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On the Application of KAM Theory to Discontinuous Dynamical Systems

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So far the application of Kolmogorov–Arnold–Moser (KAM) theory has been restricted to smooth dynamical systems. Since there are many situations which can be modeled only by differential equations containing discontinuous terms such as state-dependent jumps (e.g., in control theory or nonlinear oscillators), it is shown by a series of transformations how KAM theory can be used to analyze the dynamical behaviour of such discontinuous systems as well. The analysis is carried out for the example

$$\ddot{x} + x + a\,\mathrm{sgn}(x) = p(t)$$

with $p \in C^6$ being periodic. It is known that all solutions are unbounded for small $a > 0$. We prove that all solutions are bounded for $a > 0$ sufficiently large, and that there are infinitely many periodic and quasiperiodic solutions in this case.

1. INTRODUCTION AND MAIN RESULT

Classical Kolmogorov–Arnold–Moser (KAM) theory deals with the analysis of smooth dynamical systems, but in many applications discontinuous or non-smooth systems play an important role, e.g., in control theory, dynamical contact problems, or in oscillators where state-dependent kicks have to be included. In this paper we look at this new problem from the point of view of dynamical systems. We address similar questions as they are investigated in the smooth case $\ddot{x} + f(x) = p(t)$, i.e., we ask if all

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solutions are bounded, or whether there are "many" periodic and quasiperiodic solutions (cf. Littlewood's question [13] and [12, 14, 5, 8, 9, 10, 17, 18, 11]).

We will illustrate our results by investigating a forced oscillator equation, where a nonlinearity is introduced by a jump of the restoring force at $x=0$, i.e.,

$$\ddot{x} + f_a(x) = p(t) \tag{1}$$

with $p \in C^6(\mathbb{R})$ being ω-periodic, and for fixed $a \geqslant 0$

$$f_a(x) := \begin{cases} x+a: & x \geqslant 0 \\ x-a: & x<0, \end{cases} \tag{2}$$

so that the equation is

$$\ddot{x} + x + a\,\mathrm{sgn}(x) = p(t) \tag{3}$$

if $x(t) \neq 0$.

Using a suitable Lyapunov function approach, it follows from [7] (which is along the lines of [2]) that all solutions of (3) are unbounded in the $(x, \dot{x})$-phase plane in case that $a < |\int_0^{2\pi} p(t)\, e^{it}\, dt|/4$.

This implies that some condition on the size of a (the size of the "gap" of f_a) is needed, and we shall show the boundedness of all solutions of (3) as well as the existence of infinitely many periodic and quasiperiodic solutions for all a being sufficiently large. Here a solution is understood to be $W^{2,2}$ on every finite interval.

A direct application of Moser's invariant curve theorem requires smoothness. To describe how (3) nevertheless can be subordinated to an application of a suitable invariant curve theorem, we first remark that this equation is Hamiltonian with

$$H_a(x, y, t) = \tfrac{1}{2}y^2 + F_a(x) - xp(t), \tag{4}$$

where, letting $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$,

$$F_a(x) = \tfrac{1}{2}((x_+ + a)^2 + (x_- + a)^2 - a^2) \tag{5}$$

is continuous, but not smooth in x. It is easy to see that also $F_a(x) = F_a(-x)$. In the corresponding autonomous equation with $p=0$, for h being sufficiently large, the level set $\{(x, y)\colon \tfrac{1}{2}y^2 + F_a(x) = h\}$ is a closed curve which carries the periodic solution of (1) with period denoted by $T(h)$, cf. Fig. 1.

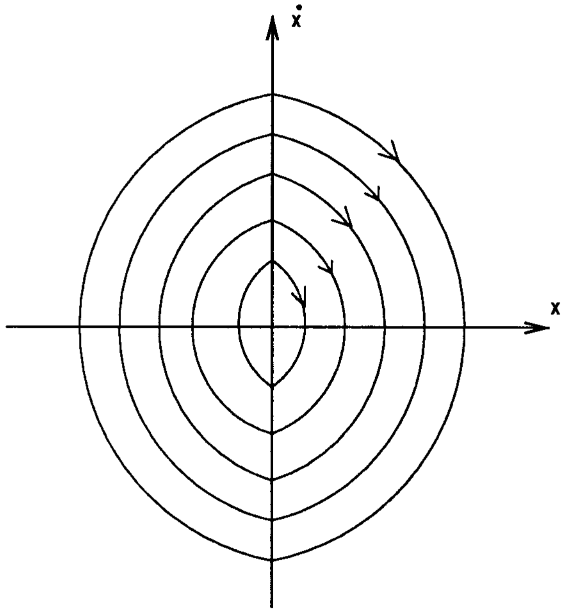

FIG. 1. Phase portrait of (1) with $p = 0$.

By introducing suitable action-angle variables (I, ϕ) (which will be defined precisely in (12), (14) below), the Hamiltonian $H_a(x, y, t)$ from (4) of the full system will be transformed into

$$H(\phi, I, t) = h_0(I) - x(\phi, I)\, p(t) \tag{6}$$

through some transformation $\Phi_1: (x, y) \mapsto (\phi, I)$. The new Hamiltonian (6) is smooth in I, but only continuous in ϕ. In order to apply KAM theory, the perturbation needs to be sufficiently smooth in the angle variable ϕ. To overcome the difficulty that $x(\phi, I)$ is only continuous with respect to ϕ, we shift the lack of regularity from $x(\phi, I)$ to the forcing $p(t)$ by exchanging the roles of the ϕ and t-variables, cf. the very illustrative [9, Fig. 1.2, p. 47]. This is achieved by the further transformation

$$\Phi_2: \theta = t,\ r = h_0(I) + x(\phi, I)\, p(t),\ \tau = \phi \tag{7}$$

where τ plays the role of new time. We note that this change of variables (and a further transformation) was made use of in [3, 9, 16] to get at least the leading term time-independent. Our purpose here is different: since p is assumed to be C^6, we have obtained a new Hamiltonian which is regular enough in (θ, r) to apply Moser's twist theorem to the corresponding Poincaré map.

In this paper we will show the following

THEOREM 1. *For every given ω-periodic function $p \in C^6(\mathbb{R})$ there is a sufficiently large a_* such that for all $a \geqslant a_*$, every solution of equation* (3) *is bounded, i.e., for every* $(t_0, x_0, \dot{x}_0)$

$$\sup_{t \in \mathbb{R}} (|x(t; t_0, x_0, \dot{x}_0)| + |\dot{x}(t; t_0, x_0, \dot{x}_0)|) < \infty.$$

Moreover, there are infinitely many periodic solutions and quasiperiodic solutions with large amplitude of the form

$$x(t) = f(\lambda t, t/\omega),$$

with f being defined on a 2-torus.

Remark that taking the gap (i.e., a in (2)) sufficiently large corresponds to making p small. In fact our system is close to a linear one, since we will show $h_0(I) \sim I$, cf. Lemma 2 below. This indicates that we cannot expect to obtain results without some smallness condition on p, because techniques like [5, Proposition 1] cannot be applied to improve the perturbation step by step. In this respect, we are in a position comparable to [18], where similar results were obtained for $\ddot{x} + f(x) = 1 + p(t)$ with f piecewise linear, but continuous at $x = 0$. Although we know that the condition $a \geqslant |\int_0^{2\pi} p(t)\, e^{it}\, dt|/4$ is necessary to have bounded solutions, we do not know whether this condition is also sufficient, or if a_* from Theorem 1 has to be "very large".

Since we wanted to show the boundedness of *all* solutions we cannot use KAM theory in a fixed compact domain, but we have to derive estimates for the perturbation. It should also be noted that it is enough to assume $p \in C^4(\mathbb{R})$ in Theorem 1, since the version of Moser's twist theorem which will be applied below (cf. Theorem 2) holds if the Poincaré-map is only C^3-regular, cf. [6].

The paper is organized as follows. In Section 2 we describe the necessary coordinate changes to transform the system into a smooth and nearly integrable one. We also state the corresponding estimates for the application of the twist theorem, postponing their proofs to the Appendix in Section 3. Although these estimates are lengthy, we consider it important to give the details since they are cruical in the procedure to fit the problem into the framework of KAM theory.

2. COORDINATE TRANSFORMATIONS, ESTIMATES, AND PROOF OF THEOREM 1

In this section we give some preliminary results and introduce in detail suitable transformations which will allow us to apply Moser's twist theorem. To simplify notations, we don't consider (3) directly, but the equivalent

$$\ddot{x} + x + \operatorname{sgn}(x) = \varepsilon p(t) \tag{8}$$

where $\varepsilon = 1/a$. In the autonomous case $p = 0$, for energies $h > 1/2$, the closed curve $H_1^0(x, y) = \frac{1}{2} y^2 + F_1(x) = h$ with F_1 from (5) carries the periodic solution of (1) with period $T(h)$. If we let $\alpha(h) = \sqrt{2h - 1} > 0$, then the intersections of this level set with the x-axis in the phase plane are $(\pm\alpha(h), 0)$. Hence symmetry of F_1 yields

$$\begin{aligned} T(h) &= 4 \int_0^{\alpha(h)} \frac{dx}{\sqrt{2(h - F_1(x))}} \\ &= 2\pi - 4 \arcsin(2h)^{-1/2} = 4 \arccos(2h)^{-1/2} \ (h > 1/2). \end{aligned} \tag{9}$$

Some properties of $T(h)$ are collected in

LEMMA 1. *$T(\cdot)$ is smooth in h, $0 \leqslant T(h) \leqslant 2\pi$, $T'(h) > 0$, and*

$$T^{-1}(\rho) = \frac{1}{2 \cos^2(\rho/4)}, \qquad 0 < \rho < 2\pi.$$

In addition, the following estimates hold:

$$c_i h^{-1/2 - i} \leqslant |D^i T(h)| \leqslant C_i h^{-1/2 - i} \qquad (i \geqslant 1) \tag{10}$$

$$\begin{aligned} c_0 \, |2\pi - \rho|^{-2} &\leqslant T^{-1}(\rho) \leqslant C_0 \, |2\pi - \rho|^{-2}, \\ c_i \, |2\pi - \rho|^{-(i+2)} &\leqslant |D^i T^{-1}(\rho)| \leqslant C_i \, |2\pi - \rho|^{-(i+2)} \qquad (i \geqslant 1) \end{aligned} \tag{11}$$

for some constants C_i, $c_i > 0$ and h sufficiently large resp. $2\pi - \rho$ sufficiently small; here D^i denotes the ith derivative. Moreover, let $\delta_n = 1/n$. Then for sufficiently large $n \in \mathbb{N}$ the intervals $[b_n^-, b_n^+] := T^{-1}([2\pi - 2\delta_n, 2\pi - \delta_n]) \subset (1/2, \infty)$ are disjoint, and $b_n^-, b_n^+, b_n^+ - b_n^- \to \infty$ as $n \to \infty$.

Proof. From (9) we have

$$T'(h) = \frac{2}{h \sqrt{2h - 1}}.$$

Thus it follows by induction that $D^i T(h) = P_{i-1}(h)\, h^{-i}(2h-1)^{1/2-i}$ for $i \geqslant 1$, with P_{i-1} being a polynomial of degree i, and hence we obtain the claimed estimates for $D^i T$. The estimates for $D^i T^{-1}$ are obtained inductively by differentiation of $T^{-1}(T(h)) \equiv h$ and by (10), cf. [9, (A1.5), p. 73] for the corresponding formula. The claim concerning the intervals $[b_n^-, b_n^+]$ can also be derived from the explicitely known T^{-1}. ∎

To construct the action-angle variables (cf. [4], [9, Section 2] for more information) and to transform the Hamiltonian into a nearly integrable one, let

$$\Phi_1 : (x, y) \mapsto (\phi, I)$$

be defined implicitly by the following two equations: For $h > 1/2$ and $|x| \leqslant \alpha(h)$ set

$$\phi(x, h) = \begin{cases} \phi_1(x, h): & x, y \geqslant 0, \\ \pi - \phi_1(x, h): & x \geqslant 0, y \leqslant 0, \\ \pi + \phi_1(-x, h): & x, y \leqslant 0, \\ 2\pi - \phi_1(-x, h): & x \leqslant 0, y \geqslant 0, \end{cases} \tag{12}$$

where for $0 \leqslant x \leqslant \alpha(h)$

$$\phi_1(x, h) = \frac{2\pi}{T(h)} \int_0^x \frac{d\xi}{\sqrt{2(h - F_1(\xi))}} = \frac{2\pi}{T(h)} \left(\arcsin \frac{1+x}{\sqrt{2h}} - \arcsin \frac{1}{\sqrt{2h}} \right), \tag{13}$$

and

$$\begin{aligned} I(h) &= 4 \int_0^{\alpha(h)} \sqrt{2(h - F_1(x))}\, dx = 4 \int_1^{1+\alpha(h)} \sqrt{2h - x^2}\, dx \\ &= 2h\pi - 2\sqrt{2h-1} - 4h \arcsin(2h)^{-1/2} = hT(h) - 2\sqrt{2h-1}. \end{aligned} \tag{14}$$

So by (12), (13) and (14) we have obtained concrete formulae for the action-angle transformation Φ_1. Geometrically, $I(h)$ is the area surrounded by $\{(x, y) \colon \frac{1}{2} y^2 + F_1(x) = h\}$.

Before turning our attention to Φ_1, we first collect some properties of $I(h)$. Here and in the sequel, positive constants not depending on important quantities are denoted by the same symbols c, c_i, C_i, ... etc.

LEMMA 2. *$I(\cdot)$ as well as its inverse $I \mapsto h(I) =: h_0(I)$ are smooth, and $I'(h) = T(h) > 0$. Moreover, for suitable positive constants and for sufficiently large h resp. I*

$$c_0 h \leqslant I(h) \leqslant C_0 h, \quad c_1 \leqslant I'(h) \leqslant C_1, \quad c_i h^{1/2-i} \leqslant D^i I(h) \leqslant C_i h^{1/2-i} \quad (i \geqslant 2) \tag{15}$$

$$c_0 I \leqslant h_0(I) \leqslant C_0 I, \quad c_1 \leqslant h_0'(I) \leqslant C_1, \quad |D^i h_0(I)| \leqslant C_i I^{1/2-i} \quad (i \geqslant 2). \tag{16}$$

Proof. First, (14) gives $I' = T$, and the estimates for I follow from (14), (9), and (10). Second, the desired estimates for h_0 are again obtained by successive differentiation of $h_0(I(h)) \equiv h$ using (15). ▮

Hence $h_0(\cdot)$ behaves like $I(\cdot)$ itsself. Now we can investigate Φ_1 in greater detail.

LEMMA 3. *Φ_1 is a homeomorphism from $\mathbb{R}^2 \backslash \{0\}$ to the cylinder $[0, 2\pi] \times (1/2, \infty)$. Moreover, if $(x, y)(\phi, I) := \Phi_1^{-1}(\phi, I)$, then $x(\phi, I)$ is smooth in I for every fixed ϕ, and*

$$|\partial_I^i x(\phi, I)| \leqslant C_i I^{1/2-i} \qquad (i \geqslant 0). \tag{17}$$

Proof. By construction of (ϕ, I) it is clear that Φ_1 is continuous, onto and one-to-one. In addition, we note that $x(\phi, I) = \tilde{x}(\phi, h_0(I))$, where we have, e.g., in the first quadrant $\phi = \phi_1(\tilde{x}, h)$ with $h = h_0(I)$ and ϕ_1 from (13). Therefore for $\phi \in (0, \pi/2)$, by solving $\phi = \phi_1(\tilde{x}, h)$ w.r.t. $\tilde{x}$,

$$\begin{aligned} \tilde{x}(\phi, h) &= -1 + \sqrt{2h} \sin\left(\frac{T(h)}{2\pi}\phi + \arcsin(2h)^{-1/2}\right) \\ &= -1 + \sqrt{2h-1} \sin\left(\frac{T(h)}{2\pi}\phi\right) + \cos\left(\frac{T(h)}{2\pi}\phi\right), \end{aligned} \tag{18}$$

where we used the formula $\sin(v + \arcsin(u)) = \sqrt{1-u^2} \sin(v) + u\cos(v)$ for $u \in [0, 1]$, $v \in \mathbb{R}$. In the same way we obtain for $\phi \in (3\pi/2, 2\pi)$

$$\tilde{x}(\phi, h) = 1 - \sqrt{2h-1} \sin\left(\frac{T(h)}{2\pi}[2\pi - \phi]\right) - \cos\left(\frac{T(h)}{2\pi}[2\pi - \phi]\right),$$

and this proves the continuity of $\tilde{x}(\phi, h)$ in ϕ at $\phi = 0 \cong 2\pi$. Hence it follows that $\tilde{x}(\phi, h)$ is continuous in (ϕ, h), and this carries over to x because of Lemma 2. Moreover, (18) shows that $\tilde{x}(\phi, h)$ is smooth in h for fixed $\phi \in [0, \pi/2]$, and thus $x(\phi, I)$ is smooth in I in view of Lemma 2.

To verify the desired estimates, we first claim that

$$|\partial_h^i \tilde{x}(\phi, h)| \leqslant C_i h^{1/2-i} \qquad (i \geqslant 0). \tag{19}$$

For $\phi \in [0, \pi/2]$ this can be derived from (18) and (10). Here we omit the details, since the required technique will be applied lateron once more, cf. the Appendix, or [5, 9, 14, 18] and related papers. Usually, these estimates can be derived by induction, using Leibniz' rule and the formulas from Lemma 8. Since $x(\phi, I) = \tilde{x}(\phi, h_0(I))$, the estimate (19) also carries over to x, by Lemma 8 and the estimates for $D^i h_0$ from (16).

Applying the action-angle transformation to the full system, the Hamiltonian $H_1(x, y, t) = \frac{1}{2} y^2 + F_1(x) - \varepsilon x p(t)$ which corresponds to (8) is transformed into a new function in (ϕ, I), that is

$$H(\phi, I, t) = h_0(I) - \varepsilon x(\phi, I)\, p(t). \tag{20}$$

From Lemma 3 it follows that $H(\phi, I, t)$ is smooth in I and continuous in ϕ. Moreover, $|\partial_I x(\phi, I)| \leqslant C I^{-1/2}$ by (17). Therefore (16) implies that $H(\phi, \cdot, t)$ is invertible for sufficiently large I.

Next we exchange the roles of ϕ and t by means of

$$\Phi_2 \colon (\phi, I, t) \mapsto (\theta, r, \tau) := (t, H(\phi, I, t), \phi),$$

cf. [9, Section 3]. This transformation again leads to a Hamiltonian system, where the new Hamiltonian is

$$\mathscr{H}(\theta, r, \tau) = [H(\tau, \cdot, \theta)]^{-1}(r). \tag{21}$$

Thus $\mathscr{H}(\theta, r, \tau)$ is ω-periodic in θ, 2π-periodic in τ, and smooth in (θ, r). We write

$$\mathscr{H}(\theta, r, \tau) = I(r) + \varepsilon \mathscr{H}_1(\theta, r, \tau), \tag{22}$$

i.e., $\mathscr{H}_1$ is defined by this formula. Some estimates of $\mathscr{H}_1$ are given in

LEMMA 4. *For r sufficiently large,*

$$|\partial_\theta^i \partial_r^j \mathscr{H}_1(\theta, r, \tau)| \leqslant C_{i,j} r^{1/2 - j} \qquad (0 \leqslant i + j \leqslant 6), \tag{23}$$

where $C_{i,j}$ depends on $|p|_{C^i([0,\omega])}$.

Proof. Cf. appendix.

Since $I' = T$, the equations of motion corresponding to the Hamiltonian $\mathscr{H}$ from (22) are

$$\frac{d\theta}{d\tau} = T(r) + \varepsilon \partial_r \mathscr{H}_1(\theta, r, \tau), \qquad \frac{dr}{d\tau} = -\varepsilon \partial_\theta \mathscr{H}_1(\theta, r, \tau), \tag{24}$$

with τ serving as time in the equation and $\mathscr{H}_1(\theta, r, \tau)$ being ω-periodic in θ and 2π-periodic in τ. Moreover, since $\partial_r \mathscr{H}_1$ resp. $\partial_\theta \mathscr{H}_1$ are C^6 resp. C^5 in θ, smooth in r, and continuous in τ, we note that in particular solutions $\tau \mapsto (\theta(\tau), r(\tau))$ of (24) exist. In addition, by Lemma 4 and the equation for

$dr/d\tau$, there are (a large) $r_* > 0$ and a constant C_* such that all solutions of (24) with $\varepsilon \leqslant 1$ and initial values $r_0 \geqslant r_*$ are defined on the whole of $[0, 2\pi]$ and satisfy there $|r(\tau) - r_0| \leqslant C_* r_0$, hence $[C_* - 1]\, r_0 \leqslant r(\tau) \leqslant [C_* + 1]\, r_0$. Thus, letting $\delta_n = 1/n$ and retaining the notation of Lemma 1, we obtain for initial values

$$r_0 \in J_n := [b_n^- /(C_* - 1), b_n^+ /(C_* + 1)] \tag{25}$$

that

$$T(r(\tau)) \in [2\pi - 2\delta_n, 2\pi - \delta_n], \qquad \tau \in [0, 2\pi]. \tag{26}$$

Also note that $b_n^- /(C_* - 1) \to \infty$, $b_n^+ /(C_* + 1) \to \infty$, and $b_n^+ /(C_* + 1) - b_n^- /(C_* - 1) \to \infty$ as $n \to \infty$. In particular we may assume that the initial values r_0 are large enough that all estimates derived so far hold. Supressing the index n, we rescale, cf. [9, Section 4.2],

$$\bar{\theta}(\tau) = \omega^{-1} \theta(\tau) \qquad \text{and} \qquad \bar{\rho}(\tau) = \delta^{-1} [T(r(\tau)) - 2\pi]. \tag{27}$$

Thus

$$\bar{\rho} \in [-2, -1] \tag{28}$$

by (26), and differentiation yields

$$\frac{d\bar{\theta}}{d\tau} = \frac{\delta}{\omega} \bar{\rho} - \frac{2\pi}{\omega} + \varepsilon f_1(\bar{\theta}, \bar{\rho}, \tau) \qquad \text{and} \qquad \frac{d\bar{\rho}}{d\tau} = \varepsilon f_2(\bar{\theta}, \bar{\rho}, \tau) \tag{29}$$

with

$$f_1(\bar{\theta}, \bar{\rho}, \tau) = \omega^{-1} \partial_r \mathscr{H}_1(\omega\bar{\theta}, r(\bar{\rho}), \tau)$$

and

$$f_2(\bar{\theta}, \bar{\rho}, \tau) = -\delta^{-1} T'(r(\bar{\rho}))\, \partial_\theta \mathscr{H}_1(\omega\bar{\theta}, r(\bar{\rho}), \tau),$$

where we let $r(\bar{\rho}) = T^{-1}(\delta\bar{\rho} + 2\pi)$. Then f_1 and f_2 are 1-periodic in $\bar{\theta}$ and 2π-periodic in τ. Estimates on f_1 and f_2 are given in

LEMMA 5. *There is a C depending on $|p|_{C^6([0, \omega])}$ (but not on δ) such that*

$$|\partial_{\bar{\theta}}^i \partial_{\bar{\rho}}^j f_1(\bar{\theta}, \bar{\rho}, \tau)| + |\partial_{\bar{\theta}}^i \partial_{\bar{\rho}}^j f_2(\bar{\theta}, \bar{\rho}, \tau)| \leqslant C\delta \qquad (0 \leqslant i + j \leqslant 5), \tag{30}$$

Proof. Cf. appendix.

Let $\bar{P}: \mathbb{R} \times [-2, 1] \to \mathbb{R} \times \mathbb{R}$ denote the time-2π-map of (29). Then this Poincaré-map is of class C^5 in $(\bar{\theta}, \bar{\rho})$, 1-periodic in $\bar{\theta}$, and it can be written in the following form

$$\bar{P}: \quad \begin{aligned} \bar{\theta}_1 &= \bar{\theta} + \alpha + \bar{\delta}\bar{\rho} + \bar{\delta}\varepsilon \bar{P}_1(\bar{\theta}, \bar{\rho}) \\ \bar{\rho}_1 &= \bar{\rho} + \bar{\delta}\varepsilon \bar{P}_2(\bar{\theta}, \bar{\rho}), \end{aligned} \tag{31}$$

i.e., we define $\bar{P}_1$ and $\bar{P}_2$ through this relations. Here $\alpha = -4\pi^2/\omega$ and $\bar{\delta} = 2\pi\delta/\omega$. Note that $\bar{P}$, $\bar{P}_1$, $\bar{P}_2$, and $\bar{\delta}$ in fact depend on n. The next result gives estimates on $\bar{P}_1$ and $\bar{P}_2$.

LEMMA 6. *For sufficiently large $n \in \mathbb{N}$, i.e., sufficiently small $\bar{\delta} = \bar{\delta}_n$,*

$$|\partial^i_{\bar{\theta}} \partial^j_{\bar{\rho}} \bar{P}_1(\bar{\theta}, \bar{\rho})| + |\partial^i_{\bar{\theta}} \partial^j_{\bar{\rho}} \bar{P}_2(\bar{\theta}, \bar{\rho})| \leqslant C \ (0 \leqslant i + j \leqslant 5), \tag{32}$$

with C depending on $|p|_{C^6([0,\omega])}$, but not on n.

Proof. Cf. appendix. ∎

Up to one more transformation, we are now in the position to apply Moser's twist theorem [15]. We state it in the form analogous to [18, Theorem 4.5], which follows from the results of [6, Section 5]. For the definition of an irrational number of constant type and the corresponding Markoff constant see also [6, 18], whereas the term "intersection property" is made precise in Lemma 7 below.

THEOREM 2 (Invariant Curve Theorem). *Let $P: \mathbb{R} \times [-6, -3] \to \mathbb{R}^2$, $(u, v) \mapsto (u_1, v_1)$, be of class C^5, one-to-one, and 1-periodic in u. In addition, assume that P has the intersection property, and that P may be written in the form*

$$u_1 = u + \beta + \bar{\delta} v + \bar{\delta} F_1(u, v), \ v_1 = v + \bar{\delta} F_2(u, v)$$

where $\bar{\delta} \in (0, 2)$, and β is an irrational of constant type with Markoff constant γ satisfying

$$\gamma \leqslant \bar{\delta} \leqslant M\gamma \tag{33}$$

for some fixed constant M. Then there is a positive constant M_*, depending only on M, such that if

$$|F_1|_{C^5(\mathbb{R}^2)} + |F_2|_{C^5(\mathbb{R}^2)} \leqslant M_*,$$

one finds $\mu \in C^3(\mathbb{R}/\mathbb{Z})$ such that the curve $\Gamma_\mu = \{(u, \mu(u)) : u \in \mathbb{R}\}$ is invariant under P, and $P|_{\Gamma_\mu}$ has rotation number β.

Now we can carry out the

Proof of Theorem 1.

The proof of the boundedness of all solutions is similar to the one of [18, Theorem 4.1]. We will no longer supress the index n, to emphasize the dependence of $\bar{P}$, $\bar{P}_1$, $\bar{P}_2$, and $\bar{\delta}$ on n. In order to apply Theorem 2, we need to approximate $\alpha = -4\pi^2/\omega$ by irrationals of constant type with suitable Markoff constants. By [18, Lemma 4.4] we can find such irrationals α_n of constant type with Markoff constants γ_n such that

$$\frac{4\pi}{\omega n} \leqslant \alpha_n - \alpha \leqslant \frac{8\pi}{\omega n} \qquad \text{and} \qquad \frac{\pi}{4\omega n} \leqslant \gamma_n \leqslant \frac{\pi}{\omega n},$$

$$\text{hence} \quad \bar{\rho} + \frac{\alpha - \alpha_n}{\bar{\delta}_n} \in [-6, -3]. \tag{34}$$

We obtain from (31) that

$$\bar{P}_{(n)}\colon \bar{\theta}_1 = \bar{\theta} + \alpha_n + \bar{\delta}_n \left[\bar{\rho} + \frac{\alpha - \alpha_n}{\bar{\delta}_n} \right] + \bar{\delta}_n \varepsilon \bar{P}_{1,(n)}(\bar{\theta}, \bar{\rho}),$$

$$\bar{\rho}_1 = \bar{\rho} + \bar{\delta}_n \varepsilon \bar{P}_{2,(n)}(\bar{\theta}, \bar{\rho}),$$

and this implies that we finally can transform

$$u = \bar{\theta}, \qquad v = \bar{\rho} + \frac{\alpha - \alpha_n}{\bar{\delta}_n} \tag{35}$$

to obtain in new coordinates (u, v) the Poincaré-maps $P_{(n)}\colon \mathbb{R} \times [-6, -3] \to \mathbb{R}^2$, $(u, v) \mapsto (u_1, v_1)$, where

$$P_{(n)}\colon u_1 = u + \alpha_n + \bar{\delta}_n v + \bar{\delta}_n \varepsilon P_{1,(n)}(u, v),\; v_1 = v + \bar{\delta}_n \varepsilon P_{2,(n)}(u, v)$$

with $P_{j,(n)}(u, v) = \bar{P}_{j,(n)}(u, v - (\alpha - \alpha_n)/\bar{\delta}_n)$ for $j = 1, 2$. We also have

LEMMA 7. *Every* $P = P_{(n)}$ *has the intersection property, i.e., if an embedded circle* $\mathscr{C}$ *in* $\mathbb{R} \times [-6, -3]$ *is homotopic to a circle* $v = \text{const}$, *then* $P(\mathscr{C}) \cap \mathscr{C} \neq \varnothing$.

Proof. Let $P_{(24)}$ be the time-2π-map of (24). Since (24) comes from a Hamiltonian system, $P_{(24)}$ has the intersection property, cf. [5, Lemma 5]. Let $\Phi_3\colon (\theta, r) \mapsto (\bar{\theta}, \bar{\rho}) \mapsto (u, v)$ denote the transformations from (27) and (35). Then $P = \Phi_3 \circ P_{(24)} \circ \Phi_3^{-1}$, i.e., P and $P_{(24)}$ are conjugated, and hence P also has the intersection property. ∎

Next we note that by (34)

$$\gamma_n \leqslant \frac{2\pi}{\omega n} = \bar{\delta}_n = M \frac{\pi}{4\omega n} \leqslant M\gamma_n \qquad \text{with} \quad M = 8.$$

Hence we may choose M_*, independent of n, such that the claim of the invariant curve theorem holds (where $\beta = \alpha_n$), since for large n we also have $\bar{\delta}_n \in (0, 2)$. Thus by Lemma 6, for sufficiently small $\varepsilon > 0$ (corresponding to sufficiently large a in (1)) and all large n, every $P_{(n)}$ has an invariant curve. Transformed back to the original system this means that we have found arbitrary large invariant tori in $(x, \dot{x}, t \bmod \omega)$ space, and this implies the boundedness of all solutions, cf. [5, 9, 18].

Finally we remark that the existence of infinitely many periodic solutions is obtained via the Poincaré–Birkhoff-theorem. This theorem may be applied to $P_{(24)}$, the area-preserving time-2π-map of the Hamiltonian system (24), just like in [5, pp. 92/93], cf. also [9, Theorem 1]. Moreover, the existence of quasiperiodic solutions may also be shown analogously to these papers. ∎

3. APPENDIX

In this appendix we give the proofs of the estimates in Lemma 4, 5, and 6 stated in Section 2. Before doing this, we first include, for convenience of the reader, a result on the differentation of chain functions.

LEMMA 8. *Let $F: \mathbb{R}^2 \to \mathbb{R}$ and $f, g: \mathbb{R}^2 \to \mathbb{R}$ be sufficiently smooth. Then for $i + j \geqslant 1$*

$$\partial_x^i \partial_y^j [F \circ (f, g)] = \sum_{\substack{(k, p) \in \mathbb{N}_0^2 :\, 1 \leqslant k + p \leqslant i + j, \\ \vec{\imath} = (i_1, ..., i_{k+p}),\, |\vec{\imath}| = i, \\ \vec{\jmath} = (j_1, ..., j_{k+p}),\, |\vec{\jmath}| = j}} c_{k, p, \vec{\imath}, \vec{\jmath}} (\partial_1^k \partial_2^p F(f, g)) \times (\partial_x^{i_1} \partial_y^{j_1} f) \cdots (\partial_x^{i_k} \partial_y^{j_k} f)(\partial_x^{i_{k+1}} \partial_y^{j_{k+1}} g) \cdots (\partial_x^{i_{k+p}} \partial_y^{j_{k+p}} g),$$

with integer coefficients $c_{k, p, \vec{\imath}, \vec{\jmath}}$ satisfying $c_{k, p, \vec{\imath}, \vec{\jmath}} = 0$ if $i_l = j_l = 0$ for some $1 \leqslant l \leqslant k + p$, i.e., no terms with $i_l = j_l = 0$ will appear. In particular, the above formula gives for $F: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{R}^2 \to \mathbb{R}$

$$\partial_x^i \partial_y^j [F \circ f] = \sum_{\substack{1 \leqslant k \leqslant i + j, \\ \vec{\imath} = (i_1, ..., i_k),\, |\vec{\imath}| = i, \\ \vec{\jmath} = (j_1, ..., j_k),\, |\vec{\jmath}| = j}} c_{k, \vec{\imath}, \vec{\jmath}} (D^k F(f)) (\partial_x^{i_1} \partial_y^{j_1} f) \cdots (\partial_x^{i_k} \partial_y^{j_k} f).$$

Proof. The proof is omitted, cf. [1, p. 3], [5, p. 88], [9] or [18, Lemma 3.8] for similar results. The claim follows inductively by use of Leibniz' rule for the differentiation of products of $n \geqslant 2$ functions. ▮

Proof of Lemma 4.

We obtain from (21) and (20) with $\mathscr{H}_1 = \mathscr{H}_1(\theta, r, \tau)$

$$r = H(\tau, I(r) + \varepsilon\mathscr{H}_1, \theta) = h_0(I(r) + \varepsilon\mathscr{H}_1) - \varepsilon x(\tau, I(r) + \varepsilon\mathscr{H}_1)\, p(\theta),$$

and thus, since $I(\cdot) = h_0^{-1}$,

$$\varepsilon\mathscr{H}_1 = I(r + \varepsilon x(\tau, I(r) + \varepsilon\mathscr{H}_1)\, p(\theta)) - I(r).$$

Consequently, differentiation of $f(\rho) := I(r + \rho\varepsilon x(\cdots)\, p(\theta))$ and $I' = T$ yield

$$\mathscr{H}_1 = x(\tau, I(r) + \varepsilon\mathscr{H}_1)\, p(\theta) \int_0^1 T(r + \rho\varepsilon x(\tau, I(r) + \varepsilon\mathscr{H}_1)\, p(\theta))\, d\rho. \tag{36}$$

From this equation the claimed estimates will be derived inductively, always assuming that r is large enough and that $\varepsilon \leqslant 1$.

For that, we consider first the case $i = j = 0$. Then (36), (17) and the boundedness of T imply $|\mathscr{H}_1| \leqslant C\,|x(\tau, \mathscr{H})| \leqslant C\,|\mathscr{H}|^{1/2}$. Consequently, by (22), and since $|I(r)| \leqslant Cr$, we find $|\mathscr{H}_1| \leqslant Cr^{1/2}$ as desired. In particular, (15) and (22) yield

$$cr \leqslant |\mathscr{H}| \leqslant Cr, \tag{37}$$

and thus

$$|\partial_I^k x(\tau, \mathscr{H})| \leqslant Cr^{1/2-k} \qquad (k \geqslant 0) \tag{38}$$

because of (17). For the induction step we assume that (23) already holds for all $0 \leqslant i + j \leqslant N$ and is to be shown for some fixed $i^* + j^* = N + 1$. We start to estimate the derivatives of the ingredients of the right-hand side of (36). First we will show

$$|\partial_\theta^i \partial_r^j [x(\tau, \mathscr{H})]| \leqslant C(r^{-1/2}\,|\partial_\theta^i \partial_r^j \mathscr{H}| + r^{1/2-j}) \qquad (0 \leqslant i + j \leqslant N + 1). \tag{39}$$

By induction assumption this will yield in particular

$$|\partial_\theta^i \partial_r^j [x(\tau, \mathscr{H})]| \leqslant C(r^{-1/2} r^{1/2-j} + r^{1/2-j}) \leqslant Cr^{1/2-j} \qquad (0 \leqslant i + j \leqslant N). \tag{40}$$

To prove (39), we write, using the second formula from Lemma 8,

$$\begin{aligned}
&\partial_\theta^i \partial_r^j [x(\tau, \mathscr{H})] \\
&\quad = C(\partial_I x(\tau, \mathscr{H}))(\partial_\theta^i \partial_r^j \mathscr{H}) \\
&\qquad + \sum_{\substack{2 \leqslant k \leqslant i+j, \\ \vec{\imath}=(i_1, ..., i_k),\, |\vec{\imath}|=i, \\ \vec{\jmath}=(j_1, ..., j_k),\, |\vec{\jmath}|=j}} c_{k, \vec{\imath}, \vec{\jmath}} (\partial_I^k x(\tau, \mathscr{H}))(\partial_\theta^{i_1} \partial_r^{j_1} \mathscr{H}) \cdots (\partial_\theta^{i_k} \partial_r^{j_k} \mathscr{H}).
\end{aligned} \tag{41}$$

Since due to (38) the first term is dominated by $Cr^{-1/2} |\partial_\theta^i \partial_r^j \mathscr{H}|$, we only have to show that every term in the $\sum$ is $\leqslant Cr^{1/2-j}$. To do this, we note that because of Lemma 8, the nonzero terms in this sum have $i_l + j_l \geqslant 1$ for every $1 \leqslant l \leqslant k$, and thus $i_l + j_l \leqslant N$, since in the opposite case one would obtain $i + j = i_l + j_l = N+1$, and thus $\vec{\imath} = ie_l \in \mathbb{R}^k$ and $\vec{\jmath} = je_l \in \mathbb{R}^k$, with e_l being the lth unit vector. But since $k \geqslant 2$, this would imply that at least either $i_{l+1} = j_{l+1} = 0$ or $i_{l-1} = j_{l-1} = 0$, a contradiction. Therefore $i_l + j_l \leqslant N$ in the non-vanishing terms, and hence we can apply the induction hypotheses to estimate $|\partial_\theta^{i_l} \partial_r^{j_l} \mathscr{H}_1| \leqslant Cr^{1/2-j_l}$ for those indices. Thus if $i_l \geqslant 1$ we obtain from (22) that $|\partial_\theta^{i_l} \partial_r^{j_l} \mathscr{H}| = \varepsilon\, |\partial_\theta^{i_l} \partial_r^{j_l} \mathscr{H}_1| \leqslant Cr^{1/2-j_l}$. On the other hand, if $i_l = 0$, then

$$|\partial_r^{j_l} \mathscr{H}| = |D^{j_l} I + \varepsilon \partial_r^{j_l} \mathscr{H}_1| \leqslant \begin{cases} C + Cr^{-1/2}: & j_l = 1 \\ Cr^{1/2-j_l} + Cr^{1/2-p}: & j_l \geqslant 2 \end{cases}$$

$$\leqslant \begin{cases} C: & j_l = 1 \\ Cr^{1/2-j_l}: & j_l \geqslant 2, \end{cases}$$

where we have used (15). To sum up, we have shown that for all non-vanishing terms in the sum in (41) one has

$$|\partial_\theta^{i_l} \partial_r^{j_l} \mathscr{H}| \leqslant \begin{cases} C: & (i_l, j_l) = (0, 1) \\ Cr^{1/2-j_l}: & (i_l, j_l) \neq (0, 1), \end{cases} \qquad 1 \leqslant l \leqslant k. \tag{42}$$

This information can be used as follows. If in one of the non-zero terms there are no index-pairs $(i_l, j_l) = (0, 1)$, then by (38) and (42)

$$\begin{aligned}
&|(\partial_I^k x(\tau, \mathscr{H}))(\partial_\theta^{i_1} \partial_r^{j_1} \mathscr{H}) \cdots (\partial_\theta^{i_k} \partial_r^{j_k} \mathscr{H})| \\
&\qquad \leqslant Cr^{1/2-k} r^{1/2-j_1} \cdots r^{1/2-j_k} = Cr^{(1-k)/2-|\vec{\jmath}|} = Cr^{(1-k)/2-j},
\end{aligned}$$

but for every appearing pair $(i_l, j_l) = (0, 1)$ the corresponding term $r^{1/2-j_l} = r^{-1/2}$ in this estimate has to be replaced by a constant. Thus, if in a general non-zero term in the sum in (41) there are $1 \leqslant M \leqslant k$ such pairs, then we obtain the estimate

$$|\cdots| \leqslant Cr^{(1-k)/2-j} r^{M/2} = Cr^{(1-k+M)/2-j}.$$

Since $1-k+M \leqslant 1$, we finally conclude that $|\cdots| \leqslant Cr^{1/2-j}$, as was desired to finish the proof of (39) and (40).

Now we intend to show that with $U(\theta, r, \tau) := x(\tau, \mathscr{H}(\theta, r, \tau))\, p(\theta)$ we also have

$$|\partial_\theta^i \partial_r^j U| \leqslant C(r^{-1/2}\, |\partial_\theta^i \partial_r^j \mathscr{H}| + r^{1/2-j}) \qquad (0 \leqslant i+j \leqslant N+1), \tag{43}$$

and therefore again by induction assumption

$$|\partial_\theta^i \partial_r^j U| \leqslant Cr^{1/2-j} \qquad (0 \leqslant i+j \leqslant N). \tag{44}$$

Estimate (43) is in fact a direct consequence of Leibniz' rule, since by (40) and (39)

$$\begin{aligned} |\partial_\theta^i \partial_r^j U| &= |\partial_\theta^i (\partial_r^j [x(\tau, \mathscr{H})]\ p)| \\ &\leqslant C\left(\sum_{k=0}^{i-1} |\partial_\theta^k \partial_r^j [x(\tau, \mathscr{H})]| + |\partial_\theta^i \partial_r^j [x(\tau, \mathscr{H})]|\right) \\ &\leqslant C\left(\sum_{k=0}^{i-1} r^{1/2-j} + r^{-1/2}\, |\partial_\theta^i \partial_r^j \mathscr{H}| + r^{1/2-j}\right) \\ &\leqslant C(r^{-1/2}\, |\partial_\theta^i \partial_r^j \mathscr{H}| + r^{1/2-j}). \end{aligned}$$

Our next step towards the estimation of $|\partial_\theta^{i^*} \partial_r^{j^*} \mathscr{H}_1|$ by means of (36) will be to prove that for fixed $\rho \in [0, 1]$

$$|\partial_\theta^i \partial_r^j [T(r+\rho\varepsilon U)]| \leqslant C(r^{-2}\, |\partial_\theta^i \partial_r^j \mathscr{H}| + r^{-1/2-j}) \qquad (0 \leqslant i+j \leqslant N+1). \tag{45}$$

Again this implies by induction assumption

$$|\partial_\theta^i \partial_r^j [T(r+\rho\varepsilon U)]| \leqslant Cr^{-1/2-j} \qquad (0 \leqslant i+j \leqslant N). \tag{46}$$

To see (45), we first remark that $|U| \leqslant C(r^{-1/2}\, |\mathscr{H}| + r^{1/2}) \leqslant Cr^{1/2}$ by (43) and (37). Therefore $|r+\rho\varepsilon U| \geqslant cr$, and this in turn yields by means of (10)

$$|D^k T(r+\rho\varepsilon U)| \leqslant C\, |r+\rho\varepsilon U|^{-1/2-k} \leqslant Cr^{-1/2-k} \qquad (k \geqslant 1). \tag{47}$$

Moreover,

$$\partial_\theta^i \partial_r^j [r+\rho\varepsilon U] \leqslant \begin{cases} 1+\rho\varepsilon(\partial_r U): & i=0,\ j=1 \\ \rho\varepsilon(\partial_\theta^i \partial_r^j U): & \text{otherwise for } i+j \geqslant 1. \end{cases}$$

First we show (45) for $i=0$, $j=1$. In this case by (47) and (43)

$$\begin{aligned}|\partial_r[T(r+\rho\varepsilon U)]| &\leqslant C\,|T'(r+\rho\varepsilon U)|\,(1+|\partial_r U|)\\ &\leqslant Cr^{-3/2}(1+r^{-1/2}\,|\partial_r \mathscr{H}|+r^{-1/2})\\ &\leqslant C(r^{-2}\,|\partial_r \mathscr{H}|+r^{-3/2}).\end{aligned}$$

So we can turn to the general case $i+j\geqslant 1$ and $(i, j)\neq(0, 1)$ where we have $|\partial_\theta^i\partial_r^j[r+\rho\varepsilon U]|\leqslant C\,|\partial_\theta^i\partial_r^j U|$. By Lemma 8 we obtain, analogously to (41)

$$\begin{aligned}&\partial_\theta^i\partial_r^j[T(r+\rho\varepsilon U)]\\ &\quad=CT'(r+\rho\varepsilon U)(\partial_\theta^i\partial_r^j[r+\rho\varepsilon U])\\ &\qquad+\sum_{\substack{2\leqslant k\leqslant i+j,\\ \vec{\imath}=(i_1,\ldots,i_k),\ |\vec{\imath}|=i,\\ \vec{\jmath}=(j_1,\ldots,j_k),\ |\vec{\jmath}|=j}} c_{k,\vec{\imath},\vec{\jmath}}\,D^kT(r+\rho\varepsilon U)(\partial_\theta^{i_1}\partial_r^{j_1}[r+\rho\varepsilon U])\cdots\\ &\qquad\times(\partial_\theta^{i_k}\partial_r^{j_k}[r+\rho\varepsilon U]).\end{aligned}$$

As a consequence of (47) and (43) the fist term may be estimated by

$$Cr^{-3/2}(r^{-1/2}\,|\partial_\theta^i\partial_r^j\mathscr{H}|+r^{1/2-j})\leqslant C(r^{-2}\,|\partial_\theta^i\partial_r^j\mathscr{H}|+r^{-1-j}).$$

Concerning the sum, using (47) and (44) we may argue completely analogous to the estimation of (41) to obtain the bound $Cr^{-1/2-j}$ for every non-zero term, and from this (45) follows.

Finally we turn to prove the claim of the induction step by means of (36). For that, we let $\tilde{T}(\rho):=T(r+\rho\varepsilon U)$. In this notation, (36) reads as $\mathscr{H}_1=U\int_0^1\tilde{T}(\rho)\,d\rho$. Therefore, by Leibniz' rule, (43), (45), (44) and (46),

$$\begin{aligned}&|\partial_\theta^{i^*}\partial_r^{j^*}\mathscr{H}_1|\\ &\quad\leqslant|\partial_\theta^{i^*}\partial_r^{j^*}U|+\int_0^1|\partial_\theta^{i^*}\partial_r^{j^*}[T(r+\rho\varepsilon U)]|\,d\rho\\ &\qquad+C\sum_{\substack{p=0,\ldots,i^*\\ k=0,\ldots,j^*\\ (p,k)\neq(0,0),\ (p,k)\neq(i^*,j^*)}}|\partial_\theta^p\partial_r^k U|\left(\int_0^1|\partial_\theta^{i^*-p}\partial_r^{j^*-k}[T(r+\rho\varepsilon U)]|\,d\rho\right)\\ &\quad\leqslant C\bigg(r^{-1/2}\,|\partial_\theta^{i^*}\partial_r^{j^*}\mathscr{H}|+r^{1/2-j^*}+r^{-2}\,|\partial_\theta^{i^*}\partial_r^{j^*}\mathscr{H}|\\ &\qquad+r^{-1/2-j^*}+\sum_{(p,k)}r^{1/2-k}r^{-1/2-[j^*-k]}\bigg)\\ &\quad\leqslant C(r^{-1/2}\,|\partial_\theta^{i^*}\partial_r^{j^*}\mathscr{H}|+r^{1/2-j^*}).\end{aligned}$$

Therefore we can choose r as large as is necessary to ensure $Cr^{-1/2} \leqslant 1/2$, and thus to obtain the claim of (23). ∎

Proof of Lemma 5.

We first remark that by (11), (28) and the definition of $r(\bar{\rho})$

$$c\delta^{-2} \leqslant c_0\, \delta^{-2}\, |\bar{\rho}|^{-2} \leqslant r(\bar{\rho}) \leqslant C_0\, \delta^{-2}\, |\bar{\rho}|^{-2} \leqslant C\delta^{-2},$$

$$|D^i[r(\bar{\rho})]| = \delta^i\, |D^i T^{-1}(\delta\bar{\rho} + 2\pi)| \leqslant C\delta^i (\delta\, |\bar{\rho}|)^{-(2+i)} \leqslant C\delta^{-2} \quad (i \geqslant 1) \tag{49}$$

Consequently (10) implies

$$|D^i T(\bar{\rho})| \leqslant C\, |\bar{\rho}|^{-1/2 - i} \leqslant C\delta^{1+2i} \qquad (i \geqslant 1).$$

Therefore in particular $|T'(\bar{\rho})| \leqslant C\delta^3$, and the second formula in Lemma 8 yields for $i \geqslant 1$

$$\begin{aligned} &|D^i[T'(r(\bar{\rho}))]| \\ &\quad = \Bigg| \sum_{\substack{1 \leqslant k \leqslant i, \\ \vec{\imath} = (i_1, ..., i_k),\ |\vec{\imath}| = i, \\ \forall 1 \leqslant l \leqslant k:\ i_l \geqslant 1}} c_{k, \vec{\imath}} D^{k+1} T(r(\bar{\rho}))(D^{i_1}[r(\bar{\rho})]) \cdots (D^{i_k}[r(\bar{\rho})]) \Bigg| \\ &\quad \leqslant C \sum_k \delta^{3+2k} \delta^{-2} \cdots \delta^{-2} \leqslant C\delta^3, \end{aligned}$$

i.e., $|D^i[T'(r(\bar{\rho}))]| \leqslant C\delta^3$ for $i \geqslant 0$. Next, it follows from the second formula in Lemma 8, Lemma 4, and (49), that

$$\begin{aligned} &|\partial^i_{\bar{\theta}} \partial^j_{\bar{\rho}} [\partial_\theta \mathscr{H}_1(\omega\bar{\theta}, r(\bar{\rho}), \tau)]| \\ &\quad = \omega^i\, |\partial^j_{\bar{\rho}} [\partial^{i+1}_\theta \mathscr{H}_1(\omega\bar{\theta}, r(\bar{\rho}), \tau)]| \\ &\quad = \omega^i \Bigg| \sum_{\substack{1 \leqslant k \leqslant j, \\ \vec{\jmath} = (j_1, ..., j_k),\ |\vec{\jmath}| = j, \\ \forall 1 \leqslant l \leqslant k:\ j_l \geqslant 1}} c_{k, \vec{\jmath}} (\partial^k_r \partial^{i+1}_\theta \mathscr{H}_1(\omega\bar{\theta}, r(\bar{\rho}), \tau))(D^{j_1}[r(\bar{\rho})]) \cdots \\ &\qquad \times (D^{j_k}[r(\bar{\rho})]) \Bigg| \\ &\quad \leqslant C \sum_k r(\bar{\rho})^{1/2 - k}\, \delta^{-2} \cdots \delta^{-2} \leqslant C\delta^{-1}. \end{aligned}$$

Since the same reasoning applies to give $|\partial^i_{\bar{\theta}} \partial^j_{\bar{\rho}} [\partial_r \mathscr{H}_1(\omega\bar{\theta}, r(\bar{\rho}), \tau)]| \leqslant C\delta$, only the estimate for $|\partial^i_{\bar{\theta}} \partial^j_{\bar{\rho}} f_2|$ is still to be verified. For that, we note that due to the above estimates and Leibniz' rule

$$\begin{aligned}|\partial_{\bar\theta}^i\partial_{\bar\rho}^j[T'(r(\bar\rho))\,\partial_\theta\mathscr{H}_1(\omega\bar\theta, r(\bar\rho), \tau)]|\\ &\leqslant C\sum_{k=0}^{j}|D^{j-k}[T'(r(\bar\rho))]|\,|\partial_{\bar\theta}^i\partial_{\bar\rho}^k[\partial_\theta\mathscr{H}_1(\omega\bar\theta, r(\bar\rho), \tau)]|\\ &\leqslant C\sum_k \delta^3\delta^{-1}\leqslant C\delta^2.\end{aligned}$$

Taking into consideration that $|\partial_{\bar\theta}^i\partial_{\bar\rho}^j f_2|$ has an extra δ^{-1} in comparison with the estimated expression, we obtain the claim. ▮

Proof of Lemma 6.

We make the ansatz, cf. [5, Lemma 4],

$$\bar\theta(\tau)=\bar\theta+\frac{\alpha}{2\pi}\tau+\frac{\bar\delta}{2\pi}\bar\rho\tau+\bar\delta\varepsilon A(\bar\theta,\bar\rho,\tau),\ \bar\rho(\tau)=\bar\rho+\bar\delta\varepsilon B(\bar\theta,\bar\rho,\tau) \tag{50}$$

for the solution $\tau\mapsto(\bar\theta(\tau),\bar\rho(\tau))$ of (29) with initial values $(\bar\theta,\bar\rho)$. Here $A(\bar\theta,\bar\rho,\tau)$ and $B(\bar\theta,\bar\rho,\tau)$ are suitable functions which are defined through (50). Since $\bar P_1(\bar\theta,\bar\rho)=A(\bar\theta,\bar\rho,2\pi)$ as well as $\bar P_2(\bar\theta,\bar\rho)=B(\bar\theta,\bar\rho,2\pi)$, we need to estimate the derivatives of A and B. For that, we first remark that due to (29)

$$\bar\delta A(\bar\theta,\bar\rho,\tau)=\frac{\bar\delta^2}{2\pi}\int_0^\tau B(\bar\theta,\bar\rho,s)\,ds+\int_0^\tau f_1(\bar\theta(s),\bar\rho(s),s)\,ds \qquad \text{and} \tag{51}$$

$$\bar\delta B(\bar\theta,\bar\rho,\tau)=\int_0^\tau f_2(\bar\theta(s),\bar\rho(s),s)\,ds. \tag{52}$$

Inserting (50) for $\bar\theta(s)$ and $\bar\rho(s)$, we have obtained a system of two integral equations for A and B, from which the desired estimates inductively can be derived as follows. If we let $\|B\|_N:=\sup_{0\leqslant i+j\leqslant N}|\partial_{\bar\theta}^i\partial_{\bar\rho}^j B|_\infty$, and analogously for A, then (52), (30) and $c\bar\delta\leqslant\delta\leqslant C\bar\delta$ imply $\|B\|_0\leqslant C$, hence (51) and (30) yield $\bar\delta\,\|A\|_0\leqslant C\bar\delta^2+C\bar\delta$, and therefore also $\|A\|_0\leqslant C$. Now we assume that we already have shown $\|A\|_N+\|B\|_N\leqslant C$ for some N, and we fix indices i^*, j^* with $1\leqslant i^*+j^*=N+1$. We have by (50)

$$\partial_{\bar\theta}[\bar\theta(\tau;\bar\theta,\bar\rho)]=1+\bar\delta\varepsilon(\partial_{\bar\theta}A),\ \partial_{\bar\theta}[\bar\rho(\tau;\bar\theta,\bar\rho)]=\bar\delta\varepsilon(\partial_{\bar\theta}B), \tag{53}$$

$$\partial_{\bar\rho}[\bar\theta(\tau;\bar\theta,\bar\rho)]=\frac{\bar\delta}{2\pi}\tau+\bar\delta\varepsilon(\partial_{\bar\rho}A),$$

$$\partial_{\bar\rho}[\bar\rho(\tau;\bar\theta,\bar\rho)]=1+\bar\delta\varepsilon(\partial_{\bar\rho}B), \qquad \text{and} \tag{54}$$

$$\partial_{\bar\theta}^i\partial_{\bar\rho}^j[\bar\theta(\tau;\bar\theta,\bar\rho)]=\bar\delta\varepsilon(\partial_{\bar\theta}^i\partial_{\bar\rho}^j A),$$

$$\partial_{\bar\theta}^i\partial_{\bar\rho}^j[\bar\rho(\tau;\bar\theta,\bar\rho)]=\bar\delta\varepsilon(\partial_{\bar\theta}^i\partial_{\bar\rho}^j B) \qquad (i+j\geqslant 2), \tag{55}$$

hence in particular

$$|\partial_{\bar\theta}^{i}\partial_{\bar\rho}^{j}[\bar\theta(\tau;\bar\theta,\bar\rho)]|+|\partial_{\bar\theta}^{i}\partial_{\bar\rho}^{j}[\bar\rho(\tau;\bar\theta,\bar\rho)]| \leqslant C(1+\|A\|_{i+j}+\|B\|_{i+j}) \qquad (i+j\geqslant 1). \tag{56}$$

Let F denote f_1 or f_2. Then by the first formula in Lemma 8

$$\begin{aligned}
&|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[F(\bar\theta(\tau),\bar\rho(\tau),\tau)]|\\
&\quad\leqslant |c_1\partial_{\bar\theta}F(\bar\theta(\tau),\bar\rho(\tau),\tau)\,\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[\bar\theta(\tau)]|\\
&\qquad+|c_2\partial_{\bar\rho}F(\bar\theta(\tau),\bar\rho(\tau),\tau)\,\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[\bar\rho(\tau)]|\\
&\qquad+\Bigg|\sum_{\substack{(k,p)\in\mathbb{N}_0^2:\,2\leqslant k+p\leqslant i^*+j^*,\\ \vec{\imath}=(i_1,\ldots,i_{k+p}),\,|\vec{\imath}|=i^*,\\ \vec{\jmath}=(j_1,\ldots,j_{k+p}),\,|\vec{\jmath}|=j^*}}(c_{k,p,\vec{\imath},\vec{\jmath}}\partial_{\bar\theta}^{k}\partial_{\bar\rho}^{p}F(\bar\theta(\tau),\bar\rho(\tau),\tau)\\
&\qquad\times(\partial_{\bar\theta}^{i_1}\partial_{\bar\rho}^{j_1}[\bar\theta(\tau)])\cdots(\partial_{\bar\theta}^{i_k}\partial_{\bar\rho}^{j_k}[\bar\theta(\tau)])\\
&\qquad\times(\partial_{\bar\theta}^{i_{k+1}}\partial_{\bar\rho}^{j_{k+1}}[\bar\rho(\tau)])\cdots(\partial_{\bar\theta}^{i_{k+p}}\partial_{\bar\rho}^{j_{k+p}}[\bar\rho(\tau)]))\Bigg|.
\end{aligned}$$

In the $\sum_{2\leqslant k+p\leqslant i^*+j^*}(\cdots)$, every non-zero term must have $i_l+j_l\geqslant 1$ for all $1\leqslant l\leqslant k+p$, by Lemma 8. This implies $i_l+j_l\leqslant N$, since $i^*+j^*=N+1$, and hence $i_l+j_l=N+1$ is impossible because both vectors $\vec{\imath}$ and $\vec{\jmath}$ under consideration have $k+p$, thus at least 2, components. Therefore in every non-zero term in the above sum for all $1\leqslant l\leqslant k$, by (56) and by induction hypotheses,

$$|\partial_{\bar\theta}^{i_l}\partial_{\bar\rho}^{j_l}[\bar\theta(\tau)]|\leqslant C(1+\|A\|_{i_l+j_l}+\|B\|_{i_l+j_l})\leqslant C(1+\|A\|_N+\|B\|_N)\leqslant C.$$

Since analogously $|\partial_{\bar\theta}^{i_l}\partial_{\bar\rho}^{j_l}[\bar\rho(\tau)]|\leqslant C$ for $k+1\leqslant l\leqslant k+p$, we obtain from (30)

$$\Bigg|\sum_{2\leqslant k+p\leqslant i^*+j^*}(\cdots)\Bigg|\leqslant C\bar\delta,$$

and consequently, again by (30), for $F=f_1$ or $F=f_2$

$$|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[F(\bar\theta(\tau),\bar\rho(\tau),\tau)]|\leqslant C\bar\delta(1+|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[\bar\theta(\tau)]|+|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[\bar\rho(\tau)]|).$$

Then it follows from (53), (54), (55), and taking also into account $\tau\in[0,2\pi]$ and w.l.o.g. $\varepsilon\leqslant 1$, that in any case

$$|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}[F(\bar\theta(\tau),\bar\rho(\tau),\tau)]|\leqslant C\bar\delta(1+\bar\delta[|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}A|_\infty+|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}B|_\infty]). \tag{57}$$

This in turn implies by (52)

$$|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}B|_\infty \leqslant C(1+\bar\delta[\,|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}A|_\infty + |\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}B|_\infty]),$$

and thus $|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}B|_\infty \leqslant C(1+\bar\delta\,|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}A|_\infty)$ if $\bar\delta$ is sufficiently small. Inserting this and (57) with $F=f_1$ in (51), we finally conclude that $|\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}A|_\infty + |\partial_{\bar\theta}^{i^*}\partial_{\bar\rho}^{j^*}B|_\infty \leqslant C$ for sufficiently small $\bar\delta$, so that we have shown $\|A\|_{N+1} + \|B\|_{N+1} \leqslant C$, and thus in particular the claim of the lemma. ∎

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