transformation. So there exist $S \in U(2m)$ and $\rho \in \mathbb{R}^m$, such that $X = S^* h(\rho) S$, and then

$$S^* H(t\rho) S = S^* \mathrm{e}^{h(t\rho)} S = \mathrm{e}^{tS^* h(\rho) S} = \mathrm{e}^{tX} \in SO(2m, \mathbb{R}), \qquad t \in \mathbb{R}.$$

Denote by $p : \mathbb{R}^m \to \mathbb{R}^m/\mathbb{Z}^m$ the standard projection. Let $T_\rho \subseteq \mathbb{R}^m/\mathbb{Z}^m$ be the closure of the set $\{p(t\rho) : t \in \mathbb{R}\}$ which is a subtorus of $\mathbb{R}^m/\mathbb{Z}^m$. For any $\phi \in p^{-1}(T_\rho)$, it is obvious that $S^* H(\phi) S \in SO(2m, \mathbb{R})$.

Without loss of generality, we assume $\rho_1, \cdots, \rho_r$ $(1 < r \leqslant m)$ are rationally independent and $\rho_i = \frac{p_{i1}}{q_{i1}}\rho_1 + \ldots + \frac{p_{ir}}{q_{ir}}\rho_r$ for $i = r+1, \ldots, m$, where $\frac{p_{ij}}{q_{ij}}$ $(i = r+1, \ldots, m,\ j = 1, \ldots, r)$ are rational numbers. Let

$$V_\rho = \{x = (x_1, \ldots, x_m) : x_1, \ldots, x_r \in \mathbb{R}, \quad x_i = \sum_{j=1}^r \frac{p_{ij}}{q_{ij}} x_j,\ i = r+1, \ldots, m.\}.$$

One can prove that $p(V_\rho) \subseteq T_\rho$. So for any $\phi \in V_\rho$ we have $S^* H(\phi) S \in SO(2m, \mathbb{R})$.

Now let $K_\rho^{(0)}=\prod_{i=r+1,\ldots,m,j=1,\ldots,r}|q_{ij}|$. For any $\phi\in V_\rho$, one can find integers $N_1,\ldots,N_r$ and $\phi_1^{(0)},\ldots,\phi_r^{(0)}\in[0,K_\rho^{(0)}]$, such that $\phi_i=\phi_i^{(0)}+N_iK_\rho^{(0)}$, $i=1,\ldots,r$. Let

$$\phi_i^{(0)}=\sum_{j=1}^{r}\frac{p_{ij}}{q_{ij}}\phi_j^{(0)},\qquad i=r+1,\ldots,m,$$

and thus $\phi=(\phi_1^{(0)},\ldots,\phi_m^{(0)})\in[-mK_\rho^{(0)},mK_\rho^{(0)}]^m\cap V_\rho$. For $(\phi-\phi^{(0)})\in\mathbb{Z}^m$, we have $H(\phi)=H(\phi^{(0)})$. □

In the following lemma we prove that the closure of $\{R^n(\Phi):n=1,2,\ldots\}$ w.r.t. C^ω topology contains (at least) a constant. Recall that

$$R^n(\Phi)=\{(1,A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}\cdot)),(\alpha_n,A_{(-1)^nq_n}(\beta_{n-1}\cdot))\}.$$

Lemma 4.4. *There exist a sequence n_k of $\mathbb{N}$, sequences L_k,M_k of vectors in $\mathbb{Z}^m$, $\varphi,\psi\in V_\rho\cap[-K_\rho,K_\rho]^m$ and $\delta>0$, such that $\alpha_{n_k}=G^{n_k}(\alpha)\in DC(\gamma,\tau)$,*

$$\alpha_\infty:=\lim_{k\to\infty}\alpha_{n_k}\in DC(\gamma,\tau),\qquad \lim_{k\to\infty}(\lambda_k\rho+L_k)=\varphi,\qquad \lim_{k\to\infty}(\mu_k\rho+M_k)=\psi,$$

where $\lambda_k=(-1)^{n_k-1}q_{n_k-1}$ and $\mu_k=(-1)^{n_k}q_{n_k}$, and in the domain $\{z\in\mathbb{C}:|\mathrm{Im}\,z|<\delta\}$ the analytic extensions of

$$A_{(-1)^{n_k-1}q_{n_{k-1}}}(\beta_{n_{k-1}}\cdot)\qquad\text{and}\qquad A_{(-1)^{n_k}q_{n_k}}(\beta_{n_k}\cdot)$$

converge uniformly to $B(0)^TS^*H(\varphi)SB(0)$ and $B(0)^TS^*H(\psi)SB(0)$, respectively.

Proof. There exists a sequence n_k of $\mathbb{N}$, such that $\alpha_{n_k}\in DC(\gamma,\tau)$ and converge to some α_∞ in $DC(\gamma,\tau)$. By lemma 4.3, one can choose sequences $L_k,M_k\in Z^m$, s.t.

$$(\lambda_k\rho+L_k),(\mu_k\rho+M_k)\in V_\rho\cap[-K_\rho,K_\rho]^m,$$

where $\lambda_k=(-1)^{n_k-1}q_{n_k-1}$ and $\mu_k=(-1)^{n_k}q_{n_k}$. Thus there exists a subsequence of n_k (assume it is n_k itself for convenience) and $\varphi,\psi\in V_\rho\cap[-K_\rho,K_\rho]^m$, s.t.

$$\lim_{k\to\infty}(\lambda_k\rho+L_k)=\varphi,\qquad \lim_{k\to\infty}(\mu_k\rho+M_k)=\psi.$$

By lemma 4.1 there exists $\delta>0$, such that in the domain $\{z\in\mathbb{C}:|\mathrm{Im}\,z|<\delta\}$ U_n and V_n admit uniformly bounded analytic extensions, where

$$U_n(\cdot)=A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}\cdot),\qquad V_n(\cdot)=A_{(-1)^nq_n}(\beta_{n-1}\cdot).$$

Thus there exists a subsequence of n_k (assume it is n_k itself for convenience), such that in the domain $\{z\in\mathbb{C}:|\mathrm{Im}\,z|<\delta\}$ the analytic extensions of $\widetilde{U}_k:=U_{n_k}$ and $\widetilde{V}_k:=V_{n_k}$ converge uniformly to some $\widetilde{U}_\infty$ and $\widetilde{V}_\infty$, respectively.

In the following, we prove that

$$\widetilde{U}_\infty=B(0)^TS^*H(\varphi)SB(0),\qquad \widetilde{V}_\infty=B(0)^TS^*H(\psi)SB(0).$$

Let

$$I(x,\varepsilon)=\{y\in\mathbb{R}:\|B(y)-B(x)\|<\varepsilon\quad\text{and}\quad\|B(y)^T-B(x)^T\|<\varepsilon\}.$$

Note that 0 has been assumed as a measurable continuity point of B and B^T. Thus for any $\varepsilon>0$ and fixed $d>1$,

$$\lim_{k\to\infty}\frac{m(I(0,\varepsilon)\cap[-\beta_{n_k-1}d,\beta_{n_k-1}d])}{2\beta_{n_k-1}d}=1.$$

By lemma 4.1, $\widetilde{U}_k=U_{n_k}$ are C^1 uniformly bounded and are thus equicontinuous. So there exists $\widehat{\delta}>0$, such that $\|\widetilde{U}_k(x)-\widetilde{U}_k(y)\|<\varepsilon$ for all k as $|x-y|<\widehat{\delta}$. As k is large enough, for any $x \in [-d+1, d-1]$, there exists $x_* = x_*(k, x) \in [-d+1, d-1] \cap (x - \widehat{\delta}, x + \widehat{\delta})$ such that $\beta_{n_k-1}x_*$, $\beta_{n_k-1}(x_*+1) \in I(0, \varepsilon)$. So $\|\widetilde{U}_k(x) - \widetilde{U}_k(x_*)\| < \varepsilon$, $\|B(\beta_{n_k-1}x_*) - B(0)\| < \varepsilon$ and $\|B(\beta_{n_k-1}(x_*+1)) - B(0)\| < \varepsilon$. Note that

$$B(\beta_{n_k-1}(x_*+1))\widetilde{U}_k(x_*)B(\beta_{n_k-1}x_*)^T = S^* H(\lambda_k \rho) S,$$

one then easily obtains

$$\|B(0)\widetilde{U}_k(x)B(0)^T - S^* H(\lambda_k \rho) S\| < 3\varepsilon.$$

Since $\widetilde{U}_k(x) \to \widetilde{U}_\infty(x)$ $(x \in \mathbb{R})$ and $H(\lambda_k \rho) \to H(\varphi)$ as $k \to +\infty$, for any $x \in [-d+1, d-1]$

$$\|B(0)\widetilde{U}_\infty(x)B(0)^T - S^* H(\varphi) S\| < 3\varepsilon.$$

Let $\varepsilon \to 0+$ and $d \to +\infty$, we then obtain $B(0)\widetilde{U}_\infty B(0)^T \equiv S^* H(\varphi) S$. The proof for $B(0)\widetilde{V}_\infty B(0)^T \equiv S^* H(\psi) S$ is similar. □

We have proved that the sequence of renormalized fibred $\mathbb{Z}^2$-action $R^n(\Phi)$ has a subsequence $R^{n_k}(\Phi)$ converging to the constant

$$\Phi_\infty := \{(1, B(0)^T S^* H(\varphi) S B(0)), (\alpha_\infty, B(0)^T S^* H(\psi) S B(0))\}.$$

Note that both $R^{n_k}(\Phi)$ and its limit Φ_∞ are not normalized. In order to pass the $\mathbb{Z}^2$-actions to cocycles we need to normalize Φ_∞ and $R^{n_k}(\Phi)$.

Lemma 4.5. *One can find $\delta_1 > 0$ and a sequence of $P_k \in C^\omega(\mathbb{R}, SO(N, \mathbb{R}))$, such that all P_k have analytic extensions in the domain*

$$\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1,\ |\mathrm{Im}\, z| < \delta_1\}$$

converging uniformly to I as $k \to \infty$, such that

$$Ad(P_k Q B(0)).R^{n_k}(\Phi) = \{(1, I), (\alpha_{n_k}, W_k)\},$$

where $Q(x) = S^* H(-x\varphi) S$, $W_k \in C^\omega(\mathbb{R}, SO(2m, \mathbb{R}))$ are all $\mathbb{Z}$-periodic and have analytic extensions in the domain $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta_1\}$ converging uniformly to a constant.

Proof. It is obvious that

$$Ad(QB(0)).\Phi_\infty = \{(1, I), (\alpha_\infty, S^* H(\psi - \alpha_\infty \varphi) S)\}.$$

Thus in the domain $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta\}$ (for δ in lemma 4.4) the extensions of $\Psi_k(1, 0)$ and $\Psi_k(0, 1)$ converge uniformly to I and $S^* H(\psi - \alpha_\infty \varphi) S$, respectively, where $\Psi_k = Ad(QB(0)).R^{n_k}(\Phi)$.

By lemma 3.1, there exist some $\delta_1 \in (0, \delta)$ and a sequence of $P_k \in C^\omega(\mathbb{R}, SO(N, \mathbb{R}))$, converging uniformly to I in the domain

$$\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1,\ |\mathrm{Im}\, z| < \delta_1\},$$

such that $\Psi_k' := Ad(P_k).\Psi_k$ is a sequence of normalized $\mathbb{Z}^2$-action $\{(1, I), (\alpha_{n_k}, W_k)\}$. For W_k all $\mathbb{Z}$-periodic, they admit analytic extensions in the domain $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta_1\}$ converging uniformly to $S^* H(\psi - \alpha_\infty \varphi) S$ as $k \to \infty$. □

We have proved that the sequence (α_{n_k}, W_k) of cocycles analytically converges to a constant. To this stage, we have reduced the reducibility of (α, A) to the reducibility of (α_{n_k}, W_k) which are arbitrarily close to a constant for large k. In the following, we further prove that (α_{n_k}, W_k) can be conjugated to (α_{n_k}, C) by a sequence of measurable conjugation. We shall also check an assumption on such a sequence of measurable conjugacy which is assumed in proposition 1.1.

Lemma 4.6. *There exist sequences* $E_k, F_k \in C^\omega(\mathbb{R}, SO(2m, \mathbb{R}))$ *such that* $E_k B(\beta_{n_k-1}\cdot)F_k$ *are all* $\mathbb{Z}$*-periodic satisfying*

$$\liminf_{k\to\infty}[E_k B(\beta_{n_k-1}\cdot)F_k] > \tfrac{1}{2},$$

$$Ad(E_k B(\beta_{n_k-1}\cdot)F_k).(\alpha_{n_k}, W_k) = (\alpha_{n_k}, S^* H((-1)^{n_k}\beta_{n_k-1}^{-1}\rho)S).$$

Proof. Note that $\beta_{n_k-1} = (q_{n_k} + \alpha_{n_k} q_{n_k-1})^{-1}$, one can obtain

$$Ad(C_k).\{(1, I), (\alpha_{n_k}, W_k)\} = \{(1, I), (\alpha_{n_k}, S^* H((-1)^{n_k}\beta_{n_k-1}^{-1}\rho)S)\},$$

where $C_k = S^* H(-\lambda_k \rho\,\cdot)SB(\beta_{n_k-1}\cdot)B(0)^T Q^T P_k^T$. Thus it is obvious that all C_k are $\mathbb{Z}$-periodic. Let $D_k(x) = S^* H(-L_k x)SC_k(x)$, then for all $x \in \mathbb{R}/\mathbb{Z}$ we have the estimates

$$||D_k(x) - I|| \leqslant ||H((\varphi - \lambda_k\rho - L_k)x) - I|| + ||B(\beta_{n_k-1}x)B(0)^T - I|| + ||P_k(x)^T - I||.$$

We know that $\lambda_k\rho + L_k \to \varphi$ and $P_k(x) \to I$ $(x \in \mathbb{T})$ as $k \to \infty$. By lemma 2.2

$$\lim_{k\to\infty}\int_{[0,1]} ||B(\beta_{n_k-1}x)B(0)^T - I||\,\mathrm{d}x = 0.$$

Thus we have

$$\lim_{k\to\infty}\int_{[0,1]} ||C_k(x) - SH(L_k x)S^*||\,\mathrm{d}x = \lim_{k\to\infty}\int_{[0,1]} ||D_k(x) - I||\,\mathrm{d}x = 0.$$

For $[SH(L_k x)S^*] = 1$, by corollary 2.1, $[C_k] > 1/2$ as k is large enough. $\square$

Proof of theorem 1.1. By lemma 4.6, lemma 4.2 and proposition 1.1, for some sufficiently large k_0, there exists $R \in C^\omega(\mathbb{R}, SO(N, \mathbb{R}))$ satisfying $R(x) = E_{k_0}(x)B(\beta_{n_{k_0-1}}x)F_{k_0}(x)$ for almost all $x \in \mathbb{R}$. Let $B_* = E_{k_0}(\beta_{n_{k_0-1}}^{-1}\cdot)^T R(\beta_{n_{k_0-1}}^{-1}\cdot)F_{k_0}(\beta_{n_{k_0-1}}^{-1}\cdot)^T$, it is analytic and satisfies $B_*(x) = B(x)$ for almost all $x \in \mathbb{R}$. Thus $B_*(x+\alpha)A(x)B_*(x)^{-1} = C$ for almost all $x \in \mathbb{R}$ as $B(\cdot+\alpha)AB^{-1} \equiv C$ by the assumptions. It follows that $B_*(x+\alpha)A(x)B_*(x)^{-1} = C$ for all $x \in \mathbb{R}$. Moreover B_* is $\mathbb{Z}$-periodic for B $\mathbb{Z}$-periodic. Such an analytic B_* conjugates (α, A) to the constant (α, C).

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