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Quasiperiodic Solutions for Nonlinear Differential Equations of Second Order with Symmetry

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Abstract. In this paper, we study the existence of quasi-periodic solutions and the boundedness of solutions for a wide class nonlinear differential equations of second order. Using the KAM theorem of reversible systems and the theory of transformations, we obtain the existence of quasi-periodic solutions and the boundedness of solutions under some reasonable conditions.

§1. Introduction

The boundedness problem of solutions for the following scalar nonlinear differential equation of second order

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t), \tag{1.1}$$

has been widely investigated in the literature since 1940's, where $p(t)$ is continuous and periodic in $t \in R^1$.

Generally, in accordance with its physical meanings, the equation (1.1) can be divided into three aspects.

a) Dissipative Systems

In this case, when the time variable t increases, the total energy of system (1.1) declines because of the presence of the positive damping factor $f(x)$ here. Hence, the uniformly ultimate boundedness of the solutions for this system is obvious in the sense of classical mechanics and physics. Generally, one can construct an absorbing compact domain in the phase space such that all solutions of Equation (1.1) always enter this domain whenever $t \geq t_0$. For example, N. Levinson[6], G.E.H.Reuter[16] and J.Greaf[3] obtained the boundedness of solutions for Equation (1.1) under some assumptions on $f(x)$ and $g(x)$. Moreover, in [3], J.Greaf gave a necessary and sufficient condition for Equation (1.1) in this case.

b) Conservative Systems

When $f(x) \equiv 0$, the Equation (1.1) is a conservative system and takes the following form

$$\ddot{x} + g(x) = p(t), \tag{1.2}$$

The total energy of this system is a constant for the time t, and the solutions of this system cannot be uniformly ultimately bounded. Hence, the method motioned above for the dissipative system cannot be used in this case. There was not a good method for

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proving the boundedness of the solutions until the KAM theorem (Kolmogorov[4]-Arnold[1]-Moser[13]) was set up. The idea for proving the boundedness for all solutions of Equation (1.2) via the KAM theorem is as follows. By means of the transformation theory, system (1.2) is, outside a large disc $D = \{(x, \dot{x}) \in R^2 | x^2 + \dot{x}^2 \leq r^2\}$ in the $(x, \dot{x})$-plane, transformed into a Hamiltonian equation with the following property. From the Liouville's theorem, it follows that the Poincare mapping of this equation is area-preserving and is closed to a so-called twist mapping in R^2/D. Then, using the KAM theorem, one can find large invariant curves diffeomorphic to circles and surrounding the origin in the $(x, \dot{x})$-plane. Every such curve is the base of a time-periodic and under the flow invariant cylinder in the phase space $(x, \dot{x}, t) \in R^2 \times R^1$, which confines the solutions in its interior and which therefore leads to a bound of these solutions. But the conditions which are required in the KAM theorem are very severe, it is difficult to apply the KAM theorem to Equation (1.2). The first contribution for the boundedness for the solutions of (1.2) is due to G.Morris[12], who proved that all solutions of Equation (1.2) are bounded for a special case: $g(x) = 2x^3$. In 1987, R.Dieckerhoff and E.Zehnder[2] studied the following differential equation

$$\ddot{x} + x^{2n+1} + \sum_{j=0}^{2n} p_j(t) x^j = 0, \quad n \geq 1, \tag{1.3}$$

where $p_j(t+1) = p_j(t), j = 1, 2, \cdots, 2n$. They proved that each solution of (1.3) is bounded if $p_j(t) (j = 1, 2, \cdots, 2n)$ are smooth functions.

Using the method developed in [2] with minor modifications, B.Liu has proved in [8] that all solutions are bounded for the following Duffing's equation

$$\ddot{x} + \beta x^{2n+1} + (a_1 + \varepsilon a(t)) x = p(t), \quad n \geq 1, \tag{1.4}$$

where $a(t)$ and $p(t)$ only have to be continuous and periodic with the same period in $t \in R^1$, and ε is a small parameter.

In 1990, J.You[17] proved that each solution is bounded for the pendulum-type equation

$$\ddot{x} + G'_x(t, x) = p(t), \tag{1.5}$$

where $G(t+1, x) = G(t, x+1) = G(t, x)$ and $p(t+1) = p(t)$.

The results in [8] and [17] give a positive answer to J.Moser's conjectures for the boundedness of the Duffing's equations and the pendulum-type equations in [14].

Recently, M.Levi[5] and J.You[18] obtained the boundedness of all solutions for Equation (1.2) for a more general class of $g(x)$. Dropping the restriction on ε in Equation (1.4), B.Liu[9] also proves that each solution of (1.4) is bounded.

On the other hand, on the basis of J. Littlewood's work [7], Y.Long in [11] constructed a counter-example to show that the boundedness of all solutions may fail in the general sublinear Duffing's Equation (1.2) without more restrictions on $g(x)$. It seems that the boundedness of all solutions for Equation (1.2) depends on the behavior of the period function of the corresponding autonomous equation: $\ddot{x} + g(x) = 0$.

c) Reversible Systems

When $f(x), g(x)$ and $p(t)$ in Equation (1.1) have certain parities, Equation (1.1) becomes a reversible system which will be defined in the next section in this paper. In this case, the above-mentioned methods in a) and b) cannot be used in this system for proving the boundedness of solutions.

Recently, J. You[19] obtained the boundedness of all solutions when $f(x)$ and $g(x)$ are odd and periodic with the same period in x, and $p(t)$ is odd and periodic in $t \in R^1$. B. Liu[10] proves that all solutions of Equation (1.1) are bounded when $f(x) = \sum_{i=0}^{m} a_i x^{2i+1}, g(x) = x^{2n+1} + \sum_{i=0}^{n-1} b_i x^{2i+1}$ with $n \geq 2(m+1)$, and $p(t)$ is odd and periodic in $t \in R^1$. The KAM theorem of reversible systems developed by J.Moser[14] and B.Sevryuk[16] and others was used in the proofs in the above indicated papers.

The purpose of this paper is to prove the boundedness of solutions for a wide class of equations

$$\ddot{x} + g(x) = p(t, x, \dot{x}). \tag{1.6}$$

The proof is based on the KAM theorem of reversible systems. In order to apply the KAM theorem of reversible systems, we will introduce the action-angle variables (λ, θ) in the phase space, and transform this system, for a larger λ, into a simpler system for which the θ-depending terms are very small.

This paper contains four sections. In Section 2, we give some basic concepts of reversible systems and mappings, the KAM theorem of reversible systems and mappings, and some spaces of smooth functions which are similar to that in [2]. In Section 3, the main proposition in this paper will be proved. This proposition is similar to the propositions in [2] and some results in [2] are used in our proof. We will prove that all solutions of Equation (1.6) are bounded under some reasonable assumptions on $g(x)$ and $p(t, x, \dot{x})$ in the last section.

§2. Some Notations and Lemmas

In this section, we will give some definitions and lemmas which will be used in Sections 3 and 4.

Consider a first order system in R^n,

$$\dot{x} = f(x, t), \tag{2.1}$$

where x is an n-dimensional vector, $f : R^n \times R^1 \to R^n$ is continuous in (x, t) and periodic in $t \in R^1$.

Definition 2.1. *The system* (2.1) *is called a reversible system if there is an involution* $G : R^n \to R^n$ *such that*

$$DG \circ f(Gx, -t) = -f(x, t).$$

Such a system is also called a reversible system with respect to G.

Definition 2.2. *Given two diffeomorphisms A and G in R^n with $G^2 = id$. The diffeomorphism A is called a reversible diffeomorphism with respect to G if the following equality:*

$$A^{-1} = GAG$$

holds.

From the above definitions, we have the following

Lemma 2.1. *Suppose that A is the Poincare mapping of the reversible* (2.1) *with respect to G. Then A is a reversible diffeomorphism with respect to G.*

The proof of this lemma is very easy and we omit it here.

For the convenience of the statements in the next two sections, we introduce the concept of G-invariant for the transformations in R^n .

Definition 2.3. *Assume that T is an invertiable transformation in R^n, and G is an involution in R^n. The transformation T is called G-invariant if the following equality:*

$$T = G^{-1}TG$$

holds.

If T is G-invariant, then so is T^{-1}.

From the above definitions, we have

Lemma 2.2. (i) *If system* (2.1) *is a reversible system with respect to G, T is a G-invariant invertiable transformation in R^n, then the transformed system of* (2.1) *under the transformation T is also a reversible system with respect to G.*

(ii) *Let A be a reversible diffeomorphism in R^n with respect to G, T be a G-invariant invertiable transformation in R^n. Then $T^{-1}AT$ is also a reversible diffeomorphism with respect to G.*

Proof. The proof is easy and we omit it here.

Now, we give the KAM theorem of reversible diffeomorphisms as a lemma.

Lemma 2.3(KAM Theorem of Reversible Diffeomorphisms). *Let $\Omega \subset R^m$ be a closed ball of radius* 1 *and $D_* \subset C^m$ be a complex neighborhood of Ω. Let $r_0, \widetilde{r}_0 \in (0,1]$. Denote by D the following domain in C^m:*

$$D = \{x = (x_1, \cdots, x_m) \in C^m | |\mathrm{Im} x_j| < r_0, 1 \le j \le m\} \times \{y \in C^m | y \in D_*\}.$$

Suppose that $\gamma, c \in (0,1]$ are fixed and the following mappings are given on D:

$$A : (x, y) \mapsto (x + \gamma y + f^1(x, y), y + f^2(x, y))$$

and

$$G : (x, y) \mapsto (-x, y),$$

where f^1 and f^2 are normal in D, that is, f^1 and f^2 are holomorphic, 2π-periodic in x and real-valued on $D \cap R^{2m}$.

Assume that $A^{-1} = GAG$ on D.

Introduce the notation

$$\Omega_{\gamma,c} = \left\{\omega \in \gamma\Omega \left| \left|\frac{(q,\omega)}{2\pi} - p\right| \ge \frac{\gamma c}{|q|^{m+1}} \right.\right\},$$

for all $q \in Z^m/\{0\}$ and $p \in Z$.

Then for each $\varepsilon > 0$, there is a $\delta > 0$, depending only on ε, D and c but not on γ, such that if $|f^1|, |f^2| < \gamma\delta$ on D, then for each $\omega \in \Omega_{\gamma,c}$, the mappings A and G have a common invariant m-dimensional manifold

$$x = \phi + \Phi^1_\omega(\phi), \quad y = \gamma^{-1}\omega + \Phi^2_\omega(\phi), \tag{$*$}$$

where Φ^1 and Φ^2 are normal in the domain

$$\left\{\phi = (\phi_1, \cdots, \phi_m) \in C^m | |\mathrm{Im}\ \phi_j| < \frac{r_0}{2}, 1 \le j \le m\right\},$$

such that the diffeomorphisms of the manifold $(*)$ *induced by the mappings* A *and* G *are* $\phi \mapsto \phi + \omega$ *and* $\phi \mapsto -\phi$*, respectively. Moreover, the following inequality holds:*

$$|\Phi_\omega^1|, |\Phi_\omega^2| < \varepsilon.$$

Proof. See [16, Ch.1].

Remark 1. In the case of C^∞-perturbations or a finite smooth perturbations, the statement of the above lemma is still true.

Remark 2. When $m = 1$, the common invariant 1-dimensional manifold $(*)$ is a closed curve in R^2.

Lastly, we introduce two spaces of functions and give some properties of them.

Definition 2.4. *Given* $r, d \in R^1$ *with* $0 < d < 1$*, we denote by* $P_{1d}(r)$ *the space of* C^∞*-functions in* (λ, θ, t)*-space* $R^+ \times T^2$ *such that if* $g(\lambda, \theta, t) \in P_1(r)$*, then*

$$\lim_{\lambda \to \infty} (\lambda^{j-(j+k)d-r-\varepsilon} |(D_\lambda^j)(D_\theta^k) g(\lambda, \theta, t)|) = 0$$

holds for $j, k \in Z^+ \cup \{0\}$ *and any constant* $\varepsilon > 0$.

Definition 2.5. *Given* $r, d \in R^1$ *with* $0 < d < 1$*, we denote by* $P_{2d}(r)$ *the set of* C^∞*-functions in* R^1 *such that if* $f(x) \in P_{2d}(r)$*, then we have*

$$\lim_{|x| \to \infty} |x|^{j-jd-r-\varepsilon} f^{(j)}(x) = 0$$

for any constant $\varepsilon > 0$*, and* $j \in Z^+ \cup \{0\}$.

We denote by $P_1(r)$ and $P_2(r)$ respectively the sets $P_{10}(r)$ and $P_{20}(r)$.

Examples.

1. $g(x) = x^{2n+1} \log(1 + x^2) \in P_2(2n + 1)$;

2. $g(x) = \dfrac{x \log(1 + x^2)}{1 + x^2} \in P_2(-1)$;

3. $f(\lambda, \theta, t) = \lambda^r \sin\theta \sin t \in P_1(r)$.

We collect some properties of $P_1(r)$ and $P_2(r)$ in the following lemmas.

Lemma 2.4. *The following conclusions hold:*

(i) *if* $r_1 < r_2$*, then* $P_{1d}(r_1) \subset P_{1d}(r_2)$;

(ii) *if* $f \in P_{1d}(r)$*, then* $(D_\lambda^j)(D_\theta^k) f \in P_{1d}(r + id + jd - j)$;

(iii) *if* $f_1 \in P_{1d}(r_1)$ *and* $f_2 \in P_{1d}(r_2)$*, then* $f_1 f_2 \in P_{1d}(r_1 + r_2)$;

(iv) *if* $f \in P_{1d}(r)$ *satisfies* $|f(\lambda, \theta, t)| \geq c\lambda^r$ *for* $\lambda \geq \lambda_0$*, then* $\frac{1}{f} \in P_{1d}(-r)$.

Proof. Similar to that in [2].

Lemma 2.5. *The following statements are valid:*

(1) *if* $r_1 < r_2$*, then* $P_2(r_1) \subset P_{2d}(r_2)$;

(2) *if* $f \in P_{2d}(r)$*, then* $f^{(j)}(x) \in P_{2d}(r + jd - j)$;

(3) *if* $f \in P_{2d}(r)$*, then* $\int_0^x f(s)ds \in P_{2d}(r + 1)$;

(4) *if* $f_1 \in P_{2d}(r)$ *and* $f_2 \in P_{2d}(s)$*, then* $f_1 f_2 \in P_{2d}(r + s)$;

(5) *if* $f(x) \in P_{2d}(r)$ *satisfies* $\lim_{|x| \to \infty} |x|^{-r+\varepsilon} |f(x)| = +\infty$ *for any constant* $\varepsilon > 0$ *, then* $\frac{1}{f} \in P_{2d}(-r)$;

(6) *if* $f \in P_{2d}(r)$ *with some* $r > 0$ *satisfies* $\lim_{|x|\to\infty} |x|^{1-r+\varepsilon}|f'(x)| = +\infty$ *for any constant* $\varepsilon > 0$, *then* $f^{-1}(x) \in P_{2d}\left(\frac{1}{r}\right)$;

(7) *if* $f(x) \in P_{2d}(r)$ *and* $g(\lambda,\theta,t) \in P_{1d}(s)$ *with* $s > 0$ *and satisfying*

$$\lim_{|\lambda|\to\infty} \lambda^{-s+\varepsilon}|g| = \infty,$$

then $f(g(\lambda,\theta,t)) \in P_{1d}(rs)$.

Proof. The proofs are similar to that of [2,8,19].

Definition 2.6. *For* λ_0, *we denote by* $A(\lambda_0)$ *the subset of the space* $(\lambda,\theta,t) \in R^+ \times T^2$,

$$A(\lambda_0) = \{(\lambda,\theta,t) \in R^+ \times T^2 | \lambda \geq \lambda_0, (\theta,t) \in T^2\}.$$

§3. Main Propositions

In this section, we consider the reversible system of the following form:

$$\begin{aligned} \dot{\theta} &= h_1(\rho) + h_2(\rho,\theta) + h_3(\rho,\theta,t), \\ \dot{\rho} &= h_4(\rho,\theta) + h_5(\rho,\theta,t), \quad (\rho,\theta,t) \in R^+ \times S^1 \times S^1, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} h_2(\rho,-\theta) &= h_2(\rho,\theta), \quad h_3(\rho,-\theta,-t) = h_3(\rho,\theta,t), \\ h_4(\rho,-\theta) &= -h_4(\rho,\theta), \quad h_5(\rho,-\theta,-t) = -h_5(\rho,\theta,t). \end{aligned} \tag{3.2}$$

Lemma 3.1. *In system* (3.1), *we assume that*

(i) $h_1(\rho) \in P_{1d}(\alpha_1)/\cup_{a<\alpha_1} P_{1d}(a), h_i \in P_{1d}(\alpha_1), i = 2,3,4,5$;

(ii) $\alpha_4 \leq \min(1+\alpha_2, \alpha_5 + \alpha_1 - \alpha_3)$;

(iii) $\alpha_1 > \max(0,\alpha_2)$,

(iv) $\alpha_1 < 0, \alpha_1 + \alpha_5 < 1$.

Then there is a coordinate transformation $\Phi : (\rho,\theta) \mapsto (\mu,\phi)$ *such that for every* $\rho_0 > 0$ *large enough,* $A(\rho_+) \subset \Phi(A(\rho_0)) \subset A(\rho_-)$ *for some* $0 < \rho_- < \rho_0 < \rho_+$, *and the transformed system is of the form*

$$\dot{\phi} = \widehat{h}_1(\mu) + \widehat{h}_2(\mu,\phi) + \widehat{h}_3(\mu,\phi,t), \quad \dot{\mu} = \widehat{h}_4(\mu,\phi) + \widehat{h}_5(\mu,\phi,t), \tag{3.3}$$

where $\widehat{h}_1 \in P_{1d}(\alpha_1)/\cup_{a<\alpha_1} P_{1d}(a), \widehat{h}_2 \in P_{1d}(\alpha_2), \widehat{h}_3 \in P_{1d}(\alpha_3), \widehat{h}_4 \in P_{1d}(-\varepsilon_0)$ *and* $\widehat{h}_5 \in P_{1d}(\alpha_5)$.

Proof. For a $\rho > 0$ large enough, let

$$\Phi_1 : \rho_1 = \rho + U(\rho,\theta), \quad \theta_1 = \theta,$$

where

$$U(\rho,\theta) = -\frac{1}{h_1(\rho)} \int_0^\theta h_4(\rho,\theta)d\theta \in P_{1d}(\alpha_4 - \alpha_1).$$

From the Condition (ii), it follows that Φ_1 is a diffeomorphism for a ρ large enough. From the equality (3.2), we have

$$\int_0^{2\pi} h_4(\rho,\theta)d\theta = 0, \quad h_4(\rho,-\theta) = -h_4(\rho,\theta).$$

Hence, $U(\rho,\theta)$ is even and 2π-periodic in θ, Φ_1 is G^0-invariant. By Lemma 2.2, the transformed system of (3.1) under the Φ_1 is reversible with respect to G^0 and of the form

$$\dot{\theta}_1 = h_1^{(1)}(\rho_1) + h_2^{(1)}(\rho_1,\theta_1) + h_3^{(1)}(\rho_1,\theta_1,t), \quad \dot{\rho}_1 = h_4^{(1)}(\rho_1,\theta_1) + h_5^{(1)}(\rho_1,\theta_1,t), \tag{3.4}$$

where

$$h_1^{(1)}(\rho_1) = h_1(\rho_1) \in P_{1d}(\alpha_1)/\cup_{a\le\alpha_1} P_{1d}(a),$$
$$h_2^{(1)}(\rho_1,\theta_1) = h_1(\rho) - h_1(\rho_1) + h_2(\rho,\theta),$$
$$h_3^{(1)}(\rho_1,\theta_1,t) = h_3(\rho,\theta,t), \quad h_4^{(1)}(\rho_1,\theta_1) = \frac{\partial U}{\partial\theta}h_2 + \frac{\partial U}{\partial\rho}h_4,$$
$$h_5^{(1)}(\rho_1,\theta_1,t) = h_5(\rho,\theta,t) + \frac{\partial U}{\partial\theta}h_3 + \frac{\partial U}{\partial\rho}h_5.$$

Obviously, we have

$$h_2^{(1)}(\rho_1,-\theta_1) = h_2^{(1)}(\rho_1,\theta_1), \quad h_3^{(1)}(\rho_1,-\theta_1,-t) = h_3^{(1)}(\rho_1,\theta_1,t),$$
$$h_4^{(1)}(\rho_1,-\theta_1) = -h_4^{(1)}(\rho_1,\theta_1), \quad h_5^{(1)}(\rho_1,-\theta_1,-t) = -h_5^{(1)}(\rho_1,\theta_1,t).$$

Similarly to the proof in [2], one can prove

$$h_2^{(1)} \in P_{1d}(\alpha_2), \quad h_3^{(1)}(\rho_1,\theta_1,t) \in P_{1d}(\alpha_3), \quad h_4^{(1)} \in P_{1d}(\alpha_4 - \delta_1), \quad h_5^{(1)} \in P_{1d}(\alpha_5),$$

where $\delta_1 = \min(\alpha_1 - \alpha_2, 1 + \alpha_1 - \alpha_4) > 0$.

Application of this statement j times leads to $h_4^{(j)} \in P_{1d}(-\varepsilon_0)$ with $\varepsilon_0 > 0$. The proof of this lemma is complete.

Lemma 3.2. *There is a diffeomorphism Ψ:*

$$\lambda = \mu, \quad \psi = \phi + V(\mu,\phi),$$

which is G^0-invariant, such that under this diffeomorphism Equation (3.3) is transformed into the following form

$$\dot{\psi} = \widetilde{h}_1(\lambda) + \widetilde{h}_2(\lambda,\psi) + \widetilde{h}_3(\lambda,\psi,t), \quad \dot{\lambda} = \widetilde{h}_4(\lambda,\psi) + \widetilde{h}_5(\lambda,\psi,t), \tag{3.5}$$

where $\widetilde{h}_1 \in P_{1d}(\alpha_1)/\cup_{a<\alpha_1} P_{1d}(a), \widetilde{h}_2 \in P_1\left(-\frac{1}{2}\right), \widetilde{h}_3 \in P_{1d}(\alpha_3), \widetilde{h}_4 \in P_{1d}(-\varepsilon_0)$ and $\widetilde{h}_5 \in P_{1d}(\alpha_5)$.

Proof. For a $\mu > 0$ large enough, let

$$\Psi_1 : \mu_1 = \mu, \quad \phi_1 = \phi + V_1(\mu,\phi),$$

where

$$V_1(\mu,\phi) = -\frac{1}{\widehat{h}_1(\mu)}\int_0^\phi \left(\widehat{h}_2(\mu,\phi) - \frac{1}{2\pi}\int_0^{2\pi}\widehat{h}_2 d\phi\right) d\phi \in P_{1d}(\alpha_2 - \alpha_1).$$

From equality (3.2) and the condition (iii), it follows that V_1 is odd and 2π-periodic in ϕ and Ψ_1 is a diffeomorphism for $\mu \gg 1$ and so Ψ is G^0-invariant. The transformed system of (3.3) under the Ψ is reversible with respect to G^0 and of the form

$$\begin{aligned} \dot{\phi}_1 &= \widehat{h}_1^{(1)}(\mu_1) + \widehat{h}_2^{(2)}(\mu_1,\phi_1) + \widehat{h}_3^{(1)}(\mu_1,\phi_1,t), \\ \dot{\mu}_1 &= \widehat{h}_4^{(1)}(\mu_1,\phi_1) + \widehat{h}_5^{(1)}(\mu_1,\phi_1,t), \end{aligned} \tag{3.6}$$

where

$$\widehat{h}_1^{(1)}(\mu_1) = \widehat{h}_1(\mu_1) + \frac{1}{2\pi}\int_0^{2\pi} \widehat{h}_2 d\phi,$$
$$\widehat{h}_2^{(1)}(\mu_1, \phi_1) = \frac{\partial V_1}{\partial \phi}\widehat{h}_2 + \frac{\partial V_1}{\partial \mu}\widehat{h}_4,$$
$$\widehat{h}_3^{(1)}(\mu_1, \phi_1, t) = \frac{\partial V_1}{\partial \phi}\widehat{h}_3 + \widehat{H}_3 + \frac{\partial V_1}{\partial \mu}\widehat{h}_5,$$
$$\widehat{h}_4^{(1)}(\mu_1, \phi_1) = \widehat{h}_4, \quad \widehat{h}_5^{(1)}(\mu_1, \phi_1, t) = \widehat{h}_5.$$

It is easy to see the following equalities

$$\widehat{h}_2^{(1)}(\mu_1, -\phi_1) = \widehat{h}_2^{(1)}(\mu_1, \phi_1), \quad \widehat{h}_3^{(1)}(\mu_1, -\phi_1, -t) = \widehat{h}_3^{(1)}(\mu_1, \phi_1, t),$$
$$\widehat{h}_4^{(1)}(\mu - 1, -\phi_1) = -\widehat{h}_4^{(1)}(\mu_1, \phi_1), \quad \widehat{h}_5^{(1)}(\mu_1, -\phi_1, -t) = -\widehat{h}_5^{(1)}(\mu_1, \phi_1, t).$$

Similarly to the proofs in [2], we know that

$$\widehat{h}_{1d}^{(1)} \in P_{1d}(\alpha_1)/\cup_{a<\alpha_1} P_{1d}(a), \quad \widehat{h}_2^{(1)} \in P_{1d}(\alpha_2 - \eta_1),$$
$$\widehat{h}_3^{(1)} \in P_{1d}(\alpha_3), \quad \widehat{h}_4^{(1)} \in P_{1d}(-\varepsilon_0), \quad \widehat{h}_5^{(1)} \in P_{1d}(\alpha_5),$$

where $\eta_1 = \min(\alpha_1 - \alpha_2, 1 + \alpha_1 + \varepsilon_0) > 0$ is a constant.

Application of this statement j times leads to $h_2^{(j)} \in P_{1d}\left(-\frac{1}{2}\right)$, with $\varepsilon_1 > 0$. We have thus completed the proof of Lemma 3.2.

Proposition 3.1. *Under the conditions of Lemma 3.1, there is a diffeomorphism Ψ depending periodically on t, which is G^0-invariant, such that under this diffeomorphism Equation (3.5) is transformed into the following form*

$$\dot{\phi} = \mu + h_2(\mu, \phi) + h_3(\mu, \phi, t), \quad \dot{\lambda} = h_4(\mu, \phi) + h_5(\mu, \phi, t), \tag{3.7}$$

where $h_2 \in P_{1d}\left(-\frac{1}{2}\right)$, $h_3 \in P_1(\alpha_3)$, $h_4 \in P_{1d}(-\varepsilon_1)$ and $h_5 \in P_{1d}(-\varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$. The system (3.7) is reversible with respect to G^0.

Proof. Let $\Psi : \mu = \widetilde{h}_1(\lambda), \phi = \psi$. Then Equation (3.5) is transformed into

$$\begin{aligned} \dot{\phi} &= \mu + \widetilde{h}_2(\lambda, \psi) + \widetilde{h}_3(\lambda, \psi, t), \\ \dot{\mu} &= \widetilde{h}_1'(\lambda) \cdot \widetilde{h}_4(\lambda, \psi) + \widetilde{h}_1'(\lambda) \cdot \widetilde{h}_5(\lambda, \psi, t), \end{aligned} \tag{3.8}$$

where $(\lambda, \psi) = \Psi^{-1}(\mu, \phi)$. Let

$$h_2(\mu, \phi) = \widetilde{h}_2(\lambda, \psi), \quad h_3(\mu, \phi, t) = \widetilde{h}_3(\lambda, \psi, t),$$
$$h_4(\mu, \phi) = \widetilde{h}_1'(\lambda) \cdot \widetilde{h}_4(\lambda, \psi), \quad h_5(\mu, \phi, t) = \widetilde{h}_1'(\lambda) \cdot \widetilde{h}_5(\lambda, \psi, t).$$

From the results in [2], one can easily obtain the conclusions of our proposition.

§4. A Boundedness Theorem

In this section, we will consider the boundedness of solutions for the following equation

$$\ddot{x} + g(x) + f(x, \dot{x}) = p(t), \tag{4.1}$$

where $xg(x) > 0, |g(x)| \to \infty, g(x) \in C^\infty(R^1), f(x,\dot{x}) = f_1(x)f_2(\dot{x}) \in C^\infty(R^2)$ and $p(t) \in C^\infty(S^1)$.

It is equivalent to a system of the following form

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x,\dot{x}) + p(t). \tag{4.2}$$

Theorem 4.1. *Assume that the following conditions are true:*

(g1) $g(x) \in P_2(b)$, *with* $b > 1$, *and* $\lim\limits_{|x|\to\infty} |x|^{-b+\varepsilon}|g(x)| = +\infty$ *for every* $\varepsilon > 0$.

(g2) $g(-x) = -g(x), \liminf\limits_{x\to\infty} \left|\frac{2G(x)g'(x) - g^2(x)}{g^2(x)}\right| = \delta > 0$.

(f1) $f(-x,\dot{x}) = -f(x,\dot{x})$.

(f2) $f(x) \in P_2(r_1), f_2(y) \in P_2(r_2)$ *with* $\frac{r_1+1}{b+1} + \frac{r_2}{2} < 1$.

(p1) $p(-t) = -p(t)$.

Then each solution $x(t)$ *of system* (4.2) *is bounded, i.e.,* $x(t)$ *exists in* R^1, *and*

$$\sup_{t\in R^1} (|x(t)| + |\dot{x}(t)|) < +\infty.$$

The method of the proof of Theorem 4.1 is to find a diffeomorphism, which transforms (4.2) into a completely integrable system with small perturbations, such that all the assumptions of Proposition 3.1 are met for the transformed system. By Lemma 2.3, we can prove the existence of an infinitely many number of invariant curves which are diffeomorphic to a circle around the origin in (x,y)-plane. Hence, one can easily obtain the boundedness of solutions of system (4.2).

Proof of Theorem 4.1. From the conditions of Theorem 4.1, one can easily verify that system (4.2) is reversible with respect to $G^0 : (x,y) \mapsto (-x,y)$. In order to prove Theorem 4.1, we first prove the following lemmas in which we assume that the conditions of Theorem 4.1 are satisfied.

Lemma 4.1. *There is a transformation* $\Psi : (x,y) \mapsto (\theta,h)$, *which transforms the system* (4.2) *into the following form*

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{T(h)} - \frac{D_h x(\theta,h)}{T(h)}(f(x(\theta,h),\dot{y}(\theta,h)) - p(t)) \\ \frac{dh}{dt} &= \frac{D_\theta x(\theta,h)}{T(h)}(f(x(\theta,h),y(\theta,h)) - p(t)) \end{aligned} \tag{4.3}$$

for a sufficiently large h.

Proof. Since system (4.2) satisfies the Condition (g_1), $\frac{1}{2}y^2 + G(x) = h$ is a simple closed curve Γ_h surrounding the origin when h is sufficiently large and $G = \int_0^x g(s)ds$. The curve Γ_h intersects the x-axis at two points, say P_1 and P_2, with coordinates $(x(h),0)$ and $(-x(h),0)$, with $x(h) > 0$ and $G(x(h)) = h$.

Since the simple curves Γ_h are trajectories of system

$$\dot{x} = y, \qquad \dot{y} = -g(x), \tag{4.4}$$

thus each solution of (4.4) with large amplitude is periodic with period

$$T(h) = 4\int_0^{x(h)} (2(h - G(s)))^{-\frac{1}{2}} ds.$$

It is easy to see that $T(h) \in C^\infty(R^+)$ if $G(x)$ is a smooth function.

Let $\frac{1}{2}y^2 + G(x) = h$ for any point $(x,y) \in R^2$. Define $\Phi(x,y)$ by

$$\Phi(x,y) = \begin{cases} \int_0^x (2(h-G(s))^{-\frac{1}{2}} ds, & \text{for } x>0, y<0 \\ \frac{1}{4}\int_0^x (2(h-G(s))^{-\frac{1}{2}} ds, & \text{for } x>0, y>0 \\ \frac{1}{2}T(h) - \int_0^x (2(h-G(s))^{-\frac{1}{2}} ds, & \text{for } x<0, y>0 \\ \frac{3}{4}T(h) - \int_0^x (2(h-G(s))^{-\frac{1}{2}} ds, & \text{for } x<0, y<0. \end{cases}$$

Denote by Ψ_1 the following transformation :

$$\theta(x,y) = \frac{\Phi(x,y)}{T(h)}, \quad I(x,y) = 4\int_0^{x(h)} (2(h-G(s))^{\frac{1}{2}} ds. \tag{4.5}$$

It is easy to show that $\theta(-x,y) = 1 - \theta(x,y)$. Thus the inverse of $\Psi_1 : (\theta, I) \mapsto (x(\theta,I), y(\theta,I))$ is G^0-invariant with $G^0 : (\theta, I) \mapsto (-\theta, I)$. Moreover, Ψ_1 is symplectic.

Under the diffeomorphism Ψ_1, the autonomous system (4.4) is transformed into the following form

$$\frac{d\theta}{dt} = \frac{1}{T(h)}, \qquad \frac{dI}{dt} = 0. \tag{4.6}$$

Moreover, system (4.2) is transformed into a system of the form:

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{T(h)} - (f(x(\theta,h), y(\theta,h)) - p(t))D_I x(\theta,h), \\ \frac{dI}{dt} &= (f(x(\theta,h), y(\theta,h)) - p(t))D_\theta x(\theta,h), \end{aligned} \tag{4.7}$$

where $x(\theta,h)$ and $y(\theta,h)$ are determined by (4.5).

Since $dI = T(h)dh$, system (4.7) can be written in the form

$$\begin{aligned} \dot\theta &= \frac{1}{T(h)} - \frac{D_h x(\theta,h)}{T(h)}(f(x(\theta,h), y(\theta,h)) - p(t)), \\ \dot h &= (f(x(\theta,h), y(\theta,h)) - p(t))\frac{D_\theta x(\theta,h)}{T(h)}. \end{aligned} \tag{4.8}$$

The proof of Lemma 4.1 is completed.

Remark. Because system (4.2) is reversible with respect to $G : (x,y) \mapsto (-x,y)$, we can verify that system (4.8) is reversible with respect to

$$G^0 : (\theta, h) \mapsto (-\theta, h).$$

Lemma 4.2. $T(h) \in P_2\left(\frac{1}{b+1} - \frac{1}{2}\right), \frac{1}{T(h)} \in P_2\left(\frac{1}{2} - \frac{1}{b+1}\right)$.

Proof. See [19].

Lemma 4.3. *Under the assumptions of Theorem 4.1, we have*

$$\lim_{h\to\infty} h^{-\frac{1}{b+1}+a(i-1,j)-\varepsilon}(D_\theta^i)(D_h^j)x(\theta,h) = 0,$$

$$\lim_{h\to\infty} h^{-\frac{1}{2}+a(i-1,j)-\varepsilon}(D_\theta^i)(D_h^j)y(\theta,h) = 0,$$

for any constant $\varepsilon > 0$ *and* $i, j \in Z^+ \cup \{0\}$, *where* $a(i,j) = j - \max\left\{\frac{i+j-b-1}{b+1}, 0\right\}$.
Proof. See [19].
Lemma 4.4.

$$f(x(\theta,h), y(\theta,h))\frac{D_h x(\theta,h)}{T(h)} \in P_{1d_0}\left(r - \frac{1}{2}\right)$$
$$f(x(\theta,h), y(\theta,h))\frac{D_\theta X(\theta,h)}{T(h)} \in P_{1d_0}\left(r + \frac{1}{2}\right),$$

with $d_0 = \frac{1}{b+1}$ *and* $r < 1 - d_0$.
Proof. This follows easily from Lemma 2.4 and Lemma 2.5.

From Lemmas 4.1–4.4, we know that all the assumptions of Proposition 3.1 are met for system (4.3). Thus, by Proposition 3.1, there is a transformation $T : (\theta, h) \mapsto (\phi, \lambda)$ which transforms system (4.3) into the following form:

$$\dot{\phi} = h_1(\lambda) + h_2(\lambda,\phi) + h_3(\lambda,\phi,t), \quad \dot{\lambda} = h_4(\lambda,\phi) + h_5(\lambda,\phi,t), \tag{4.9}$$

where $h_1 \in P_1\left(\frac{1}{2} - \frac{1}{b+1}\right), h_2 \in P_{1d_0}\left(-\frac{1}{2}\right), h_3 \in P_{2d_0}(-M), h_4 \in P_{1d_0}\left(\frac{1}{2}\right)$ and $h_5 \in P_{1d_0}(-M)$. Moreover, system (4.9) is reversible with respect to $G_1 : (\lambda, \phi) \mapsto (\lambda, -\phi)$.

In order to apply Lemma 2.3, we define a diffeomorphism $(\lambda, \phi) \mapsto (\mu, \phi)$ by

$$\mu = h_1(\lambda), \quad \phi = \phi.$$

Then, under this diffeomorphism, system (4.9) is transformed into the form

$$\dot{\phi} = \mu + f_1(\mu,\phi) + f_2(\mu,\phi,t), \dot{\mu} = g_1(\mu,\phi) + g_2(\mu,\phi,t), \tag{4.10}$$

where $f_1 \in P_{1d_0}(-\delta_1), f_2 \in P_{1d_0}(-M_1), g_1 \in P_{1d_0}(-\delta_2)$ and $g_2 \in P_{1d_0}(-M_2)$ with $\delta_1, \delta_2 > 0$, and sufficiently large M_1, M_2.
Lemma 4.5. *The Poincare mapping* Φ^1 *of system* (4.10) *is of the form*

$$\phi_1 = \phi + 2\pi\mu + \widehat{f}(\mu,\phi), \quad \mu_1 = \mu + \widehat{g}(\mu,\phi). \tag{4.11}$$

where $\widehat{f} \in P_{1d_0}(-M_1)$ *and* $\widehat{g} \in P_{1d_0}(-M_2)$ *with sufficiently large* M_1, M_2.
Proof. See [2].

Similarly to the proofs in [2] and in [8], for every point $P = (x, y) \in R^2$, we can find an invariant curve of the Poincare mapping of Equation (4.1) such that the point P is in its interior. This curve goes around the origin. Every solution is therefore confined in the interior of some time periodic in the space $(x, \dot{x}, t)$ generated by some invariant curve in the $(x, \dot{x})$-plane and hence is bounded. This completes the proof of Theorem 4.1.
Example. We consider the following differential equation:

$$\ddot{x} + \sum_{i=1}^{m} a_i x^{2i+1}\dot{x} + \sum_{j=0}^{n} b_j x^{2j+1}\ln(1+x^2) = \sin(t),$$

where $b_n > 0$, and $n \geq 2(m+1)$.

From Theorem 4.1, it follows that every solution of the above equation is bounded for all $t \in R^1$, i.e.,

$$\sup_{t\in R^1}(|x(t)| + |\dot{x}(t)|) < \infty.$$

References

[1] Arnold, V. I., *Proof of a theorem of A.N.Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian,* Russ. Math. Surv., **18**(1963), 9–36.

[2] Dieckerhoff, R. & Zehnder, E., *Boundedness of solutions via the twist-theorem,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **14**(1987), 79–95.

[3] Graef, J. R., *On the generalized Lienard equation with negative damping,* J. Differential Equations, **12**(1972), 34–62.

[4] Kolmogorov, A.N., *On the conservation of quasi-periodic motions for a small chang in the Hamiltonian function,* DAN **98**, 4(1954), 527–530.

[5] Levi, M., *Quasiperiodic motions in superquadratic time-periodic potentials,* Commun. Math. Phys., Vol.134, No.1, 1991.

[6] Levinson, N., *On the existence of periodic solutions for second order differential equations with a forcing term,* J.Math. Phys., **32**(1943), 41–48.

[7] Littlewood, J. E., *Unbounded solutions of equation $\ddot{y}+g(y)=p(t)$ with $p(t)$ periodic and bounded, and $g(y)/y \to \infty$ as $y=\infty$,* J. London Math. Soc., **41**(1966), 497–507.

[8] Liu, B., *Boundedness for solutions of nonlinear Hill's equations with periodic forcing terms via Moser's twist theorem,* J. Differential Equations, **2**(1989), 304–315.

[9] Liu, B., *Boundedness for solutions of nonlinear periodic differential equations via Moser's twist theorem,* Acta Math. Sinica, New Series, **8**(1992), 91–99.

[10] Liu, B., *Boundedness for solutions of a non-conservative system,* Science in China, **34**:9 (1991).

[11] Long, Y., *An unbounded solution of a superlinear Duffing's equations,* Acta Math. Sinica, **7**:4 (1991), 360–369.

[12] Morris, G. R., *A case of boundedness in Littlewood's problem on oscillatory differential equations,* Bll. Astral. Math. Soc., **14**(1976), 71–93.

[13] Moser, J., *On invariant curves of area-preserving mappings of annulus,* Nachr. Akad. Wiss Gottingen Math. -Phys. kl. II(1962), 1–20.

[14] Moser, J., Stable and Random Motions in Dynamical Systems, Ann. of Math. Studies, Number 77, Princeton Press, N. J. 1973.

[15] Reuter, G. E.H., *A boundedness theorem for nonlinear differential equations of the second order,* Proc. Cambridge Phil. Soc., **47**(1951), 49–54.

[16] Sevryuk, M., *Reversible Systems,* Lect. Notes Math., **1211**(1986).

[17] You, J., *Invariant tori and Lagrange stability of pendulum-type equations,* J. Differential Equations, **85**:1 (1990), 54–65.

[18] You, J., *Boundedness for solutions of superlinear Duffing's equations via Moser's twist theorem,* Science in China, **35**:4 (1992).

[19] You, J., *Boundedness for solutions of non-conservative pendulum-type equations,* Chinese bulletin of Science, **36**:21 (1991).

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