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Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems

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Abstract In this paper, we prove that a quasi-periodic linear differential equation in $sl(2,\mathbb{R})$ with two frequencies $(\alpha, 1)$ is almost reducible provided that the coefficients are analytic and close to a constant. In the case that α is Diophantine we get the non-perturbative reducibility. We also obtain the reducibility and the rotations reducibility for an arbitrary irrational α under some assumption on the rotation number and give some applications for Schrödinger operators. Our proof is a generalized KAM type iteration adapted to all irrational α.

1 Introduction

Let $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. A quasi-periodic (q-p for short) linear system on $\mathbb{T}^d$ with coefficients in $sl(2,\mathbb{R})$ is a q-p linear skew-product ODE defined as

$$\begin{cases} \dot{x} = A(\theta)x, \\ \dot{\theta} = \omega, \end{cases} \tag{1.1}$$

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where $x \in \mathbb{R}^2$, $\theta \in \mathbb{T}^d$, $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$ ($\omega_1, \dots, \omega_d$ are usually called the frequencies) and $A : \mathbb{T}^d \to sl(2, \mathbb{R})$ is a C^r ($r = 0, 1, \dots, \infty$) or analytic map.

A typical example of q-p linear systems comes from the (continuous-time) q-p Schrödinger operator, which is defined on $L^2(\mathbb{R})$ as

$$(\mathcal{L}y)(t) = -y''(t) + q(\theta + \omega t)y(t), \tag{1.2}$$

where $q : \mathbb{T}^d \to \mathbb{R}$ is called the potential and $\theta \in \mathbb{T}^d$ is called the phase. The spectrum set of $\mathcal{L}$ is known to be independent of the phase when ω is rational independent. It is closely related to the dynamics of Schrödinger equations

$$(\mathcal{L}y)(t) = -y''(t) + q(\theta + \omega t)y(t) = Ey(t) \tag{1.3}$$

($E \in \mathbb{R}$ is called the energy), or equivalently the dynamics of one parameter family of linear systems

$$\begin{cases} \dot{x} = V_{E,q}(\theta)x, \\ \dot{\theta} = \omega, \end{cases} \tag{1.4}$$

where

$$V_{E,q}(\theta) = \begin{pmatrix} 0 & 1 \\ q(\theta) - E & 0 \end{pmatrix} \in sl(2, \mathbb{R}).$$

System (1.1) is said to be *reducible*, if there exists an $SL(2, \mathbb{R})$-valued function B defined on $2\mathbb{T}^d = \mathbb{R}^d / 4\pi\mathbb{Z}^d$ such that the change of variables $x \mapsto B(\theta)x$ transforms system (1.1) into a constant system, i.e., a linear system with constant coefficient (or we say that B conjugates system (1.1) to a constant system and B is called the conjugation map). If B is C^r (or analytic), we say that system (1.1) is C^r (or analytically) reducible. When applying to the Schrödinger operator, the reducibility in the spectrum often implies the existence of the Bloch waves and that of the absolutely continuous spectrum.

Due to its importance in the theory of dynamical systems and the spectrum theory of q-p Schrödinger operators, the reducibility problem of q-p linear systems has received much attention. For $d = 1$, i.e., the periodic case, the classical Floquet theory shows that there always exists a periodic change of variables so that the transformed system is a constant system. For $d > 1$, i.e., q-p case, system (1.1) is not always reducible (see for example [15]).

Perturbative reducibility The reducibility of q-p linear system (1.4) and its applications in the spectrum theory were initiated by Dinaburg and Sinai [14] who proved that (1.4) is reducible for a positive Lebesgue measure set of

$E > E^*(q, \omega)$, where ω is fixed and satisfies the Diophantine conditions:

$$|\langle k, \omega\rangle| > \frac{\gamma^{-1}}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d, \tag{1.5}$$

with fixed $\gamma, \tau > 1$. Here (γ, τ) are called the Diophantine constants of ω. Denote by $DC(\gamma, \tau)$ the set of all (γ, τ)-type Diophantine ω and $DC = \bigcup_{\gamma,\tau>1} DC(\gamma, \tau)$ (DC is of full measure). Dinaburg and Sinai's reducibility result implies the existence of some absolutely continuous spectrum of the Schrödinger operator (1.2). The result was generalized by Rüssmann [29] to the case that ω satisfies the Bruno condition, an arithmetic condition on ω which is slightly weaker than Diophantine. The reducibility of q-p linear systems with coefficients in $gl(n, \mathbb{R})$ was considered by Jorba and Simó [20].

Since Dinaburg and Sinai [14] there have been several breakthroughs on this problem. The first one was the work of Eliasson [15] where a *full measure* reducibility result for q-p linear Schrödinger equation was proved. More precisely, Eliasson [15] proved that (1.4) is reducible for almost all $E > E^*(q, \omega)$ in the Lebesgue measure sense, when ω satisfies fixed Diophantine condition. Based on his reducibility result, Eliasson also proved that, when the potential is analytic and small, the spectrum of (1.2) is purely absolutely continuous for all phase θ. This suggests an important role for the reducibility of q-p linear systems in the study of the spectrum of q-p Schrödinger operators. Eliasson's proof is based on an earlier KAM method (that goes back to Dinaburg and Sinai [14]) and a crucial resonance-cancelation technique which was introduced by Moser and Pöschel [27]. We remark that the full-measure reducibility result holds for more general system (1.1) with analytic A close to some constant matrix. Krikorian [21–23] generalized the full measure reducibility result to linear systems with coefficients in Lie algebra of compact semi-simple Lie group. Her-You [17] and Chavaudret [11] established the full measure reducibility with coefficients in other groups. We also remark that all the above mentioned results are *perturbative*, i.e., E^* (or the closeness of A to some constant matrix) depends on the frequencies ω through the Diophantine constants (γ, τ). Eliasson's perturbative reducibility result is optimal when $d > 2$, as shown by a counter example of Bourgain [9]. However, when $d = 2$, one could expect more. In the following we shall restrict our attention to this case.

Non-perburbative reducibility The non-perburbative reducibility means that the smallness of the perturbation does not depend on the Diophantine constants (γ, τ) of ω. The non-perturbative reducibility has been proved for Schrödinger cocycles

$$(\theta, x) \mapsto \left(\theta + \alpha, \begin{pmatrix} E - v(n\alpha + \theta) & -1 \\ 1 & 0 \end{pmatrix} x\right), \quad (\theta, x) \in \mathbb{T}^1 \times \mathbb{R}^2, \tag{1.6}$$

with one frequency $\alpha \in \mathbb{R}$. The proof, which is an indirect argument, comes from the study of the discrete-time Schrödinger operator

$$(Hy)_n = y_{n+1} + y_{n-1} + v(\theta + n\alpha)y_n, \quad (y_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}).$$

More precisely, Puig [28] proved a non-perturbative extension of Eliasson's result by using the Aubry duality [1] and the Anderson localization results, see e.g., Bourgain and Jitomirskaya [10]. A more precise extension of the Eliasson's perturbative reducibility result was given by Avila and Jitomirskaya [4] by further developing techniques in localization. In [4], some beautiful results on the continuity of the spectrum of the discrete-time q-p Schrödinger operators are deduced from the non-perturbative reducibility. The Aubry duality and the localization only work for the discrete-time Schrödinger operators, they do not seem to apply to more general $SL(2,\mathbb{R})$ cocycles or continuous-time cases.

In this paper, we give an analogous non-perturbative result of [28] and [4] for q-p system (1.1) with $d = 2$, which generalizes Eliasson's reducibility result in a non-perturbative neighborhood (a neighborhood which is independent of the Diophantine constants of ω) of constant systems.

Before stating the results, let us introduce the rotation number. Assume that $\Phi(\theta, t)$ is the basic matrix solution of system (1.1), we define the rotation number as

$$\rho = \lim_{t\to+\infty} \frac{arg(\Phi(\theta,t)x)}{t},$$

where $0 \neq x \in \mathbb{R}^2$ and arg denotes the angle. ρ is well-defined and is independent of θ and x [18]. ρ is said to be rational with respect to (w.r.t. for short) ω if $\rho = \frac{1}{2}\langle k_0, \omega\rangle$ for some $k_0 \in \mathbb{Z}^2$, and to be Diophantine w.r.t. ω, with constants $\gamma, \tau > 1$, if

$$|\langle k, \omega\rangle - 2\rho| \geq \frac{\gamma^{-1}}{|k|^\tau}, \quad k \in \mathbb{Z}^2.$$

We denote by $DC_\omega(\gamma, \tau)$ the set of all such ρ. It is well known that the union $DC_\omega = \bigcup_{\gamma,\tau>1} DC_\omega(\gamma, \tau)$ is a full measure subset of $\mathbb{R}$.

We now mainly focus our attention on the case $d = 2$ and the system of the form

$$\begin{cases} \dot{x} = (A + F(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{1.7}$$

where $A \in sl(2,\mathbb{R})$, $F : \mathbb{T}^2 \to sl(2,\mathbb{R})$ is analytic and small. We will say that $F \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$ if $F : \mathbb{T}^2 \to sl(2,\mathbb{R})$ admits an analytic extension in the complex neighborhood $|\mathrm{Im}\,\theta| < h$ of $\mathbb{T}^2$, where $|\mathrm{Im}\,\theta| = |\mathrm{Im}\,\theta_1| + |\mathrm{Im}\,\theta_2|$. In this paper, we mainly consider the case $\omega = (\alpha, 1)$ with irrational $\alpha \in (0, 1)$.

Theorem 1.1 (Non-perturbative reducibility) *Let $h > 0$ and $\omega = (\alpha, 1)$ with irrational $\alpha \in (0, 1)$. Let $A \in sl(2, \mathbb{R})$ and $F \in C_h^{\omega}(\mathbb{T}^2, sl(2, \mathbb{R}))$. Then there exists $\delta = \delta(A, h) > 0$ depending on A, h but not on α, such that system (1.7) is analytic reducible if $\alpha \in DC$, the rotation number of (1.7) is Diophantine or rational w.r.t. ω and*

$$\sup_{|\mathrm{Im}\,\theta|<h} |F(\theta)| < \delta$$

($|\cdot|$ *also stands for the usual matrix norm in this paper*).

Remark 1.1 All other results in [15] remain true in a non-perturbative neighborhood of A. For example, when $\alpha \in DC$, $\sup_{|\mathrm{Im}\,\theta|<h} |F(\theta)| < \delta$ and the rotation number of (1.7) is neither Diophantine nor rational w.r.t. ω, for any given $\phi \in \mathbb{T}^2$, we have

$$\liminf_{t\to\pm\infty} |\Phi(\phi, t) - \Phi(\phi, 0)| < \frac{1}{2}|\Phi(\phi, 0)| \quad \text{and} \quad \lim_{t\to\pm\infty} \frac{|\Phi(\phi, t)|}{t} = 0,$$

where $\Phi(\phi, \cdot)$ is the basic matrix solution of system (1.7) with $\theta(0) = \phi$.

Almost reducibility The concept of reducibility is too strong in the sense that system (1.1) might not be reducible even if ω is Diophantine and A is analytic and arbitrarily close to constants. In the following we recall the concept of almost reducibility, which is weaker than the reducibility but already has enormous implications both in the theory of dynamical systems and in the spectrum theory. Roughly speaking, a q-p linear system is said to be almost reducible if, by a sequence of q-p changes of variables, the transformed q-p linear systems are closer and closer to constant systems (the precise definition will be given later).

The almost reducibility in the modern sense was first considered by Eliasson who proved that all q-p systems are almost reducible provided that ω is Diophantine and the system is close to a constant (closeness depends on the Diophantine constants of ω).[1] A recent progress was made by Avila and Jitomirskaya [4] who gave a non-perturbative (α is still assumed to be Diophantine, but the smallness does not depend on the Diophantine constants) almost reducibility result for Schrödinger cocycles with a single frequency $\alpha \in DC$ and small potentials. Moreover, in [4], the authors used their almost reducibility result to obtain the non-perburbative reducibility result (which we have mentioned above).

[1] We remark that the convergence to constants obtained by Eliasson occurs on analyticity strips of width going to zero. In [12], Chavaudret proved that one can get convergence on strips of fixed width. Similar result for discrete-time case was also obtained earlier by Avila and Jitomirskaya [4] in some cases by different methods.

An interesting question is if the non-perturbative almost reducibility holds true for the continuous-time case. And an even more interesting problem is if almost reducibility holds for all irrational α. Note that the Avila-Jitomirskaya's non-perturbative result essentially holds for Diophantine α [4] (more precisely $\beta(\alpha)=0$ which is defined in (1.9)). Moreover the proof in [4] is an indirect argument and relies on Aubry duality and the localization neither of which works for the continuous-time case, so giving a direct proof is also an interesting question. In the following we shall give an almost reducibility result for continuous-time systems with *arbitrary* irrational α by a direct argument.

We now give the precise definition of Eliasson's almost reducibility. We will say that system (1.1) is *almost reducible* if there exist a sequence of positive h_n (maybe decrease to zero) and a sequence of $B_n : 2\mathbb{T}^2 \to SL(2,\mathbb{R})$ which admit analytic extensions in the complex neighborhood $|\mathrm{Im}\,\theta| < h_n$ of $2\mathbb{T}^2$, such that system (1.1) can be transformed by the change of variables $x \mapsto B_n(\theta)x$ into

$$\begin{cases} \dot{x} = (A_n + F_n(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{1.8}$$

i.e., the conjugation map B_n conjugate system (1.1) to system (1.8). Moreover, $A_n \in sl(2,\mathbb{R})$ is bounded and $F_n \in C^{\omega}_{h_n}(\mathbb{T}^2, sl(2,\mathbb{R}))$ satisfies

$$\lim_{n\to\infty} \frac{1}{h_n^{\chi}} \sup_{|\mathrm{Im}\,\theta|<h_n} |F_n(\theta)| = 0,$$

for any $\chi \geq 1$.

Theorem 1.2 (Almost reducibility) *Let $h>0$ and $\omega=(\alpha,1)$ with irrational $\alpha \in (0,1)$. Let $A \in sl(2,\mathbb{R})$ and $F \in C^{\omega}_h(\mathbb{T}^2, sl(2,\mathbb{R}))$. Then system (1.7) is almost reducible provided that*

$$\sup_{|\mathrm{Im}\,\theta|<h} |F(\theta)| < \delta,$$

where $\delta = \delta(A,h) > 0$ depending on A, h but not on α.

Remark 1.2 The precise dependence of δ on A and h in Theorems 1.1 and 1.2 will be given explicitly in the proof. We emphasize that the smallness does not depend on A if it is in the real normal forms $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ or $\begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$.

Rotations reducibility Now we investigate the Liouvillean case. When the frequency α is Liouvillean, one should not expect the reducibility without further assumptions. However, one could expect the rotations reducibility: system (1.1) is said to be L^2, $C^r (r=0,1,\ldots,\infty)$ or analytically *rotations* *reducible*, if there exists a L^2, C^r or analytic q-p transformation $y = B(\theta)x$, such that system (1.1) is transformed into a rotation system, i.e., a linear system with $so(2,\mathbb{R})$-valued coefficients (or we say that the conjugation map B conjugates system (1.1) to a rotation system).

Theorem 1.3 (Rotations reducibility) *Suppose that all the assumptions of Theorem 1.2 hold. Then, for the same δ as in Theorem 1.2, system (1.7) is analytically rotations reducible, if the rotation number of system (1.7) is Diophantine w.r.t. ω and*

$$\sup_{|\mathrm{Im}\,\theta|<h} |F(\theta)| < \delta.$$

Remark 1.3 Theorem 1.3 says that almost surely (i.e., for almost all rotation numbers) systems in a non-perturbative neighborhood of a constant system is rotations reducible. Similar result for cocycles has been obtained by Fayad and Krikorian [16], Avila, Fayad and Krikorian [6].

Remark 1.4 Rotations reducibility implies that all solutions of the system are bounded. Thus one should not expect the rotations reducibility for all rotation numbers since in general the solutions of a system with Liouvillean rotation number are unbounded [15]. In this sense, Theorem 1.3 is optimal.

Remark 1.5 For any fixed irrational $\alpha \in (0,1)$, Theorem 1.3 implies that the rotations reducible systems are dense in a non-perturbative neighborhood of a non-hyperbolic constant system, and thus the reducible systems are dense in a non-perturbative neighborhood of any constant system. One can refer to Avila [2], Fayad and Krikorian [16] for related results of cocycles.

Reducibility for Liouvillean α Theorem 1.3 has consequences on the reducibility when α is Liouvillean.

Let $\alpha \in (0,1)$ be irrational and $\frac{p_n}{q_n}$ be its continued fractional approximation. Define

$$\beta(\alpha) = \limsup \frac{\ln q_{n+1}}{q_n}. \tag{1.9}$$

$0 \le \beta \le +\infty$ measures how Liouvillean α is. The following two theorems are reducibility results for Liouvillean α.

Theorem 1.4 (Reducibility for $\beta = 0$) *Suppose that all the assumptions of Theorem 1.2 hold, and, in addition, suppose that $\beta(\alpha) = 0$. Then, for the same δ as in Theorem 1.2, system (1.7) is reducible, if its rotation number is Diophantine w.r.t. ω and*

$$\sup_{|\mathrm{Im}\,\theta|<h} |F(\theta)| < \delta.$$

Remark 1.6 The assumption that $\beta(\alpha)=0$ is weaker than that α is Diophantine or Bruno. Thus, when the rotation number is Diophantine w.r.t. ω, Theorem 1.4 is stronger than the corresponding result in Theorem 1.1.

Theorem 1.4 is no longer true in the case that $\beta(\alpha)>0$. However, we have the following reducibility result for a majority (more precisely positive measure set) of rotation numbers.

Theorem 1.5 (Reducibility for $\beta(\alpha)>0$) *Under the assumptions of Theorem 1.2, assume that $h>3\beta(\alpha)$ and that the rotation number is in $DC_\omega(\gamma,\tau)$. Then there exists $\delta=\delta(h,A,h-3\beta,\gamma,\tau)>0$, such that system (1.7) is reducible if*

$$\sup_{|\mathrm{Im}\,\theta|<h}|F(\theta)|<\delta.$$

Remark 1.7 This result is a generalization of Dianburg-Sinai's result to the Liouvillean case. The proof of this theorem applies essentially unchanged to the discrete case, thus same result holds true for $SL(2,\mathbb{R})$-cocycles.

Remark 1.8 The assumption that $h>3\beta(\alpha)$ is not optimal. One could expect the same result by assuming that $h>\beta(\alpha)$. From the proof, one sees that this is indeed the case if we assume that the rotation number $\rho\in[-1,1]$.

Remark 1.9 As a matter of fact, in Theorem 1.5, the smallness does not depend on A if it is in the real normal forms $\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}$ or $\begin{pmatrix}0 & \rho\\ -\rho & 0\end{pmatrix}$.

Applications to the continuous time Schrödinger operators Theorems 1.1–1.5 can be applied to the q-p Schrodinger equations. The following is the same result as that in [15] in the non-perturbative region.

Theorem 1.6 *Let $h>0$ and $\omega=(\alpha,1)$ with irrational $\alpha\in(0,1)$. Assume that $q:\mathbb{T}^2\to\mathbb{R}$ admit an analytic extension in the complex neighborhood $|\mathrm{Im}\,\theta|<h$ of $\mathbb{T}^2$. For any $\delta>0$, we define C_δ as*

$$C_\delta(s)=\begin{cases}-\infty, & \text{as } 0\le s\le\delta,\\ (s/\delta)^2, & \text{as } s>\delta.\end{cases}$$

Then there exists $\delta=\delta(h)>0$ such that whenever $E\ge E_*(h,q)\triangleq C_{\delta(h)}(\sup_{|\mathrm{Im}\,\theta|<h}|q(\theta)|)$, the conclusions of Theorems 1.1–1.4 are all true for (1.4).

Remark 1.10 Suppose that $0<\beta(\alpha)<h/3$ and the rotation number of (1.4) is in $DC_\omega(\gamma,\tau)$ for some $\gamma,\tau>1$. Then there exists a $\delta_1=\delta_1(h,3\beta-$

$h, \gamma, \tau) > 0$, such that if $E \geq C_{\delta_1}(\sup_{|\mathrm{Im}\,\theta|<h} |q(\theta)|)$ the result of Theorem 1.5 is true for (1.4).

The following result shows that when the energy E is sufficiently large or the potential q is analytically small, the Schrödinger system is either uniformly hyperbolic or has zero Lyapunov exponent.

Corollary 1.1 *The Lyapunov exponent of the Schrödinger system is zero in the big end of spectrum. If the potential $q(\theta)$ is small (the smallness does not depend on α), the Lyapunov exponent of Schrödinger system (1.4) is always zero in the spectrum.*

Remark 1.11 For discrete-time Schrödinger operators, the above result is a consequence of the non-perturbative reducibility [4, 28] and the continuity of Lyapunov component [10].

One can use the well known Moser-Pöschel argument, which has been used in a variety of contexts, such as [15, 26–28] and etc., to obtain the generic cantor spectrum for q in a non-perturbative neighborhood of zero.

Corollary 1.2 *For generic small real analytic potentials the spectrum is a Cantor set.*

Remark 1.12 Invoking the recent work of Ben Hadj Amor [7], one can also prove the $\frac{1}{2}$-Hölder continuous of the IDS in the non-perturbative region when α is Diophantine.

Global reducibility In some special cases, such as q-p almost Mathieu operators with one frequency, non-perturbative reducibility results allow rather big (even optimal) perturbations. However, the smallness assumption on general perturbations is necessary for proofs of all the above mentioned works. If the system is far from a constant, the famous Kotani theory [25, 30] implies that both Schrödinger system (1.4) and Schrödinger cocycle (1.6) are L^2 rotations reducible for almost all $E \in \{E \in R : \lambda(E) = 0\}$ where $\lambda(E)$ is the Lyapunov exponent. Using a renormalization type technique, Avila and Krikorian [5] proved that the Schrödinger cocycle with analytic potential is in fact analytically reducible for almost all $E \in \{E \in R : \lambda(E) = 0\}$ provided that α is recurrent Diophantine (this is a slightly stronger condition than being Diophantine, see [5]). In a recent paper, Avila [3] systematically studied q-p cocycles with one frequency. This represents a major breakthrough in the study of reducibility. Global reducibility of uniformly hyperbolic case (corresponding to that E is in the resolvent set for Schrödinger case) is relatively easier, see the early works of Bogoljubov et al. [8], Coppel [13], Johnson and Sell

[19]. The global theory for continuous-time case is an interesting problem, but so far little is known.

On the proof and generalization Our proof is based on KAM. It has been a prevalent opinion that KAM is incompatible with non-perturbative results since it depends heavily on the arithmetical condition of α (through the Diophantine constants). We shall show in this paper that it is not the case. In fact, we shall utilize a kind of KAM like iteration based on fractional expansion of α to prove our theorems. We shall see that the KAM method does work well in the non-perturbative neighborhood if one adds some new ingredients to the traditional KAM machinery (more precisely adding Floquet theory to Eliasson's KAM machinery [15]).

Theorem 1.2 and 1.3 are the main results of the paper. Let us sketch their proofs. Theorem 1.2 is proved by iteration. Each iteration step contains four sub-steps: (1) Removing all non-resonant terms (the corresponding denominators are not very small) in the Fourier expansion of the perturbation; (2) If the rotation number is resonant w.r.t. the frequency, we invoke the resonance-cancelation technique of Moser and Pöschel, as done by Eliasson in [15]; (3) After the previous two steps, all the remaining resonant terms are "in a line", i.e., the corresponding $k \in \mathbb{Z}^2$ in the Fourier expansion are all of the form $\{l(q, -p) : l \in \mathbb{Z}\}$ for some fixed $p, q \in \mathbb{Z}$ defined by the fractional expansion of α; (4) At the end, the special structure of the resonances allows us to apply a quantitative Floquet theory to cancel it, and thus finish a step of iteration. In the above four sub-steps, the analysis of the structure of the resonant terms in the step 3 is the most important (see Corollary 4.1).

The proof of Theorem 1.3 contains two key ingredients: (1) The small divisor related to the off-diagonals terms, up to a large truncation of the Fourier expansion of the perturbation (see Lemma 6.1), is controllable. As a result, one can always remove all off-diagonals θ-dependent terms at each iteration step. (2) Finding a way to solve the homological equations which is scalar but θ-dependent.

The differences between our work and previous ones are the following: (i) Our proof is direct and is developed from the traditional KAM method; (ii) Our method has a potential generalization to q-p systems in $gl(n, \mathbb{R})$ or other groups (with two frequencies); (iii) Our method adapts to all irrational α, no matter whether α is Diophantine or Liouvillean.

We organize this paper as follows. In Sect. 2, we prove Theorem 1.1, assuming Theorem 1.2 and using Eliasson's perturbative result. Sections 3 and 4 contain some technical lemmas needed for the proof of Theorem 1.2. In Sect. 5, we give a proof of Theorem 1.2 by employing the KAM like iteration. Theorem 1.3, and consequently, Theorems 1.4–1.5, are proved in Sect. 6. The appendices include proofs of some of the lemmas that are used in the main part of the paper.

2 Proof of Theorem 1.1

Theorem 1.1 and the claim in Remark 1.1 can be obtained from Theorem 1.2 and its proof, combined with the following perturbative result of Eliasson [15].

Proposition 2.1 (H. Eliasson [15]) *Let $h > 0$. In system (1.7), we assume that $\alpha \in DC(\gamma, \tau)$, $A \in sl(2, \mathbb{R})$ is of normal forms $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ or $\begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$ and $F \in C_h^{\omega}(\mathbb{T}^2, sl(2, \mathbb{R}))$. Then there exist constants $C = C(\gamma, \tau) > 0$ and $\chi = \chi(\gamma, \tau) > 1$, such that if*

$$\sup_{|\mathrm{Im}\,\theta| < h} |F(\theta)| < C h^{\chi},$$

we have the following:

(a) *If furthermore the rotation number is Diophantine or rational w.r.t. ω, then system (1.7) is analytically reducible.*
(b) *Otherwise, if the rotation number is neither Diophantine nor rational w.r.t. ω, then for any given $\phi \in \mathbb{T}^2$, we have*

$$\liminf_{t \to \pm\infty} |\Phi(\phi, t) - \Phi(\phi, 0)| < |\Phi(\phi, 0)| \quad \textit{and} \quad \lim_{t \to \pm\infty} \frac{|\Phi(\phi, t)|}{t} = 0,$$

where $\Phi(\phi, \cdot)$ is the basic matrix solution of system (1.7) with $\theta(0) = \phi$.

Proof By Theorem 1.2 and the proof of it, there exist a sequence $h_n > 0$, a sequence of bounded $A_n \in sl(2, \mathbb{R})$ in normal forms $\begin{pmatrix} \lambda_n & 0 \\ 0 & -\lambda_n \end{pmatrix}$ or $\begin{pmatrix} 0 & \rho_n \\ -\rho_n & 0 \end{pmatrix}$, and a sequence of $F_n : \mathbb{T}^2 \to sl(2, \mathbb{R})$ analytic in $|\mathrm{Im}\,\theta| < h_n$ satisfying

$$\lim_{n \to \infty} \frac{1}{h_n^{\chi}} \sup_{|\mathrm{Im}\,\theta| < h_n} |F_n(\theta)| = 0,$$

for any $\chi \geq 1$, such that system (1.7) is conjugated to

$$\begin{cases} \dot{x} = (A_n + F_n(\theta))x, \\ \dot{\theta} = \omega. \end{cases} \tag{2.1}$$

Then, for any fixed χ, $\sup_{|\mathrm{Im}\,\theta| < h_n} |F_n(\theta)| < C(h_n)^{\chi}$ as n large enough. Note that, by Lemma 9.1, the rotation number of (2.1) is still Diophantine or rational w.r.t. ω provided that the rotation number of system (1.7) is so. The proof of Theorem 1.1 and the claim in Remark 1.1 is then completed by applying Proposition 2.1 to system (2.1). □

3 A basic lemma

The following four sections are devoted to the proofs of Theorems 1.2–1.5. It is well known that KAM iteration is a procedure of cancelation of lower order non-resonant terms. In this section, we will prove a basic lemma which will play an important role in the proofs of Theorems 1.2–1.5.

Let M_2 denote the space of all 2×2 matrices and I denote the identity one. For any $h>0$ and matrix function $F:\mathbb{T}^2\to M_2$ with the Fourier expansion

$$F(\theta)=\sum_{k\in\mathbb{Z}^2}\widehat{F}(k)e^{i\langle k,\theta\rangle},$$

we introduce the definition

$$|F|_h\triangleq\sum_{k\in\mathbb{Z}^2}|\widehat{F}(k)|e^{|k|h},$$

where $|k|=|k_1|+|k_2|$ for $k=(k_1,k_2)$. Denote by $\mathfrak{B}_h$ the set of all analytic $F:\mathbb{T}^2\to sl(2,\mathbb{R})$ with $|F|_h<+\infty$, which is a Banach algebra under norm $|\cdot|_h$. The union $\mathfrak{B}=\bigcup_{h>0}\mathfrak{B}_h$ includes all analytic $sl(2,\mathbb{R})$-valued functions on $\mathbb{T}^2$.

For any $N>0$, we define the truncating operators $\mathcal{T}_N$ on $\mathfrak{B}$ as

$$(\mathcal{T}_NF)(\theta)=\sum_{k\in\mathbb{Z}^2,|k|<N}\widehat{F}(k)e^{i\langle k,\theta\rangle}$$

and $\mathcal{R}_N$ as

$$(\mathcal{R}_NF)(\theta)=\sum_{k\in\mathbb{Z}^2,|k|\geq N}\widehat{F}(k)e^{i\langle k,\theta\rangle}.$$

It is obvious that $\mathcal{T}_NF+\mathcal{R}_NF=F$ and for any $h>0$, $\mathcal{T}_N\mathfrak{B}_h,\mathcal{R}_N\mathfrak{B}_h\subseteq\mathfrak{B}_h$. For any $0<h_+<h$, by a simple computation one can check that

$$|\mathcal{R}_NF|_{h_+}\leq\frac{36}{\min\{1,(h-h_+)\}^2}|F|_he^{-\frac{N(h-h_+)}{2}}.\tag{3.1}$$

Remark 3.1 For any $F\in\mathfrak{B}_h$, $F\in C_h^{\omega}(\mathbb{T}^2,sl(2,\mathbb{R}))$ and

$$\sup_{|\mathrm{Im}\,\theta|<h}|F(\theta)|\leq|F|_h.$$

In general the converse is not true. However, for any $F\in C_h^{\omega}(\mathbb{T}^2,sl(2,\mathbb{R}))$, we have $F\in\widetilde{\mathfrak{B}}_{h_+}$ for any $0<h_+<h$ with the estimate

$$|F|_{h_+}\leq\frac{36}{\min\{1,(h-h_+)\}^2}\sup_{|\mathrm{Im}\,\theta|<h}|F(\theta)|.\tag{3.2}$$

To find a conjugation map that is close to the identity I and transforms the system

$$\begin{cases} \dot{x} = (A + F(\theta))x, \\ \dot{\theta} = \omega \end{cases}$$

into a constant system, we need to find a constant matrix $\widetilde{A}$ and $sl(2,\mathbb{R})$-valued function $Y(\theta)$ which solves the cohomological equation

$$\partial_\omega e^Y = (A + F)e^Y - e^Y \widetilde{A}. \tag{3.3}$$

For perturbative problems, the solution of (3.3) will be inductively constructed by the Newtonian iteration. For this purpose, we firstly solve the linearized equation of (3.3)

$$\partial_\omega Y - [A, Y] = F \tag{3.4}$$

(Lie bracket $[\cdot,\cdot]$ is defined as $[X_1, X_2] = X_1X_2 - X_2X_1$), which is called the linearized co-homological equation. A solution Y of (3.4) together with a suitable bound is necessary to make the iteration work. To control the norm of the inverse of the linear operator

$$\partial_\omega \cdot -[A, \cdot], \tag{3.5}$$

one encounters the small divisor problem.

One step in our proofs is to remove all the non-resonant components in the Fourier expansion of F. For any given $h > 0$, $\omega \in \mathbb{R}^2$ and $A \in sl(2,\mathbb{R})$, we decompose $\mathfrak{B}_h = \mathfrak{B}_h^{(nre)} \oplus \mathfrak{B}_h^{(re)}$ (the decomposition depends on A, ω, η) in such a way that for any $Y \in \mathfrak{B}_h^{(nre)}$

$$\partial_\omega Y, [A, Y] \in \mathfrak{B}_h^{(nre)}, \quad |\partial_\omega Y - [A, Y]|_h \geq \eta |Y|_h. \tag{3.6}$$

Let $\mathbb{P}_{nre}$ ($\mathbb{P}_{re}$) be the standard projection from $\mathfrak{B}_h$ onto $\mathfrak{B}_h^{(nre)}$ ($\mathfrak{B}_h^{(re)}$). We call $\mathfrak{B}_h^{(nre)}$ ($\mathfrak{B}_h^{(nre)}$) the η-nonresonant (η-resonant) subspace. With the above assumptions, one can prove the following Lemma.

Lemma 3.1 *Let $\varepsilon \in (0, (1/10)^8)$ and in (3.6) $\eta \geq \varepsilon^{\frac{1}{4}}$. Then for any $F \in \mathfrak{B}_h$ satisfying $|F|_h \leq \varepsilon$, there exist $Y \in \mathfrak{B}_h$ and $F^{(re)} \in \mathfrak{B}_h^{(re)}$ (we remark that $F^{(re)}$ is not necessarily $\mathbb{P}_{re}F$), such that*

$$\partial_\omega e^Y = (A + F)e^Y - e^Y(A + F^{(re)}),$$

i.e., the system

$$\begin{cases} \dot{x} = (A + F(\theta))x, \\ \dot{\theta} = \omega \end{cases} \tag{3.7}$$

can be conjugated to the system

$$\begin{cases} \dot{x} = (A + F^{(re)}(\theta))x, \\ \dot{\theta} = \omega \end{cases} \tag{3.8}$$

by the conjugation map e^Y, with the estimates

$$|Y|_h \leq \varepsilon^{\frac{1}{2}}, \qquad |F^{(re)}|_h \leq 2\varepsilon.$$

Proof We prove it by iteration. Let $\varepsilon_0 = \varepsilon$, $A_0 = A$, $F_0^{(re)} = \mathbb{P}_{re}F$, $F_0^{(nre)} = \mathbb{P}_{nre}F$. Assume that for $j = 1, 2, \dots, n$, there are Y_{j-1}, $F_j^{(nre)} \in \mathfrak{B}_h^{(nre)}$ and $F_j^{(re)} \in \mathfrak{B}_h^{(re)}$ satisfying the estimates

$$|Y_{j-1}|_h < \varepsilon_{j-1}^{\frac{5}{8}}, \qquad |F_j^{(re)} - F_{j-1}^{(re)}|_h < \varepsilon^{1/8}\varepsilon_{j-1}, \qquad |F_j^{(nre)}|_h < \varepsilon_j,$$

where $\varepsilon_j = \varepsilon^{(9/8)^j} = \varepsilon_{j-1}^{9/8}$, such that Y_{j-1} solves

$$\partial_\omega e^{Y_{j-1}} = (A + F_{j-1}^{(re)} + F_{j-1}^{(nre)})e^{Y_{j-1}} - e^{Y_{j-1}}(A + F_j^{(re)} + F_j^{(nre)}).$$

One can check easily that $|F_j^{(re)}|_h \leq 2\varepsilon$.

For any $Y \in \mathfrak{B}_h^{(nre)}$, from (3.6), $\partial_\omega Y, [A, Y] \in \mathfrak{B}_h^{(nre)}$ and

$$|\partial_\omega Y - \mathbb{P}_{nre}[A + F_n^{(re)}, Y]|_h \geq \eta |Y|_h - |F_n^{(re)}|_h |Y|_h \geq \frac{1}{2}\varepsilon^{\frac{1}{4}}|Y|_h.$$

So, restricted on $\mathfrak{B}_h^{(nre)}$, the linear operator

$$\partial_\omega \cdot - \mathbb{P}_{nre}[A + F_n^{(re)}, \cdot]$$

is invertible and the norm of its inverse is bounded by $2\varepsilon^{-\frac{1}{4}}$. One then can find $Y_n \in \mathfrak{B}_h^{(nre)}$, such that

$$\partial_\omega Y_n - \mathbb{P}_{nre}[A + F_n^{(re)}, Y_n] = F_n^{(nre)}$$

with the estimate

$$|Y_n|_h \leq 2\varepsilon^{-\frac{1}{4}}|F_n^{(nre)}|_h.$$

Let $G_n = F_n^{(nre)} - \mathbb{P}_{re}[F_n^{(re)}, Y_n]$. We have (recall that $\mathbb{P}_{re}[A, Y_n] = 0$)

$$\partial_\omega Y_n - [A + F_n^{(re)}, Y_n] = G_n.$$

Moreover, one can inductively prove that, for any power Y_n^k $(k = 2, 3, \dots)$ of Y_n,

$$\partial_\omega (Y_n^k) - [A + F_n^{(re)}, Y_n^k] = Y_n^{k-1} G_n + Y_n^{k-2} G_n Y_n + \cdots + G_n Y_n^{k-1} \triangleq G_n^{(k)}.$$

Thus we have

$$\partial_\omega e^{Y_n} - [A + F_n^{(re)}, e^{Y_n}] = F_n^{(nre)} - \mathbb{P}_{re}[F_n^{(re)}, Y_n] + \sum_{k=2}^{\infty} \frac{1}{k!} G_n^{(k)}.$$

It follows that

$$\partial_\omega e^{Y_n} = (A + F_n^{(re)} + F_n^{(nre)}) e^{Y_n} - e^{Y_n} (A + F_{n+1}^{(re)} + F_{n+1}^{(nre)}),$$

where

$$F_{n+1}^{(re)} = F_n^{(re)} + \mathbb{P}_{re} \left\{ e^{-Y_n} \left\{ F_n^{(nre)} (e^{Y_n} - I) + \mathbb{P}_{re}[F_n^{(re)}, Y_n] - \sum_{k=2}^{\infty} \frac{1}{k!} G_n^{(k)} \right\} \right\}$$

and

$$\begin{aligned} F_{n+1}^{(nre)} &= \mathbb{P}_{nre} \left\{ e^{-Y_n} \left\{ F_n^{(nre)} (e^{Y_n} - I) + \mathbb{P}_{re}[F_n^{(re)}, Y_n] - \sum_{k=2}^{\infty} \frac{1}{k!} G_n^{(k)} \right\} \right\} \\ &= \mathbb{P}_{nre} \left\{ e^{-Y_n} \left\{ F_n^{(nre)} (e^{Y_n} - I) - \sum_{k=2}^{\infty} \frac{1}{k!} G_n^{(k)} \right\} \right. \\ &\qquad \left. + (e^{-Y_n} - I) \mathbb{P}_{re}[F_n^{(re)}, Y_n] \right\}. \end{aligned}$$

Furthermore, we have the estimates

$$|F_{n+1}^{(re)} - F_n^{(re)}|_h < \varepsilon^{1/8} \varepsilon_n, \qquad |F_{n+1}^{(nre)}|_h < \varepsilon_n^{9/8} = \varepsilon_{n+1}.$$

Define Y by $e^Y = \prod_{n=0}^{\infty} e^{Y_n}$. $Y \in \mathfrak{B}_h$ is obviously well-defined. Moreover,

$$|Y_n|_h \le 2\varepsilon_n^{\frac{3}{4}} \le \varepsilon_n^{\frac{5}{8}},$$

which implies that

$$|Y|_h < \varepsilon^{\frac{1}{2}}.$$

Let $F^{(re)} = \lim_{n\to+\infty} F_n^{(re)}$, we then have $F^{(re)} \in \mathfrak{B}_h^{(re)}$ with the estimate

$$|F^{(re)}|_h \le 2\varepsilon.$$

One can check that

$$\partial_\omega e^Y = (A+F)e^Y - e^Y(A+F^{(re)}). \qquad \square$$

The definition of the sub-space $\mathfrak{B}_h^{(nre)}$ and $\mathfrak{B}_h^{(re)}$ depends on A. When A is in normal form $\left(\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right)$, we have the following more precise characterization of $\mathfrak{B}_h^{(nre)}$ and $\mathfrak{B}_h^{(re)}$. Let Λ_1 and Λ_2 be two subsets of $\mathbb{Z}^2$ with $\Lambda_j = -\Lambda_j$ $(j = 1, 2)$ such that we have that

$$k \in \Lambda_1 \Rightarrow |\langle k, \omega\rangle| \ge \eta \quad \text{and} \quad k \in \Lambda_2 \Rightarrow |2\lambda \pm i\langle k, \omega\rangle| \ge \eta.$$

$\mathfrak{B}_h^{(nre)}$ is defined to be the space of all $F \in \mathfrak{B}_h$ of the form

$$F(\theta) = \sum_{k\in\Lambda_1} \begin{pmatrix}\widehat{F}_{11}(k) & 0\\ 0 & -\widehat{F}_{11}(k)\end{pmatrix} e^{i\langle k,\theta\rangle} + \sum_{k\in\Lambda_2} \begin{pmatrix}0 & \widehat{F}_{12}(k)\\ \widehat{F}_{21}(k) & 0\end{pmatrix} e^{i\langle k,\theta\rangle} \tag{3.9}$$

and $\mathfrak{B}_h^{(re)}$ is defined to be the space of all $F \in \mathfrak{B}_h$ of the form

$$F(\theta) = \sum_{k\in\Lambda_1^c} \begin{pmatrix}\widehat{F}_{11}(k) & 0\\ 0 & -\widehat{F}_{11}(k)\end{pmatrix} e^{i\langle k,\theta\rangle} + \sum_{k\in\Lambda_2^c} \begin{pmatrix}0 & \widehat{F}_{12}(k)\\ \widehat{F}_{21}(k) & 0\end{pmatrix} e^{i\langle k,\theta\rangle}, \tag{3.10}$$

where $\Lambda_1^c = \mathbb{Z}^2\backslash\Lambda_1$, $\Lambda_2^c = \mathbb{Z}^2\backslash\Lambda_2$. Clearly, $\mathfrak{B}_h = \mathfrak{B}_h^{(nre)} \oplus \mathfrak{B}_h^{(re)}$.

Corollary 3.1 *Let $\varepsilon \in (0, (1/10)^8)$, $\eta \ge \varepsilon^{\frac14}$ and $A = \left(\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right)$ $(\lambda \in \mathbb{R})$. Then all conclusions in Lemma 3.1 are true with $F^{(re)}$ in the form of (3.10).*

Proof For any $Y \in \mathfrak{B}_h^{(nre)}$, we have that

$$\begin{aligned}\partial_\omega Y(\theta) &= \sum_{k\in\Lambda_1} i\langle k,\omega\rangle \begin{pmatrix}\widehat{Y}_{11}(k) & 0\\ 0 & -\widehat{Y}_{11}(k)\end{pmatrix} e^{i\langle k,\theta\rangle}\\ &\quad + \sum_{k\in\Lambda_2} i\langle k,\omega\rangle \begin{pmatrix}0 & \widehat{Y}_{12}(k)\\ \widehat{Y}_{21}(k) & 0\end{pmatrix} e^{i\langle k,\theta\rangle} \in \mathfrak{B}_h^{(nre)},\\ [A, Y](\theta) &= \sum_{k\in\Lambda_2} \begin{pmatrix}0 & 2\lambda\widehat{Y}_{12}(k)\\ -2\lambda\widehat{Y}_{21}(k) & 0\end{pmatrix} e^{i\langle k,\theta\rangle} \in \mathfrak{B}_h^{(nre)},\end{aligned}$$

and then

$$\begin{aligned}
&(\partial_\omega Y - [A, Y])(\theta) \\
&\quad = \sum_{k\in\Lambda_1} \begin{pmatrix} i\langle k,\omega\rangle \widehat{Y}_{11}(k) & 0 \\ 0 & -i\langle k,\omega\rangle \widehat{Y}_{11}(k) \end{pmatrix} e^{i\langle k,\theta\rangle} \\
&\qquad + \sum_{k\in\Lambda_2} \begin{pmatrix} 0 & (i\langle k,\omega\rangle - 2\lambda)\widehat{Y}_{12}(k) \\ (i\langle k,\omega\rangle + 2\lambda)\widehat{Y}_{21}(k) & 0 \end{pmatrix} e^{i\langle k,\theta\rangle}.
\end{aligned}$$

So one can easily check that

$$|\partial_\omega Y - [A, Y]|_h \geq \eta |Y|_h. \qquad \square$$

In the case that $A = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$, we have the following precise characterization of $\mathfrak{B}_h^{(nre)}$ and $\mathfrak{B}_h^{(re)}$.

Let $M = \frac{1}{1+i}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$, then $MAM^{-1} = \begin{pmatrix} i\rho & 0 \\ 0 & -i\rho \end{pmatrix}$. Note that any $F \in \mathfrak{B}_h$ can be uniquely re-written as

$$\begin{aligned}
F(\theta) &= \sum_{k\in\mathbb{Z}^2} \begin{pmatrix} 0 & \widehat{F}_-(k) \\ -\widehat{F}_-(k) & 0 \end{pmatrix} e^{i\langle k,\theta\rangle} + \sum_{k\in\mathbb{Z}^2} \begin{pmatrix} \widehat{F}_{11}(k) & \widehat{F}_+(k) \\ \widehat{F}_+(k) & -\widehat{F}_{11}(k) \end{pmatrix} e^{i\langle k,\theta\rangle} \\
&= \sum_{k\in\mathbb{Z}^2} M^{-1} \begin{pmatrix} i\widehat{F}_-(k) & 0 \\ 0 & -i\widehat{F}_-(k) \end{pmatrix} M e^{i\langle k,\theta\rangle} \\
&\quad + \sum_{k\in\mathbb{Z}^2} M^{-1} \begin{pmatrix} 0 & \widehat{F}_{11}(k) - i\widehat{F}_+(k) \\ \widehat{F}_{11}(k) + i\widehat{F}_+(k) & 0 \end{pmatrix} M e^{i\langle k,\theta\rangle},
\end{aligned}$$

where

$$\widehat{F}_\pm(k) = \frac{1}{2}\big(\widehat{F}_{12}(k) \pm \widehat{F}_{21}(k)\big).$$

We assume that Λ_1 and Λ_2 are two subsets of $\mathbb{Z}^2$ with $\Lambda_j = -\Lambda_j$ $(j = 1, 2)$, such that

$$k \in \Lambda_1 \Rightarrow |\langle k,\omega\rangle| \geq \eta \quad \text{and} \quad k \in \Lambda_2 \Rightarrow |2\rho \pm \langle k,\omega\rangle| \geq \eta.$$

We define $\mathfrak{B}_h^{(nre)}$ to be the space of all $F \in \mathfrak{B}_h$ of the form

$$F(\theta) = \sum_{k\in\Lambda_1} \begin{pmatrix} 0 & \widehat{F}_-(k) \\ -\widehat{F}_-(k) & 0 \end{pmatrix} e^{i\langle k,\theta\rangle} + \sum_{k\in\Lambda_2} \begin{pmatrix} \widehat{F}_{11}(k) & \widehat{F}_+(k) \\ \widehat{F}_+(k) & -\widehat{F}_{11}(k) \end{pmatrix} e^{i\langle k,\theta\rangle} \tag{3.11}$$

and $\mathfrak{B}_h^{(re)}$ to be the space of all $F \in \mathfrak{B}_h$ of the form

$$F(\theta) = \sum_{k\in\Lambda_1^c} \begin{pmatrix} 0 & \widehat{F}_-(k) \\ -\widehat{F}_-(k) & 0 \end{pmatrix} e^{i\langle k,\theta\rangle} + \sum_{k\in\Lambda_2^c} \begin{pmatrix} \widehat{F}_{11}(k) & \widehat{F}_+(k) \\ \widehat{F}_+(k) & -\widehat{F}_{11}(k) \end{pmatrix} e^{i\langle k,\theta\rangle}. \tag{3.12}$$

It is obvious that $\mathfrak{B}_h = \mathfrak{B}_h^{(nre)} \oplus \mathfrak{B}_h^{(re)}$.

Corollary 3.2 *Let $\varepsilon \in (0, (1/10)^8)$, $\eta \geq \varepsilon^{\frac{1}{4}}$ and $A = \left(\begin{smallmatrix} 0 & \rho \\ -\rho & 0 \end{smallmatrix}\right)$ $(\rho \in \mathbb{R})$. Then all conclusions in Lemma 3.1 are true with $F^{(re)}$ in the form (3.12).*

Proof Note that any $Y \in \mathfrak{B}_h^{(nre)}$ has the form

$$\begin{aligned} Y(\theta) &= \sum_{k\in\Lambda_1} M^{-1} \begin{pmatrix} i\widehat{Y}_-(k) & 0 \\ 0 & -i\widehat{Y}_-(k) \end{pmatrix} M e^{i\langle k,\theta\rangle} \\ &\quad + \sum_{k\in\Lambda_2} M^{-1} \begin{pmatrix} 0 & \widehat{Y}_{11}(k) - i\widehat{Y}_+(k) \\ \widehat{Y}_{11}(k) + i\widehat{Y}_+(k) & 0 \end{pmatrix} M e^{i\langle k,\theta\rangle}, \end{aligned}$$

which implies that

$$\begin{aligned} &\partial_\omega Y(\theta) \\ &\quad = \sum_{k\in\Lambda_1} i\langle k,\omega\rangle M^{-1} \begin{pmatrix} i\widehat{Y}_-(k) & 0 \\ 0 & -i\widehat{Y}_-(k) \end{pmatrix} M e^{i\langle k,\theta\rangle} \\ &\qquad + \sum_{k\in\Lambda_2} i\langle k,\omega\rangle M^{-1} \begin{pmatrix} 0 & \widehat{Y}_{11}(k) - i\widehat{Y}_+(k) \\ \widehat{Y}_{11}(k) + i\widehat{Y}_+(k) & 0 \end{pmatrix} M e^{i\langle k,\theta\rangle} \\ &\quad \in \mathfrak{B}_h^{(nre)}, \\ &[A, Y](\theta) \\ &\quad = \sum_{k\in\Lambda_2} M^{-1} \begin{pmatrix} 0 & 2i\rho\{\widehat{Y}_{11}(k) - i\widehat{Y}_+(k)\} \\ -2i\rho\{\widehat{Y}_{11}(k) + i\widehat{Y}_+(k)\} & 0 \end{pmatrix} M e^{i\langle k,\theta\rangle} \\ &\quad \in \mathfrak{B}_h^{(nre)}, \end{aligned}$$

and hence

$$\begin{aligned} &(\partial_\omega Y - [A, Y])(\theta) \\ &\quad = \sum_{k\in\Lambda_1} i\langle k,\omega\rangle M^{-1} \begin{pmatrix} i\widehat{Y}_-(k) & 0 \\ 0 & -i\widehat{Y}_-(k) \end{pmatrix} M e^{i\langle k,\theta\rangle} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\in\Lambda_2} M^{-1}\\
&\times\begin{pmatrix} 0 & i(\langle k,\omega\rangle-2\rho)\{\widehat{Y}_{11}(k)-i\widehat{Y}_{+}(k)\}\\ i(\langle k,\omega\rangle+2\rho)\{\widehat{Y}_{11}(k)+i\widehat{Y}_{+}(k)\} & 0\end{pmatrix}\\
&\times Me^{i\langle k,\theta\rangle}.
\end{aligned}
$$

One then can easily check that

$$
|\partial_\omega Y-[A,Y]|_h\geq\eta|Y|_h. \qquad \square
$$

4 Resonances and non-resonances

Let $\alpha\in(0,1)$ be irrational with the continued fractional expansion

$$
\alpha=\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}},
$$

We set $\alpha_0=\alpha$, and

$$
\alpha_n=\cfrac{1}{a_{n+1}+\cfrac{1}{a_{n+2}+\cdots}}.
$$

In fact, $\alpha_n=G^n(\alpha)$ where G is the Gauss map. The integers a_n are given by $a_n=[\alpha_{n-1}^{-1}]$ ($[\cdot]$ denotes the integer part). We also set $a_0=0$ for convenience.

Let $\beta_n=\prod_{j=0}^n\alpha_j$. Define

$$
\begin{aligned}
Q_0&=\begin{bmatrix} q_0 & p_0\\ q_{-1} & p_{-1}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\\
Q_n&=\begin{bmatrix} q_n & p_n\\ q_{n-1} & p_{n-1}\end{bmatrix}=\begin{bmatrix}a_n&1\\1&0\end{bmatrix}\begin{bmatrix} q_{n-1} & p_{n-1}\\ q_{n-2} & p_{n-2}\end{bmatrix}.
\end{aligned}
$$

It is easy to see that $Q_n=U(\alpha_n)\cdots U(\alpha_1)$ where

$$
U(x)=\begin{bmatrix}[x^{-1}]&1\\1&0\end{bmatrix}.
$$

Thus $\det(Q_n)=q_np_{n-1}-p_nq_{n-1}=(-1)^n$. Note that

$$
\beta_n=(-1)^n(q_n\alpha-p_n)=\frac{1}{q_{n+1}+\alpha_{n+1}q_n},
$$

$$\frac{1}{q_n + q_{n+1}} < \beta_n < \frac{1}{q_{n+1}}.$$

Moreover, for any $p, q \in \mathbb{Z}$ satisfying $|q| < q_{n+1}$, we have

$$|q\alpha - p| \geq \beta_n > \frac{1}{q_n + q_{n+1}}. \tag{4.1}$$

The following fact is important for our proof.

Lemma 4.1 *For any $k = (k_1, k_2) \in \mathbb{Z}^2$ which satisfies*

$$\text{(a)} \quad |k| := |k_1| + |k_2| \leq \frac{1}{6} q_{n+1}, \quad \textit{and} \quad \text{(b)} \quad k \neq l(q_n, -p_n), \quad l \in \mathbb{Z},$$

we have

$$|\langle k, \omega \rangle| \geq \frac{1}{7q_n},$$

where $\omega = (\alpha, 1)$.

Proof For any $k = (k_1, k_2) \in \mathbb{Z}^2$, there exist $s, l \in \mathbb{Z}$, s.t.

$$k = (k_1, k_2) = s(q_{n-1}, -p_{n-1}) + l(q_n, -p_n). \tag{4.2}$$

It is obvious that $|s| \geq 1$ if $k \neq l(q_n, -p_n)$, $l \in \mathbb{Z}$, and

$$|k| \geq |k_1| \geq |l| q_n - |s| q_{n-1},$$

and thus

$$|l| \leq \frac{|k|}{q_n} + |s| \frac{q_{n-1}}{q_n}. \tag{4.3}$$

So, by (4.2) and (4.3), we have

$$\begin{aligned}
|\langle k, \omega \rangle| &\geq |s||q_{n-1}\omega - p_{n-1}| - |l||q_n\omega - p_n| \\
&\geq \frac{|s|}{q_{n-1} + q_n} - \frac{1}{q_{n+1}} \left(\frac{|k|}{q_n} + |s| \frac{q_{n-1}}{q_n} \right) \\
&= \left(\frac{1}{q_{n-1} + q_n} - \frac{q_{n-1}}{q_n q_{n+1}} \right) |s| - \frac{|k|}{q_n q_{n+1}} \\
&\geq \left(\frac{1}{q_{n-1} + q_n} - \frac{q_{n-1}}{q_n q_{n+1}} \right) - \frac{|k|}{q_n q_{n+1}}.
\end{aligned}$$

Since $|k| \le \frac{1}{6}q_{n+1}$, one has

$$|\langle k, \omega\rangle| \ge \left(\frac{1}{q_{n-1}+q_n} - \frac{q_{n-1}}{q_n q_{n+1}}\right) - \frac{1}{6q_n}.$$

If $q_{n+1} \ge 6q_n$, we have

$$\frac{q_{n-1}}{q_n q_{n+1}} \le \frac{q_{n-1}}{6q_n^2} \le \frac{1}{3(q_{n-1}+q_n)},$$

and then

$$\begin{aligned} |\langle k, \omega\rangle| &\ge \frac{2}{3(q_{n-1}+q_n)} - \frac{1}{6q_n} \\ &\ge \frac{1}{3q_n} - \frac{1}{6q_n} \\ &= \frac{1}{6q_n}. \end{aligned}$$

Otherwise, $q_{n+1} < 6q_n$, by (4.1), we have

$$|\langle k, \omega\rangle| \ge \frac{1}{q_n + q_{n+1}} \ge \frac{1}{7q_n}.$$

□

Corollary 4.1 (The structure of resonances) *Suppose that $k \in \mathbb{Z}^2$ satisfies $|k| < \frac{1}{6}q_{n+1}$ and $|\langle k, \omega\rangle| < \frac{1}{7q_n}$. Then $k = l(q_n, -p_n)$ for some $l \in \mathbb{Z}$.*

5 Proof of Theorem 1.2

In this section, we shall inductively prove Theorem 1.2.

We begin with the system

$$\begin{cases} \dot{x} = (A + F(\theta))x, \\ \dot{\theta} = \omega. \end{cases} \tag{5.1}$$

Let

$$\Gamma \triangleq \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix} : \rho \in \mathbb{R} \right\} \subseteq sl(2, \mathbb{R}).$$

For simplicity, we assume that $h \le 1$, $F \in \mathfrak{B}_h$ and $A \in \Gamma$ with the estimate[2]

$$|F|_h < \min\left\{\left(\frac{1}{1000}\right)^{1000}, h^{16}\right\}. \tag{5.2}$$

We give the iteration sequences:

1. $q_0, q_1, \ldots$ and $p_0, p_1, \ldots$ are defined by the Fractional expansion of α.
2. Let $(h_0, \varepsilon_0) = (h, |F|_h)$, and

$$\begin{cases} (h_n, \varepsilon_n) = (h_{n-1}, \varepsilon_{n-1}), & \text{if } \varepsilon_{n-1} \le e^{-\frac{q_n h_{n-1}}{65\times 21}}; \\ (h_n, \varepsilon_n) = (\frac{1}{4}h_{n-1}, \varepsilon_{n-1}^{33/32} e^{-\frac{q_n h_{n-1}}{65\times 21\times 4}}), & \text{when } \varepsilon_{n-1} > e^{-\frac{q_n h_{n-1}}{65\times 21}}. \end{cases} \tag{5.3}$$

By definition, it is obvious that ε_n and h_n are decreasing. Moreover, we have the following simple and important facts:

1. We always have the inequality,

$$\varepsilon_n \le \min\left\{h_n^{16}, e^{-\frac{q_n h_n}{65\times 21}}\right\}. \tag{5.4}$$

The inequality $\varepsilon_n \le e^{-\frac{q_n h_n}{65\times 21}}$ can be obtained easily from the definition (5.3). The inequality $\varepsilon_n \le h_n^{16}$ can be proved inductively. Firstly $\varepsilon_0 < h_0^{16}$ is the case. We assume $\varepsilon_{n-1} < h_{n-1}^{16}$. By definition, one just need to check that when $\varepsilon_{n-1} > e^{-\frac{q_n h_{n-1}}{65\times 21}}$, there is

$$\varepsilon_n \le \varepsilon_{n-1}^{33/32} = \varepsilon_{n-1}^{1/32} \cdot \varepsilon_{n-1} \le \varepsilon_{n-1}^{1/32} \cdot h_{n-1}^{16} \le \varepsilon_{n-1}^{1/32} \cdot 4^{16} \cdot h_n^{16} \le h_n^{16}.$$

2. We always have

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \varepsilon_n = 0, \tag{5.5}$$

and for any $\chi \ge 1$,

$$\lim_{n\to\infty} \varepsilon_n h_n^{-\chi} = 0. \tag{5.6}$$

We argue by contradiction. If (5.5) is not true, by the iterative definition of (h_n, ε_n), there exists n_0 such that $(h_n, \varepsilon_n) = (h_{n_0}, \varepsilon_{n_0})$ when $n > n_0$, then it

[2] If A is of general form in the system (5.1), one can find some $P \in SL(2, \mathbb{R})$, such that $PAP^{-1} \in \Gamma$ or $PAP^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (parabolic case). For the parabolic case, one can further use $P_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{pmatrix}$ ($\epsilon > 0$) to conjugate A to $D = \begin{pmatrix} 0 & \epsilon^2 \\ 0 & 0 \end{pmatrix}$ and then $D + P_1 PFP^{-1} P_1^{-1}$ are treated as the new perturbation. In this case, the smallness of the perturbation will also depend on A through $\min\{|P| : PAP^{-1} \in \Gamma \text{ or } PAP^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$.

follows that

$$0 < \varepsilon_{n_0} = \varepsilon_n \leq e^{-\frac{q_n h_{n-1}}{65\times 21}} = e^{-\frac{q_n h_{n_0}}{65\times 21}} \to 0.$$

This is a contradiction. Condition (5.6) can then be obtained from the definition (5.3).

Assume that, after n steps of iteration, we arrive at

$$\begin{cases} \dot{x} = (A^{(n)} + F^{(n)}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.7}$$

where $A^{(n)} \in \Gamma$ and $F^{(n)} \in \mathfrak{B}_{h_n}$, with the estimate

$$|F^{(n)}|_{h_n} \leq \varepsilon_n.$$

We will construct a conjugation map $B^{(n)}$ which conjugates (5.7) to

$$\begin{cases} \dot{x} = (A^{(n+1)} + F^{(n+1)}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.8}$$

with $A^{(n+1)} \in \Gamma$ and $F^{(n+1)} \in \mathfrak{B}_{h_{n+1}}$ satisfying the estimates

$$|F^{(n+1)}|_{h_{n+1}} \leq \varepsilon_{n+1}.$$

If $\varepsilon_n \leq e^{-\frac{q_{n+1} h_n}{65\times 21}}$, one needs to do nothing at this step. We just simply let

$$A^{(n+1)} = A^{(n)}, \qquad F^{(n+1)} = F^{(n)}.$$

Otherwise, i.e., $\varepsilon_n > e^{-\frac{q_{n+1} h_n}{65\times 21}}$. The proof is decomposed into the following four lemmas. For simplicity of notations, in the following $A^{(n)}$, $F^{(n)}$ are written as A, F respectively, $h_n, \varepsilon_n, q_n, p_n, h_{n+1}, \varepsilon_{n+1}, q_{n+1}$ are written as $h, \varepsilon, q, p, h_+, \varepsilon_+, q_+$ respectively. Now, by (5.4), we have

$$e^{-\frac{q_+ h}{65\times 21}} < \varepsilon \leq \min\left\{h^{16}, e^{-\frac{qh}{65\times 21}}\right\}. \tag{5.9}$$

We emphasis that condition (5.9) is the starting point of the following proof.

We first handle the elliptic case, i.e., $A = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix} \in sl(2, \mathbb{R})$, which is the most complicated. Let

$$\Lambda_1 = \{k \in \mathbb{Z}^2 : |\langle k, \omega\rangle| \geq \varepsilon^{1/4}\}, \qquad \Lambda_2 = \{k \in \mathbb{Z}^2 : |2\rho \pm \langle k, \omega\rangle| \geq \varepsilon^{1/4}\}.$$

As in Sect. 3, we use M to denote the matrix $\frac{1}{1+i}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$. By Corollary 3.2, there is a $Y \in \mathfrak{B}_h$ satisfying

$$|Y|_h \leq \varepsilon^{1/2},$$

and a $F^{(re)} \in \mathfrak{B}_h$ of the form

$$
\begin{aligned}
F^{(re)}(\theta) &= \sum_{k\in\Lambda_1^c} \begin{pmatrix} 0 & \widehat{F_-^{(re)}}(k) \\ -\widehat{F_-^{(re)}}(k) & 0 \end{pmatrix} e^{i\langle k,\theta\rangle} \\
&\quad + \sum_{k\in\Lambda_2^c} \begin{pmatrix} \widehat{F_{11}^{(re)}}(k) & \widehat{F_+^{(re)}}(k) \\ \widehat{F_+^{(re)}}(k) & -\widehat{F_{11}^{(re)}}(k) \end{pmatrix} e^{i\langle k,\theta\rangle} \\
&= \sum_{k\in\Lambda_1^c} M^{-1} \begin{pmatrix} i\widehat{F_-^{(re)}}(k) & 0 \\ 0 & -i\widehat{F_-^{(re)}}(k) \end{pmatrix} M e^{i\langle k,\theta\rangle} \\
&\quad + \sum_{k\in\Lambda_2^c} M^{-1} \begin{pmatrix} 0 & \widehat{F_{11}^{(re)}}(k) - i\widehat{F_+^{(re)}}(k) \\ \widehat{F_{11}^{(re)}}(k) - i\widehat{F_+^{(re)}}(k) & 0 \end{pmatrix} \\
&\quad \times M e^{i\langle k,\theta\rangle}
\end{aligned} \tag{5.10}
$$

(recall that $\widehat{F_\pm^{(re)}}(k) = \frac{1}{2}\{\widehat{F_{12}^{(re)}}(k) \pm \widehat{F_{21}^{(re)}}(k)\}$), such that

$$\partial_\omega e^Y = (A+F)e^Y - e^Y(A+F^{(re)}),$$

with the estimate

$$|F^{(re)}|_h \le 2\varepsilon.$$

We then obtain the following conclusion.

Lemma 5.1 (Eliminating the non-resonant terms) *System* (5.7) *is conjugated to*

$$
\begin{cases} \dot{x} = (A + F^{(re)}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.11}
$$

with $F^{(re)}$ *of the form* (5.10) *satisfying*

$$|F^{(re)}|_h \le 2\varepsilon.$$

Note that $\varepsilon^{1/8} < 1/7q$ since (by (5.9))

$$7q\varepsilon^{1/8} \le 7qhe^{-\frac{qh}{65\times 21\times 32}}\varepsilon^{1/32} \le 7\times 65\times 21\times 32\varepsilon^{1/32} < 1.$$

Thus, by Corollary 4.1, we obtain the following fact

$$\left\{k\in\mathbb{Z}^2:|\langle k,\omega\rangle|\le\varepsilon^{1/8}\text{ and }|k|<\frac{q_+}{6}\right\}$$
$$\subseteq\left\{k=l(q,-p):l\in\mathbb{Z}\text{ and }|k|<\frac{q_+}{6}\right\},\tag{5.12}$$

which will be frequently used in the following.

Firstly, from (5.12), we obviously have

$$\Lambda_1^c\cap\left\{k\in\mathbb{Z}^2:|k|<\frac{q_+}{6}\right\}\subseteq\left\{k=l(q,-p):l\in\mathbb{Z}\text{ and }|k|<\frac{q_+}{6}\right\}.\tag{5.13}$$

Now we consider different cases of Λ_2^c:

If $\Lambda_2^c\cap\{k\in\mathbb{Z}^2:|k|<\frac{q_+}{6}\}=\emptyset$, $\mathcal{T}_{\frac{q_+}{6}}F_{12}^{(re)}=-\mathcal{T}_{\frac{q_+}{6}}F_{21}^{(re)}$, i.e.,

$$\mathcal{T}_{\frac{q_+}{6}}F^{(re)}(\theta)=\sum_{k\in\Lambda_1^c,|k|<\frac{q_+}{6}}M^{-1}\begin{pmatrix}i\widehat{F_{12}^{(re)}}(k)&0\\0&-i\widehat{F_{12}^{(re)}}(k)\end{pmatrix}Me^{i\langle k,\theta\rangle}$$

one can use $B_*\triangleq e^E$, with

$$E(\theta)=\sum_{k\in\Lambda_1^c,0<|k|<\frac{q_+}{6}}M^{-1}\begin{pmatrix}i\widehat{F_{12}^{(re)}}(k)&0\\0&-i\widehat{F_{12}^{(re)}}(k)\end{pmatrix}M\frac{e^{i\langle k,\theta\rangle}}{i\langle k,\omega\rangle}\in\mathcal{T}_{\frac{q_+}{6}}\mathfrak{B}_h$$

to conjugate (5.11) to

$$\begin{cases}\dot{x}=(A_*+F_*(\theta))x,\\\dot{\theta}=\omega,\end{cases}$$

where $A_*\triangleq A+\widehat{F^{(re)}}(0)$ and $F_*\triangleq B_*(\mathcal{R}_{\frac{q_+}{6}}F^{(re)})B_*^{-1}$. Note that

$$|B_*|_h\le e^{|E|_h}\le e^{4q_+\varepsilon}.$$

By (5.9), there is

$$\begin{aligned}|F_*|_{h_+}=|F_*|_{\frac{h}{4}}&\le\varepsilon e^{-\frac{3q_+h}{2\times6\times4}}e^{8q_+\varepsilon}\\&\le\varepsilon\cdot e^{-\frac{q_+h}{16}}e^{8q_+h\varepsilon^{15/16}}\\&\le\varepsilon\cdot e^{-\frac{q_+h}{32}}e^{-\frac{q_+h}{32}(1-8\times32\varepsilon^{15/16})}\\&\le\varepsilon e^{-\frac{q_+h}{32}}\end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon e^{-\frac{q_+h}{65\times 21}} e^{-\frac{q_+h}{65\times 21\times 4}}\\
&\le \varepsilon^2 e^{-\frac{q_+h}{65\times 21\times 4}}\\
&\le \varepsilon^{33/32} e^{-\frac{q_+h_n}{65\times 21\times 4}} = \varepsilon_+.
\end{aligned}
$$

For this simple case, a step of iteration is thus finished.

If $\Lambda_2^c \cap \{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6}\} \neq \emptyset$, let $k_* \in \Lambda_2^c$ satisfy $|k_*| = \min\{|k| : k \in \Lambda_2^c\}$.

If $|k_*| = 0$, we have $0 \in \Lambda_2^c$. Since

$$\Lambda_2 = \{k \in \mathbb{Z}^2 : |2\rho \pm \langle k, \omega\rangle| \ge \varepsilon^{1/4}\},$$

we therefore have $|\rho| < \frac{1}{2}\varepsilon^{1/4}$. By (5.12), we have

$$
\begin{aligned}
\Lambda_2^c \cap \left\{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6}\right\} &\subseteq \left\{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6} \text{ and } |2\rho_n - \langle k, \omega\rangle| < \varepsilon^{1/4}\right\}\\
&\subseteq \left\{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6} \text{ and } |\langle k, \omega\rangle| < 2\varepsilon^{1/4}\right\}\\
&\subseteq \left\{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6} \text{ and } |\langle k, \omega\rangle| < \varepsilon^{1/8}\right\}\\
&\subseteq \left\{k = l(q, -p) : l \in \mathbb{Z} \text{ and } |k| < \frac{q_+}{6}\right\}.
\end{aligned}
$$

$\mathcal{T}_{\frac{q_+}{6}} F^{(re)}(\theta)$ is of the form

$$\mathcal{T}_{\frac{q_+}{6}} F^{(re)}(\theta) = \sum_{k=l(q,-p), |k| < \frac{q_+}{6}} \widehat{F^{(re)}}(k) e^{i\langle k, \theta\rangle}$$

and the following Lemma 5.3 applies.

If $|k_*| \neq 0$, we invoke the following lemma to put the system into a better normal form.

Lemma 5.2 (Rotation) *By some $4\pi\mathbb{Z}$-periodic analytic conjugation map*, (5.11) *is conjugated to*

$$
\begin{cases}
\dot{x} = (A_1 + F_1(\theta))x,\\
\dot{\theta} = \omega,
\end{cases}
\tag{5.14}
$$

where

$$A_1 = \begin{pmatrix} 0 & \rho - \frac{\langle k_*,\omega\rangle}{2} \\ -\rho + \frac{\langle k_*,\omega\rangle}{2} & 0 \end{pmatrix},$$

$$\mathcal{T}_{\frac{q_+}{6}} F_1 = \sum_{k=l(q,-p),|k|<\frac{q_+}{6}} \widehat{F}_1(k) e^{i\langle k,\theta\rangle}$$

with the estimates

$$|A_1| \le \varepsilon^{\frac{1}{4}}, \qquad |F_1|_{\frac{h}{3}} \le 2\varepsilon^{3/4}.$$

Proof By the definition of k_*, $\Lambda_2^c \cap \{k \in \mathbb{Z}^2 : |k| < |k_*|\} = \emptyset$, which implies that

$$\begin{aligned} &\mathcal{T}_{|k_*|} F^{(re)}(\theta) \\ &\quad = \sum_{k=l(q,-p)\in\Lambda_1^c,|k|<|k_*|} M^{-1} \begin{pmatrix} i\widehat{F_{12}^{(re)}}(k) & 0 \\ 0 & -i\widehat{F_{12}^{(re)}}(k) \end{pmatrix} M e^{i\langle k,\theta\rangle}. \end{aligned} \tag{5.15}$$

By a direct computation, one sees that the $4\pi\mathbb{Z}$-periodic rotation

$$Q(\theta) = \begin{pmatrix} \cos\frac{\langle k_*,\theta\rangle}{2} & -\sin\frac{\langle k_*,\theta\rangle}{2} \\ \sin\frac{\langle k_*,\theta\rangle}{2} & \cos\frac{\langle k_*,\theta\rangle}{2} \end{pmatrix} = M^{-1} \begin{pmatrix} e^{-\frac{\langle k_*,\theta\rangle}{2}i} & 0 \\ 0 & e^{\frac{\langle k_*,\theta\rangle}{2}i} \end{pmatrix} M$$

conjugates (5.11) to

$$\begin{cases} \dot{x} = (A_1 + \widetilde{F}_1(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.16}$$

with $\widetilde{F}_1 = QF^{(re)}Q^{-1}$ and

$$A_1 = (\partial_\omega Q)Q^{-1} + QAQ^{-1} = \begin{pmatrix} 0 & \widetilde{\rho} \\ -\widetilde{\rho} & 0 \end{pmatrix},$$

where $\widetilde{\rho} = \rho - \langle k_*,\omega\rangle/2$. Note that $k_* \in \Lambda_2^c$ implies $|\widetilde{\rho}| < \frac{1}{2}\varepsilon^{1/4}$.

Since $\mathcal{T}_{|k_*|}F^{(re)}$ commutes with Q, i.e., $Q\mathcal{T}_{|k_*|}F^{(re)}Q^{-1} = \mathcal{T}_{|k_*|}F^{(re)}$, we have

$$\widetilde{F}_1 = \mathcal{T}_{|k_*|}F^{(re)} + Q\mathcal{R}_{|k_*|}F^{(re)}Q^{-1}.$$

By the estimates $|\mathcal{T}_{|k_*|}F^{(re)}|_{h_n} \le 2\varepsilon \le \frac{1}{2}\varepsilon^{3/4}$ and

$$|Q\mathcal{R}_{|k_*|}F^{(re)}Q^{-1}|_{\frac{h}{3}} \le \frac{9\times 36}{2h^2}\varepsilon e^{-\frac{|k_*|h}{3}} e^{\frac{|k_*|h}{3}} \le 5\times 36\varepsilon^{7/8} \le \frac{1}{2}\varepsilon^{3/4},$$

one has

$$|\widetilde{F}_1|_{\frac{h}{3}} \le |\mathcal{T}_{|k_*|}F^{(re)}|_{\frac{h}{3}} + |Q\mathcal{R}_{|k_*|}F^{(re)}Q^{-1}|_{\frac{h}{3}} \le \varepsilon^{3/4} \triangleq \widetilde{\varepsilon}.$$

A simple computation, together with (5.12) and the fact that $|\widetilde{\rho}| < \frac{1}{2}\varepsilon^{1/4}$, leads to

$$\begin{aligned}(\widetilde{\Lambda}_1^c \cup \widetilde{\Lambda}_2^c) \cap \left\{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6}\right\} &\subseteq \left\{k \in \mathbb{Z}^2 : |k| < \frac{q_+}{6} \text{ and } |\langle k, \omega\rangle| < \varepsilon^{1/8}\right\}\\ &\subseteq \left\{k = l(q, -p) : l \in \mathbb{Z} \text{ and } |k| < \frac{q_+}{6}\right\},\end{aligned}$$

where

$$\widetilde{\Lambda}_1 = \{k \in \mathbb{Z}^2 : |\langle k, \omega\rangle| \ge (\widetilde{\varepsilon})^{1/4}\}, \qquad \widetilde{\Lambda}_2 = \{k \in \mathbb{Z}^2 : |2\widetilde{\rho} \pm \langle k, \omega\rangle| \ge (\widetilde{\varepsilon})^{1/4}\}.$$

Thus by Corollary 3.2, one can find a $Y_1 \in \mathfrak{B}_{\frac{h}{3}}$, such that e^{Y_1} conjugates system (5.16) to system (5.14) with $F_1 \in \mathfrak{B}_{\frac{h}{3}}$ of the form

$$\mathcal{T}_{\frac{q_+}{6}} F_1(\theta) = \sum_{k=l(q,-p), |k|<\frac{q_+}{6}} \widehat{F}_1(k) e^{i\langle k, \theta\rangle}$$

and satisfying $|F_1|_{\frac{h}{3}} \le 2\widetilde{\varepsilon} = 2\varepsilon^{3/4}$. The proof is then completed. □

Remark 5.1 It is obvious that the degree of Q in the above proof is k_* (one can refer to Appendix C for precise definition of degree) with $|k_*| < q_+/6$.

In any case, we are in the position to use the following lemma. We remark that the special form of $\mathcal{T}_{\frac{q_+}{6}} F_1$ is very important, while the special form of A_1 is irrelevant in the following lemma. Let

$$\begin{cases} \dot{x} = (A_1 + \mathcal{T}_{\frac{q_+}{6}} F_1(\theta))x = (A_1 + \sum_{k=l(q,-p), |k|<\frac{q_+}{6}} \widehat{F}_1(k) e^{i\langle k, \theta\rangle})x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.17}$$

with the estimates

$$|A_1| \le \varepsilon^{\frac{1}{4}}, \qquad |F_1|_{\frac{h}{3}} \le 2\varepsilon^{3/4}.$$

Lemma 5.3 (Floquet) *System* (5.14) *is conjugated to*

$$\begin{cases} \dot{x} = (A_2 + F_2(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.18}$$

where $A_2 \in sl(2,\mathbb{R})$ *and* $F_2 \in \mathfrak{B}_{h/4}$ *satisfies*

$$|F_2|_{\frac{h}{4}} \le \varepsilon^{1/8} e^{-\frac{q_+h}{25\times 6}}, \qquad |A_2| \le \frac{1}{2\pi q_+} e^{24\pi q_+ \varepsilon^{1/4}} \quad \textit{and} \quad |spec(A_2)| \le \frac{1}{q_+}.$$

Proof By Lemma 7.1 in Appendix A, which is based on Floquet Theory, one can find some $4\pi\mathbb{Z}$-periodic analytic conjugation map B, which is analytic in $|\mathrm{Im}\,\theta| < h/3$, to conjugate system (5.17) to

$$\begin{cases} \dot{x} = A_2 x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.19}$$

where A_2 is a constant matrix, with the estimates (notice that $|A_1| = |spec(A_1)| \le \varepsilon^{1/4}$)

$$\sup_{|\mathrm{Im}\,\theta|<\frac{h}{3}} |B(\theta)| \le e^{3q_+\varepsilon^{1/4}(8\pi+qh)},$$

$$|A_2| \le \frac{1}{2\pi q_+} e^{24\pi q_+ \varepsilon^{1/4}} \quad \text{and} \quad |spec(A_2)| \le \frac{1}{q_+}.$$

The inequality

$$3q_+\varepsilon^{1/4}(8\pi + qh) \le 30\pi q_+ \varepsilon^{3/32}$$

follows from the fact that

$$qh\varepsilon^{1/4} \le \varepsilon^{1/8} qh e^{-\frac{qh}{65\times 21\times 8}} \le 65\times 21\times 8\cdot \varepsilon^{1/8} \le \varepsilon^{3/32}.$$

Thus, by the inequality (3.2), we have

$$|B|_{\frac{h}{4}} \le \frac{36\times 144}{h^2} e^{30\pi q_+ \varepsilon^{3/32}}. \tag{5.20}$$

Note that $F_2 = B\mathcal{R}_{\frac{q_+}{6}} F_1 B^{-1}$ and

$$|\mathcal{R}_{\frac{q_+}{6}} F_1|_{\frac{h}{4}} \le \frac{144\times 36}{h^2}|F_1|_{\frac{h}{3}} \le \frac{144\times 72}{h^2}\varepsilon^{3/4} e^{-\frac{q_+h}{24\times 6}} \le \varepsilon^{1/2} e^{-\frac{q_+h}{24\times 6}}.$$

It follows that

$$\begin{aligned} |F_2|_{\frac{h}{4}} &\le \frac{144\times 36}{h^2}\varepsilon^{1/2} e^{-\frac{q_+h}{24\times 6}} e^{60\pi q_+\varepsilon^{3/32}} \\ &\le \varepsilon^{1/4} e^{-\frac{q_+h}{25\times 6}} e^{-\frac{q_+h}{24\times 25\times 6}(1-24\times 25\times 6\times 60\pi\varepsilon^{1/32})} \\ &\le \varepsilon^{1/4} e^{-\frac{q_+h}{25\times 6}}. \end{aligned}$$

□

Remark 5.2 By (5.20) and Corollary 9.1 in Appendix C, the degree of the conjugation map B in the above proof is no more than

$$\frac{c}{h^{\sigma}}e^{\sigma q_+ \varepsilon^{1/\sigma}}$$

where $c, \sigma > 1$ are two universal constants.

Finally we normalize A_2.

Lemma 5.4 (Normalization) *One can conjugate system* (5.18) *to*

$$\begin{cases} \dot{x} = (A_3 + F_3(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{5.21}$$

with $A_3 \in \Gamma$ *and* $F_3 \in \mathfrak{B}_{h_+}$, *such that*

$$|A_3| \le \frac{1}{q_+} + \varepsilon_+, \qquad |F_3|_{h_+} \le \varepsilon_+.$$

Proof Let $\epsilon = |F_2|_{\frac{h}{4}}^{1/7} = |F_2|_{h_+}^{1/7}$. The estimate

$$\epsilon|A_2| \le e^{-\frac{q_+h}{25\times 42}}e^{24\pi q_+\varepsilon^{1/4}} \le e^{-\frac{q_+h}{25\times 42}(1-24\pi\times 25\times 42\varepsilon^{3/16})} < 1$$

and Lemma 8.2 together imply that there exist $P_2 \in SL(2,\mathbb{R})$ satisfying $|P_2| \le 2\epsilon^{-3}$, $D_3 \in sl(2,\mathbb{R})$ satisfying $|D_3| \le 2\epsilon$ and $A_3 \in \Gamma$ satisfying

$$|A_3| \le |spec(A_2)| + \epsilon \le \frac{1}{q_+} + \epsilon,$$

such that $P_2 A_2 P_2^{-1} = A_3 + D_3$. Let $F_3 = D_3 + P_2 F_2 P_2^{-1}$, then P_2 conjugates the system (5.18) to the system (5.21) with the estimate

$$\begin{aligned} |F_3|_{h_+} = |F_3|_{\frac{h}{4}} \le 6\epsilon \le 6\varepsilon^{1/28}e^{-\frac{q_+h}{25\times 42}} \\ \le \varepsilon^{1/32}e^{-\frac{q_+h}{25\times 42}} \\ \le \varepsilon^{1/32}e^{-\frac{q_+h}{65\times 21}}e^{-\frac{q_+h}{65\times 21\times 4}} \\ \le \varepsilon^{33/32}e^{-\frac{q_+h}{65\times 21\times 4}} \\ = \varepsilon^{33/32}e^{-\frac{q_+h_+}{65\times 21}} = \varepsilon_+. \end{aligned}$$

□

The proof for the elliptic case is therefore finished. The case of $A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ with $\lambda \in \mathbb{R}$, is easier. In this case, we do not need Lemma 5.2. In fact, by

Corollary 3.1, one can obtain a result analogous to Lemma 5.1 and then obtain a similar $F^{(re)}$. Note that now $\mathcal{T}_{\frac{q_+}{6}} F_{12}^{(re)} = \mathcal{T}_{\frac{q_+}{6}} F_{21}^{(re)} = 0$ if $|\lambda| \geq \frac{1}{2}\varepsilon^{1/4}$, i.e., the matrix $\mathcal{T}_{\frac{q_+}{6}} F^{(re)}$ is diagonal. Otherwise $k_* = 0$. In latter cases, we arrive at the normal form (5.17) and then switch to Lemma 5.3. Note that now $A_1 = A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, but the difference is irrelevant.

In summary, we use some conjugation map $B^{(n)}$ to conjugate system (5.7) to system (5.8) with $A^{(n+1)} \in \Gamma$ and $F^{(n+1)} \in \mathfrak{B}_{h_{n+1}}$ satisfying the estimates

$$|F^{(n+1)}|_{h_{n+1}} \leq \varepsilon_{n+1}.$$

At the end, we remark from the proof that

$$\begin{cases} |A^{(n+1)}| = |A^{(n)}|, & \text{when } \varepsilon_n \leq e^{-\frac{q_{n+1}h_n}{65\times 21}}; \\ |A^{(n+1)}| \leq \max\{|A^{(n)}|, \frac{1}{q_{n+1}}\} + 2\varepsilon_n, & \text{when } \varepsilon_n > e^{-\frac{q_{n+1}h_n}{65\times 21}}. \end{cases}$$

That is to say, the sequence $A^{(n)}$, $n = 1, 2, \ldots$ is bounded.

We thus complete the proof of Theorem 1.2.

Remark 5.3 If in system (5.1), $F \in \mathfrak{B}_h$ $(h > 0)$ satisfies $|F|_h \leq 10^{-8}$, $A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ with $\lambda \in \mathbb{R}$ and $|\lambda| \geq \frac{1}{2}|F|_h^{1/4}$, the proof is very simple. In fact, by Corollary 3.1, one can conjugates it to

$$\begin{cases} \dot{x} = (A + \widetilde{F}(\theta))x, \\ \dot{\theta} = \omega, \end{cases}$$

with $\widetilde{F} \in \mathfrak{B}_h$ in diagonal form, and it is thus uniformly hyperbolic. Then the almost reducibility is easily obtained by conjugating it to

$$\begin{cases} \dot{x} = (A + \widehat{\widetilde{F}}(0) + \mathcal{R}_n \widetilde{F}(\theta))x, \\ \dot{\theta} = \omega, \end{cases}$$

by a sequence of conjugation maps $\exp\{-\sum_{0<|k|<n} \frac{\widehat{\widetilde{F}}(k)}{i\langle k,\omega\rangle} e^{i\langle k,\theta\rangle}\}$.

Remark 5.4 When $\varepsilon_n \leq e^{-\frac{q_{n+1}h_n}{65\times 21}}$, the degree of the conjugation map $B^{(n)}$ is zero for $B^{(n)} = I$. When $\varepsilon_n > e^{-\frac{q_{n+1}h_n}{65\times 21}}$, the nonzero degree could only occurs in the procedure of Lemma 5.2 or Lemma 5.3. By Remark 5.1 and Remark 5.2, the degree of the conjugation map $B^{(n)}$ is no more than

$$\frac{cq_{n+1}}{h_n^\sigma} e^{\sigma q_{n+1}\varepsilon_n^{1/\sigma}},$$

where $c, \sigma > 1$ are two universal constants.

6 Proofs of Theorems 1.3–1.5

Firstly, we prove Theorem 1.3, i.e., systems which are close to a non-hyperbolic constant system can be conjugated to *rotations* of the form

$$\begin{cases} \dot{x} = \varphi(\theta) J x \\ \dot{\theta} = \omega \end{cases} \tag{6.1}$$

under some assumptions on the rotation number, where φ is analytic and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M^{-1} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} M$ (recall that M denotes the matrix $\frac{1}{1+i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$).

Let $\mathcal{C}_h$ ($h > 0$) be the set of all analytic $f : \mathbb{T}^2 \to \mathbb{R}$ with the Fourier expansion

$$f(\theta) = \sum_{k \in \mathbb{Z}^2} \widehat{f}(k) e^{i\langle k, \theta \rangle}$$

satisfying

$$|f|_h \triangleq \sum_{k \in \mathbb{Z}^2} |\widehat{f}(k)| e^{|k|h} < +\infty.$$

For any $N > 0$, we also use the notations $\mathcal{T}_N$ and $\mathcal{R}_N$ to denote the operators on $\mathcal{C}_h$ defined as

$$(\mathcal{T}_N f)(\theta) = \sum_{|k| < N} \widehat{f}(k) e^{i\langle k, \theta \rangle}, \qquad (\mathcal{R}_N f)(\theta) = \sum_{|k| \geq N} \widehat{f}(k) e^{i\langle k, \theta \rangle}.$$

Moreover, for any $\omega \in \mathbb{R}^2$, we define the formal linear operator Π_ω as

$$(\Pi_\omega f)(\theta) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{\widehat{f}(k)}{i\langle k, \omega \rangle} e^{i\langle k, \theta \rangle}.$$

It is obvious that $\Pi_\omega \partial_\omega f = \partial_\omega \Pi_\omega f = f - \widehat{f}(0)$ for any $f \in \mathcal{C}_h$. In the end, we use $\mathcal{C} = \bigcup_{h>0} \mathcal{C}_h$ to denote the set of all analytic $f : \mathbb{T}^2 \to \mathbb{R}$.

Let $\omega = (\alpha, 1)$ with $\alpha \in (0,1)$ being irrational. Recall that $q_1, q_2, \ldots$ and $p_1, p_2, \ldots$ are defined in Sect. 3 by the fractional expansion of α and

$$\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.$$

Remark 6.1 System (6.1) is analytically reducible if $\varphi \in \mathcal{C}_h$ and $\beta(\alpha) < h$. In fact the analytic conjugation map $e^{\Pi_\omega \varphi J}$ (note that $\Pi_\omega \varphi \in \mathcal{C}$ as $\beta(\alpha) < h$) conjugates (6.1) to

$$\begin{cases} \dot{x} = \widehat{\varphi}(0) J x, \\ \dot{\theta} = \omega. \end{cases} \tag{6.2}$$

For general systems close to a constant one, we should not expect the rotations reducibility for all rotation numbers (even under the assumption that the rotation number is not rational w.r.t. ω) since the rotations reducibility implies that all solutions of the linear system are bounded (which is not always the case as shown in [15]). The rotations reducibility is closely related to the rotation number.

We need the following lemma on the rotation number, which says that the divisor is big enough even for very large k (comparing with q_n).

Lemma 6.1 *Let $\gamma,\tau\geq 1$ and $\rho\in DC_\omega(\gamma,\tau)\cap[-1,1]$. Then for any $k\in\mathbb{Z}^2$ satisfying $|k|\leq q_{n+1}/\gamma(8q_n)^\tau$, we have*

$$|\langle k,\omega\rangle-2\rho|\geq\frac{1}{\gamma(8q_n)^\tau}.$$

Proof We just need to consider all $k\in\mathbb{Z}^2$ satisfying $|k|\leq\frac{q_{n+1}}{\gamma(8q_n)^\tau}$ and $|\langle k,\omega\rangle-2\rho|<1$. For such k, one can find $l\in\mathbb{Z}$ and $\widehat{k}=(\widehat{k}_1,\widehat{k}_2)\in\mathbb{Z}^2$, such that $k=l(q_n,-p_n)+(\widehat{k}_1,\widehat{k}_2)$, with $|l|\leq\frac{q_{n+1}}{\gamma(8q_n)^\tau q_n}$ and $|\widehat{k}_1|<q_n$. Thus from

$$1>|\langle k,\omega\rangle-2\rho|=|l(q_n\alpha-p_n)+\widehat{k}_1\alpha+\widehat{k}_2-2\rho|,$$

we obtain that $|\widehat{k}_2|\leq q_n+2|\rho|+2$, and $|\widehat{k}|\leq 2q_n+2|\rho|+2\leq 4q_n$. The assumption that $\rho\in DC_\omega(\gamma,\tau)$ implies that $|\langle\widehat{k},\omega\rangle-2\rho|\geq\frac{\gamma^{-1}}{4^\tau q_n^\tau}=\frac{2^\tau}{\gamma(8q_n)^\tau}$, which, in turn, implies that

$$\begin{aligned}|\langle k,\omega\rangle-2\rho|&=|l(q_n\alpha-p_n)+\langle\widehat{k},\omega\rangle-2\rho|\\&\geq\frac{2^\tau}{\gamma(8q_n)^\tau}-\frac{q_{n+1}}{\gamma(8q_n)^\tau q_n}\cdot\frac{1}{q_{n+1}}\\&\geq\frac{1}{\gamma(8q_n)^\tau}.\end{aligned}$$

This completes the proof. □

The following theorem gives a result which is slightly weaker than Theorem 1.3. More precisely, it is a rotations reducibility result for a positive measure set of rotation numbers.

Theorem 6.1 *Consider*

$$\begin{cases}\dot{x}=(\rho_0J+F(\theta))x,\\ \dot{\theta}=\omega.\end{cases}\tag{6.3}$$

Assume that the rotation number of (6.3) $\rho \in DC_\omega(\gamma, \tau)$ *and that* $F \in \mathfrak{B}_h$ *for* $h > 0$. *For any* $\overline{h} \in (0, \frac{h}{3})$, *if*

$$|F|_h \leq \delta_0 \left(\frac{1}{200\gamma\tau} \min\left\{ \frac{h - 3\overline{h}}{3}, 1 \right\} \right)^{120\tau^2}, \tag{6.4}$$

where δ_0 *is some sufficiently small positive universal constant, one can find* $\varphi \in \mathcal{C}_{\overline{h}}$ *with* $\widehat{\varphi}(0) = \rho$, $Y \in \mathfrak{B}_{\overline{h}}$, *such that system* (6.3) *is conjugated by* e^Y *to*

$$\begin{cases} \dot{x} = \varphi(\theta) J x, \\ \dot{\theta} = \omega. \end{cases} \tag{6.5}$$

Proof We firstly consider the case $\rho \in [-1, 1]$. Let $\widetilde{F} = F + (\rho_0 - \rho) J$. Then $|\widetilde{F}|_h \leq 2c|F|_h$ for $\rho_0 - \rho \leq c|F|_h$, where $c > 1$ is some universal constant. The system (6.3) can be rewritten as

$$\begin{cases} \dot{x} = (\rho J + \widetilde{F}(\theta)) x, \\ \dot{\theta} = \omega. \end{cases} \tag{6.6}$$

Without loss of generality, we assume that $h \in (0, 1)$.

For $n = 0, 1, 2, \ldots$, we define

$$K_n = 8^{2\tau^2} \gamma^{3\tau} q_n^{2\tau^2},$$

and

$$\begin{cases} L_n = 1, & \text{when} \quad q_{n+1} \leq (8^\tau \gamma q_n^\tau)^2; \\ L_n = 8^\tau \gamma q_n^\tau, & \text{when } q_{n+1} > (8^\tau \gamma q_n^\tau)^2. \end{cases}$$

One can easily check that

$$q_{n+1}/L_n \geq q_{n+1}^{1/2}. \tag{6.7}$$

For any $k \in \mathbb{Z}^2$ satisfying $|k| < q_{n+1}/L_n$, by (4.1) we have

$$|\langle k, \omega \rangle| \geq 1/2q_{n+1}, \tag{6.8}$$

and by Lemma 6.1 we have

$$|\langle k, \omega \rangle - 2\rho| \geq 1/K_n. \tag{6.9}$$

We also give some other iteration sequences inductively as follows: Let $\varepsilon_0 = 2c|F|_h$, $h_0 = h$ and $r_0 = \frac{1}{2}(h - \overline{h})$. Suppose that ε_n, h_n, r_n are defined. We then define $h_{n+1}, \varepsilon_{n+1}, r_{n+1}$ as follows:

(a) If $\varepsilon_n < e^{-\frac{q_{n+1}r_n}{4L_n}}$, let

$$\varepsilon_{n+1} = \varepsilon_n, \qquad h_{n+1} = h_n, \quad \text{and} \quad r_{n+1} = r_n.$$

(b) If $\varepsilon_n \geq e^{-\frac{q_{n+1}r_n}{4L_n}}$, let

$$r_{n+1} = \frac{r_n}{2}$$

and then let

$$h_{n+1} = h_n - \frac{r_n}{2} = h_n - r_{n+1}, \quad \text{and} \quad \varepsilon_{n+1} = \varepsilon_n e^{-\frac{q_{n+1}r_n}{8L_n}} = \varepsilon_n e^{-\frac{q_{n+1}r_{n+1}}{4L_n}}.$$

Under the assumption (6.4), one can prove inductively that

$$\varepsilon_n \leq \min\left\{e^{-\frac{q_n r_n}{4L_{n-1}}}, \left(\frac{r_n}{50\gamma}\right)^{60\tau^2}\right\} \leq \min\{e^{-\frac{1}{4}q_n^{1/2}r_n}, r_n^2\}. \tag{6.10}$$

Thus we have

$$\begin{aligned}\max\{K_n^5, L_n^5\}\varepsilon_n &\leq (8^{2\tau^2}\gamma^{3\tau}q_n^{2\tau^2})^5\varepsilon_n \\ &\leq \varepsilon_n^{1/2}(8^{2\tau^2}\gamma^{3\tau})^5(q_n^{1/2}e^{-\frac{q_n^{1/2}r_n}{80\tau^2}})^{20\tau^2} \\ &\leq \varepsilon_n^{1/2}\left(\frac{50\gamma\tau}{r_n}\right)^{60\tau^2} \leq 1,\end{aligned}$$

which implies that

$$\varepsilon_n \leq \min\{K_n^{-5}, L_n^{-5}\}. \tag{6.11}$$

We iteratively prove the conclusion as in the proof of Theorem 1.2. Let $\varphi_0 = \rho$, $F_0 = \widetilde{F}$. For $|\Pi_\omega\varphi_0|_{h_0} = |\Pi_\omega\rho|_{h_0} = 0$, it is obvious that

$$\varepsilon_0 \leq e^{-18|\Pi_\omega\varphi_0|_{h_0}} = 1. \tag{6.12}$$

Assume that we have obtained the system

$$\begin{cases}\dot{x} = (\varphi_n(\theta)J + F_n(\theta))x, \\ \dot{\theta} = \omega,\end{cases} \tag{6.13}$$

with $\varphi_n \in \mathcal{T}_{q_n}\mathcal{C}_{h_n}$ and $F_n \in \mathfrak{B}_{h_n}$, satisfying $\widehat{\varphi}_n(0) = \rho$ which is the rotation number of the system (6.13) and

$$|F_n|_{h_n} \leq \varepsilon_n \leq e^{-18|\Pi_\omega\varphi_n|_{h_n}}. \tag{6.14}$$

We shall prove that the system (6.13) can be conjugated to

$$\begin{cases} \dot{x} = (\varphi_{n+1}(\theta) J + F_{n+1}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.15}$$

with desired estimates on F_{n+1} and φ_{n+1}. The proof is summarized in the following lemma:

Lemma 6.2 *There exists $Y_n \in \mathfrak{B}_{h_{n+1}}$ such that system (6.13) is conjugated to system (6.15) by e^{Y_n}, where $\varphi_{n+1} \in \mathcal{T}_{q_{n+1}} \mathcal{C}_{h_{n+1}}$, $F_{n+1} \in \mathfrak{B}_{h_{n+1}}$ and*

$$\widehat{\varphi}_{n+1}(0) = \rho, \qquad |\varphi_{n+1} - \varphi_n|_{h_{n+1}} \leq 2\varepsilon_n^{8/9},$$
$$|F_{n+1}|_{h_{n+1}} \leq \varepsilon_{n+1} \leq e^{-18|\Pi_\omega \varphi_{n+1}|_{h_{n+1}}}, \qquad |Y_n|_{h_{n+1}} \leq \varepsilon_n^{1/4}.$$

Proof If $\varepsilon_n < e^{-\frac{q_{n+1} r_n}{4L_n}}$, we shall do nothing. In other words, we just let $Y_n = 0$, $\varphi_{n+1} = \varphi_n$, $F_{n+1} = F_n$.

So we only need to consider the case $\varepsilon_n \geq e^{-\frac{q_{n+1} r_n}{4L_n}}$. Note that by (6.7), (6.10), (6.11), the definitions and the assumptions at the nth step, we have the inequality

$$e^{-\frac{q_{n+1} r_n}{4L_n}} \leq \varepsilon_n \leq \min\{K_n^{-5}, L_n^{-5}, r_n^2\}. \tag{6.16}$$

Firstly, let $\psi_n = \Pi_\omega \varphi_n$, then system (6.13) is conjugated by $e^{\psi_n J}$ to

$$\begin{cases} \dot{x} = (\rho J + F_{n,1}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.17}$$

where $F_{n,1} = e^{-\psi_n J} F_n e^{\psi_n J}$. By (6.14), there is

$$e^{2|\Pi_\omega \varphi_n|_{h_n}} \leq \varepsilon_n^{-1/18} \tag{6.18}$$

and then

$$|F_{n,1}|_{h_n} \leq \varepsilon_n e^{2|\psi_n|_{h_n}} \leq \varepsilon_n e^{2|\Pi_\omega \varphi_n|_{h_n}} \leq \varepsilon_n^{8/9}.$$

Since $1/K_n \geq (\varepsilon_n^{8/9})^{1/4} = \varepsilon_n^{2/9}$, by Corollary 3.2, there exists $Y_{n,1} \in \mathfrak{B}_{h_n}$ such that system (6.17) is conjugated by $e^{Y_{n,1}}$ to

$$\begin{cases} \dot{x} = (\rho J + \widetilde{\varphi}_n(\theta) J + F_{n,2}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.19}$$

where $\widetilde{\varphi}_n \in \mathcal{T}_{q_{n+1}/L_n} \mathcal{C}_h$ and $F_{n,2} \in \mathcal{R}_{q_{n+1}/L_n} \mathfrak{B}_{h_n}$, with the estimates

$$|Y_{n,1}|_{h_n} \leq \varepsilon^{4/9} \quad \text{and} \quad |\widetilde{\varphi}_n|_{h_n}, |F_{n,2}|_{h_n} \leq 2\varepsilon_n^{8/9}.$$

Moreover, we have the estimate

$$\begin{aligned}\widetilde{\varepsilon}_n \triangleq |F_{n,2}|_{h_{n+1}} &\le \frac{36}{(h_n-h_{n+1})^2}\varepsilon_n^{8/9}e^{-\frac{q_{n+1}(h_n-h_{n+1})}{2L_n}}\\ &\le \frac{1}{2}\varepsilon_n^{7/9}e^{-\frac{q_{n+1}(h_n-h_{n+1})}{2L_n}}\\ &\le \frac{1}{2}\varepsilon_n^{7/9}e^{-\frac{q_{n+1}r_n}{4L_n}}.\end{aligned}$$

Since $e^{Y_{n,1}}$ is close to I, the rotation number of system (6.19) is still ρ (see Appendix C). Thus the difference between ρ and the rotation number of the system

$$\begin{cases}\dot{x} = (\rho J + \widetilde{\varphi}_n(\theta)J)x,\\ \dot{\theta} = \omega\end{cases} \tag{6.20}$$

is no more than $c|e^{-\Pi_\omega\widetilde{\varphi}_n J}F_{n,2}e^{\Pi_\omega\widetilde{\varphi}_n J}|_{C^0} \le c|F_{n,2}|_{C^0} \le c\widetilde{\varepsilon}_n$, which means that $|\widehat{\widetilde{\varphi}}_n(0)| \le c\widetilde{\varepsilon}_n$. Let $\varphi_{n,3} = \widetilde{\varphi}_n - \widehat{\widetilde{\varphi}}_n(0)$ and $F_{n,3} = F_{n,2} + \widehat{\widetilde{\varphi}}_n(0)J$, thus the system (6.19) can be rewritten as

$$\begin{cases}\dot{x} = (\rho J + \varphi_{n,3}(\theta)J + F_{n,3}(\theta))x,\\ \dot{\theta} = \omega,\end{cases} \tag{6.21}$$

with the estimates

$$|\varphi_{n,3}|_{h_n} \le |\widetilde{\varphi}_n|_{h_n} \le 2\varepsilon_n^{8/9} \quad \text{and} \quad |F_{n,3}|_{h_{n+1}} \le 2c\widetilde{\varepsilon}_n.$$

Finally, we use $e^{-\psi_n J}$ to conjugate system (6.21) back to

$$\begin{cases}\dot{x} = (\varphi_{n+1}(\theta)J + F_{n+1}(\theta))x,\\ \dot{\theta} = \omega,\end{cases} \tag{6.22}$$

where $\varphi_{n+1} = \varphi_n + \varphi_{n,3}$ (note that $\widehat{\varphi}_{n+1}(0) = \rho$) and $F_{n+1} = e^{\psi_n J}F_{n,3}e^{-\psi_n J}$, with

$$|\varphi_{n+1} - \varphi_n|_{h_{n+1}} \le |\varphi_{n,3}|_{h_{n+1}} \le 2\varepsilon_n^{8/9},$$

and (by (6.7))

$$\begin{aligned}|F_{n+1}|_{h_{n+1}} &\le 2c\widetilde{\varepsilon}_n e^{2|\psi_n|_{h_n}}\\ &\le c\varepsilon_n^{2/3}e^{-\frac{q_{n+1}r_n}{4L_n}}\\ &\le \varepsilon_n^{1/2}e^{-\frac{q_{n+1}r_n}{4L_n}}\\ &\le \varepsilon_n^{1/2}e^{-\frac{q_{n+1}r_n}{8L_n}}e^{-\frac{q_{n+1}r_n}{8L_n}}\\ &\le \varepsilon_n e^{-\frac{q_{n+1}r_n}{8L_n}} = \varepsilon_{n+1}.\end{aligned}$$

To complete the proof, we need to check two more facts. Firstly, by (6.18), there is

$$\begin{aligned}|e^{-\psi_n J} e^{Y_{n,1}} e^{\psi_n J} - I|_{h_{n+1}} &= |e^{-\psi_n J}(e^{Y_{n,1}} - I)e^{\psi_n J}|_{h_{n+1}} \\ &\le 2\varepsilon_n^{4/9} e^{2|\psi_n|_{h_n}} \le 2\varepsilon_n^{1/3}.\end{aligned}$$

Thus there exists $Y_n \in \mathfrak{B}_{h_{n+1}}$, such that $e^{Y_n} = e^{-\psi_n J} e^{Y_{n,1}} e^{\psi_n J}$, with $|Y_n|_{h_{n+1}} \le \varepsilon_n^{1/4}$. Secondly,

$$\begin{aligned}|F_{n+1}|_{h_{n+1}}^{1/18} e^{|\Pi_\omega \varphi_{n+1}|_{h_{n+1}}} &\le |F_{n+1}|_{h_{n+1}}^{1/18} e^{|\psi_n|_{h_n} + |\Pi_\omega \varphi_{n,3}|_{h_{n+1}}} \\ &\le \varepsilon_n^{1/18} e^{-\frac{q_{n+1} r_n}{8\times 18 L_n}} e^{|\psi_n|_{h_n}} e^{4q_{n+1}\varepsilon_n^{8/9}} \\ &= \varepsilon_n^{1/18} e^{|\psi_n|_{h_n}} e^{-\frac{q_{n+1} r_n}{8\times 18 L_n}} e^{4q_{n+1}\varepsilon_n^{8/9}} \\ &\le e^{-\frac{q_{n+1} r_n}{8\times 18 L_n}} e^{4q_{n+1}\varepsilon_n^{8/9}} \\ &\le e^{-\frac{q_{n+1}}{8\times 18 L_n}\{r_n - 32\times 18 L_n \varepsilon_n^{8/9}\}} \\ &\le e^{-\frac{q_{n+1}}{8\times 18 L_n}\{r_n - \varepsilon_n^{1/2}\}} \le 1\end{aligned}$$

since $|\Pi_\omega \varphi_{n,3}|_{h_{n+1}} \le 2q_{n+1}|\varphi_{n,3}|_{h_n}$. □

Finally, we define

$$\varphi = \lim_{n\to\infty} \varphi_n, \tag{6.23}$$

and define Y so that

$$e^Y = \lim_{n\to\infty} e^{Y_0} e^{Y_1} \cdots e^{Y_n}. \tag{6.24}$$

Thus the conclusion is proved as $\rho \in [-1, 1]$.

If ρ is not in $[-1, 1]$, there is $k_* \in \mathbb{Z}^2$ such that

$$|\langle k_*, \omega\rangle - 2\rho| < 1, \quad \text{and} \quad |\langle k_*, \omega\rangle - 2\rho| \ge 1 \quad \text{for } 0 < |k| < |k_*|.$$

Then, by Lemma 3.1, there exists $Y \in \mathfrak{B}_h$ which satisfies $|Y|_h \le \varepsilon^{1/2}$ and e^Y conjugates system (6.6) to[3]

$$\begin{cases}\dot{x} = (\rho J + \varphi(\theta) J + F_*(\theta))x, \\ \dot{\theta} = \omega,\end{cases} \tag{6.25}$$

[3]Here we assume that $\hat{\varphi}(0) = \hat{F}_*(0) = 0$ for simplicity.

where $\varphi \in \mathcal{T}_{|k_*|}\mathcal{C}_h$ and $F_* \in \mathcal{R}_{|k_*|}\mathfrak{B}_h$, with the estimate

$$|\varphi|_h, |F_*|_h \le 2\varepsilon.$$

Furthermore, we have

$$|F_*|_{h/3} \le \frac{36 \times 9}{4h^2}\varepsilon e^{-\frac{|k_*|h}{3}} \le \frac{1}{2}\varepsilon^{1/2} e^{-\frac{|k_*|h}{3}}.$$

Now one can use $\exp\{-\frac{k_*\theta}{2}J\}$ to conjugate system (6.25) to

$$\begin{cases} \dot{x} = ((\rho - \frac{\langle k_*, \omega\rangle}{2})J + \varphi(\theta)J + \widetilde{F}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.26}$$

where $\widetilde{F} = \exp\{-\frac{k_*\theta}{2}J\}F_*(\theta)\exp\{\frac{k_*\theta}{2}J\} \in \mathfrak{B}_{h/2}$. Thus

$$|\varphi J + \widetilde{F}|_{h/3} \le |\varphi|_h + e^{\frac{|k_*|h}{3}}|F_*|_{h/3} \le 2\varepsilon + \frac{1}{2}\varepsilon^{1/2} \le \varepsilon^{1/2}.$$

Let $\widetilde{\rho}$ be the rotation number of the system (6.26), we have

$$|\widetilde{\rho}| \le \left|\rho - \frac{\langle k_*, \omega\rangle}{2}\right| + \varepsilon^{1/2} < \frac{1}{2} + \varepsilon^{1/2} \le 1.$$

The system is reduced to the case we have studied. $\square$

Now we are in the position to prove Theorems 1.3–1.5. Let us consider the system

$$\begin{cases} \dot{x} = (A + F(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.27}$$

where $A \in sl(2, \mathbb{R})$ is a general matrix which is not necessarily a rotation. Recall that the rotation number is Diophantine w.r.t. ω as we assumed in Theorems 1.3–1.5.

Proof of Theorem 1.3 We use Theorem 1.2 and also its proof in Sect. 5 to do it. There exists $\delta > 0$ such that if $|F|_h \le \delta$, system (6.27) can be conjugated to

$$\begin{cases} \dot{x} = (A^{(n)} + F^{(n)}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.28}$$

where $F^{(n)} \in \mathfrak{B}_{h_n}$ satisfies $|F^{(n)}|_{h_n} \le \varepsilon_n$ and $A^{(n)}$ is either of the form $\rho_n J$ ($\rho_n \in \mathbb{R}$, elliptic case), or of the form $\begin{pmatrix} \lambda_n & 0 \\ 0 & -\lambda_n \end{pmatrix}$ ($\lambda_n \in \mathbb{R}$, hyperbolic case). For hyperbolic $A^{(n)}$, we always have

$$|\lambda_n| \leq \frac{1}{2}\varepsilon_n^{1/4},$$

then $A+F$ can be all viewed as a perturbation to $0 = 0J$. Otherwise, by Remark 5.3, system (6.28) is uniformly hyperbolic, which contradicts the assumption that the rotation number is irrational w.r.t. ω (by Lemma 9.1 in Appendix C, the rotation number is invariant modulo $\{\frac{1}{2}\langle k, \omega\rangle : k \in \mathbb{Z}^2\}$). In any case system (6.28) can be rewritten as

$$\begin{cases} \dot{x} = (\rho_{n,1} + F^{(n,1)}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.29}$$

where $\rho_{n,1} \in \mathbb{R}$ and $F^{(n,1)} \in \mathfrak{B}_{h_n}$ satisfies $|F^{(n,1)}|_{h_n} \leq \varepsilon_n^{1/4}$.

Moreover, from the iteration procedure of the proof in Sect. 5, one can choose a subsequence $\{n_j\}_{j=0}^{\infty}$ of $\mathbb{N}$, such that for any $j \geq 0$, there is

$$\big(A^{(n_j)}, F^{(n_j)}\big) = \big(A^{(n_j+1)}, F^{(n_j+1)}\big) = \cdots = \big(A^{(n_{j+1}-1)}, F^{(n_{j+1}-1)}\big).$$

Now we let $\widetilde{q}_j = q_{n_j}$, $\widetilde{q}_j^+ = q_{n_j+1}$, $\widetilde{\varepsilon}_j = \varepsilon_{n_j}$, $\widetilde{h}_j = h_{n_j}$. Note that we always have

$$(\widetilde{\varepsilon}_{j+1}, \widetilde{h}_{j+1}) = \left(\widetilde{\varepsilon}_j^{33/32} \exp\left\{-\frac{\widetilde{q}_j^+ \widetilde{h}_j}{65 \times 21 \times 4}\right\}, \widetilde{h}_j/4\right), \tag{6.30}$$

and for any $\chi > 1$,

$$\lim_{j\to\infty} \frac{\widetilde{\varepsilon}_j}{\widetilde{h}_j^{\chi}} = 0. \tag{6.31}$$

In summary, system (6.27) can be conjugated to a sequence of systems of the form

$$\begin{cases} \dot{x} = (\widetilde{\rho}_j^{(0)} J + \widetilde{F}_j(\theta))x, \\ \dot{\theta} = \omega \end{cases} \quad (j\text{th system}) \tag{6.32}$$

where $\widetilde{\rho}_j^{(0)} = \rho_{n_j,1}$ and $\widetilde{F}_j = F^{(n_j,1)}$ (in system (6.29)). Note that $\widetilde{F}_j$ satisfies

$$|\widetilde{F}_j|_{\widetilde{h}_j} \leq \widetilde{\varepsilon}_j^{1/4}.$$

Denote by $\widetilde{\rho}_j$ the rotation number of the jth system of (6.32).

We will prove in the end that one can choose a sequence of $\gamma_j > 1$, such that

$$\widetilde{\rho}_j \in DC(\gamma_j, \tau) \quad \text{and} \quad \gamma_{j+1} \le \left(\frac{c\widetilde{q}_j^{+}}{\widetilde{h}_j^{\sigma}} \exp\{\sigma \widetilde{q}_j^{+} \widetilde{\varepsilon}_j^{1/\sigma}\} \right)^{\tau} \gamma_j, \tag{6.33}$$

where $c, \sigma > 1$ are some universal constants. Now in condition (6.4) of Theorem 6.1, we choose $h = \widetilde{h}_j$ (note that $\widetilde{h}_j \le 1$ as we assume in Sect. 5) and $\overline{h} = \widetilde{h}_j/6$ to obtain a sequence of conditions

$$\nu_j \triangleq \delta_0 \left(\frac{\widetilde{h}_j}{1200\gamma_j \tau} \right)^{120\tau^2}.$$

By Theorem 6.1, to complete the proof, we then just need to find j, such that

$$|\widetilde{F}_j|_{\widetilde{h}_j} < \nu_j. \tag{6.34}$$

By (6.30) and (6.31),

$$\begin{aligned}
\frac{|\widetilde{F}_{j+1}|_{\widetilde{h}_{j+1}}/\nu_{j+1}}{|\widetilde{F}_j|_{\widetilde{h}_j}/\nu_j} &= \frac{|\widetilde{F}_{j+1}|_{\widetilde{h}_{j+1}}/|\widetilde{F}_j|_{\widetilde{h}_j}}{\nu_{j+1}/\nu_j} \\
&\le C \left(\frac{\widetilde{q}_j^{+}}{\widetilde{h}_j} \right)^{L} \exp\{L\widetilde{q}_j^{+} \widetilde{\varepsilon}_j^{1/L}\} \widetilde{\varepsilon}_j^{1/L} \exp\left\{ -\frac{\widetilde{q}_j^{+} \widetilde{h}_j}{L} \right\} \\
&= C \frac{\widetilde{\varepsilon}_j^{1/L}}{\widetilde{h}_j^{L}} \left(\widetilde{q}_j^{+} \exp\left\{ -\frac{\widetilde{q}_j^{+} \widetilde{h}_j}{2L^2} \right\} \right)^{L} \exp\left\{ L\widetilde{q}_j^{+} \widetilde{\varepsilon}_j^{1/L} - \frac{\widetilde{q}_j^{+} \widetilde{h}_j}{2L} \right\} \\
&\le C_1 \left(\frac{\widetilde{\varepsilon}_j}{\widetilde{h}_j^{2L^2}} \right)^{1/L} \exp\left\{ -L\widetilde{q}_j^{+} \left(\frac{\widetilde{h}_j}{2L^2} - \widetilde{\varepsilon}_j^{1/L} \right) \right\}
\end{aligned}$$

decreases to zero as $j \to \infty$, where constants C, C_1 and L depend only on τ. This fact implies that (6.34) can be verified for sufficiently large j.

Now we choose $\gamma_0, \gamma_1, \ldots$ and verify (6.33). Assume that $\rho_0 = \rho \in DC_\omega(\gamma, \tau)$. Firstly let $\gamma_0 = \gamma$. We assume that γ_j have been chosen. Let r_j be the degree of the conjugation map $B^{(n_j)}$ from the jth system to $(j+1)$th system in the form (6.32). By Lemma 9.1 in Appendix C, there is

$$\rho_{j+1} = \rho_j + \frac{\langle r_j, \omega \rangle}{2}.$$

If $r_j = (0,0)$, $\rho_{j+1} = \rho_j$ and we can choose $\gamma_{j+1} = \gamma_j$. If $r_j \neq (0,0)$, for $\rho_j \in DC_\omega(\gamma_j, \tau)$,

$$|2\rho_j - \langle k, \omega\rangle| \geq \frac{\gamma_j^{-1}}{|k|^\tau}, \quad \text{for any } 0 \neq k \in \mathbb{Z}^2.$$

We then have

$$|2\rho_{j+1} - \langle k, \omega\rangle| = |2\rho_j - \langle k + r_j, \omega\rangle| \geq \frac{\gamma_j^{-1}}{|r_j|^\tau |k|^\tau}, \quad \text{for any } 0 \neq k \in \mathbb{Z}^2.$$

So we can choose $\gamma_{j+1} = \gamma_j |r_j|^\tau$ and $\rho_{j+1} \in DC_\omega(\gamma_{j+1}, \tau)$. By Remark 5.4, there is

$$|r_j| \leq \frac{c\widetilde{q}_j}{\widetilde{h}_j^\sigma} \exp\{\sigma \widetilde{q}_j^{+} \widetilde{\varepsilon}_j^{1/\sigma}\}.$$

In this way, the sequence $\gamma_0, \gamma_1, \ldots$ satisfying (6.33) is chosen. □

Proof of Theorem 1.4 By Remark 6.2, we get Theorem 1.4 from Theorem 1.3. □

Theorem 6.1 remains true if we consider the perturbations to arbitrary constant system. In fact, we have the following conclusion:

Theorem 6.2 (Positive measure rotations reducibility) *Suppose that in* (6.27) $F \in \mathfrak{B}_h$ *and the rotation number* ρ *is in* $DC_\omega(\gamma, \tau)$. *For any* $\overline{h} \in (0, \frac{h}{3})$, *there exists* $\delta = \delta(A, h, 3h - \overline{h}, \gamma, \tau) > 0$ *such that system* (6.27) *can be conjugated to system* (6.5) *with* $\varphi \in \mathcal{C}_{\overline{h}}$ *if*

$$|F|_h \leq \delta.$$

Proof By footnote 2, we only need to consider the elliptic case $A = \rho J$ with $\rho \in \mathbb{R}$, and the hyperbolic case $A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ with $\lambda \in \mathbb{R}$. Now we assume that $\varepsilon = |F|_h$. For hyperbolic case, we always have

$$|\lambda| \leq \frac{1}{2}\varepsilon^{1/4},$$

then $A + F$ can be all viewed as a perturbation to $0 = 0J$. Otherwise, by Remark 5.3 the system is uniformly hyperbolic, which contradicts the assumption that the rotation number is irrational w.r.t. ω. In any case, system (6.27) can be rewritten as

$$\begin{cases} \dot{x} = (\widetilde{\rho} J + \widetilde{F}(\theta))x, \\ \dot{\theta} = \omega, \end{cases} \tag{6.35}$$

where $\widetilde{\rho} \in \mathbb{R}$, and $\widetilde{F} \in \mathfrak{B}_h$ satisfying $|\widetilde{F}|_h \leq \varepsilon^{1/4}$. Then it follows from Theorem 6.1. □

Proof of Theorem 1.5 Theorem 1.5 can be obtained from Remark 6.1 and Theorem 6.2. □

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Appendix A: Floquet theory

In this section, we present a result based on Floquet Theory and its proof.

Lemma 7.1 *We consider system*

$$\begin{cases} \dot{x} = F(\theta)x, \\ \dot{\theta} = \omega = (\alpha, 1), \end{cases} \tag{7.1}$$

where $F \in \mathfrak{B}_h$ satisfies $|F|_h < \varepsilon$ and is of the form

$$F(\theta_1, \theta_2) = \sum_{l \in \mathbb{Z}} \widehat{F}(lq, -lp) e^{il(q\theta_1 - p\theta_2)},$$

with $(q, -p) \in \mathbb{Z}^2$ fixed. Then one can find analytic $B : 2\mathbb{T}^2 \to SL(2, \mathbb{R})$ and $C \in sl(2, \mathbb{R})$ with the following properties:

(1) *$B(\theta)$ admits analytic extension in $|\mathrm{Im}\,\theta| < h$ with*

$$|B|_{C^0} \leq \exp\left\{\frac{8\pi |F|_h}{|\tau|}\right\} \quad \textit{and}$$

$$\sup_{|\mathrm{Im}\,\theta| < h} |B(\theta)| \leq \exp\left\{\frac{|F|_h}{|\tau|}(8\pi + (|q| + |p|)h)\right\}$$

where $\tau = q\alpha - p$;

(2) *C satisfies the estimates*

$$|C| < \frac{|\tau|}{2\pi} \exp\left\{\frac{4\pi |F|_h}{|\tau|}\right\} \quad \textit{and} \quad |spec(C)| \leq \frac{|\tau|}{4\pi} \max\{|F|_h, 2\pi\};$$

(3) *In* $|\mathrm{Im}\,\theta| < h$, *there is*

$$\partial_\omega B(\theta) \equiv CB(\theta) - B(\theta)F(\theta),$$

i.e., B *conjugates* (7.1) *to constant system*

$$\begin{cases} \dot{x} = Cx, \\ \dot{\theta} = \omega = (\alpha, 1). \end{cases}$$

Proof Let $\phi = q\theta_1 - p\theta_2$, $\widetilde{h} = (|q| + |p|)h$ and

$$G(\phi) = \sum_{l\in\mathbb{Z}} \widehat{F}(lq, -lp)e^{il\phi}.$$

Thus we have $F(\theta_1, \theta_2) = G(q\theta_1 - p\theta_2)$ and

$$\begin{aligned} \sup_{\phi\in\mathbb{T}^1, |\mathrm{Im}\,\phi|\le\widetilde{h}} |G(\phi)| &\le \sum_{l\in\mathbb{Z}} |\widehat{F}(lq, -lp)|e^{|l|\widetilde{h}} \\ &= \sum_{l\in\mathbb{Z}} |\widehat{F}(lq, -lp)|e^{|l|(|q|+|p|)h} = |F|_h. \end{aligned}$$

Let us consider the equation

$$\frac{dx}{dt} = G(\tau t)x, \tag{7.2}$$

with the basic matrix solution $\Phi(t)$ satisfying $\Phi(0) = I$. We have the estimate (by Gronwall inequality)

$$\sup_{0\le t\le \frac{4\pi}{|\tau|}} |\Phi(t)| \le \exp\left\{\frac{4\pi}{|\tau|}|F|_h\right\}.$$

The fact that G is $\frac{2\pi}{|\tau|}$-periodic implies that $\Phi(\frac{4\pi}{|\tau|}) = \Phi(\frac{2\pi}{|\tau|})^2$. Thus one can choose $C \in sl(2, \mathbb{R})$ satisfying $\Phi(\frac{4\pi}{|\tau|}) = e^{\frac{4\pi}{|\tau|}C}$ with the estimates

$$\left|\frac{4\pi}{|\tau|}C\right| \le 2\exp\left\{\frac{4\pi}{|\tau|}|F|_h\right\} \quad \text{and} \quad \left|spec\left(\frac{4\pi}{|\tau|}C\right)\right| \le \max\{|F|_h, 2\pi\},$$

i.e.,

$$|C| \le \frac{|\tau|}{2\pi}\exp\left\{\frac{4\pi}{|\tau|}|F|_h\right\} \quad \text{and} \quad |spec(C)| \le \frac{|\tau|}{4\pi}\max\{|F|_h, 2\pi\}.$$

Define $B_1(t) = e^{Ct}\Phi(t)^{-1}$, then $B_1(t) = B_1(t + \frac{4\pi}{|\tau|})$.

Moreover, Φ has analytic extension in $|\mathrm{Im}\, t| < \widetilde{h}/|\tau|$. In fact, it is easy to see that $\Phi(t+\mathrm{i}s)$, for any fixed $|s| < \widetilde{h}/|\tau|$, is the basic matrix solution of the equation

$$\frac{dx}{dt} = G(\tau t + \mathrm{i}s)x$$

with $\Phi(0+\mathrm{i}s) = \Phi(\mathrm{i}s)$. As in the case $s=0$, we have the estimate

$$\sup_{0\le t\le \frac{4\pi}{\tau}} |\Phi(t+\mathrm{i}s)| \le \exp\left\{\frac{4\pi}{|\tau|}|F|_h\right\}|\Phi(\mathrm{i}s)|.$$

In the same way, from the equation

$$\frac{dx}{d(\mathrm{i}s)} = G(\mathrm{i}s)x,$$

we obtain the estimate

$$|\Phi(\mathrm{i}s)| \le \exp\{|s||F|_h\}.$$

Hence,

$$\sup_{0\le t\le \frac{4\pi}{|\tau|}} |\Phi(t+\mathrm{i}s)| \le \exp\left\{\left(\frac{4\pi}{|\tau|}+|s|\right)|F|_h\right\}.$$

In other words, the matrix function $\Phi(t)$ satisfies the estimate

$$\sup_{0\le|\mathrm{Re}\, t|\le 4\pi/|\tau|, |\mathrm{Im}\, t|<\widetilde{h}/|\tau|} |\Phi(t)| \le \exp\left\{\frac{|F|_h}{|\tau|}\{4\pi + (|q|+|p|)h\}\right\}.$$

Thus $B_1(t) = e^{Ct}\Phi(t)^{-1}$ has analytic extension in $|\mathrm{Im}\, t| < \widetilde{h}/|\tau|$ with the estimate (notice that B_1 is $\frac{4\pi}{|\tau|}$-periodic and $|M| = |M^{-1}|$ for any $M \in SL(2,\mathbb{C})$)

$$|B_1|_{C^0} \le \exp\left\{\frac{8\pi|F|_h}{|\tau|}\right\} \quad \text{and}$$

$$\sup_{|\mathrm{Im}\, t|<\widetilde{h}/|\tau|} |B_1(t)| \le \exp\left\{\frac{|F|_h}{|\tau|}(8\pi + (|q|+|p|)h)\right\}.$$

One can check easily that B_1 conjugates (7.2) to some equation with constant coefficient. More precisely, we have

$$\frac{d}{dt}B_1(t)B_1(t)^{-1} + B_1(t)G(\tau t)B_1(t)^{-1} \equiv C$$

in the domain $|\mathrm{Im}\, t| < \widetilde{h}/|\tau|$. In other words,

$$\tau \frac{d}{d\phi}\widetilde{B}_1(\phi)\widetilde{B}_1(\phi)^{-1} + \widetilde{B}_1(\phi)G(\phi)\widetilde{B}_1(\phi)^{-1} \equiv C,$$

where $\widetilde{B}_1(\phi) = B_1(\phi/\tau)$, in the domain $|\mathrm{Im}\, t| < \widetilde{h}$. It is obvious that $\widetilde{B}_1$ is 4π-periodic.

Let $B_2(\theta_1, \theta_2) = \widetilde{B}_1(q\theta_1 - p\theta_2)$. We see that B_2 is analytic in $|\mathrm{Im}\, \theta| < h$ and

$$|B_2|_{C^0} \le \exp\left\{\frac{8\pi |F|_h}{|\tau|}\right\} \quad \text{and}$$

$$\sup_{|\mathrm{Im}\,\theta|<h} |B_2(\theta)| \le \exp\left\{\frac{|F|_h}{|\tau|}\{8\pi + (|q| + |p|)h\}\right\}.$$

Moreover,

$$\partial_\omega B_2(\theta)B_2(\theta)^{-1} + B_2(\theta)F(\theta)B_2(\theta)^{-1} \equiv C$$

in the domain $|\mathrm{Im}\, \theta| < h$. □

Appendix B: Normal form

In this section, we give some estimates on conjugating a matrix to normal form.

Lemma 8.1 *Let $A \in sl(2,\mathbb{R})$ satisfy $spec(A) = \{\mathrm{i}\rho, -\mathrm{i}\rho\}$ with $0 \ne \rho \in \mathbb{R}$. There exists $P \in SL(2,\mathbb{R})$ such that $|P| \le 2(|A|/\rho)^{1/2}$ and that $PAP^{-1} = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$.*

Proof There exist $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ with $|u| = |v| = 1$ such that $Au = -\rho v$ and $Av = \rho u$. Let $\theta = \angle(u, v)$, $0 \le \theta \le \pi/2$. Without loss of generality, we assume that $\langle u, v\rangle = \cos\theta$.

For any $t \in \mathbb{R}$, we have that $\langle u + tv, u + tv\rangle = 1 + t^2 + 2t\cos\theta$ and $\langle A(u + tv), A(u + tv)\rangle = \rho^2(1 + t^2 - 2t\cos\theta)$. Thus

$$|A| = \sup_{t\in\mathbb{R}} |\rho|\sqrt{\frac{1 + t^2 - 2t\cos\theta}{1 + t^2 + 2t\cos\theta}} \triangleq |\rho| \sup_{t\in\mathbb{R}} \sqrt{f(t)} = |\rho| \sqrt{\sup_{t\in\mathbb{R}} f(t)}.$$

By simple computation, one can see that

$$\sup_{t\in\mathbb{R}} f(t) = f(-1) = \frac{1 + \cos\theta}{1 - \cos\theta} = \frac{(1 + \cos\theta)^2}{\sin^2\theta}.$$

Thus we have

$$\sin\theta = \frac{|\rho|}{|A|}(1+\cos\theta) \geq \frac{|\rho|}{|A|}.$$

Now let $\widetilde{P} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$, then $\widetilde{P}A\widetilde{P}^{-1} = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$. Recall that $|\det\widetilde{P}| = |u||v||\sin\theta| = \sin\theta$, so $P = \pm(\sin\theta)^{-\frac{1}{2}}\widetilde{P} \in SL(2,\mathbb{R})$ satisfies $|P| \leq 2(|A|/|\rho|)^{1/2}$ and $PAP^{-1} = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$. □

Lemma 8.2 *Let $\epsilon > 0$. Assume that $A \in sl(2,\mathbb{R})$ and $|A| \leq 1/\epsilon$. Then there exist $P \in SL(2,\mathbb{R})$ with $|P| \leq 2(1/\epsilon)^3$, $D \in sl(2,\mathbb{R})$ with $|D| \leq 2\epsilon$ and $\Delta = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ or $\Delta = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$, such that $PAP^{-1} = \Delta + D$. Here $\lambda, \rho \in \mathbb{R}$ satisfy $||spec(\Delta)| - |spec(A)|| \leq \epsilon$, i.e., $||\lambda| - |spec(A)|| \leq \epsilon$ or $||\rho| - |spec(A)|| \leq \epsilon$.*

Proof We consider firstly the case of $spec(A) = \{\lambda, -\lambda\} \subseteq \mathbb{R}$. In this case, one can find some $M \in SO(2,\mathbb{R})$, such that

$$MAM^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \triangleq \Delta + C.$$

It is obvious that $|c| \leq |A| \leq 1/\epsilon$. Now let

$$K = \begin{pmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{pmatrix},$$

then we have

$$KMAM^{-1}K^{-1} = K(\Delta + C)K^{-1} = \Delta + KCK^{-1},$$

where

$$D \triangleq KCK^{-1} = \begin{pmatrix} 0 & \epsilon^2 c \\ 0 & 0 \end{pmatrix}.$$

It is obvious that $|KM| \leq |K| \leq 1/\epsilon$ and $|D| \leq \epsilon$.

When $spec(A) = \{\mathrm{i}\rho, -\mathrm{i}\rho\}$ with $0 \neq \rho \in \mathbb{R}$, without loss of generality, we assume that

$$A = \begin{pmatrix} \lambda & u \\ -v & -\lambda \end{pmatrix}$$

with $0 < v \leq u \leq |A| \leq 1/\epsilon$.

If $v < \epsilon^3$, let

$$A_1 = \begin{pmatrix} \lambda & u \\ 0 & -\lambda \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 0 \\ -v & 0 \end{pmatrix}.$$

For $spec(A_1) = \{\lambda, -\lambda\} \subseteq \mathbb{R}$, there exist $K_1 \in SL(2,\mathbb{R})$ satisfying $|K_1| \leq 1/\epsilon$ and $D_1 \in sl(2,\mathbb{R})$ satisfying $|D_1| \leq \epsilon$, such that

$$K_1 A_1 K_1^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} + D_1 \triangleq \Delta + D_1.$$

Let

$$D = D_1 + K_1 C_1 K_1^{-1},$$

we then have $|D| \leq |D_1| + |K_1|^2 |C_1| \leq 2\epsilon$. In this case, one can check that $||\lambda| - |\rho|| \leq \epsilon$.

If $v \geq \epsilon^3$, let

$$L_1 = \begin{pmatrix} (v/u)^{1/4} & 0 \\ 0 & (u/v)^{1/4} \end{pmatrix}.$$

It is obvious that $|L_1| \leq (|A|/\epsilon^3)^{1/4} \leq 1/\epsilon$. One can see that

$$E_1 \triangleq L_1 A L_1^{-1} = \begin{pmatrix} \lambda & \sqrt{uv} \\ -\sqrt{uv} & -\lambda \end{pmatrix}.$$

Now we have $uv = \lambda^2 + \rho^2$. Moreover, it is obvious that $\min\{|\lambda - \sqrt{uv}|, |\lambda + \sqrt{uv}|\} \leq \rho$.

Under the assumption $v \geq \epsilon^3$, there are furthermore two cases:

If $\rho < \epsilon^3$, there is $|d| < \epsilon^3$, such that

$$E_1 = \lambda \begin{pmatrix} 1 & \pm 1 \\ \mp 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \triangleq E_2 + G,$$

For $spec(E_2) = \{0\} \subseteq \mathbb{R}$, there exist $K_2 \in SL(2,\mathbb{R})$ satisfying $|K_2| \leq 1/\epsilon$. Let $D_2 = K_2 E_2 K_2^{-1}$. Then there is $|D_2| \leq \epsilon$. Now we let

$$\Delta = 0 \quad \text{and} \quad D = D_2 + K_2 G K_2^{-1}.$$

We then have $|D| \leq |D_2| + |K_2|^2 |G| \leq 2\epsilon$.

Otherwise, if $\rho \geq \epsilon^3$, by Lemma 8.1 there exists $P_1 \in SL(2,\mathbb{R})$ such that $|P_1| \leq 2(|A|/\rho)^{1/2} \leq 2(1/\epsilon)^2$ and let

$$\Delta = P_1 E_1 P_1^{-1} = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix} \quad \text{and} \quad D = 0. \qquad \square$$

Appendix C: Degree and rotation number

In this section we discuss the impact of conjugation maps on the rotation number. A continuous map $B : 2\mathbb{T}^2 \to SL(2,\mathbb{R})$ is said to be of degree $r =$

$(r_1, r_2) \in \mathbb{Z}^2$, if $B(\cdot, 0)$ is homotopic to $R(\cdot)^{r_1}$ and $B(0, \cdot)$ is homotopic to $R(\cdot)^{r_2}$, where

$$R(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Recall that the fundamental group of $SL(2, \mathbb{R})$ is $\mathbb{Z}$ with generator R.

Lemma 9.1 *Let $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ be irrational. Assume that a C^1 conjugation map $B : 2\mathbb{T}^2 \to SL(2, \mathbb{R})$ of degree r conjugates system*

$$\begin{cases} \dot{x} = A(\theta)x, \\ \dot{\theta} = \omega, \end{cases} \tag{9.1}$$

with the rotation number ρ to system

$$\begin{cases} \dot{x} = \widetilde{A}(\theta)x, \\ \dot{\theta} = \omega, \end{cases} \tag{9.2}$$

with rotation number $\widetilde{\rho}$. Then we have

$$\widetilde{\rho} = \rho + \frac{1}{2}\langle r, \omega\rangle.$$

Proof Without loss of generality, assume that $\omega_2 > 0$. Let $\Phi(\cdot, t)$ and $\widetilde{\Phi}(\cdot, t)$ be basic matrix solutions of (9.1) and (9.2) respectively. Consider the Poincaré Cocycles

$$\left(\frac{4\pi\omega_1}{\omega_2}, \Psi\right) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2,$$

$$(\phi, x) \mapsto \left(\phi + \frac{4\pi\omega_1}{\omega_2}, \Phi\left((\phi, 0), \frac{4\pi}{\omega_2}\right)x\right)$$

and

$$\left(\frac{4\pi\omega_1}{\omega_2}, \widetilde{\Psi}\right) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2,$$

$$(\phi, x) \mapsto \left(\phi + \frac{4\pi\omega_1}{\omega_2}, \widetilde{\Phi}\left((\phi, 0), \frac{4\pi}{\omega_2}\right)x\right).$$

By the definition of the rotation number of cocycles [5, 24], $(\frac{4\pi\omega_1}{\omega_2}, \Psi)$ and $(\frac{4\pi\omega_1}{\omega_2}, \widetilde{\Psi})$ have rotation numbers $\frac{4\pi\rho}{\omega_2}$ and $\frac{4\pi\widetilde{\rho}}{\omega_2}$ respectively. Moreover $W(\phi) = B(\phi, 0) = B(\phi, \frac{4\pi}{\omega_2})$ defined on $2\mathbb{T}$ conjugates $(\frac{4\pi\omega_1}{\omega_2}, \Psi)$ to

$(\frac{4\pi\omega_1}{\omega_2}, \widetilde{\Psi})$. Recall that $W(\phi)$ is homotopic to $R(\cdot)^{r_1}$. By Proposition 1.1 in [24] and its remarks, we have

$$\frac{4\pi\widetilde{\rho}}{\omega_2} = \frac{4\pi\rho}{\omega_2} + \frac{2r_1\pi\omega_1}{\omega_2} \mod 2\pi\mathbb{Z},$$

i.e.,

$$\widetilde{\rho} = \rho + \frac{r_1\omega_1 + l\omega_2}{2}, \quad \text{for some } l \in \mathbb{Z}.$$

In the same way, one can show that

$$\widetilde{\rho} = \rho + \frac{\widetilde{l}\omega_1 + r_2\omega_2}{2}, \quad \text{for some } \widetilde{l} \in \mathbb{Z}.$$

Recall that ω is irrational, we thus have $\widetilde{l} = r_1$ and $l = r_2$, i.e.,

$$\widetilde{\rho} = \rho + \frac{\langle r, \omega\rangle}{2}.$$

□

The following Lemma shows that the degree of a conjugation map is controlled by its C^1-norm.

Lemma 9.2 *For any C^1 map $B : 2\mathbb{T}^2 \to SL(2, \mathbb{R})$ with the degree r, we have*

$$|r| \le c|B|_{C^1}^{\sigma},$$

where $c, \sigma > 1$ are some universal constants.

Proof We only need to prove that, for any C^1 map $W : 2\mathbb{T} \to SL(2, \mathbb{R})$ which is homotopic to R^s,

$$|s| \le c|W|_{C^1}^{\sigma},$$

where $c, \sigma > 1$ are universal constants. In fact, we define the path $w : 2\mathbb{T} \to \mathbb{S}^1 \subseteq \mathbb{R}^2$, where $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, as

$$w(\phi) = \frac{(1, 0)W(\phi)}{|(1, 0)W(\phi)|},$$

(the norm $|\cdot|$ here is the standard Euclidean norm of $\mathbb{R}^2$). It is obvious that w is homotopic to the path $e : 2\mathbb{T} \to \mathbb{S}^1 \subseteq \mathbb{R}^2$ defined as

$$e(\phi) = \frac{(1, 0)R(\phi)^s}{|(1, 0)R(\phi)^s|} = \left(\cos\frac{s\phi}{2}, \sin\frac{s\phi}{2}\right).$$

Thus the degree of w is also s, so the length of it is not less than $2\pi|s|$, i.e.,

$$4\pi|w'|_{C^0} \geq \int_0^{4\pi} |w'(\phi)|d\phi \geq 2\pi|s|,$$

which implies $|s| \leq 2|w'|_{C^0} \leq c|W|_{C^1}^{\sigma}$, where $c, \sigma > 1$ are some universal constants. □

By Cauchy inequality, we have a corollary as follows:

Corollary 9.1 *Let $h > 0$. For any map $B : 2\mathbb{T}^2 \to SL(2,\mathbb{R})$ analytic in $|\mathrm{Im}\,\theta| < h$ with the degree r, we have*

$$|r| \leq \frac{c}{\min\{1, h^{\sigma}\}} \Big(\sup_{|\mathrm{Im}\,\theta|<h} |B(\theta)| \Big)^{\sigma},$$

where $c, \sigma > 1$ are some universal constants.

Added in the Proof With minor modification in the proof of Lemma 5.2, one can get almost reducibility (Theorem 1.2) in a stronger sense: the analytic radius h_n of F_n does not tend to zero. More precisely, $h_n > h - \delta$ for arbitrarily small δ (see the definition before Theorem 1.2).

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