SIAM J. MATH. ANAL.
Vol. 36, No. 6, pp. 1965–1990

QUASI-PERIODIC SOLUTIONS FOR 1D SCHRÖDINGER EQUATIONS WITH HIGHER ORDER NONLINEARITY*

ZHENGUO LIANG† AND JIANGONG YOU†

Abstract. In this paper, one-dimensional (1D) nonlinear Schrödinger equations

$$iu_t - u_{xx} + mu + \nu|u|^4 u = 0,$$

with Dirichlet boundary conditions are considered. It is proved that for all real parameters m, the above equation admits small-amplitude quasi-periodic solutions corresponding to b-dimensional invariant tori of an associated infinite-dimensional dynamical system. The proof is based on infinite-dimensional KAM theory, partial normal form, and scaling skills.

Key words. quasi-periodic solutions, infinite-dimensional KAM theory, partial normal form

AMS subject classifications. 37K55, 35Q55

DOI. 10.1137/S0036141003435011

1. Introduction and main result. In this paper, we will prove that one-dimensional (1D) nonlinear Schrödinger equation

$$iu_t - u_{xx} + mu + \nu|u|^4 u = 0 \tag{1.1}$$

subject to Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0, \tag{1.2}$$

admits small-amplitude quasi-periodic solutions for all m. Equation (1.1) with $m = 0$ and negative ν is called "focusing" while (1.1) with $m = 0$ and positive ν is called "defocusing." Under some initial-boundary conditions they have been considered by many authors (see [2, 3, 4, 11]). Throughout this paper, we suppose $\nu > 0$ in (1.1). As we will see later, the sign of ν is immaterial for our results.

We study the equation (1.1) as a Hamiltonian system on $\mathcal{P} = W_0^1([0,\pi])$, the Sobolev space of all complex valued L^2-functions on $[0,\pi]$ with an L^2-derivative and vanishing boundary values. Let

$$\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin jx, \lambda_j = j^2 + m, j \geq 1$$

be the basic modes and their frequencies for the linear equation $iu_t = u_{xx} - mu$ with Dirichlet boundary conditions. Then every solution is the superposition of oscillations of the basic modes, with the coefficients moving on circles,

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), q_j(t) = q_j^0 e^{i\lambda_j t}.$$

*Received by the editors September 20, 2003; accepted for publication (in revised form) July 2, 2004; published electronically June 30, 2005. The work was supported by the National Natural Science Foundation of China (19925107) and the Special Funds for Major State Basic Research Projects of China (973 projects).

http://www.siam.org/journals/sima/36-6/43501.html

†Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China (jyou@nju.edu.cn).

Together they move on a rotational torus of finite or infinite dimension, depending on how many modes are excited. In particular, for every choice

$$J = \{j_1 < j_2 < \cdots < j_b\} \subset \mathbb{N}$$

of b basic modes there is an invariant linear space E_J of complex dimension b which is completely foliated into rotational tori,

$$E_J = \{u = q_1\phi_{j_1} + \cdots + q_b\phi_{j_b} : q \in C^b\} = \bigcup_{I \in \overline{P^b}} \mathcal{T}_J(I),$$

where $P^b = \{I : I_j > 0 \text{ for } 1 \le j \le b\}$ and

$$\mathcal{T}_J(I) = \{u = q_1\phi_{j_1} + \cdots + q_b\phi_{j_b} : |q_j|^2 = 2I_j \text{ for} 1 \le j \le b\}.$$

In addition, each such torus is linearly stable and all solutions have vanishing Lyapunov exponents. This is the linear situation.

Upon restoration of the nonlinearity $\nu|u|^4 u$, we show that there exists a Cantor set $\mathcal{C} \subset \mathcal{P}^b$, a specially chosen index set $\mathcal{I} = n_1 < n_2 < \cdots < n_b \subset \mathbb{N}$ (we will call it an admissible set, for more specific see section 3) and a family of b-tori

$$\mathcal{T}_{\mathcal{I}}[\mathcal{C}] = \cup_{I \in \mathcal{C}} \mathcal{T}_{\mathcal{I}}(I) \subset E_{\mathcal{I}}$$

over $\mathcal{C}$, and a Whitney smooth embedding

$$\Phi : \mathcal{T}_{\mathcal{I}}[\mathcal{C}] \hookrightarrow \mathcal{P},$$

such that the restriction of Φ to each $\mathcal{T}_{\mathcal{I}}(I)$ in the family is an embedding of a rotational b-torus for the nonlinear equation. In [10], The image $\mathcal{E}_{\mathcal{I}}$ of $\mathcal{T}_{\mathcal{I}}[\mathcal{C}]$ is called a Cantor manifold of rotational b-tori given by the embedding $\Phi : \mathcal{T}_{\mathcal{I}}[\mathcal{C}] \to \mathcal{E}_{\mathcal{I}}$.

THEOREM 1 (main theorem). *Consider the* 1*D nonlinear Schrödinger equation* (1.1) *with* (1.2). *Then for any admissible index set* $\mathcal{I} = \{n_1 < n_2 < \cdots < n_b\} \subset \mathbb{N}$ *and* $m \in R$*, there exists a positive-measure Cantor manifold* $\mathcal{E}_{\mathcal{I}}$ *of real analytic, linearly stable, Diophantine b-tori for the nonlinear Schrödinger equation given by a Whitney smooth embedding* $\Phi : \mathcal{T}_{\mathcal{I}}[\mathcal{C}] \to \mathcal{E}_{\mathcal{I}}$*, which is a higher order perturbation of the inclusion map* $\Phi_0 : E_{\mathcal{I}} \hookrightarrow \mathcal{P}$ *restricted to* $\mathcal{T}_{\mathcal{I}}[\mathcal{C}]$.

Remark 1. The existence of admissible sets will be proved in the appendix. In fact there exist infinite admissible index sets $\mathcal{I}$.

Remark 2. The result remains true for more general nonlinearities $f(|u|^2)u$, where $f(0) = f'(0) = 0, f''(0) \neq 0$. Our method essentially applies to the nonlinearities $f(|u|^2)u$, where $f(0) = f^{(1)}(0) = \cdots = f^{(k-1)}(0), f^{(k)}(0) \neq 0, k \geq 1$, but the proof would be much more complicated.

Remark 3. The frequencies of the diophantine tori are also under control. They are $\omega(I) = (\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_b}) + \frac{1}{\pi^2}(10I_1^2 + 18I_2^2 + \cdots + 18I_b^2 + 36I_1(I_2 + \cdots + I_b) + 48(I_2I_3 + \cdots + I_{b-1}I_b), \ldots, 18I_1^2 + \cdots + 18I_{b-1}^2 + 10I_b^2 + 36I_b(I_1 + \cdots + I_{b-1}) + 48(I_1I_2 + \cdots + I_{b-2}I_{b-1})) + O(||I||^{\frac{13}{6}})$.

Remark 4. The technique of this paper is not restricted to the nonlinear Schrodinger equation. It applies equally well to the nonlinear 1D beam equations

$$u_{tt} + u_{xxxx} = f(u)$$

with hinged boundary conditions, where f is a real analytic, odd function of u of the form $f(u) = au^3 + \sum_{k \geq 5} f_k u^k,\ a \neq 0$. Our result is an improvement on [6]. Details will be given in another paper. Unfortunately, our technique can't be applied to the complete resonant 1D wave equation

$$u_{tt} + u_{xx} = u^3 \tag{1.3}$$

with Dirichlet boundary conditions. From the proof, one sees that that superlinear growth of the eigenvalues $\lambda_j \sim j^2$ is crucial. For (1.3), the admissible set does not exist and one can't obtain the desired partial Birkhoff normal form by eliminating all the unpleasant terms, which include 2 or 3 tangential coordinates.

The rest of the paper is organized as follows. In section 2 the Hamiltonian function is written in infinitely many coordinates, which is then put into partial normal form in section 3. In section 4 we improve an infinite dimensional KAM theorem, which is developed by many people (see Kuksin [7, 8, 9], Wayne [16], Pöschel [13], Chierchia and You [5]). Measure estimates are given in section 5. Some propositions are proved in the appendix.

2. The Hamiltonian. For simplicity, we choose $\nu = 1$. Other cases can be rescaled into this case. The Hamiltonian of the nonlinear Schrödinger equation is

$$H = \frac{1}{2}\langle Au, u\rangle + \frac{1}{6}\int_0^\pi |u|^6 dx, \tag{2.1}$$

where $A = -d^2/dx^2 + m$. We rewrite H as a Hamiltonian in infinitely many coordinates by making the ansatz

$$u(x) = \sum_{j\geq 1} q_j\phi_j, \quad \phi_j = \sqrt{\frac{2}{\pi}}\sin jx, \quad j \geq 1.$$

The coordinates are taken from the Hilbert spaces $\mathcal{H}^{a,\rho}$ of all complex-valued sequences $q = (q_1, q_2, \ldots)$ with

$$\|q\|_{a,\rho}^2 = \sum_{j\geq 1} |q_j|^2 j^{2a} e^{2j\rho} < \infty.$$

Fix $\rho > 0$ and $a \geq 0$ later. One then obtains the Hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1}\lambda_j|q_j|^2 + \frac{1}{6}\int_0^\pi |u|^6 dx \tag{2.2}$$

on the phase space $\mathcal{H}^{a,\rho}$ with symplectic structure $\frac{i}{2}\sum_j dq_j \wedge d\bar{q}_j$. Its equations of motion are

$$\dot{q}_j = 2i\frac{\partial H}{\partial \bar{q}_j}, \quad j \geq 1. \tag{2.3}$$

They are the classical Hamiltonian equations of motion for the real and imaginary parts of $q_j = x_j + iy_j$ written in complex notation. Rather than discussing the above formal validity, we shall, following [10] or [5], use the following elementary observation.

Lemma 1. *Let I be an interval and let*

$$t \in I \to q(t) \equiv (\{q_j(t)\}_{j\geq 1})$$

be an analytic solution of (2.3) *such that*

$$\sup_{t\in I}\sum_{j\geq 1}|q_j(t)|^2 j^{2a}e^{2j\rho}<\infty \tag{2.4}$$

for some $\rho>0$ *and* $a\geq 0$*. Then*

$$u(t,x)\equiv\sum_{j\geq 1}q_j(t)\phi_j(x),$$

is an analytic solution of (1.1).

For the proof, refer to Lemma 1 in [10].

Next, we consider the regularity of the gradient of G. To this end, let $\mathcal{H}_b^2$ and L^2, respectively, be the Hilbert spaces of all bi-infinite, square summable sequences with complex coefficients and all square-integrable complex valued functions on $[-\pi,\pi]$. Let

$$\mathcal{F}:\mathcal{H}_b^2\to L^2,\quad q\mapsto\mathcal{F}q=\frac{1}{\sqrt{2\pi}}\sum_j q_j e^{\mathrm{i}jx}$$

be the inverse discrete Fourier transform, which defines an isometry between the two spaces. The subspaces $\mathcal{H}_b^{a,\rho}\subset\mathcal{H}_b^2$ consist, by definition, of all bi-infinite sequences with finite norm

$$\|q\|_{a,\rho}^2=|q_0|^2+\sum_j|q_j|^2|j|^{2a}e^{2|j|\rho}.$$

Through $\mathcal{F}$ they define subspaces $W^{a,\rho}\subset L^2$ that are normed by setting $\|\mathcal{F}q\|_{a,\rho}=\|q\|_{a,\rho}$.

LEMMA 2. *For* $a>\frac{1}{2}$ *and* $\rho\geq 0$*, the space* $\mathcal{H}_b^{a,\rho}$ *is a Hilbert algebra with respect to convolution of sequences, and*

$$\|q*p\|_{a,\rho}\leq c\|q\|_{a,\rho}\|p\|_{a,\rho}$$

with a constant c *depending only on* a*. Consequently,* $W^{a,\rho}$ *is a Hilbert algebra with respect to a multiplication of functions.*

For the proof, see [10].

LEMMA 3. *For* $a>\frac{1}{2}$ *and* $\rho\geq 0$*, the gradient* G_q *is real analytic as a map from some neighborhood of the origin in* $\mathcal{H}^{a,\rho}$ *into* $\mathcal{H}^{a,\rho}$*, with*

$$\|G_q\|_{a,\rho}=O(\|q\|_{a,\rho}^5).$$

The proof is similar with Lemma 3 in [10], which we omit.

By the elementary computation, one can get

$$\begin{aligned}G&=\frac{1}{6}\int_0^\pi|u(x)|^6\,dx\\&=\frac{1}{6}\sum_{i,j,k,l,m,n}G_{ijklmn}q_iq_jq_k\bar{q}_l\bar{q}_m\bar{q}_n\end{aligned}$$

with

$$G_{ijklmn}=\int_0^\pi\phi_i\phi_j\phi_k\phi_l\phi_m\phi_n\,dx. \tag{2.5}$$

It is not difficult to verify that $G_{ijklmn} = 0$ unless $i \pm j \pm k \pm l \pm m \pm n = 0$, for some combination of plus and minus signs. For simplicity, we denote $G_{ijk} = G_{iijjkk}, G_i = G_{iiiiii}$. If we choose $n_1, n_2, \ldots, n_b$ satisfying

$$n_i \neq n_j + n_k,\ \forall i, j, k \in \{1, 2, \ldots, b\}, \tag{2.6}$$

one can get

$$G_{n_1} = \cdots = G_{n_b} = \frac{5}{2\pi^2}, \quad G_{n_i n_j n_j} = \frac{3}{2\pi^2}, \quad G_{n_i n_j n_k} = \frac{1}{\pi^2}$$

and

$$G_{n_i n_j l} = \frac{1}{4\pi^2}(4 - \delta^{n_j}_{n_i+l} - \delta^{n_i}_{n_j+l} - \delta^{l}_{n_i+n_j}), \quad G_{n_i n_i l} = \frac{1}{4\pi^2}(6 - \delta^{2n_i}_{l}),$$

where $i \neq j,\ j \neq k,\ k \neq i,\ i, j, k \in \{1, 2, \ldots, b\},\ l \notin \{n_1, n_2, \ldots, n_b\}$, and for $v \in \mathbb{Z}$

$$\delta^v_i = \begin{cases} 1, & i = v, \\ 0, & \text{otherwise.} \end{cases}$$

3. Partial Birkhoff normal form. We shall use the KAM iteration to get the desired result. Since the quadratic part of the Hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1} \lambda_j |q_j|^2 + \frac{1}{6}\sum_{i\pm j\pm k\pm l\pm m\pm n=0} G_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n$$

does not provide any "twist" required by KAM theory, we shall use the normal form technique to get the "twisted" integrable terms from the sixth order terms. To get finite dimensional KAM tori, we shall first fix finite many sites $\{n_1, n_2, \ldots, n_b\}$, and call $q = (q_{n_1}, \ldots, q_{n_b})$ tangential variables. All the other variables, denoted by w, are called normal variables. For our purpose, the sixth order terms with at most two normal variables

$$q_i q_j q_k q_l \bar{q}_m \bar{q}_n, \qquad q_i q_j q_k \bar{q}_l q_m w_n, \qquad q_i q_j q_k q_l w_m w_n$$

must be put into normal form, i.e., the terms that remain after normal form procedure must have the form of $|q_i|^2|q_j|^2|q_k|^2$ or $|q_i|^2|q_j|^2|w_k|^2$. The other sixth order terms are left since they can be scaled into higher perturbations. Such kind of normal form is called a partial Birkhoff normal form since we don't normalize all sixth order terms. In order to get the desired partial Birkhoff normal form, we have to carefully choose $\{n_1, n_2, \ldots, n_b\}$.

For fixed $\{n_1, n_2, \ldots, n_b\}$, we define the index sets $\Delta_*, * = 0, 1, 2$ and Δ_3 in the following way: Δ_* is the set of index (i, j, k, l, m, n) such that there exist right $*$ components not in $\{n_1, n_2, \ldots, n_b\}$. Δ_3 is the set of the index (i, j, k, l, m, n) such that there exist at least three components not in $\{n_1, n_2, \ldots, n_b\}$. Define the resonance sets $\mathcal{N} =\{(i, j, k, i, j, k)\} \cap \Delta_2$ and $\mathcal{M} =\{(i, j, k, i, j, k)\} \cap \Delta_2$. For our convenience, rewrite $G = G^0 + G^1 + G^2 + \hat{G}$, where

$$G^* = \frac{1}{6}\sum_{i\pm j\pm k\pm l\pm m\pm n=0,(i,j,k,l,m,n)\in\Delta_*} G_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n, \tag{3.1}$$

and $* = 0, 1, 2$.

DEFINITION 1. *The index set $\mathcal{I}=\{n_1<n_2<\cdots<n_b\}$ is said to be admissible if and only if $n_1,n_2,\ldots,n_b$ satisfy the following Assumptions A,B,C and* (2.6).

A. If $i\pm j\pm k\pm l\pm m\pm n=0,(i,j,k,l,m,n)\in\Delta_0\setminus\mathcal{N}$, then

$$\lambda_i+\lambda_j+\lambda_k-\lambda_l-\lambda_m-\lambda_n\neq 0.$$

B. If $i\pm j\pm k\pm l\pm m\pm n=0,(i,j,k,l,m,n)\in\Delta_1$, then

$$\lambda_i+\lambda_j+\lambda_k-\lambda_l-\lambda_m-\lambda_n\neq 0.$$

C. If $i\pm j\pm k\pm l\pm m\pm n=0,(i,j,k,l,m,n)\in\Delta_2\setminus\mathcal{M}$, then

$$\lambda_i+\lambda_j+\lambda_k-\lambda_l-\lambda_m-\lambda_n\neq 0.$$

PROPOSITION 1. *There exist infinite many admissible index sets.*

When $b=2$, we can construct some of the admissible index sets clearly. Denote

$$\mathcal{S}=\{n_1\leq n_2|n_1\equiv 5 \text{ or } 9(\text{mod}14),\ n_2\equiv 8(\text{mod}14), n_2\geq 11n_1^2\}.$$

PROPOSITION 2. *If $b=2$, any element in $\mathcal{S}$ is the admissible index set.*

The proofs of Propositions 1 and 2 are given in the appendix.

Next we transform the Hamiltonian (2.2) into the partial Birkhoff form of order six so that the infinite KAM Theorem (see section 4) can be applied.

LEMMA 4. *For any given admissible index set $\{n_1<n_2<\cdots<n_b\}$, there exists a real analytic, symplectic change of coordinates X_F^1 in some neighborhood of the origin that takes the Hamiltonian $H=\Lambda+G$ into*

$$H\circ X_F^1=\Lambda+\overline{G}+\hat{G}+K,$$

where $X_{\overline{G}},X_{\hat{G}}$ and X_K are real analytic vector fields in a neighborhood of the origin in $\mathcal{H}^{a,\rho}$,

$$\begin{aligned}\overline{G}=&\frac{5}{12\pi^2}(|q_{n_1}|^6+\cdots+|q_{n_b}|^6)\\&+\frac{9}{4\pi^2}(|q_{n_1}|^4|q_{n_2}|^2+\cdots+|q_{n_1}|^4|q_{n_b}|^2\\&+|q_{n_2}|^4|q_{n_1}|^2+|q_{n_2}|^4|q_{n_3}|^2+\cdots+|q_{n_2}|^4|q_{n_b}|^2+\cdots+|q_{n_b}|^4|q_{n_{b-1}}|^2)\\&+\frac{6}{\pi^2}(|q_{n_1}|^2|q_{n_2}|^2|q_{n_3}|^2+\cdots+|q_{n_{b-2}}|^2|q_{n_{b-1}}|^2|q_{n_b}|^2)\\&+\frac{3}{2}\left(\sum_{i\neq n_1,n_2,\ldots,n_b}G_{n_1n_1i}|q_{n_1}|^4|q_i|^2+\cdots+\sum_{i\neq n_1,n_2,\ldots,n_b}G_{n_bn_bi}|q_{n_b}|^4|q_i|^2\right)\\&+6\left(\sum_{i\neq n_1,n_2,\ldots,n_b}G_{n_1n_2i}|q_{n_1}|^2|q_{n_2}|^2|q_i|^2+\ldots+\sum_{i\neq n_1,n_2,\ldots,n_b}G_{n_{b-1}n_bi}|q_{n_{b-1}}|^2|q_{n_b}|^2|q_i|^2\right),\end{aligned}$$

$|K|=O(||q||_{a,\rho}^8)$ *and* $\hat{G}=\frac{1}{6}\sum_{i\pm j\pm k\pm l\pm m\pm n=0,(i,j,k,l,m,n)\in\Delta_3}G_{ijklmn}q_iq_jq_k\bar{q}_l\bar{q}_m\bar{q}_n$.

Proof. Let $\Gamma=X_F^t|_{t=1}$ be the time 1-map of the flow of the Hamiltonian vector field X_F given by the Hamiltonian

$$\begin{aligned} F &= F^0 + F^1 + F^2 \\ &= \frac{1}{6}\Big\{ \sum_{i,j,k,l,m,n} F^0_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n \\ &\quad + \sum_{i,j,k,l,m,n} F^1_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n \\ &\quad + \sum_{i,j,k,l,m,n} F^2_{ijklmn} q_i q_j q_k \bar{q}_l \bar{q}_m \bar{q}_n \Big\} \end{aligned}$$

with coefficients

$$\mathrm{i}F^0_{ijklmn} = \begin{cases} \frac{G_{ijklmn}}{\lambda_i+\lambda_j+\lambda_k-\lambda_l-\lambda_m-\lambda_n} & i \pm j \pm k \pm l \pm m \pm n = 0, (i,j,k,l,m,n) \in \Delta_0 \backslash \mathcal{N}, \\ 0 & otherwise, \end{cases}$$

$$\mathrm{i}F^1_{ijklmn} = \begin{cases} \frac{G_{ijklmn}}{\lambda_i+\lambda_j+\lambda_k-\lambda_l-\lambda_m-\lambda_n} & i \pm j \pm k \pm l \pm m \pm n = 0, (i,j,k,l,m,n) \in \Delta_1, \\ 0 & otherwise, \end{cases}$$

$$\mathrm{i}F^2_{ijklmn} = \begin{cases} \frac{G_{ijklmn}}{\lambda_i+\lambda_j+\lambda_k-\lambda_l-\lambda_m-\lambda_n} & i \pm j \pm k \pm l \pm m \pm n = 0, (i,j,k,l,m,n) \in \Delta_2 \backslash \mathcal{M}, \\ 0 & otherwise. \end{cases}$$

Note our Assumptions A, B, C, the remained proof is just a copy of Lemma 4 of [10]. □

Now our Hamiltonian is $H = \Lambda + \overline{G} + \hat{G} + K$. Introduce the symplectic polar and complex coordinates by setting

$$q_j = \begin{cases} \sqrt{2(\xi_j + y_j)}e^{-\mathrm{i}x_j}, & j = n_1, n_2, \dots, n_b \\ \sqrt{2}z_j, & j \neq n_1, n_2, \dots, n_b \end{cases}$$

depending on parameters $\xi \in \Pi = [0,1]^b$. The precise domain will be specified later. In order to simplify the expression, we substitute $\xi_{n_j}, j = 1, 2, \dots, b$ by $\xi_j, j = 1, 2, \dots, b$. Then one gets

$$\frac{i}{2} \sum_{j \geq 1} dq_j \wedge d\bar{q}_j = \sum_{j = n_1, n_2, \dots, n_b} dx_j \wedge dy_j + i \sum_{j \neq n_1, n_2, \dots, n_b} dz_j \wedge d\bar{z}_j.$$

The new Hamiltonian

$$H = \Lambda + \overline{G} + \hat{G} + K = \langle \omega(\xi), y \rangle + \langle \Omega(\xi) z, \bar{z} \rangle + \tilde{G} + \hat{G} + K$$

with frequencies $\omega(\xi) = \alpha' + A(\xi)$, $\Omega(\xi) = \beta' + B(\xi)$, where

$$\alpha' = (\lambda_{n_1}, \lambda_{n_2}, \dots, \lambda_{n_b}), \beta' = (\lambda_i)_{i \neq n_1, \dots, n_b},$$

$$\begin{aligned} A(\xi) = \frac{1}{\pi^2}(&10\xi_1^2 + 18\xi_2^2 + \cdots + 18\xi_b^2 + 36\xi_1(\xi_2 + \cdots + \xi_b) + 48(\xi_2\xi_3 + \cdots + \xi_{b-1}\xi_b), \dots, \\ &18\xi_1^2 + \cdots + 18\xi_{b-1}^2 + 10\xi_b^2 + 36\xi_b(\xi_1 + \cdots + \xi_{b-1}) + 48(\xi_1\xi_2 + \cdots + \xi_{b-2}\xi_{b-1})), \end{aligned}$$

$$B(\xi) = (12G_{n_1n_1i}\xi_1^2 + \cdots + 12G_{n_bn_bi}\xi_b^2 + 48G_{n_1n_2i}\xi_1\xi_2 + \cdots + 48G_{n_{b-1}n_bi}\xi_{b-1}\xi_b)_{i \neq n_1, \dots, n_b},$$

and the remainder $\tilde{G} = O(|y|^3) + O(|\xi||y|^2) + O(|\xi||y||z|^2_{a,\rho}) + O(|y|^2|z|^2_{a,\rho}), \hat{G} = O(|\xi|^{\frac{3}{2}}|z|^3_{a,\rho}), K = O(|\xi|^4)$. Rescaling ξ by $\epsilon^6\xi$,$z,\bar{z}$ by $\epsilon^4 z, \epsilon^4\bar{z}$, and y by $\epsilon^8 y$, one obtains a Hamiltonian given by the rescaled Hamiltonian

$$\begin{aligned}\tilde{H}(x,y,z,\bar{z},\xi) &= \epsilon^{-20}H(x,\epsilon^8 y,\epsilon^4 z,\epsilon^4\bar{z},\epsilon^6\xi,\epsilon)\\ &= \langle\tilde{\omega}(\xi),y\rangle + \langle\tilde{\Omega}(\xi)z,\bar{z}\rangle + \epsilon\tilde{P}(x,y,z,\bar{z},\xi,\epsilon),\end{aligned}$$

where $\tilde{\omega}(\xi) = \epsilon^{-12}\alpha' + A(\xi),\ \tilde{\Omega} = \epsilon^{-12}\beta' + B(\xi),\ \xi \in [1,2]^b$. For simplicity, we rewrite $\tilde{H}$ by H, $\tilde{\omega}$ by ω, $\tilde{\Omega}$ by Ω, and $\tilde{P}$ by P.

In what follows, we use the KAM iteration which involves infinite many steps of coordinate transformations to prove the existence of the KAM tori. To make this quantitative we introduce the following notations and spaces.

Define

$$D(r,s) = \{(x,y,z,\bar{z}) : |Imx| < s, |y| < r^2, ||z||_{a,\rho} < r, ||\bar{z}||_{a,\rho} < r\}$$

a complex neighborhood of $\mathbb{T}^b \times \{y=0\} \times \{z=0\} \times \{\bar{z}=0\}$, where $|\cdot|$ denotes the sup-norm for complex vectors. For a p ($p \geq 1$) order Whitney smooth function $F(\xi)$, define

$$||F||^* = \max\left\{\sup_{\xi\in\Pi}|F|,\ldots,\sup_{\xi\in\Pi}\left|\frac{\partial^p F}{\partial\xi^p}\right|\right\},$$

$$||F||_* = \max\left\{\sup_{\xi\in\Pi}\left|\frac{\partial F}{\partial\xi}\right|,\ldots,\sup_{\xi\in\Pi}\left|\frac{\partial^p F}{\partial\xi^p}\right|\right\}.$$

If $F(\xi)$ is a vector function from ξ to $\mathcal{H}^{a,\rho}(R^n)$ which is p order Whitney smooth on ξ, define $||F||^*_{a,\rho} = ||(||F_i(\xi)||^*)_i||_{a,\rho}(||F||^*_{R^n} = \max_i(||F_i(\xi)||^*))$. If $F(\eta,\xi)$ is a vector function from $D\times\Pi$ to $\mathcal{H}^{a,\rho}$, define $||F||^*_{a,\rho,D} = \sup_{\eta\in D}||F||^*_{a,\rho}$. We usually omit D for brevity. For functions F, associate a Hamiltonian vector field defined as $X_F = \{F_y, -F_x, iF_{\bar{z}}, -iF_z\}$. Denote the weighted norm for X_F by letting

$$|X_F|^*_{r,D(r,s)} = ||F_y||^* + \frac{1}{r^2}||F_x||^* + \frac{1}{r}\|F_z\|^*_{a,\rho} + \frac{1}{r}\|F_{\bar{z}}\|^*_{a,\rho}.$$

4. An infinite dimensional KAM theorem. Theorem 1 is a direct result of Theorem 2 and measures estimates in section 5. Consider small perturbations of an infinite dimensional Hamiltonian in the parameter dependent normal form

$$N = \langle\omega(\xi),y\rangle + \langle\Omega(\xi)z,\bar{z}\rangle$$

on a phase space

$$\mathcal{P}^{a,\rho} = \mathbb{T}^n \times \mathbb{R}^n \times \mathcal{H}^{a,\rho} \times \mathcal{H}^{a,\rho} \ni (x,y,z,\bar{z}),$$

where

$$\omega_j = \frac{j^d + \cdots}{\epsilon^t} + O(\xi^p)^1, \quad \Omega_j = \frac{j^d + \cdots}{\epsilon^t} + O(\xi^p),$$

$t,p\in\mathbb{N}, \rho > 0, a \geq 0$. Suppose that $||\omega||_* \leq M_1, ||\Omega_j||_* \leq M_2, M_1 + M_2 \geq 1$. Define $M = (M_1+M_2)^p$. The parameter set Π is $[1,2]^n$.

[1] $O(\xi^p)$ means pth order terms in $\xi_1,\ldots,\xi_b$

For the Hamiltonian $H = N + P$, there exists n-dimensional, linearly stable torus $\mathcal{T}_0^n = \mathbb{T}^n \times \{0,0,0\}$ with frequencies $\omega(\xi)$ when $P = 0$. Our aim is to prove the persistence of a large portion of this family of linearly stable rotational tori under small perturbations. Suppose that the perturbation P is real analytic in the space variables, C^p in ξ, and for each $\xi \in \Pi$ its Hamiltonian vector field $X_P = (P_y, -P_x, iP_{\bar{z}}, -iP_z)^T$ defines near $\mathcal{T}_0^n$ a real analytic map $X_P : \mathcal{P}^{a,\rho} \to \mathcal{P}^{a,\rho}$. Under the previous assumptions, we have the following theorem.

THEOREM 2. *Suppose that $H = N + P$ satisfies*

$$|X_P|^*_{r,D(s,r)} \leq \gamma s^{2(1+\mu)}, \tag{4.1}$$

where γ depends on n, p, τ and M, $\mu = (p+1)\tau + p + \frac{n}{2}$. Then there exists a Cantor set $\Pi_\epsilon \subset \Pi$, a Whitney smooth family of torus embeddings $\Phi : \mathbb{T}^n \times \Pi_\epsilon \to \mathcal{P}^{a,\rho}$, and a Whitney smooth map $\omega_* : \Pi_\epsilon \to \mathbb{R}^n$, such that for each $\xi \in \Pi_\epsilon$, the map Φ restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of a rotational torus with frequencies $\omega_*(\xi)$ for the Hamiltonian H at ξ.

Each embedding is real analytic on $|\mathrm{Im} x| < \frac{s}{2}$, and

$$\|\Phi - \Phi_0\|^*_r \leq c\epsilon^{\frac{1}{2}},$$
$$||\omega_* - \omega||^* \leq c\epsilon,$$

uniformly on that domain and Π_ϵ, where Φ_0 is the trivial embedding $\mathbb{T}^n \times \Pi \to \mathcal{T}_0^n$. Moreover, there exist Whitney smooth maps ω_m and Ω_m on Π for $m \geq 1$ satisfying $\omega_1 = \omega$, $\Omega_1 = \Omega$ and

$$||\omega_m - \omega||^* \leq c\epsilon, \tag{4.2}$$

$$\|\Omega_m - \Omega\|^* \leq c\epsilon. \tag{4.3}$$

Remark. Note that in the theorem, we didn't claim that the measure of Π_ϵ is positive. For positive measure, one needs further information of the frequencies $\omega(\xi)$ and $\Omega(\xi)$. We shall come back to this point in section 5.

Since the proof of Theorem 2 is essentially standard, we only state the main step of KAM iteration. The more detailed steps can be found in [13] and other papers.

4.1. Solving the linearized equations and KAM step. At each step of KAM iteration, the symplectic coordinate change Φ is obtained as the time 1-map $X_F^t|_{t=1}$ of the flow of Hamiltonian vector field X_F. Its generating function F and some normal correction $\hat{N}$ to the given normal form N are solutions of the linear equation

$$\{F, N\} + \hat{N} = R, \tag{4.4}$$

where

$$R = \sum_{2m+|q+\bar{q}|\leq 2} R_{kmq\bar{q}} y^m z^q \bar{z}^{\bar{q}} e^{i\langle k,x\rangle}, R_{kmq\bar{q}} = P_{kmq\bar{q}},$$

and the coefficients $R_{kmq\bar{q}}$ depend on ξ such that $X_R : P^{a,\rho} \to P^{a,\rho}$ is real analytic and Whitney smooth in ξ. Below we solve the linear equation and estimate the generating function F.

LEMMA 5. *Suppose that uniformly on* $\Pi_+ \subset \Pi$,

$$|\langle k,\omega\rangle| \geq \frac{\epsilon^\beta}{A_k} \text{ for } k \neq 0, \tag{4.5}$$

$$|\langle k,\omega\rangle + \Omega_i| \geq \frac{\epsilon^\beta}{A_k}, \tag{4.6}$$

$$|\langle k,\omega\rangle + \Omega_i + \Omega_j| \geq \frac{\epsilon^\beta(|i-j|+1)}{A_k}, \tag{4.7}$$

$$|\langle k,\omega\rangle + \Omega_i - \Omega_j| \geq \frac{\epsilon^\beta(|i-j|+1)}{A_k}, i \neq j. \tag{4.8}$$

Then the linear equation has solution F *and* $\hat{N}$, *which satisfy* $[F]=0$, $[\hat{N}]=\hat{N}$. *Moreover,*

$$|X_{\hat{N}}|^*_{r,D(s,r)} \leq |X_R|^*_{r,D(s,r)}, |X_F|^*_{r,D(s-\sigma,r)} \leq \frac{cM}{\epsilon^{(p+1)\beta}\sigma^\mu}|X_R|^*_{r,D(s,r)}, \tag{4.9}$$

where $A_k = 1+|k|^\tau$, β *will be denoted later.*

For the proof, refer to [13].

LEMMA 6. *If* $|X_F|^*_{r,D(s-\sigma,r)} \leq \sigma$, *then for any* $\xi \in \Pi_+$, *the flow* $X_F^t(\cdot,\xi)$ *exists on* $D(s-2\sigma,\frac{r}{2})$ *for* $|t|\leq 1$ *and maps* $D(s-2\sigma,\frac{r}{2})$ *into* $D(s-\sigma,r)$. *Moreover, for* $|t|\leq 1$,

$$|X_F^t - id|^*_{r,D(s-2\sigma,\frac{r}{2})}, \sigma||DX_F^t - Id||^*_{r,r,D(s-3\sigma,\frac{r}{4})} \leq c|X_F|^*_{r,D(s-\sigma,r)},$$

where D *is the differentiation operator with respect to* $(x,y,z,\bar{z})$, *id and* Id *are identity mapping and unit matrix, and the operator norm*

$$||A(\xi,\eta)||_{\bar{r},r,D(s,r)} = \sup_{\eta\in D(s,r)}\sup_{w\neq 0}\frac{||A(\xi,\eta)w||_{a,\bar{r}}}{||w||_{a,r}},$$

$$||A||^*_{r,r} = \max\left\{||A||_{r,r},\ldots,\left|\left|\frac{\partial^p A}{\partial \xi^p}\right|\right|_{r,r}\right\}.$$

For the proof, see [14].

Below we consider the new perturbation under the sympletic transformation $\Phi = X_F^t|_{t=1}$. Let $|X_P|^*_{r,D(s,r)} \leq \epsilon$. From the above we have

$$R = \sum_{2|m|+|q+\bar{q}|\leq 2} R_{kmq\bar{q}} y^m z^q \bar{z}^{\bar{q}} e^{i\langle k,x\rangle}.$$

Thus $|X_R|^*_{r,D(s,r)} \leq \cdot|X_P|^*_{r,D(s,r)} \leq \cdot\epsilon$, and for $\eta \leq \frac{1}{8}$,

$$|X_{P-R}|^*_{\eta r,D(s,4\eta r)} \leq \cdot\eta|X_P|^*_{r,D(s,r)} \leq \cdot\eta\epsilon. \tag{4.10}$$

Since

$$\hat{N} = \sum_{2|m|+|q+\bar{q}|\leq 2, q=\bar{q}} P_{0mq\bar{q}} y^m z^q \bar{z}^{\bar{q}} e^{i\langle k,x\rangle},$$

the new normal form is

$$N_+ = N + \hat{N} = \langle \omega_+, y\rangle + \langle \Omega_+ z, \bar{z}\rangle.$$

By Lemma 5, one has $|X_{\hat{N}}|^*_{r,D(s,r)} \le \cdot\epsilon$. Therefore,

$$||\omega_+ - \omega||^*, ||\Omega_+ - \Omega||^* \le \cdot\epsilon, \tag{4.11}$$

where $||\Omega||^* = \max_{j\ge 1} ||\Omega_j||^*$. If $\frac{cM\epsilon^{1-\beta(p+1)}}{\sigma^{\mu+1}} \le 1$, by Lemmas 5 and 6, it follows that for $|t| \le 1$,

$$\frac{1}{\sigma}|X_F^t - id|^*_{r,D(s-2\sigma,\frac{r}{2})}, ||DX_F^t - Id||^*_{r,r,D(s-3\sigma,\frac{r}{4})} \le \frac{cM\epsilon^{1-(p+1)\beta}}{\sigma^{\mu+1}}. \tag{4.12}$$

Under the transformation $\Phi = X_F^1$, $(N+R)\circ\Phi = N_+ + R_+$, where $R_+ = \int_0^1 \{(1-t)\hat{N} + tR, F\}\circ X_F^t$. Thus, $H\circ\Phi = N_+ + R_+ + (P-R)\circ\Phi = N_+ + P_+$, where the new perturbation

$$P_+ = R_+ + (P-R)\circ\Phi = (P-R)\circ\Phi + \int_0^1 \{\bar{R}(t), F\}\circ X_F^t dt,$$

where $\bar{R}(t) = (1-t)\hat{N} + tR$. Hence, the Hamiltonian vector field of the new perturbation is

$$X_{P_+} = (X_F^1)^*(X_{P-R}) + \int_0^1 (X_F^t)^*[X_{\bar{R}(t)}, X_F]dt.$$

For the estimate of X_{P_+}, we need the following lemma.

LEMMA 7. *If the Hamiltonian vector field $W(\cdot,\xi)$ on $V = D(s-4\sigma, 2\eta r)$ depends on the parameter $\xi \in \Pi_+$ with $||W||^*_{r,V} < +\infty$, and $\Phi = X_F^t : U = D(s-5\sigma, \eta r) \to V$, then $\Phi^* W = (D\Phi)^{-1} W\circ\Phi$ and if $\frac{cM\epsilon^{1-(p+1)\beta}}{n^2\sigma^{\mu+1}} \le 1$, we have $||\Phi^*W||^*_{\eta r,U} \le c||W||^*_{\eta r,V}$.*

For the proof, see [14].

Now we estimate X_{P_+}. By Lemma 7, if $\frac{cM\epsilon^{1-(p+1)\beta}}{n^2\sigma^{\mu+1}} \le 1$,

$$|X_{P_+}|^*_{\eta r,D(s-5\sigma,\eta r)} \le c|X_{P-R}|^*_{\eta r,D(s-4\sigma,2\eta r)} + c\int_0^1 |[X_{\bar{R}(t)}, X_F]|^*_{\eta r,D(s-4\sigma,2\eta r)}dt.$$

By Cauchy's inequality and Lemma 6, one obtains

$$\begin{aligned}|[X_{\bar{R}(t)}, X_F]|^*_{\eta r,D(s-4\sigma,2\eta r)} &\le \frac{cM\epsilon^{2-(p+1)\beta}}{\eta^2\sigma^{\mu+1}} \\ &= cM\eta\epsilon,\end{aligned}$$

where one chooses $\eta^3 = \frac{\epsilon^{1-(p+1)\beta}}{\sigma^{\mu+1}}$. Combining (4.10) we have

$$|X_{P_+}|^*_{\eta r,D(s-5\sigma,\eta r)} \le cM\eta\epsilon.$$

4.2. Iteration and proof of Theorem 2. To iterate the KAM step infinitely we must choose suitable sequences. For $m \ge 1$ set

$$\epsilon_{m+1} = \frac{cM(m)\epsilon_m^{\frac{4}{3}-\frac{1}{3}(p+1)\beta}}{\sigma_m^{\frac{1}{3}(1+\mu)}}, \quad \sigma_{m+1} = \frac{\sigma_m}{2}, \quad \eta_m^3 = \frac{\epsilon_m^{1-(p+1)\beta}}{\sigma_m},$$

where $\beta = \frac{1}{2(p+1)}$. Furthermore, $s_{m+1} = s_m - 5\sigma_m$, $r_{m+1} = \eta_m r_m$, $M(m) = (M_1 + M_2 + 2c(\epsilon_1 + \cdots + \epsilon_{m-1}))^p$, and $D_m = D(s_m, r_m)$. As initial value fix $\sigma_1 = \frac{s_1}{20} \leq \frac{1}{2}$. Choose

$$\epsilon_1 \leq \frac{\gamma_0^6 \sigma_1^{2(\mu+1)}}{c^6 M^{6p}}, \tag{4.13}$$

where $\gamma_0 \leq \frac{1}{c(M+1)^p 2^{6p+4\mu}}$. Finally, let $K_m = K_1 2^{m-1}$ with

$$K_1^{\tau+1} = c^{5-6\beta} M^{6p(1-\beta)} 2^{2(1+\mu)(1-\beta)-3} \gamma_0^{6(\beta-1)}. \tag{4.14}$$

LEMMA 8. *Suppose $H_m = N_m + P_m$ is given on $D_m \times \Pi_m$, where $N_m = \langle \omega_m(\xi), y\rangle + \langle \Omega_m, z\bar{z}\rangle$ is a normal form satisfying*

$$|\langle k, \omega_m\rangle| \geq \frac{\epsilon_m^\beta}{A_k} \text{ for } k \neq 0, \tag{4.15}$$

$$|\langle k, \omega_m\rangle + \Omega_{m,i}| \geq \frac{\epsilon_m^\beta}{A_k}, \tag{4.16}$$

$$|\langle k, \omega_m\rangle + \Omega_{m,i} + \Omega_{m,j}| \geq \frac{\epsilon_m^\beta(|i-j|+1)}{A_k}, \tag{4.17}$$

$$|\langle k, \omega_m\rangle + \Omega_{m,i} - \Omega_{m,j}| \geq \frac{\epsilon_m^\beta(|i-j|+1)}{A_k}, i \neq j, \tag{4.18}$$

for any $\xi \in \Pi_m$, and

$$|X_{P_m}|^*_{r_m, D_m} \leq \epsilon_m.$$

Then there exists a Whitney smooth family of real analytic symplectic coordinate transformations $\Phi_{m+1} : D_{m+1} \times \Pi_m \to D_m$ and a closed subset

$$\Pi_{m+1} = \Pi_m \setminus \bigcup_{|k| > K_m} R_{kl}^{m+1}(\epsilon_{m+1})$$

of Π_m, where

$$R_{kl}^{m+1}(\epsilon_{m+1}) = A_{k1}^{m+1} \cup A_{k2}^{m+1} \cup A_{k3}^{m+1} \cup A_{k4}^{m+1},$$

and

$$A_{k1}^{m+1} = \left\{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle| < \frac{\epsilon_{m+1}^\beta}{A_k}\right\},$$

$$A_{k2}^{m+1} = \bigcup_i B_{ki}^{m+1,1} = \bigcup_i \left\{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle + \Omega_{m+1,i}| < \frac{\epsilon_{m+1}^\beta}{A_k}\right\},$$

$$A_{k3}^{m+1} = \bigcup_{i,j} B_{kij}^{m+1,11} = \bigcup_{i,j} \left\{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle + \Omega_{m+1,i} + \Omega_{m+1,j}| < \frac{\epsilon_{m+1}^\beta(|i-j|+1)}{A_k}\right\},$$

$$A_{k4}^{m+1} = \bigcup_{i\neq j} B_{kij}^{m+1,12} = \bigcup_{i\neq j} \left\{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle + \Omega_{m+1,i} - \Omega_{m+1,j}| < \frac{\epsilon_{m+1}^\beta(|i-j|+1)}{A_k}\right\},$$

such that for $H_{m+1} = H_m \circ \Phi_{m+1} = N_{m+1} + P_{m+1}$ *the same assumptions are satisfied with* $m+1$ *in place of* m.

Proof. Note the value for p_1, ϵ_1, β and σ_1, one verifies that

$$\frac{M_{m+1}\epsilon_{m+1}^{1-(p+1)\beta}}{\sigma_{m+1}^{1+\mu}} \leq \frac{1}{2}\frac{M_m \epsilon_m^{1-(p+1)\beta}}{\sigma_m^{1+\mu}} \tag{4.19}$$

for all $m \geq 1$. So the smallness condition of the KAM step is satisfied. For the remained proof, see the iterative lemma in [13]. □

With (4.11) and (4.12), we also obtain the following estimate.

LEMMA 9. *For* $m \geq 1$*,*

$$\frac{1}{\sigma_m}||\Phi_{m+1} - id||^*_{r_m, D_{m+1}}, ||D\Phi_{m+1} - I||^*_{r_m, r_m, D_{m+1}} \leq \frac{cM(m)\epsilon_m^{1-(p+1)\beta}}{\sigma_m^{\mu+1}} \tag{4.20}$$

$$||\omega_{m+1} - \omega_m||^*_{\Pi_m}, ||\Omega_{m+1} - \Omega_m||^*_{\Pi_m} \leq c\epsilon_m. \tag{4.21}$$

Proof of Theorem 2. The smallness condition is

$$\epsilon_1 \leq \frac{\gamma_0^6}{20^{2(1+\mu)(cM^p)^6}} s_1^{2(1+\mu)}. \tag{4.22}$$

To apply Lemma 8 with $m = 1$, set $s_1 = s$, $r_1 = r, \ldots, N_1 = N$, $P_1 = P$,

$$\gamma = \frac{\gamma_0^6}{20^{2(1+\mu)(cM^p)^6}} \text{ and } \epsilon_1 = \gamma s_1^{2(1+\mu)}.$$

The smallness condition is satisfied, because

$$|X_{P_1}|^*_{r_1, D(s_1, r_1)} = |X_P|^*_{r, D(r,s)} \leq \gamma s^{2(1+\mu)} = \epsilon_1.$$

The small divisor conditions are satisfied by setting $\Pi_1 = \Pi \setminus \cup_{kl} R^1_{kl}(\epsilon)$, where $k \neq 0$ for A^1_{k1}, and $\Pi_0 = \Pi$. Then the iterative lemma applies. □

Remark. For the rescaled Hamiltonian H, we fix $r = 1$. Then

$$|X_{\epsilon P}|^*_{1, D(s,1)} \leq |X_{\epsilon P}|^*_{1, D(1,1)} \leq c\epsilon \leq \gamma s^{2(1+\mu)},$$

for ϵ small enough. If fix $\rho > 0$ and $a > \frac{1}{2}$ arbitrarily, Theorem 2 can be applied to the rescaled Hamiltonian.

5. Measure estimates. The remaining job is to estimate the measure. We first give the measure estimates for the first step. In our case, the tangent frequencies $\omega_i = \lambda_i + O(\xi^2)(i = n_1, \ldots, n_b)$ and normal frequencies $\Omega_j = \lambda_j + O(\xi^2)(j \neq n_1, \ldots, n_b)$ are second orders in ξ while the ones appeared in the papers such as [10] and [12] are linear in ξ. This is another main difference between our paper and others. To obtain the measure estimates, we have to control the higher order derivatives for $\langle k, \omega\rangle \pm \Omega_i \pm \Omega_j$ etc. One finds that more information from $O(\xi^2)$ is needed to exclude the degenerate cases. The measure estimates in the subsequent steps are based on the techniques developed in [14] and [15].

5.1. Measure estimates in the first step. The thrown parameter sets in the first step are $(\cup_{k\neq 0}A^1_{k1})\bigcup(\cup_k(A^1_{k2}\cup A^1_{k3}\cup A^1_{k4}))$, where

$$A^1_{k1}=\left\{\xi\in\Pi:|\langle k,\omega\rangle|<\frac{\epsilon^\beta}{A_k}\right\}, \tag{5.1}$$

$$A^1_{k2}=\bigcup_i B^{1,1}_{ki}=\bigcup_i\left\{\xi\in\Pi:|\langle k,\omega\rangle+\Omega_i|<\frac{\epsilon^\beta}{A_k}\right\}, \tag{5.2}$$

$$A^1_{k3}=\bigcup_{i,j} B^{1,11}_{kij}=\bigcup_{i,j}\left\{\xi\in\Pi:|\langle k,\omega\rangle+\Omega_i+\Omega_j|<\frac{\epsilon^\beta(|i-j|+1)}{A_k}\right\}, \tag{5.3}$$

$$A^1_{k4}=\bigcup_{i\neq j} B^{1,12}_{kij}=\bigcup_{i\neq j}\left\{\xi\in\Pi:|\langle k,\omega\rangle+\Omega_i-\Omega_j|<\frac{\epsilon^\beta(|i-j|+1)}{A_k}\right\}. \tag{5.4}$$

It is obvious that $|A^1_{02}\cup A^1_{03}\cup A^1_{04}|=0$.

LEMMA 10. *Suppose that $g(x)$ is an* m*th differentiable function on the closure $\bar I$ of I, where $I\subset\mathbb{R}$ is an interval. Let $I_h=\{x||g(x)|<h\}, h>0$. If for some constant $d>0$, $|g^m(x)|\geq d$ for any $x\in I$, then $|I_h|\leq ch^{\frac{1}{m}}$, where $|I_h|$ denotes the Lebesgue measure of I_h and $c=2(2+3+\cdots+m+d^{-1})$.*

For the proof, see [15]. The similar method can be found in [1] and [14].

LEMMA 11. *For $\tau>2b+5$, $|\bigcup_{k\neq 0}A^1_{k3}|=O(\epsilon^{\frac{\beta}{2}})$.*

Proof. Suppose $i\geq j$ without losing generalities. When $i\geq c|k|$, one obtains $\frac{|\Omega_i+\Omega_j|}{1+i-j}\geq\frac{c|k|}{8\epsilon^{12}}$. But we know $\frac{|\langle k,\omega\rangle|}{1+(i-j)}\leq\frac{c'|k|}{\epsilon^{12}}$. If c is large enough, then $\frac{|\Omega_i+\Omega_j+\langle k,\omega\rangle|}{1+(i-j)}\geq 1$. This means $A^1_{k3}=\bigcup_{\max\{i,j\}\leq c|k|}B^{1,11}_{kij}$. Define

$$\begin{aligned}f(\xi)=&\frac{k_1}{\pi^2}(10\xi_1^2+18\xi_2^2+\cdots+18\xi_b^2+36\xi_1(\xi_2+\cdots+\xi_b)+48(\xi_2\xi_3+\cdots+\xi_{b-1}\xi_b))+\cdots\\&+\frac{k_b}{\pi^2}(18\xi_1^2+18\xi_2^2+\cdots+10\xi_b^2+36\xi_b(\xi_1+\cdots+\xi_{b-1})+48(\xi_1\xi_2+\cdots+\xi_{b-2}\xi_{b-1}))\\&+(12\xi_1^2G_{n_1n_1i}+\cdots+12\xi_b^2G_{n_bn_bi}+48G_{n_1n_2i}\xi_1\xi_2+\cdots+48G_{n_{b-1}n_bi}\xi_{b-1}\xi_b)\\&+(12\xi_1^2G_{n_1n_1j}+\cdots+12\xi_b^2G_{n_bn_bj}+48G_{n_1n_2j}\xi_1\xi_2+\cdots+48G_{n_{b-1}n_bj}\xi_{b-1}\xi_b).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\pi^2}{2}\frac{\partial^2 f}{\partial\xi_1^2}&=10k_1+18k_2+\cdots+18k_b+3(c_1+c_1')\\&\vdots\\\frac{\pi^2}{2}\frac{\partial^2 f}{\partial\xi_n^2}&=18k_1+18k_2+\cdots+10k_b+3(c_b+c_b'),\end{aligned}$$

where $c_i,c_i'=5$ or $6, i=1,2,\ldots,b$. We will prove the inequality

$$\max\left(\frac{\pi^2}{2}\left|\frac{\partial^2 f}{\partial\xi_1^2}\right|,\ldots,\frac{\pi^2}{2}\left|\frac{\partial^2 f}{\partial\xi_b^2}\right|\right)\geq 1$$

always holds. If it is not true, one gets that

$$k_1=\frac{-3}{8(9b-4)}((13-9b)(c_1+c_1')+9(c_2+c_2'+\cdots+c_b+c_b')).$$

One can draw the contradictions from the following three cases.

Case 1. Two "5s" in $\{c_1, c_2, \ldots, c_b, c_1', c_2', \ldots, c_b'\}$. In this case, we discuss it from different possibilities.

Subcase a: $c_1 = 5,\ c_1' = 6$. One obtains $k_1 = -\frac{3(9b+26)}{72b-32}$. It is obvious that $k_1 \notin Z$. It is similar for the case $c_1 = 6,\ c_1' = 5$.

Subcase b: $c_1 = c_1' = 6$. One gets $k_1 = -\frac{45}{4(9b-4)}$. It is impossible.

Subcase c: $c_1 = c_1' = 5$. One gets $|k_1| = \frac{3(18b+22)}{8(9b-4)}$. When $b \geq 6$, $0 < |k_1| < 1$. For $b = 2, \ldots, 5$, one can get $k_1 \notin Z$(check directly).

Case 2. Only one "5" in $\{c_1, c_2, \ldots, c_b, c_1', c_2', \ldots, c_b'\}$.

Subcase a: $c_1 = 5$ or $c_1' = 5$. One obtains $|k_1| = \frac{105+27b}{72b-32}$. If $b \geq 4$, then $0 < |k_1| < 1$. It is impossible. If $b = 2, 3$, one has $k_1 \notin Z$(check directly). It also contradicts with the previous assumption.

Subcase b: $c_{k_0} = 5$ or $c_{k_0}' = 5 (k_0 \neq 1)$. One gets $k_1 = -\frac{117}{8(9b-4)}$. It can't happen.

Case 3. No "5s" in $\{c_1, c_2, \ldots, c_b, c_1', c_2', \ldots, c_b'\}$.

One gets $k_1 = -\frac{18}{9b-4}$. If $b \geq 3$, one can get $0 < |k_1| < 1$. When $b = 2$, we obtain $k_1 \notin Z$ directly. It is impossible.

Hence, for any $k \neq 0, i$, there exists some $k_0 \in \{1, 2, \ldots, b\}$, *s.t.*, $|\frac{\pi^2}{2}\frac{\partial^2 f}{\partial \xi_{k_0}^2}| \geq 1$. Then one obtains

$$\begin{aligned}
\left|\bigcup_{k\neq 0} A_{k3}^1\right| &= \left|\bigcup_{k\neq 0}\bigcup_{i,j\leq c|k|} B_{kij}^{1,11}\right|, \\
&\leq \cdot \sum_{k\neq 0}\left(\frac{\epsilon^{\beta}|k|}{A_k}\right)^{\frac{1}{2}}|k|^2, \\
&\leq \cdot \sum_{l=1}^{+\infty}\frac{1}{l^{\frac{\tau-2b-3}{2}}}\epsilon^{\frac{\beta}{2}}, \\
&= O(\epsilon^{\frac{\beta}{2}}). \qquad \square
\end{aligned}$$

LEMMA 12. *For* $\tau > 2b+5$, $|\bigcup_{k\neq 0} A_{k4}^1| = O(\epsilon^{\frac{\beta}{2}})$.

Proof. By the same methods, one obtains that $A_{k4}^1 = \bigcup_{\max\{i,j\}\leq c|k|} B_{kij}^{1,12}$. Following the similar way, we can get

$$k_l = \frac{-3}{8(9b-4)}\left((13-9b)(c_l - c_l') + 9\left(\sum_{m\neq l} c_k - \sum_{m\neq l} c_k'\right)\right),$$

where $l, m \in \{1, \ldots, b\}$.

Case 1. Two "5s" in $\{c_1, c_2, \ldots, c_b, c_1', c_2', \ldots, c_b'\}$.

If for any i, we have $c_i = c_i'$. One gets $k = 0$ in this case. It is impossible. If $\exists i_0, c_{i_0} \neq c_{i_0}'$, there exist two cases. One is $c_{i_0} = 5,\ c_{i_0}' = 6$. The other is $c_{i_0} = 6,\ c_{i_0}' = 5$. In any case, one can get $|k_{i_0}| = \frac{3}{8}$.

Case 2. One "5" in $\{c_1, c_2, \ldots, c_b, c_1', c_2', \ldots, c_b'\}$.

Case 3. No "5s" in $\{c_1, c_2, \ldots, c_b, c_1', c_2', \ldots, c_b'\}$.

We omit the proof for the two cases. The measure estimate is similar as before. We also omit it. $\square$

The following conclusions are obvious according to the above methods.

Lemma 13. *For* $\tau > 2b+2$, $|\bigcup_{k\neq 0} A^1_{k2}| = O(\epsilon^{\frac{\beta}{2}})$.

Lemma 14. *For* $\tau > 2b$, $|\bigcup_{k\neq 0} A^1_{k1}| = O(\epsilon^{\frac{\beta}{2}})$.

Lemma 15. *For* $\tau > 2b+5$, $|(\cup_{k\neq 0} A^1_{k1}) \bigcup (\cup_k (A^1_{k2} \cup A^1_{k3} \cup A^1_{k4}))| = O(\epsilon^{\frac{\beta}{2}})$.

5.2. The total measure. In order to estimate the total measure of the parameter sets Π_ϵ which is thrown in all the steps, we must estimate the measure in the subsequent steps. The thrown parameter set in $m+1$ step is $\bigcup_{|k|>K_m} R^{m+1}_{kl}(\epsilon_{m+1})$, where $\xi \in \Pi_m$. In fact, we may extend ω_m and Ω_m defined in Π_m to Π. The following ω_m and Ω_m are both defined in Π.

Lemma 16. *For* $\tau > 2b+4$ *and* $K_m \geq \frac{80b}{c_1}$,

$$\left| \bigcup_{|k|>K_m} A^{m+1}_{k4} \right| = \left| \bigcup_{|k|>K_m} \bigcup_{i\neq j} B^{m+1,12}_{k,ij} \right| = O\left(\epsilon^{\frac{\beta}{2}}_{m+1}\right),$$

where c_1 *is a constant which depends on* b *and will be defined in the following.*

Proof. For our convenience, we write ω' and Ω' for ω_{m+1} and Ω_{m+1}. Define $v_1 = (1,0,\ldots,0)^T$ and $v_b = (0,0,\ldots,1)^T$. Define $S = \{(x_1,x_2,\ldots,x_b) \in \mathbb{R}^b : |x_1|+|x_2|+\cdots+|x_b| = 1\}$. Write $A(\xi) = (D^2_{v_1}\omega, D^2_{v_2}\omega, \ldots, D^2_{v_b}\omega)^T$. It is easy to check that $|A(\xi)| = c > 0$, for any $\xi \in \Pi$. For any $(\xi,\upsilon) \in \Pi\times S$, $|A(\xi)\upsilon|_1 \geq c_1 > 0$. Thus for any $(\xi,\upsilon) \in \Pi \times S$, there exists a open neighborhood S_υ of υ in S, such that for some i, $|\langle D^2_{v_i}\omega, \upsilon'\rangle| \geq \frac{c_1}{2b}$, for any $(\xi,\upsilon') \in \Pi\times S_\upsilon$. Since $\{\Pi\times S_\upsilon\}$ covers the compact set $\Pi\times S$, there exist finite covers: $\Pi\times S_1,\ldots,\Pi\times S_{k_0}$ such that $\bigcup_{i=1}^{k_0} \Pi\times S_i \supset \Pi\times S$ and for any $(\xi,\upsilon) \in \Pi\times S_i$,

$$|\langle D^2_{\bar v}\omega, \upsilon\rangle| \geq \frac{c_1}{2b},$$

where $\bar v \in \{v_1, v_2, \ldots, v_b\}$.

Now fix $k \neq 0$ and suppose $\frac{k}{|k|} \in S_i$. Then for any $\xi \in \Pi$,

$$\left|\left\langle D^2_{\bar v}\omega, \frac{k}{|k|}\right\rangle\right| \geq \frac{c_1}{2b} > 0. \tag{5.5}$$

Define $f(\xi) = \langle k, \omega'\rangle + \Omega'_i - \Omega'_j$. Note

$$\begin{aligned} D^2_{\bar v}\frac{f(\xi)}{|k|} &= \left\langle \frac{k}{|k|}, D^2_{\bar v}(\omega)\right\rangle + \frac{D^2_{\bar v}(\Omega_i-\Omega_j)}{|k|} + \frac{D^2_{\bar v}(\Omega'_i-\Omega_i)}{|k|} \\ &\quad + \frac{D^2_{\bar v}(\Omega_j-\Omega'_j)}{|k|} + \left\langle \frac{k}{|k|}, D^2_{\bar v}(\omega'-\omega)\right\rangle. \end{aligned} \tag{5.6}$$

We estimate every term in (5.6). From (4.2) and (4.3), one obtains

$$\left|\left\langle \frac{k}{|k|}, D^2_{\bar v}(\omega'-\omega)\right\rangle\right| \leq |D^2_{\bar v}(\omega'-\omega)| \leq c\epsilon, \tag{5.7}$$

$$\frac{|D^2_{\bar v}(\Omega'_i-\Omega_i)|}{|k|} \leq \frac{c\epsilon}{|k|} \leq \frac{1}{|k|}, \tag{5.8}$$

$$\frac{|D^2_{\bar v}(\Omega_j-\Omega'_j)|}{|k|} \leq \frac{c\epsilon}{|k|} \leq \frac{1}{|k|}. \tag{5.9}$$

Note $\frac{|D^2_{\bar{v}}(\Omega_i-\Omega_j)|}{|k|} \leq \frac{8}{|k|}$ and (5.7), (5.8), (5.9), (5.5), we arrive at $|D_{\bar{v}}\frac{f(\xi)}{|k|}| \geq \frac{c_1}{4b}$ when $|k| \geq \frac{80b}{c_1}$. We will show in what follows that when $\max\{i,j\} \geq c|k|$,

$$\frac{|\langle k,\omega'\rangle + \Omega'_i - \Omega'_j|}{|i-j|+1} \geq 1. \tag{5.10}$$

The proof is similar as before. First,

$$\begin{aligned}
\frac{|\Omega'_i - \Omega'_j|}{|i-j|+1} &\geq \frac{|\Omega_i-\Omega_j|}{2|i-j|} - \frac{|\Omega'_i-\Omega_i|}{2|i-j|} - \frac{|\Omega'_j-\Omega_j|}{2|i-j|} \\
&\geq \frac{c|k|}{2\epsilon^{12}} - M_9 - c_*\epsilon \\
&\geq \frac{c|k|}{4\epsilon^{12}}.
\end{aligned}$$

Moreover,

$$|\langle k,\omega'\rangle| \leq |\langle k,\omega\rangle| + |\langle k,\omega'-\omega\rangle| \leq \frac{c'|k|}{\epsilon^{12}}.$$

Therefore, when c is large enough and $\max\{i,j\} \geq c|k|$, (5.10) holds. So when $K_m \geq \frac{80b}{c_1}$ and $\tau > 2b+4$,

$$\begin{aligned}
\left|\bigcup_{|k|>K_m}\bigcup_{i\neq j} B^{m+1,12}_{k,ij}\right| &= \left|\bigcup_{|k|>K_m}\bigcup_{i,j\leq c|k|} B^{m+1,12}_{k,ij}\right| \\
&\leq \sum_{|k|\geq|K_m|}\sum_{i,j\leq c|k|}\left(\frac{|i-j|}{A_k|k|}\right)^{\frac{1}{2}} O(\epsilon^{\frac{\beta}{2}}_{m+1}), \\
&\leq \sum_{l=1}^{+\infty}\frac{1}{l^{\frac{\tau}{2}-b-1}}O(\epsilon^{\frac{\beta}{2}}_{m+1}) \\
&= O(\epsilon^{\frac{\beta}{2}}_{m+1}). \qquad \square
\end{aligned}$$

Lemma 17. *For $\tau > 2b+4$ and $K_m \geq \frac{80b}{c_1}$,*

$$\left|\bigcup_{|k|>K_m} A^{m+1}_{k3}\right| = \left|\bigcup_{|k|>K_m}\bigcup_{i,j} B^{m+1,11}_{k,ij}\right| = O(\epsilon^{\frac{\beta}{2}}_{m+1}).$$

Lemma 18. *For $\tau > 2b+1$ and $K_m \geq \frac{80b}{c_1}$,*

$$\left|\bigcup_{|k|>K_m} A^{m+1}_{k2}\right| = \left|\bigcup_{|k|>K_m}\bigcup_{i} B^{m+1,1}_{k,i}\right| = O(\epsilon^{\frac{\beta}{2}}_{m+1}).$$

Lemma 19. *For $\tau > 2b-1$,*

$$\left|\bigcup_{k\neq 0} A^{m+1}_{k1}\right| = O(\epsilon^{\frac{\beta}{2}}_{m+1}).$$

Lemma 20. *For $\tau > 2b+4$ and $K_m \geq \frac{80b}{c_1}$,*

$$\left|\bigcup_{|k|>K_m} R_{kl}^{m+1}(\epsilon_{m+1})\right| = O(\epsilon_{m+1}^{\frac{\beta}{2}}).$$

In order to estimate the value for K_1, a series of constants have to be chosen. We know $p=2$, $\beta=\frac{1}{6}$. One fixes $M, \tau > 2b+5$. It is easy to see that one obtains $K_1 \geq \frac{80b}{c_1}$ when γ_0 is small enough. Now we compute the total measure of the parameter sets Π_ϵ which is thrown in all the steps

$$\begin{aligned}|\Pi_\epsilon| &\leq O(\epsilon_1^{\frac{1}{12}}) + O(\epsilon_2^{\frac{1}{12}}) + \cdots \\ &\leq O(\epsilon_1^{\frac{1}{12}}) = O(\epsilon^{\frac{1}{12}}).\end{aligned}$$

6. Appendix. The existence of infinite admissible index sets isn't obvious since the corresponding tangential frequencies have to satisfy infinite many nonresonance conditions. The main idea of the proof is as follows: Suppose that our conclusions hold when $b = d-1$, we prove that there exists at least one n_d in $[x, x+\sqrt{\frac{x}{9n_{d-1}}}]$($x, n_1$ is large enough) such that $n_1, \ldots, n_{d-1}, n_d$ satisfy all the nonresonance assumptions (see section 3). The idea is to estimate the total number of integers n in $[x, x+\sqrt{\frac{x}{9n_{d-1}}}]$ such that $n_1, n_2, \ldots, n_{d-1}, n$ conflicts with one of our nonresonance assumptions. In fact we can prove that the total number is far less than $\sqrt{\frac{x}{9n_{d-1}}}$. This shows the existence of n_d. In case $d=2$, we explicitly construct the admissible index sets. The proof of Proposition 1 requires a couple of lemmas. For our convenience, we introduce the set $K^2 = \{k^2 | k \in Z\}$ and define $L = \sqrt{\frac{n_d}{9n_{d-1}}}$.

Lemma 21. *For any given $n_1, n_2, \ldots, n_{d-1}$ with $n_1 < n_2 < \cdots < n_{d-1}$, $\{n_1, n_2\} \in \mathcal{S}$ and n_d large enough, there exists at most $\frac{L}{8d}$ integers $x_d \in [n_d, n_d+L] \cap Z$ satisfying $5x_d^2 + 2kx_d - 3k^2 \in K^2, k \in \{n_1, n_2, \ldots, n_{d-1}\}$.*

Proof. Note $\sqrt{5} \notin Q$, the conclusion is obvious. □

Similarly, we have the following lemma.

Lemma 22. *For any given $n_1, n_2, \ldots, n_{d-1}$ with $n_1 < n_2 < \cdots < n_{d-1}$, $\{n_1, n_2\} \in \mathcal{S}$ and n_d large enough, there exist at most $\frac{L}{8d}$ integers $x_d \in [n_d, n_d+L] \cap Z$ satisfying $5x_d^2 - 2kx_d - 3k^2 \in K^2, k \in \{n_1, n_2, \ldots, n_{d-1}\}$.*

For the following two lemmas, it is easy to draw the contradictions from the contrary.

Lemma 23. *For any given $n_1, n_2, \ldots, n_d$ where $n_1 < n_2 < \cdots < n_{d-1} < n_d$, $n_d >> n_{d-1}^2$ and $\{n_1, n_2\} \in \mathcal{S}$, there exists at most one $x_{ij} \in [n_d, n_d+L] \cap Z$ satisfying*

$$4x_{ij}(n_j+n_i) + (n_j - 3n_i)(n_j+n_i) \in K^2,$$

where $i, j \in \{1, 2, \ldots, d-1\}$.

Lemma 24. *For any given $n_1, n_2, \ldots, n_d$ with $n_1 < n_2 < \cdots < n_{d-1} < n_d$, $n_d >> n_{d-1}^2$ and $\{n_1, n_2\} \in \mathcal{S}$, there exists at most one $x_{ij} \in [n_d, n_d+L] \cap Z$ satisfying*

$$4x_{ij}(n_j-n_i) + (n_j + 3n_i)(n_j-n_i) \in K^2,$$

where $1 \leq i < j \leq d-1, i, j \in Z$.

LEMMA 25. *For any given* $n_1, n_2, \dots, n_{d-1}, n_d$*, where* $n_1 < n_2 < \cdots < n_{d-1} < n_d$*,* $n_d >> n_{d-1}^3$ *and* $(n_1, n_2) \in \mathcal{S}$*, there exists at most* $\frac{12L}{\sqrt{n_j - n_i}}$ *integers* x *belonging to* $[n_d, n_d + L]$ *so that* $\frac{x^2 - n_i^2}{n_j - n_i} \in Z$*, where* $1 \le i < j \le d-1, i, j \in Z$.

Proof. Rewrite $n_j - n_i = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, where $p_1, \dots, p_s$ are different prime numbers, $k_1, k_2, \dots, k_s \in Z^+$. For $x + n_i, x \in [n_d, n_d + L]$, it is apparent that there exist at most $\frac{L}{p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}} + 2$ integers including $p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$ as factor, where $0 \le l_i \le k_i, i = 1, 2, \dots, s$. For our convenience, we use $A^{l_1 l_2 \cdots l_s}$ representing the event that $x + n_i$ includes $p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$ as factor. Similarly, $B^{l_1 l_2 \cdots l_s}$ represents the event that $x - n_i$ includes $p_1^{k_1 - l_1} p_2^{k_2 - l_2} \cdots p_s^{k_s - l_s}$ as factor. C represents the event $\frac{x^2 - n_i^2}{n_j - n_i} \in Z$. It is apparent that $\cup_{l_1 \cdots l_s} A^{l_1 l_2 \cdots l_s} B^{l_1 l_2 \cdots l_s} = C$. Then the probability of C is

$$\begin{aligned} P(C) &= \sum_{l_1 \cdots l_s} P(A^{l_1 l_2 \cdots l_s} B^{l_1 l_2 \cdots l_s}) \\ &\le \sum_{l_1 \cdots l_s} P(A^{l_1 l_2 \cdots l_s}) P(B^{l_1 l_2 \cdots l_s}) \\ &\le \sum_{l_1 \cdots l_s} \left(\frac{\frac{L}{p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}} + 2}{L} \right) \left(\frac{\frac{L}{p_1^{k_1 - l_1} p_2^{k_2 - l_2} \cdots p_s^{k_s - l_s}} + 2}{L} \right) \\ &\le 2 \sum_{l_1 \cdots l_s} \frac{1}{p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}} \\ &\le \frac{2(k_1 + 1) \cdots (k_s + 1)}{p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}}. \end{aligned}$$

We know that $l + 1 \le p^{\frac{l}{2}} (p \ge 4)$, $l + 1 \le 3^{\frac{l}{2} + 1}$, and $l + 1 \le 2^{\frac{l}{2} + 1}$, for any $l \ge 1$. Then $P(C) \le \frac{2(k_1 + 1) \cdots (k_s + 1)}{n_j - n_i} \le \frac{12}{\sqrt{n_j - n_i}}$. Now it is easy to see that our conclusion holds. ☐

Similarly, we have the following lemma.

LEMMA 26. *For any given* $n_1, n_2, \dots, n_d$ *where* $n_1 < n_2 < \cdots < n_{d-1} < n_d$*,* $n_d >> n_{d-1}^3$ *and* $\{n_1, n_2\} \in \mathcal{S}$*, there exists at most* $\frac{12L}{\sqrt{n_j + n_i}}$ *integers* x *belonging to* $[n_d, n_d + L]$ *so that* $\frac{x^2 - n_i^2}{n_j + n_i} \in Z$*, where* $L = \sqrt{\frac{n_d}{9 n_{d-1}}}, 1 \le i \le j \le d-1, i, j \in Z$.

The proof of Proposition 1.

We first admit that Proposition 2 holds (the proof will be delayed to the end). This means that Proposition 1 holds for $b = 2$. Suppose that Proposition 1 holds for $b = d - 1 \ge 2$, we will show that it also holds for d. When $b = d - 1$, one can choose one admissible set made of $n_1, n_2, \dots, n_{d-1}$. Our aim is to construct n_d so that $\{n_1 < n_2 < \cdots < n_d\}$ is an admissible set for $b = d$. We first construct n_d to satisfy Assumption A. In fact, it is enough when $n_d >> n_i, i \le d - 1$. Otherwise one gets

$$\begin{cases} i^2 + j^2 + k^2 = l^2 + m^2 + n^2 \\ i \pm j \pm k \pm l \pm m \pm n = 0, \end{cases}$$

where $i, j, k, l, m, n \in \{n_1, n_2, \dots, n_d\}$. One can induce the contradictions from different cases. We only prove the case in which there exist two $n_d's$ in $\{i, j, k, l, m, n\}$. For any more or less n_d(at least one n_d), the proof is similar. Note $n_d >> n_i (i \le d - 1)$, one gets $\{i,j,k\}\cap\{l,m,n\}\supset n_d$. Hence, one obtains

$$\begin{cases} j^2+k^2=l^2+m^2 \\ n_d\pm n_d\pm j\pm k\pm l\pm m=0. \end{cases} \tag{6.1}$$

We know that $\pm n_d\pm n_d=0,\pm 2n_d$. For the preceding, from Lemma 5 in [10], it contradicts with our choice of $\{i,j,k,l,m,n\}$. For the last, it is apparent that $|j\pm k\pm l\pm m|<2n_d$. This leads a contradiction to (6.1). If none of $n_d's$ is in $\{i,j,k,l,m,n\}$, this contradicts with the choice of $n_1,n_2,\dots,n_{d-1}$.

In fact, n_d also satisfies Assumption B under the same condition. If this is not true, then

$$x^2+j^2+k^2-l^2-m^2-n^2=0.$$

The unique index which is different with $n_1,n_2,\dots,n_d$ is denoted by x. We only prove the case in which there exist three $n_d's$ in $\{j,k,l,m,n\}$. For the other cases (at least one n_d), the method is similar. One can induce the contradictions from the following three cases.

Case 1.

$$\begin{cases} x^2+j^2+k^2=3n_d^2 \\ x\pm j\pm k\pm n_d\pm n_d\pm n_d=0. \end{cases}$$

From $x^2+j^2+k^2=3n_d^2$, we conclude that $x\approx\sqrt{3}n_d$. But from $x\pm j\pm k\pm n_d\pm n_d\pm n_d=0$, we know that $|x|\approx 3n_d$ or n_d. It is impossible.

Case 2.

$$\begin{cases} x^2+n_d^2=m^2+n^2 \\ x\pm n_d\pm n_d\pm n_d\pm m\pm n=0. \end{cases}$$

From $n_d^2>>m^2+n^2$, we know it can't happen.

Case 3.

$$\begin{cases} x^2+j^2=n_d^2+m^2 \\ x\pm j\pm m\pm n_d\pm n_d\pm n_d=0. \end{cases} \tag{6.2}$$

From $x^2+j^2=n_d^2+m^2$, we can get $x\approx n_d$. Hence, (6.2) holds only when $\pm n_d\pm n_d\pm n_d=\pm n_d$. But at this case, from Lemma 5 of [10], one can get $\{x,j\}=\{n_d,m\}$. It can't happen.

If none n_d in $\{j,k,l,m,n\}$, this contradicts with the choice of $n_1,n_2,\dots,n_{d-1}$.

For Assumption C, one must place much heavier restrictions on n_d. From Lemmas 21–26, we will prove that there exist many integer points x belonging to $[n_d,n_d+\sqrt{\frac{n_d}{9n_{d-1}}}]$ so that $n_1,\dots,x$ fulfill our Assumption C when n_d and n_1 is large enough. If it isn't true, then

$$i^2+j^2+k^2=l^2+m^2+n^2.$$

The other two indexes different from $n_1,n_2,\dots,n_d$ are denoted by x,y,

Case 1.

$$\{x,y\}\subset\{i,j,k\} \text{ or } \{x,y\}\subset\{l,m,n\}.$$

Without losing generality, one gets

$$\begin{cases} x^2+y^2=l^2+m^2+n^2-k^2 \\ x\pm y=\pm k\pm l\pm m\pm n. \end{cases}$$

We have to consider several different subcases. For our convenience, we introduce the notation "$|\cdot|$." The equality $|\{k,l,m,n\}|=t, t=1,2,3,4$, means there exist exactly t $n_d's$ in $\{k,l,m,n\}$.

Subcase a.

$$|\{k,l,m,n\}|=1.$$

It is easy to see that the case $x^2+y^2+n_d^2=l^2+m^2+n^2$ can't happen. So only the following case need be considered:

$$\begin{cases} x^2+y^2+k^2=l^2+m^2+n_d^2 \\ x\pm y\pm k\pm l\pm m\pm n_d=0. \end{cases}$$

We only consider the case when

$$x=y\pm k\pm l\pm m\pm n_d. \tag{6.3}$$

For $x=-y\pm k\pm l\pm m\pm n_d$, it is similar. From (6.3), one obtains

$$2y^2+2(\pm k\pm l\pm m\pm n_d)y+(\pm k\pm l\pm m\pm n_d)^2+k^2-l^2-m^2-n_d^2=0.$$

Write $a=\pm k\pm l\pm m$. Note

$$\triangle=4(n_d-a)^2+8(l^2+m^2-a^2-k^2),$$

and $y\in Z$, one gets

$$l^2+m^2=a^2+k^2. \tag{6.4}$$

At the same time, one obtains

$$y=\frac{-(a\pm n_d)\pm(n_d-a)}{2}.$$

By further computations, one knows $|x|=n_d$ or $|y|=n_d$. It is impossible.

Subcase b.

$$|\{k,l,m,n\}|=2.$$

We must consider different cases.

Case I.

$$\begin{cases} x^2+y^2=m^2+n^2 \\ x\pm y\pm m\pm n\pm n_d\pm n_d=0. \end{cases} \tag{6.5}$$

If $\pm n_d\pm n_d=0$, we arrive at $\{x,y\}=\{m,n\}$ from Lemma 5 of [10]. It is impossible. When $\pm n_d\pm n_d=\pm 2n_d$, we get $|x\pm y|<<n_d$ from (6.5). Hence the equality $x\pm y\pm m\pm n\pm 2n_d=0$ can't hold.

Case II.

$$\begin{cases} x^2+y^2+k^2=2n_d^2+n^2 \\ x\pm y\pm k\pm n\pm n_d\pm n_d=0. \end{cases} \tag{6.6}$$

Write $\pm k\pm n=a$. If $\pm n_d\pm n_d=0$, one gets

$$2y^2+2ay+(a^2+k^2-2n_d^2-n^2)=0$$

and

$$\triangle=16n_d^2+8n^2-8k^2-4a^2.$$

From $y\in Z$, one obtains $8n^2-8k^2-4a^2=0$. Then we have $3k^2\pm 2kn-n^2=0$. Hence, $3k=n$ or $k=n$. Only the last case need be considered. But at this case, we get $|y|=n_d$. It is impossible.

If $\pm n_d\pm n_d=2n_d$, from (6.6), one gets $|x|<<\sqrt{3}n_d$. Then the equality $x\pm y\pm k\pm n+2n_d=0$ can't hold. Similarly, the equality $x-y\pm k\pm n-2n_d=0$ can't be true. So the only case need be considered is

$$x+y\pm k\pm n-2n_d=0. \tag{6.7}$$

Denote $\pm k\pm n=a$. From (6.6) and (6.7), one gets

$$2y^2-2y(a+2n_d)+2n_d^2+4an_d+a^2+k^2-n^2=0.$$

If $a=-k-n$, we obtain $\triangle=4(n_i+n_j)(4n_d+n_j-3n_i)=4\triangle_1, i,j\in\{1,2,\dots,d-1\}$. If $a=-k+n, k\neq n$, we obtain $\triangle=4(n_j-n_i)(4n_d+3n_i+n_j)=4\triangle_2$. Other cases can't happen. In order to draw the contradictions, one removes all the integers belonging to $[n_d,n_d+L]$ which satisfy $\triangle_1\in K^2,\triangle_2\in K^2$. Thanks to Lemmas 23 and 24, we throw at most $2(d-1)^2$ integer points. Then $y\notin Z$.

Subcase c.

$$|\{k,l,m,n\}|=3.$$

Case I.

$$\begin{cases} x^2+y^2=n_d^2+n^2 \\ x\pm y\pm n\pm n_d\pm n_d\pm n_d=0. \end{cases} \tag{6.8}$$

If $\pm n_d\pm n_d\pm n_d=\pm n_d$, from (6.8) and Lemma 5 of [10], one gets $\{x,y\}=\{n_d,n\}$. It is impossible. If $\pm n_d\pm n_d\pm n_d=\pm 3n_d$, from $x^2+y^2=n_d^2+n^2$ one obtains $|\pm x\pm y\pm n|<<\frac{5}{2}n_d$. Hence the equality $x\pm y\pm n\pm 3n_d=0$ can't hold.

Case II.

$$\begin{cases} x^2+y^2+k^2=3n_d^2 \\ x\pm y\pm k\pm n_d\pm n_d\pm n_d=0. \end{cases} \tag{6.9}$$

If $\pm n_d\pm n_d\pm n_d=\pm 3n_d$, from $x^2+y^2+k^2=3n_d^2$, one gets $|x\pm y|^2\le 2(x^2+y^2)\le 6n_d^2$. Hence, one knows $|x\pm y\pm k|<<\sqrt{7}n_d$. The inequality $x\pm y\pm k\pm 3n_d=0$ can't hold. For the case when $\pm n_d\pm n_d\pm n_d=\pm n_d$, we throw all the integers belonging to $[n_d,n_d+L]$ which satisfy $5n_d^2\pm 2kn_d-3k^2\in K^2, k=n_1,\dots,n_{d-1}$. From Lemmas 21 and 22, the thrown integers are at most $\frac{L}{4}$. Then $y\notin Z$.

Subcase d.

$$|\{k,l,m,n\}| = 4.$$

$$\begin{cases} x^2+y^2 = 2n_d^2 \\ x \pm y \pm n_d \pm n_d \pm n_d \pm n_d = 0. \end{cases}$$

The discussion is trivial. We omit it.

Subcase e. If which is no n_d in $\{k,l,m,n\}$, this contradicts with the choice of $n_1, n_2, \ldots, n_{d-1}$.

Case 2.

$$\{x,y\} \cap \{i,j,k\} \neq \{x,y\} \text{ and } \{x,y\} \cap \{l,m,n\} \neq \{x,y\}.$$

In this case, one obtains

$$\begin{cases} x^2-y^2 = m^2+n^2-i^2-j^2 \\ x \pm y \pm i \pm j \pm m \pm n = 0. \end{cases} \tag{6.10}$$

We have to discuss it in several subcases.

Subcase a′.

$$|\{i,j,m,n\}| = 1.$$

In this case, (6.10) is

$$\begin{cases} x^2-y^2 = m^2+n_d^2-i^2-j^2 \\ x \pm y \pm i \pm j \pm m \pm n_d = 0. \end{cases} \tag{6.11}$$

Without losing generality, we suppose that $x = y \pm i \pm j \pm m \pm n_d$. From (6.11), one gets

$$2y(\pm i \pm j \pm m \pm n_d) + (\pm i \pm j \pm m \pm n_d)^2 + i^2 + j^2 - m^2 - n_d^2 = 0.$$

Write $a = \pm i \pm j \pm m$. If $x = y + a + n_d$, one has

$$y = -a + \frac{a^2+m^2-i^2-j^2}{2(a+n_d)}.$$

If $n_d >> n_{d-1}^2$ and $y \in Z$, one obtains $a^2+m^2-i^2-j^2 = 0$ and $y = -a$. Hence $|x| = n_d$. It is impossible. If $x = y + a - n_d$, the proof is similar.

Subcase b′. $|\{i,j,m,n\}| = 2$.

If $\{x,i,j\} \cap \{y,m,n\} = \{n_d\}$, then

$$\begin{cases} x^2+i^2 = y^2+m^2 \\ x \pm y \pm i \pm m \pm n_d \pm n_d = 0. \end{cases} \tag{6.12}$$

When $\pm n_d \pm n_d = 0$, from Lemma 5 of [10], one gets $\{x,i,n_d\} = \{y,m,n_d\}$. It contradicts with our assumptions. When $\pm n_d \pm n_d = \pm 2n_d$, write $a = \pm i \pm m$. Without losing generality, we suppose that

$$x = y - a \pm 2n_d. \tag{6.13}$$

From (6.13) and (6.12), one gets

$$y = \frac{m^2 - i^2 - (-a \pm 2n_d)^2}{2(-a \pm 2n_d)}.$$

Note that $y > 0$, we have

$$y = \frac{1}{2}(a + 2n_d) + \frac{m^2 - i^2}{-2(a + 2n_d)}.$$

If $\{\frac{1}{2}(a + 2n_d)\} = \frac{1}{2}$, one gets $y \notin Z$. It is impossible. If $\{\frac{1}{2}(a + 2n_d)\} = 0$ and $m^2 - i^2 \neq 0$, we know $y \notin Z$. It is also impossible. If $\{\frac{1}{2}(a + 2n_d)\} = 0$ and $m^2 = i^2$, one gets $x = y$. It can't happen.

If $\{x, i, j\} \cap \{y, m, n\} = \emptyset$, we have

$$\begin{cases} x^2 + 2n_d^2 = y^2 + m^2 + n^2 \\ x \pm y \pm m \pm n \pm n_d \pm n_d = 0. \end{cases} \tag{6.14}$$

When $\pm n_d \pm n_d = \pm 2n_d$, write $\pm m \pm n = a$. If $x = y + a \pm 2n_d$, from (6.14) and $y > 0$, we get

$$y = \frac{6n_d - a}{4} + \frac{2m^2 + 2n^2 - a^2}{4(a - 2n_d)}.$$

If $\{\frac{6n_d - a}{4}\} \neq 0$, one gets $y \notin Z$. It is impossible. Only when $\{\frac{6n_d - a}{4}\} = 0$ and $a = 2m$ or $a = -2m$, we gets $y \in Z$. But by further computation, one gets $x < 0$. It is impossible. If $x = -y + a \pm 2n_d$, one get $x < 0$ by similar method. When $\pm n_d \pm n_d = 0$, we throw all the integers x belonging to $[n_d, n_d + L]$ which satisfy $\frac{x^2 - m^2}{n - m} \in Z (1 \leq m < n \leq d - 1)$ or $\frac{x^2 - m^2}{n + m} \in Z (1 \leq m \leq n \leq d - 1)$. From Lemmas 25 and 26, the thrown integers are at most $\frac{24L(d-1)^2}{\sqrt{2n_1}}$. Then $y \notin Z$.

Subcase c'. $|\{i, j, m, n\}| = 3$.

In this case, we get

$$\begin{cases} x^2 + i^2 = y^2 + n_d^2 \\ x \pm y \pm i \pm n_d \pm n_d \pm n_d = 0. \end{cases} \tag{6.15}$$

When $\pm n_d \pm n_d \pm n_d = \pm n_d$, from Lemma 5 of [10], one obtains $\{x, i\} = \{y, n_d\}$. It is impossible. When $\pm n_d \pm n_d \pm n_d = \pm 3n_d$, we suppose $x = y \pm i \pm 3n_d$. For $x = -y \pm i \pm 3n_d$, the method is similar. From (6.15) and $y > 0$, we get

$$y = \frac{4}{9}(3n_d + i) + \frac{4i^2}{9(3n_d - i)} - i,$$

or

$$y = \frac{4}{9}(3n_d - i) + \frac{4i^2}{9(3n_d + i)} + i.$$

Both can't be integers. It is impossible.

Subcase d'. $|\{i, j, m, n\}| = 4$.

We easily get $x = y$. It means $\{x, n_d, n_d\} = \{y, n_d, n_d\}$. It contradicts with our assumptions.

Subcase e′. If there is no n_d in $\{i, j, m, n\}$, this contradicts with the choice of $n_1, n_2, \ldots, n_{d-1}$.

Now we declare that there exist many integers x belonging to $[n_d, n_d + L]$ so that $n_1, \ldots, x$ fulfill our Assumptions A, B, C and (2.6) when n_d is large enough and $n_1 \geq 18432d^2$. In fact the thrown integers are at most

$$\frac{24L(d-1)^2}{\sqrt{2n_1}} + \frac{L}{4} + 2(d-1)^2. \tag{6.16}$$

If $n_1 \geq 18432d^2$, one can get (6.16)$\leq \frac{L}{2}$. Then there exist many x satisfying Assumptions A, B, C and (2.6). From Proposition 2, Proposition 1 is complete. □

The proof of Proposition 2 also requires a couple of lemmas. Since the proof is elementary, we give them without proof as follows.

LEMMA 27. *If* $n_1, n_2 \in \mathbb{N}, n_1 < n_2$, *then* $-7n_2^2 + n_1^2 \pm 6n_1n_2 \notin K^2$.

LEMMA 28. *If* $n_1 \equiv 2$ or $5(mod 7), n_2 \equiv 1(mod 7)$, *then* $-7n_1^2 + n_2^2 \pm 6n_1n_2 \notin K^2$.

LEMMA 29. *If* $n_1, n_2 \in \mathbb{N}, n_2 > 11n_1^2$, *then* $-3n_1 \pm n_2 \dagger n_2^2 - n_1^2, 3n_2 \pm n_1 \dagger n_2^2 - n_1^2$, *where the notation* $a \dagger b$ *means that* a *is not a factor of* b.

LEMMA 30. *If* $n_1 \in 2\mathbb{N} - 1, n_2 \in 2\mathbb{N},$, *then* $5n_1^2 - 3n_2^2 \pm 2n_1n_2 \notin K^2$, $5n_2^2 - 3n_1^2 \pm 2n_1n_2 \notin K^2$, $n_1(2n_2 - n_1) \notin K^2$, $n_2^2 - n_1^2 \notin K^2$.

Since the proof of Proposition 2 is similar with Proposition 1, we omit it. □

Acknowledgments. The authors thank the anonymous referees for several kind suggestions. It is also a pleasure to thank Dr. Jiansheng Geng for many fruitful discussions on this subject.

REFERENCES

[1] D. BAMBUSI, *Birkhoff normal form for some nonlinear PDEs*, Comm. Math. Phys., 234 (2003), pp. 253–283.

[2] J. BOURGAIN, *Nonlinear Schrödinger equations (Park City Lectures)*, AMS, Providence, RI, 1999.

[3] J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO, *Global well-posedness for Schrödinger equations with derivative*, SIAM J. Math. Anal., 33 (2001), pp. 649–669.

[4] J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO, *A refined global well-posedness result for Schrödinger equations with derivative*, SIAM J. Math. Anal., 34 (2002), pp. 64–86.

[5] L. CHIERCHIA AND J. YOU, *KAM tori for* 1*D nonlinear wave equations with periodic boundary conditions*, Comm. Math. Phys., 211 (2000), pp. 497–525.

[6] J. GENG AND J. YOU, *KAM tori of Hamiltonian perturbations of* 1*D linear beam equations*, J. Math. Anal. Appl., 277 (2003), pp. 104–121.

[7] S. B. KUKSIN, *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes in Mathematics 1556, Springer-Verlag, Berlin, 1993.

[8] S. B. KUKSIN, *A KAM-theorem for equations of the Korteweg-de Vries type*, Rev. Math. Math. Phys., 10 (1998), pp. 1–64.

[9] S. B. KUKSIN, *Analysis of Hamiltonian PDEs*, Oxford University Press, Oxford, 2000.

[10] S. B. KUKSIN AND J. PÖSCHEL, *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation*, Ann. of Math., 143 (1996), pp. 149–179.

[11] F. MERLE, *On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödiger equation with critical exponent and critical mass*, Comm. Pure Appl. Math., 45 (1992), pp. 203–254.

[12] J. PÖSCHEL, *Quasi-periodic solutions for a nonlinear wave equation*, Comment. Math. Helv., 71 (1996), pp. 269–296.

[13] J. PÖSCHEL, *A KAM-theorem for some nonlinear partial differential equations*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., 23 (1996), pp. 119–148.

[14] J. Xu , J. You, and Q. Qiu, *A KAM theorem of degenerate infinite-dimensional Hamiltonian systems*. I. II. Sci. China Ser. A, 39 (1996), pp. 372–394.
[15] J. Xu, J. You, Q. Qiu, *Invariant tori for nearly integrable Hamiltonian systems with degeneracy*, Math. Z., 226 (1997), pp. 375–387.
[16] C. E. Wayne, *Periodic and quasi-periodic solutions for nonlinear wave equations via KAM theory*, Commun. Math. Phys., 127 (1990), pp. 479–528.